Multiparametric linear programming: support set and optimal partition invariancy

Milan Hladík
Charles University, Faculty of Mathematics and Physics,
Malostranské nám. 25, 118 00, Prague, Czech Republic,
e-mail: milan.hladik@matfyz.cz.

Abstract

This paper reflects the renascence of sensitivity and parametric analysis in linear programming and extends single-parametric results to the case when there are multiple parameters in the objective function and in the right-hand side of equations. Multiparametric approach enables us to study more complex perturbation occurring in linear programs than the simpler sensitivity analysis does. Interior point methods in linear optimization suppressed the traditional parametric analysis based on preserving optimal basis and gave rise to new ones. In this paper there is presented a description of the set of admissible parameters under so called support set invariancy and optimal partition invariancy and compared with the classical optimal basis concept. Such a description can be used e.g. for tolerance analysis.

Keywords: *Linear programming, sensitivity analysis, tolerance analysis, parametric optimization.*

1 Introduction

Sensitivity analysis and parametric programming are basic tools for studying perturbations in optimization problems and they are still in focus of research even for linear programming problems [2, 3, 4, 9, 10, 11, 16, 19]. Perturbations occur due to measuring errors or just to answer managerial questions “What if ...”. The more number of parameters in the model the more amount of perturbations we are able to deal with. In the past, only few authors were concerned with multiparametric programming, among others [2, 4, 6, 7, 13, 17]. So far, various kinds of invariancies [3, 8, 10, 11] were used mainly in single-parametric analysis. Our aim is to extend them to the multiparametric case.

Let us consider the linear program

$$\min\ c^T x \ \text{ subject to }\ Ax = b,\ x \geq 0 \tag{P}$$

and its dual in the form

$$\max\ b^T y \ \text{ subject to }\ A^T y \leq c, \tag{D}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. By

$$\mathcal{P} := \{x \mid Ax = b,\ x \geq 0\},$$
$$\mathcal{D} := \{y \mid A^T y \leq c\}$$

we denote the feasible sets of the primal and dual problem, respectively, and by $\mathcal{P}^*$ and $\mathcal{D}^*$ the corresponding optimal solution sets. The support set of a nonnegative vector x is defined as

$$\sigma(x) := \{i \mid x_i > 0\}.$$

The index set $\{1, \dots, n\}$ can be disjointly partitioned into two subsets

$$\mathcal{B} := \{i \mid x_i > 0 \text{ for some } x \in \mathcal{P}^*\},$$
$$\mathcal{N} := \{i \mid c_i - A^T_{\bullet i} y > 0 \text{ for some } y \in \mathcal{D}^*\},$$

which is known as *optimal partition* [1, 3, 8, 9, 10, 11, 12] and is unique. A primal feasible vector x^0 and a dual feasible vector y^0 are called *a pair of strictly complementary solutions* if they satisfy $x^0 + c - A^T y^0 > 0$. Such a pair always exists; see [8, 11] and references there. Clearly, $\sigma(x^0) = \mathcal{B}$.

Let μ be a k-dimensional vector of parameters. We introduce the general parametrization of the primal problem by

$$\min\ c(\mu)^T x \ \text{ subject to }\ A(\mu)x = b(\mu),\ x \geq 0, \tag{P$_\mu$}$$

that is, the matrix $A(\mu)$ and vectors $b(\mu)$, $c(\mu)$ depend on the vector of parameters μ. Its dual is

$$\min\ b(\mu)^T y \ \text{ subject to } \ A(\mu)^T y \leq c(\mu). \qquad (\mathrm{D}_\mu)$$

Let μ^0 be fixed and let x^* and y^* be optimal solutions of (P_{μ^0}) and (D_{μ^0}), respectively. The optimal partition corresponding to x^* is denoted by $(\mathcal{B}, \mathcal{N})$.

We follow Hadigheh et al. [3, 11] and categorize parametric analysis in three types of invariancies:

- **Optimal basis invariancy:** Suppose that x^* is a basic optimal solution with the basis B. We want to compute the set $\Upsilon_B(x^*)$ of parameters μ for which B remains the optimal basis.

- **Support set invariancy:** The aim is to compute the set of parameters μ such that there is an optimal solution x^*_μ of (P_μ) for which $\sigma(x^*_\mu) = \sigma(x^*)$. Moreover, more general cases can be considered:

  1. $\sigma(x^*_\mu) = \sigma(x^*)$,
  2. $\sigma(x^*_\mu) \supseteq \sigma(x^*)$,
  3. $\sigma(x^*_\mu) \subseteq \sigma(x^*)$.

  The corresponding sets of parameters are denoted by $\Upsilon_1(x^*)$, $\Upsilon_2(x^*)$ and $\Upsilon_3(x^*)$, respectively. We refer to Hadigheh and Terlaky [11] for the economic interpretation of these cases.

- **Optimal partition invariancy:** We want to compute the set Υ_p of parameters μ for which the problem (P_μ) has the optimal partition $(\mathcal{B}, \mathcal{N})$.

The sets $\Upsilon_B(x^*)$, $\Upsilon_1(x^*)$, $\Upsilon_2(x^*)$, $\Upsilon_3(x^*)$, and Υ_p are referred to as *critical regions*, corresponding to particular invariancies.

Optimal basis invariancy [5, 6, 7, 17] is used when solving linear programs by the simplex method. But interior point methods (solving linear programs in polynomial time) yield generally nonbasic solutions and new kinds of invariancies had to be developed. These include optimal partition invariancy [1, 2, 8, 9, 11, 12] and recently appeared support set invariancy [3, 10, 11]. Both of them overcome the main drawback of optimal basis invariancy—possible degeneracy of the optimal solution.

