Locally injective graph homomorphism: Lists guarantee dichotomy

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Abstract. We prove that in the List version, the problem of deciding the existence of a locally injective homomorphism to a parameter graph H performs a full dichotomy. Namely we show that it is polynomially time solvable if every connected component of H has at most one cycle and NP-complete otherwise.

1 Introduction

We consider finite undirected graphs without loops or multiple edges. A recently intensively studied notion, for its algebraic motivation and connections as well as being a natural generalization of graph coloring, is the notion of *graph homomorphisms*. A homomorphism $f : G \to H$ from a graph G to a graph H is an edge-preserving vertex mapping, i.e., a mapping $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. (For a recent monograph on graph homomorphisms the reader is referred to [14].) It follows from the definition that the neighborhood of every vertex is mapped into the neighborhood of its image, formally $f(N_G(x)) \subseteq N_H(f(x))$ for all $x \in V(G)$. Properties of these restricted mappings, local constraints, lead to the definition of *locally constrained homomorphisms.* The homomorphism f is called *locally injective* (*bijective, surjective*, resp.) if for every $x \in V(G)$, the restricted mapping $f : N_G(x) \to N_H(f(x))$ is injective (bijective, surjective, resp.). All three of these notions have been studied on their own with different motivations. Locally surjective homomorphisms

* Supported by the Ministry of Education of the Czech Republic as project 1M0021620808.

correspond to so called role assignment graphs studied in sociological applications [11], locally bijective ones correspond to graph covers well known from topological graph theory [2, 15] and theory of local computation [1, 3]. Locally injective homomorphisms are closely related to generalized $L(2,1)$-labelings of graphs and the Frequency Assignment Problem [8, 9].

From the computational complexity point of view we are interested in the decision problem if an input graph G allows a homomorphism of certain type into a fixed target graph H. As these problems are parametrized by the graph H, we use the notation H-Hom (when asking for the existence of any homomorphism), H-LIHom, H-LBHom and H-LSHom (when asking for locally injective, bijective or surjective homomorphisms, respectively).

The complexity of H-Hom is fully understood and dichotomy was proved by Hell and Nešetřil in [13]. The problem is polynomially solvable if H is bipartite and NP-complete otherwise. The complexity of H-LSHom was studied in [17] and completed by Fiala and Paulusma in [11]. This problem (for connected graph H) is solvable in polynomial time if H has at most 2 vertices and NP-complete otherwise. Several papers have been devoted to studying the complexity of locally injective and bijective homomorphisms, but only partial results are known [8, 10, 16]. Though conjectured at least for the case of locally bijective homomorphisms, the Polynomial/NP-completeness dichotomy has been proved in neither of the last two cases. However, this question is natural especially in view of the fact that locally constrained homomorphisms can be expressed as Constraint Satisfaction Problem, for which the dichotomy was conjectured in the fully general case by Feder and Vardy [6].

The CSP view also suggests considering the List versions of the problems, since lists correspond to unary relations (cf. next section). The input to the List version of a homomorphism problem is a graph G together