We focus on two leading types of parametrization: vector of parameters in the objective function and in the right-hand side of equations. For both types we propose a description for all mentioned types of invariancies and show relationships between them.

Notation

$A_{i\bullet}$	the i-th row of a matrix A
$A_{\bullet j}$	the j-th column of a matrix A
$u \lneqq v$	at once $u \leq v$ and $u \neq v$
A_P	submatrix of A consisting of the columns indexed by P
A_P^T	transposition of A_P, i.e. $(A_P)^T$
$conv\,\mathcal{M}$	convex hull of a set $\mathcal{M}$
$cl\,\mathcal{M}$	closure of a set $\mathcal{M}$
$rel\,int\,\mathcal{M}$	relative interior of a set $\mathcal{M}$
$\mathcal{L}^{\perp}$	orthogonal complement of a linear space $\mathcal{L}$

2 Objective function perturbation

Let us consider the special case of parametrization, when parameters are situated in the objective function, i.e.

$$\min\ \lambda^T x \ \text{ subject to } \ Ax = b,\ x \geq 0, \tag{P$_\lambda$}$$

where λ is the n-dimensional vector of parameters. The dual problem is

$$\max\ b^T y \ \text{ subject to } \ A^T y \leq \lambda. \tag{D$_\lambda$}$$

Let $\lambda^0 \in \mathbb{R}^n$ be fixed and x^* be an optimal solution of (P$_{\lambda^0}$). In the following, we give a description for all proposed kinds of invariancies with the starting point λ^0.

2.1 Optimal basis invariancy

Suppose that x^* is an optimal basic solution and let B be the corresponding optimal basis (which is not unique in general). Denote by $N = \{1, \ldots, n\}\backslash B$ the set of nonbasic indices.

It is well known [5, 7, 12, 17] that the critical region of optimal basis invariancy has the description

$$\Upsilon_B(x^*) = \{\lambda \in \mathbb{R}^n \mid \lambda_N^T - \lambda_B^T A_B^{-1} A_N \geq 0\}. \tag{1}$$

The optimal value function is $\lambda^T x^*$ on $\Upsilon_B(x^*)$.

2.2 Support set invariancy

Recall that x^* is any optimal solution of (P_{λ^0}). Let $P := \sigma(x^*)$ and $Z := \{1, \ldots, n\} \setminus P$. Firstly we consider the first case of support set invariancy and derive a description of the set $\Upsilon_1(x^*)$.

Theorem 1. *Let h_i, $i \in I$, be a basis of the lineality space $\mathcal{L}_Z := \{x \mid Ax = 0, x_Z = 0\}$ and let g_j, $j \in J$, be all extremal directions of the convex polyhedral cone $\{x \mid Ax = 0, x_Z \geq 0\} \cap \mathcal{L}_Z^{\perp}$. Then*

$$\Upsilon_1(x^*) = \big\{\lambda \in \mathbb{R}^n \mid h_i^T \lambda = 0\ \forall i \in I,\ g_j^T \lambda \geq 0\ \forall j \in J\big\}. \tag{2}$$

Proof. The tangent cone to the primal feasible set $\mathcal{P} = \{x \mid Ax = b, x \geq 0\}$ at the point x^* is $\mathcal{T}(x^*) = \{x \mid Ax = 0, x_Z \geq 0\}$. Moreover, for each primal feasible point x satisfying $\sigma(x) = \sigma(x^*)$ the corresponding tangent cone is equal to $\mathcal{T}(x^*)$. According to the theory of normal (polar) cones [13, 14, 15] the vector $(-\lambda)$ is included in the normal cone to $\mathcal{T}(x^*)$ at the point x^*, from which the description (2) of the region $\Upsilon_1(x^*)$ follows. $\square$

Notice that if x^* is nondegenerate optimal basic solution, then $\Upsilon_B(x^*) = \Upsilon_1(x^*)$. Without the condition of nondegeneracy, we have only the inclusion $\Upsilon_B(x^*) \subseteq \Upsilon_1(x^*)$.

A basis of the lineality space $\mathcal{L}_Z$ can be found efficiently. But computation of all extremal directions of the convex polyhedral cone is a hard (exponential) problem; the methods which are used includes simplex algorithm and Nožička method [13] (via constructing a convex basis of a cone). Let us note that $\Upsilon_1(x^*)$ has an alternative description by means of generators (instead of inequalities) as the convex hull by

$$\Upsilon_1(x^*) = conv\,\{\pm A_{i\bullet}\ \forall i,\ e_i\ \forall i \in Z\},$$

where e_i stands for the i-th unit vector. Although this characterization of $\Upsilon_1(x^*)$ does not require any additional computations, the formula (2) is much more useful from the practical viewpoint.

Theorem 2. *We have $\Upsilon_2(x^*) = \Upsilon_1(x^*)$.*

Proof. The inclusion $\Upsilon_2(x^*) \supseteq \Upsilon_1(x^*)$ holds trivially from the definition.

Let $\lambda \in \mathbb{R}^n$ and let x be an optimal solution of (P_λ) such that $\sigma(x) \supseteq \sigma(x^*)$. Then there exists an optimal solution y of (D_λ) and the complementary slackness condition $x^T(\lambda - A^T y) = 0$ holds. Clearly, the equation $x^{*T}(\lambda - A^T y) = 0$ holds as well and hence x^* is also optimal solution of (P_λ). This proves $\Upsilon_2(x^*) \subseteq \Upsilon_1(x^*)$. $\square$

The optimal value function is $\lambda^T x^*$ on $\Upsilon_1(x^*) = \Upsilon_2(x^*)$ and the optimal solution is x^*.

We characterized two cases of support set invariancy. The remaining is the most complex one. Let $\lambda \in \mathbb{R}^n$ and let x be an optimal solution of (P_λ) such that $\sigma(x) \subseteq \sigma(x^*)$. Then there also exists a basic optimal solution x^0 of (P_λ) with property $\sigma(x^0) \subseteq \sigma(x^*)$. The point x^0 is situated in the convex polyhedral set $M_Z := \{x \mid Ax = b, x_Z = 0, x_P \geq 0\}$. Conversely, any optimal solution x of (P_λ) such that $x \in M_Z$ satisfies $\sigma(x) \subseteq \sigma(x^*)$. Therefore, it is sufficient to compute all vertices of M_Z and $\Upsilon_3(x^*)$ is formed by the union of the corresponding critical regions. More formally, let x^i, $i \in V$, be all vertices of M_Z. Then $\Upsilon_1(x^i)$ are the corresponding critical regions of the first case support set invariancy. Therefore,

$$\Upsilon_3(x^*) = \bigcup_{i \in V} \Upsilon_1(x^i).$$

In particular, if x^* is a basic solution, then $\Upsilon_3(x^*) = \Upsilon_1(x^*)$. Nevertheless, the set $\Upsilon_3(x^*)$ is nonconvex in general (see Example 1 below) and the optimal value function is not linear (even for single-parametric programming; cf. [1, 3, 10, 11]).

Summing up the previous theorems and remarks, we conclude:

$$\Upsilon_1(x^*) = \Upsilon_2(x^*) \subseteq \Upsilon_3(x^*).$$

2.3 Optimal partition invariancy

Let $(\mathcal{B}, \mathcal{N})$ be the optimal partition of (P_{λ^0}) and suppose that x^* and y^* is a pair of strictly complementary solutions. Then $P = \mathcal{B}$ and the critical region Υ_p can be easily computed from the equation (3).

Theorem 3. *We have*

$$\begin{aligned}\Upsilon_p &= rel\, int \Upsilon_1(x^*) \\ &= \big\{\lambda \in \mathbb{R}^n \mid h_i^T\lambda = 0 \ \forall i \in I,\ g_j^T\lambda > 0 \ \forall j \in J\big\},\end{aligned} \tag{3}$$

where $h_i,\ i \in I$, and $g_j,\ j \in J$, are defined like in Theorem 1.

Proof. We want to identify the set of λ, for which $(\mathcal{B}, \mathcal{N})$ is the optimal partition. Looking on the dual problem, the vector y^* is a solution of the system

$$A_{\mathcal{B}}^T y = \lambda_{\mathcal{B}},\ A_{\mathcal{N}}^T y < \lambda_{\mathcal{N}}. \tag{4}$$

Conversely, every solution y of this system provides a pair of strictly complementary solutions x^* and y. The system (4) is solvable for a fixed $\lambda \in \mathbb{R}^n$ if and only if the linear problem

$$\max\ 0^T y \ \text{ subject to } \ A_{\mathcal{B}}^T y = \lambda_{\mathcal{B}},\ A_{\mathcal{N}}^T y \leq \lambda_{\mathcal{N}} - \varepsilon$$

has an optimal solution for a sufficiently small vector $\varepsilon > 0$. From duality theory in linear programming it is equivalent to the optimality of the dual problem

$$\min\ \lambda_{\mathcal{B}}^T x_{\mathcal{B}} + (\lambda_{\mathcal{N}} - \varepsilon)^T x_{\mathcal{N}} \ \text{ subject to } \ Ax = 0, x_{\mathcal{N}} \geq 0.$$

Proceeding like in the proof of Theorem 1, we obtain that λ fulfills the system

$$\begin{aligned}\lambda^T h_i - \varepsilon^T (h_i)_{\mathcal{N}} &= 0 \ \forall i \in I, \\ \lambda^T g_j - \varepsilon^T (g_j)_{\mathcal{N}} &\geq 0 \ \forall j \in J.\end{aligned}$$

Since $\varepsilon^T (h_i)_{\mathcal{N}} = 0$ for all $i \in I$ and $\varepsilon^T (g_j)_{\mathcal{N}} > 0$ for all $j \in J$, we have equivalently

$$\begin{aligned}h_i^T \lambda &= 0 \ \forall i \in I, \\ g_j^T \lambda &> 0 \ \forall j \in J.\end{aligned}$$

□

Let us remark that the knowledge of x^* is not necessary for computing Υ_p, the only what we need is the optimal partition $(\mathcal{B}, \mathcal{N})$.

For $\lambda \in \Upsilon_p$ the optimal value of (P$_\lambda$) is equal to $\lambda^T x^*$ and hence is linear on Υ_p. However, Υ_p is not generally the maximal subset of $\mathbb{R}^n$, where the objective function is linear; for a single-parametric case see e.g. [1, 3, 10, 11]. This happens especially if the solution x^* is nonbasic, and then the region Υ_p is not fully dimensional. One may wish to enlarge Υ_p to the larger (n-dimensional) set. One possible way is to find a basic optimal solution x^0 and replace x^* by it. Then $cl\,\Upsilon_p \subsetneqq \Upsilon_1(x^0)$ and we obtain a larger region.

All vertices of

$$\{x \mid A_P x_P = b,\ x_P \geq 0,\ x_Z = 0\} \tag{5}$$

are basic optimal solutions. We are seeking for only one of them, so we can use some linear programming solver of polynomial complexity (e.g. one of interior point methods) on the problem

$$\min\ r^T x \ \text{ subject to } \ (5),$$

where r is a random vector. The resulting solution is almost surely basic. Another polynomial procedure to find a vertex of (5) is the following algorithm.

Algorithm 1. (Finding a basic solution)

1. Check whether x^* is a basic solution: If the columns of A_P are linearly independent, then stop; x^* is a basic solution. Otherwise go to step 2.

2. By a simple Gauss elimination to a reduced row echelon form of A_P find two disjoint index sets $P^+, P^- \subseteq P$ (at least one nonempty) and a vector $z > 0$ such that

$$A_{P^+} z_{P^+} - A_{P^-} z_{P^-} = 0. \tag{6}$$

   Without loss of generality assume that $P^+ \neq \emptyset$ (otherwise take into account P^-). Define

$$k := \operatorname*{argmin}_{j \in P^+} \frac{x_j^*}{z_j}$$

   and set new values of components of x^* by

$$x_i^* := \begin{cases} x_i^* & i \notin P^+ \cup P^-, \\ x_i^* - \frac{x_k^*}{z_k} z_i & i \in P^+, \\ x_i^* + \frac{x_k^*}{z_k} z_i & i \in P^-. \end{cases}$$

   Set $P := \sigma(x^*)$ and go to step 1.

The algorithm finishes after at most $|P|-1$ iterations, since in each iteration at least the index k is extracted from P. Denote $P_k^+ := P^+ \setminus \{k\}$ and $P^0 := P \setminus \big(P^+ \cup P^-\big)$. From (6) it follows that

$$A_{\bullet k} = -\frac{1}{z_k}\big(A_{P_k^+} z_{P_k^+} - A_{P^-} z_{P^-}\big).$$

If we substitute $A_{\bullet k}$ to (5), we obtain

$$-\frac{1}{z_k}\big(A_{P_k^+} z_{P_k^+} - A_{P^-} z_{P^-}\big)x_k^* + A_{P_k^+} x^*_{P_k^+} + A_{P^-} x^*_{P^-} + A_{P^0} x^*_{P^0} = b,$$

or

$$A_{P_k^+}\left(x^*_{P_k^+} - \frac{x_k^*}{z_k} z_{P_k^+}\right) + A_{P^-}\left(x^*_{P^-} + \frac{x_k^*}{z_k} z_{P^-}\right) + A_{P^0} x^*_{P^0} = b.$$

Therefore in each step the new version of x^* satisfies (5) and due to the definition of k is also nonnegative.

2.4 Illustrative example

Example 1. Consider the problem

$$\min\ \lambda^T x \ \text{ subject to } \ 6x_1 + 3x_2 + 2x_3 = 6,\ x_1, x_2, x_3 \geq 0.$$

and the initial value $\lambda^0 = (3,2,1)^T$. The optimal solution set is a segment with endpoints $x^1 = (1,0,0)^T$ and $x^2 = (0,0,3)^T$; see Figure 1.

1. (Basis invariancy) For the basic solution x^1, the formula (1) yields the critical region

   $$\Upsilon_B(x^1) = \{\lambda \in \mathbb{R}^3 \mid -\lambda_1 + 2\lambda_2 \geq 0,\ -\lambda_1 + 3\lambda_3 \geq 0\}$$

   and the optimal value function is $\lambda^T x^1 = \lambda_1$. For the basic solution x^2 we obtain

   $$\Upsilon_B(x^2) = \{\lambda \in \mathbb{R}^3 \mid \lambda_1 - 3\lambda_3 \geq 0,\ 2\lambda_2 - 3\lambda_3 \geq 0\},$$

   the optimal value function is $\lambda^T x^2 = 3\lambda_3$.

2. (Support set invariancy) For any optimal basic solution, the critical region does not differ from the previous one, since each basic solution is nondegenerate.

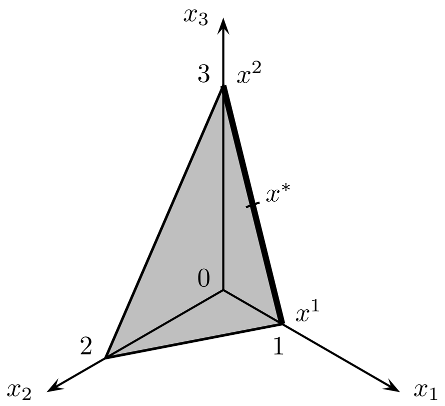

Figure 1: (Example 1) the feasible set (gray color) and the optimal set (bold line).

Choose a nonbasic solution, for instance $x^* = (\frac{1}{2}, 0, \frac{3}{2})^T$. Thus $P = (1, 3)$, $Z = (2)$ and we compute $\Upsilon_1(x^*)$ by the virtue of Theorem 1. The lineality space

$$\mathcal{L}_Z = \{x \mid 6x_1 + 3x_2 + 2x_3 = 0,\ x_2 = 0\}$$

represents a line the direction of which is $h_1 = (1, 0, -3)^T$. The convex polyhedral cone

$$\{x \mid Ax = 0, x_Z \geq 0\} \cap \mathcal{L}_Z^{\perp} = \\ \{x \mid 6x_1 + 3x_2 + 2x_3 = 0,\ x_1 - 3x_3 = 0,\ x_2 \geq 0\}$$

represents a ray in the direction $g_1 = (-9, 20, -3)^T$. Hence

$$\Upsilon_1(x^*) = \{\lambda \in \mathbb{R}^3 \mid \lambda_1 - 3\lambda_3 = 0,\ -9\lambda_1 + 20\lambda_2 - 3\lambda_3 \geq 0\}.$$

Now we turn our attention to $\Upsilon_3(x^*)$. First we compute all vertices of the convex polyhedral set

$$\begin{aligned} M_Z &= \{x \mid Ax = b, x_Z = 0, x_P \geq 0\} \\ &= \{x \mid 6x_1 + 3x_2 + 2x_3 = 0,\ x_2 = 0,\ x_1, x_3 \geq 0\}. \end{aligned}$$

These vertices are x^1 and x^2, and therefore

$$\begin{aligned}\Upsilon_3(x^*) &= \Upsilon_B(x^1) \cup \Upsilon_B(x^2)\\ &= \{\lambda \mid -\lambda_1 + 2\lambda_2 \geq 0,\ -\lambda_1 + 3\lambda_3 \geq 0\} \cup\\ &\quad\ \{\lambda \mid \lambda_1 - 3\lambda_3 \geq 0,\ 2\lambda_2 - 3\lambda_3 \geq 0\}\\ &= \{\lambda \mid -\lambda_1 + 2\lambda_2 \geq 0\} \cup \{\lambda \mid 2\lambda_2 - 3\lambda_3 \geq 0\}.\end{aligned}$$

Note that for the basic solution x^1 we get $\Upsilon_3(x^1) = \Upsilon_B(x^1)$ and likewise for x^2.

3. (Optimal partition invariancy) Optimal partition is $\mathcal{B} = (1, 3)$, $\mathcal{N} = (2)$, and hence $\Upsilon_p = \mathit{rel\,int}\,\Upsilon_1(x^*) = \{\lambda \mid \lambda_1 - 3\lambda_3 = 0,\ -9\lambda_1 + 20\lambda_2 - 3\lambda_3 > 0\}$. The optimal value function is $\lambda^T x^* = \frac{1}{2}\lambda_1 + \frac{3}{2}\lambda_3$.

3 Right-hand side perturbation

In this section we consider another special case of parametrization, when parameters are situated in the right-hand side of equations, i.e.

$$\min\ c^T x \ \text{ subject to } \ Ax = \delta,\ x \geq 0, \tag{P$_\delta$}$$

where δ is the m-dimensional vector of parameters. Its dual is

$$\max\ \delta^T y \ \text{ subject to } \ A^T y \leq c. \tag{D$_\delta$}$$

Let $\delta^0 \in \mathbb{R}^m$ be arbitrarily chosen and fixed. Let x^* be an optimal solution of (P$_{\delta^0}$) and let y^* be an optimal solution of the dual problem (D$_{\delta^0}$). We discuss all kinds of invariancies with the starting point δ^0.

3.1 Optimal basis invariancy

Suppose that x^* is an optimal basic solution and let B be a corresponding optimal basis. Denote by $N = \{1, \ldots, n\} \setminus B$ the set of nonbasic indices. It is well known [5, 12, 17] that the optimal basis B remains optimal on the set

$$\Upsilon_B(x^*) = \{\delta \in \mathbb{R}^n \mid A_B^{-1}\delta \geq 0\}. \tag{7}$$

The optimal value of (P$_\delta$) is $c_B^T A_B^{-1}\delta$ and the optimal solution is the vector $x(\delta)$ for which $x(\delta)_B = A_B^{-1}\delta$, $x(\delta)_N = 0$ holds.

3.2 Support set invariancy

Let $P = \sigma(x^*)$ and $Z = \{1, \ldots, n\} \setminus P$. First we give a description of the critical region $\Upsilon_3(x^*)$.

Theorem 4. *Let h_i, $i \in I$, be a basis of the lineality space $\mathcal{L}_P := \{y \mid A_P^T y = 0\}$ and let g_j, $j \in J$, be all extremal directions of the convex polyhedral cone $\{y \mid A_P^T y \leq 0\} \cap \mathcal{L}_P^{\perp}$. Then*

$$\Upsilon_3(x^*) = \{\delta \in \mathbb{R}^m \mid h_i^T \delta = 0 \ \forall i \in I,\ g_j^T \delta \leq 0 \ \forall j \in J\}.$$

Proof. First we show that it is necessary and sufficient to consider only such vectors $\delta \in \mathbb{R}^m$ for which the system

$$A_P x_P = \delta,\ x_P \geq 0 \tag{8}$$

is solvable. If $\delta \in \Upsilon_3(x^*)$, then there is an optimal solution x of (P_δ) such that $\sigma(x) \subseteq \sigma(x^*)$. Hence x_P fulfills (8). On the other hand, if some x_P fulfills (8), then the vector $\widetilde{x}$ defined as $\widetilde{x}_P = x_P$, $\widetilde{x}_Z = 0$ forms an optimal solution of (P_δ), because of the complementary slackness condition $\widetilde{x}^T(y^* - A^T y) = 0$.

The system (8) is solvable if and only if the linear program

$$\min\ 0^T x_P \ \text{ subject to } \ A_P x_P = \delta,\ x_P \geq 0$$

has an optimal solution. According to duality theorems in linear programming it is equivalent to the optimality of the dual problem

$$\max\ \delta^T y \ \text{ subject to } \ A_P^T y \leq 0.$$

This problem has an optimal solution if and only if the vector δ lies in the normal cone to the feasible set $\{y \mid A_P^T y \leq 0\}$ at the point 0. According to the theory of normal cones [13, 14, 15] the description of $\Upsilon_3(x^*)$ follows. $\square$

When x^* is a nondegenerate optimal basis solution, then $\Upsilon_B(x^*) = \Upsilon_3(x^*)$. Without the condition of nondegeneracy, we have only $\Upsilon_B(x^*) \supseteq \Upsilon_3(x^*)$.

Like in Section 2.2 we note that $\Upsilon_3(x^*)$ has an alternative description by means of the convex hull generators as follows

$$\Upsilon_3(x^*) = conv\,\{A_{\bullet i}\ \forall i \in P\},$$

The following theorem characterizes the critical region $\Upsilon_1(x^*)$.

Theorem 5. *We have*

$$\begin{aligned}\Upsilon_1(x^*) &= rel\,int\,\Upsilon_3(x^*) \\ &= \{\delta \in \mathbb{R}^m \mid h_i^T\delta = 0\ \forall i \in I,\ g_j^T\delta < 0\ \forall j \in J\},\end{aligned} \tag{9}$$

where $h_i,\ i \in I$, and $g_j,\ j \in J$, are defined like in Theorem 4.

Proof. For similar reasons like in the proof of Theorem 4 it is necessary and sufficient to consider only such vectors $\delta \in \mathbb{R}^m$ for which the system

$$A_P x_P = \delta,\ x_P > 0$$

is solvable. This system is solvable if and only if the linear problem

$$\min\ 0^T x_P\ \text{ subject to }\ A_P x_P = \delta,\ x_P \geq \varepsilon$$

has an optimal solution for a sufficiently small $\varepsilon > 0$. From duality theorems in linear programming it is equivalent to the optimality of the dual problem

$$\max\ \delta^T y + \varepsilon^T z\ \text{ subject to }\ A_P^T y + z \leq 0,\ z \geq 0.$$

The term $\varepsilon^T z$ is maximal for $z = -A_P^T y$ (which is nonnegative) and hence the dual problem can be rewritten as

$$\max\ (\delta^T - \varepsilon^T A_P^T)y\ \text{ subject to }\ A_P^T y \leq 0.$$

Again, we use theory of normal cones, from which it follows that $\Upsilon_1(x^*)$ is described by the system

$$\begin{aligned} h_i^T(\delta - A_P\varepsilon) &= 0\ \forall i \in I, \\ g_j^T(\delta - A_P\varepsilon) &\leq 0\ \forall j \in J, \end{aligned}$$

Since $h_i^T A_P = 0$ for all $i \in I$ and $0 \neq g_j^T A_P \leq 0$ for all $j \in J$, we have equivalently

$$\begin{aligned} h_i^T\delta &= 0\ \forall i \in I, \\ g_j^T\delta &< 0\ \forall j \in J, \end{aligned}$$

which proves the equation (9). ☐

The last case characterization follows from Theorem 6. For this purpose assume that the rows of the matrix A are linearly independent.

Theorem 6. *Let y^j, $j \in V$, be all vertices of the convex polyhedral set $\mathcal{F}_P := \{y \mid A_P^T y = c_P, A_Z^T y \le c_Z\}$. By h_j^k, $k \in H_j$, denote vectors in directions of all edges of the dual feasible set $\mathcal{D}$ coming from y^j and satisfying $A_P^T h_j^k = 0$. Similarly by g_j^k, $k \in G_j$, denote vectors in directions of all edges $\mathcal{D}$ coming from y^j and satisfying $A_P^T g_j^k \lneqq 0$. Then*

$$\Upsilon_2(x^*) = \bigcup_{j \in V} \big\{\delta \in \mathbb{R}^m \mid \delta^T h_j^k \le 0 \ \forall k \in H_j,\ \delta^T g_j^k < 0 \ \forall k \in G_j\big\}. \tag{10}$$

Proof. We are seeking for all $\delta \in \mathbb{R}^m$ such that there is an optimal partition $(\mathcal{B},\mathcal{N})$ of the problems (P_δ), (D_δ) with the property $\mathcal{B} \supseteq P$ and $\mathcal{N} \subseteq Z$. In other words, the dual problem (D_δ) has optimal solutions only in the face of $\mathcal{D}$ specified by the system

$$A_P^T y = c_P,\ A_Z^T y \le c_Z. \tag{11}$$

If the dual problem has an optimal solution, then at least one of them is a vertex of $\mathcal{D}$ (due to the condition on linear independency of row of the matrix A). Let y^j, $j \in V$ be any vertex of $\mathcal{F}_P$. Then y^j remains optimal for all δ satisfying

$$\delta^T h_j^k \le 0 \ \forall k \in H_j,\ \delta^T g_j^k \le 0 \ \forall k \in G_j,$$

but only for all δ satisfying

$$\delta^T h_j^k \le 0 \ \forall k \in H_j,\ \delta^T g_j^k < 0 \ \forall k \in G_j,$$

the set of optimal solutions lies in the face (11) of the dual feasible set $\mathcal{D}$. □

The optimal value function on regions $\Upsilon_1(x^*)$ and $\Upsilon_3(x^*)$ is $\delta^T y^*$. The critical region $\Upsilon_2(x^*)$ is not convex in general. However, each part of the union (10) is convex polyhedral set and the optimal value on it is $\delta^T y^j$.

In summary, we have: $cl\, \Upsilon_1(x^*) = \Upsilon_3(x^*)$, and $\Upsilon_1(x^*) \subseteq \Upsilon_2(x^*)$. The former equation was already addressed in Borrelli et al. [2], and thre latter one follows simply from the definition.

The regions $\Upsilon_2(x^*)$ and $\Upsilon_3(x^*)$ are generally incomparable—no one is larger than the other; see Example 2 below. We state only that $\Upsilon_3 \subseteq cl\, \Upsilon_2$. It is true, since $cl\, \Upsilon_2$ is the set of all δ for which (D_δ) has an optimal solution in the face $\mathcal{F}_P$, and Υ_3 contains δ for which the whole face $\mathcal{F}_P$ is optimal.

3.3 Optimal partition invariancy

Let $(\mathcal{B},\mathcal{N})$ be the optimal partition of (P_{δ^0}) and suppose that x^* and y^* is a pair of strictly complementary solutions. Then $P = \mathcal{B}$ and the critical region Υ_p can be easily computed from Theorem 7.

Theorem 7. *We have* $\Upsilon_p = \Upsilon_1(x^*)$.

Proof. The inclusion $\Upsilon_p \subseteq \Upsilon_1(x^*)$ holds trivially.

Let $\delta \in \Upsilon_1(x^*)$. Then there is an optimal solution x of (P_δ) with $\sigma(x) = \mathcal{B}$. Due to the complementary slackness condition $x^T(c^T - A^T y^*) = 0$, the pair x and y^* forms a pair of strictly complementary solutions. Hence $(\mathcal{B},\mathcal{N})$ is the optimal partition of (P_δ) and the inclusion $\Upsilon_p \supseteq \Upsilon_1(x^*)$ is proven. □

For every $\delta \in \Upsilon_p$ the optimal value of (P_δ) is equal to $\delta^T y^*$ and hence is linear on Υ_p.

If x^* represents a basic nondegenerate solution, then $cl\Upsilon_p = \Upsilon_B(x^*)$. If x^* is a nonbasic solution, the set of optimal solutions (see the proof of Theorem 4)

$$A_P x_P = \delta^0,\ x_P \geq 0,\ x_Z = 0 \tag{12}$$

contains at least one vertex. The closure $cl\,\Upsilon_p$ comprises all critical regions $\Upsilon_B(x)$ for all vertices x of (12) and corresponding optimal bases $B \subseteq P$. Therefore critical regions of this kind of invariancy and the support set one are quite large, which is their major advantage in contrary to optimal basis invariancy.

3.4 Remark and example

Remark 1. In real-life situations, parameters δ comes from some admissible set Δ instead the whole space $\mathbb{R}^n$. The resulting critical region then simply equals $\Delta \cap \Upsilon$, where Υ is the appropriate critical region for the unconstrained $\delta \in \mathbb{R}^m$.

Next, the right-hand side function sometimes depends on parameters as follows [2, 4, 5, 6, 13]

$$b(\nu) = b^0 + B\nu,$$

where $b^0 \in \mathbb{R}^m$, $B \in \mathbb{R}^{m\times l}$ are known and ν is the l-dimensional vector of parameters. If the critical region Υ for the unconstrained δ is described as $D\delta \leq d$, then the resulting critical region is obtained by substitution $\delta = b^0 + B\nu$ and has a simple description

$$D(b^0 + B\nu) \leq d,$$

or

$$DB\nu \leq d - Db^0.$$

Example 2. Consider the problem

$$\min\ 2x_2 - 3x_3$$
$$\text{subject to}\ \ 3x_1 + x_2 = \delta_1,\ 3x_1 - x_2 + 3x_3 = \delta_2,\ x_1, x_2, x_3 \geq 0.$$

and the initial value $\delta^0 = (3,3)^T$. The feasible set equals the optimal solution set and is represented by a segment with endpoints $x^1 = (1,0,0)^T$ and $x^2 = (0,3,2)^T$; see Figure 2.

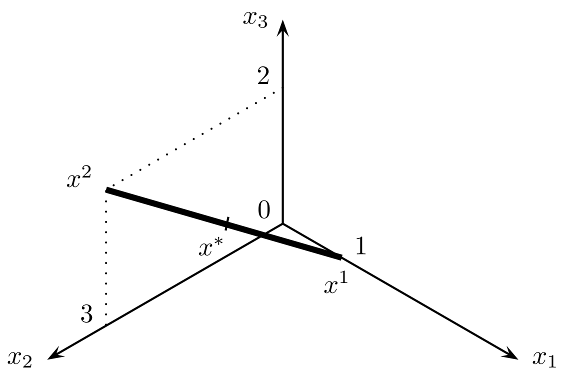

Figure 2: (Example 2) the bold segment illustrate the feasible set, which identical to the optimal set.

1. (Basis invariancy) To the optimal (degenerate) basic solution x^1 there correspond two optimal bases: $B^1 = (1,2)$ and $B^2 = (1,3)$. The former basis yields (according to (7)) the critical region

$$\Upsilon_{B^1}(x^1) = \{\delta \in \mathbb{R}^2 \mid \delta_1 + \delta_2 \geq 0,\ \delta_1 - \delta_2 \geq 0\},$$

and the latter basis yields

$$\Upsilon_{B^2}(x^1) = \{\delta \in \mathbb{R}^2 \mid \delta_1 \geq 0,\ -\delta_1 + \delta_2 \geq 0\}.$$

The optimal value function is $\delta_1 - \delta_2$, the same for both critical regions.

2. (Support set invariancy) For the solution x^1 we have $P = (1)$, $Z = (2, 3)$ and we compute $\Upsilon_3(x^1)$ according to Theorem 4. The lineality space $\mathcal{L}_P = \{y \mid 3y_1 + 3y_2 = 0\}$ represents a line in direction $h_1 = (1, -1)^T$. The convex polyhedral cone

$$\{y \mid A_P^T y \leq 0\} \cap \mathcal{L}_P^{\perp} = \{y \mid y_1 + y_2 \leq 0,\ y_1 - y_2 = 0\}$$

represents a ray in the direction $g_1 = (-1, -1)^T$. Hence

$$\Upsilon_3(x^1) = \{\delta \in \mathbb{R}^2 \mid \delta_1 - \delta_2 = 0,\ -\delta_1 - \delta_2 \leq 0\}.$$

Using Theorem 5, we obtain

$$\Upsilon_1(x^1) = \{\delta \in \mathbb{R}^2 \mid \delta_1 - \delta_2 = 0,\ -\delta_1 - \delta_2 < 0\}.$$

To give the description of $\Upsilon_2(x^1)$ by Theorem 6 we need to compute all vertices of

$$\mathcal{F}_P = \{y \mid 3y_1 + 3y_2 = 0,\ y_1 - y_2 \leq 2,\ 3y_2 \leq -3\}.$$

There is only one such a vertex $y^1 = (1, -1)^T$, and there are two edges of $\mathcal{D}$ coming from y^1: one in direction of $g_1^1 = (-1, 0)^T$ and the second one in direction of $g_1^2 = (-1, -1)^T$. Hence

$$\Upsilon_2(x^1) = \{\delta \in \mathbb{R}^2 \mid -\delta_1 < 0,\ -\delta_1 - \delta_2 < 0\}.$$

Now consider some nonbasic solution, for instance $x^* = (\frac{1}{2}, \frac{3}{2}, 1)^T$. Then $P = (1, 2, 3)$, $Z = \emptyset$. The lineality space $\mathcal{L}_P = \{y \mid 3y_1 + 3y_2 = 0, y_1 - y_2 = 0, 3y_2 = 0\} = \{(0, 0)\}$ has empty basis. The convex polyhedral cone

$$\begin{gathered}\{y \mid A_P^T y \leq 0\} \cap \mathcal{L}_P^{\perp} = \\ \{y \mid 3y_1 + 3y_2 \leq 0,\ y_1 - y_2 \leq 0,\ 3y_2 \leq 0\}\end{gathered}$$

has two edges $g_1 = (-1, 0)^T$ and $g_2 = (-1, -1)^T$. Hence

$$\Upsilon_3(x^*) = \{\delta \in \mathbb{R}^2 \mid -\delta_1 \leq 0,\ -\delta_1 - \delta_2 \leq 0\}.$$

By the way, this critical region consists of the union $\Upsilon_{B^1}(x^1) \cup \Upsilon_{B^2}(x^1)$. The region $\Upsilon_1(x^*)$ is characterized as follows

$$\Upsilon_1(x^*) = \{\delta \in \mathbb{R}^2 \mid -\delta_1 < 0,\ -\delta_1 - \delta_2 < 0\}.$$

It remains to compute the region $\Upsilon_2(x^*)$. The convex polyhedral set

$$\mathcal{F}_P = \big\{y \mid 3y_1 + 3y_2 = 0,\ y_1 - y_2 = 2,\ 3y_2 = -3\big\}.$$

has only one vertex $y^1 = (1, -1)^T$. In $\mathcal{D}$, two edges in directions of $g_1^1 = (-1, 0)^T$ and $g_1^2 = (-1, -1)^T$ come from y^1. Hence

$$\Upsilon_2(x^*) = \{\delta \in \mathbb{R}^2 \mid -\delta_1 < 0,\ -\delta_1 - \delta_2 < 0\}.$$

The optimal value function is $\delta_1 - \delta_2$ and is the same for all mentioned regions.

3. (Optimal partition invariancy) Optimal partition is $\mathcal{B} = (1, 2, 3)$, $\mathcal{N} = \emptyset$, and hence $\Upsilon_p = \Upsilon_1(x^*) = \{\delta \in \mathbb{R}^2 \mid -\delta_1 < 0,\ -\delta_1 - \delta_2 < 0\}$. The pair of strictly complementary solutions consists of x^* for the primal problem and$y^* = (1, -1)$ for the dual problem. The optimal value function is again $\delta^T y^* = \delta_1 - \delta_2$.

4 Concluding remarks

We studied parametric analysis under support set and optimal partition invariancy and compared it with the classical optimal basis invariancy. The resulting critical regions are always characterized by means of linear equations and inequalities (and unions of these systems) and hence represent polyhedral set, in most cases convex.

Perturbing objective function ceofficients, the new types of invariancies yield the same or better (larger) critical regions as long as the optimal solution is unique. Otherwise, the critical region is usually degenerate (has no full dimension). On the contrary, perturbing the right-hand side, the new types of invariancies usually yield better critical regions if the optimal solution is not unique and yield worse ones if the optimal (basic) solution is degenerate.

The proposed general form of parametrization can be used for special kinds of multiparametric programming (see Remark 1) and for the tolerance analysis developed by Wendell et al. [17, 18, 19].

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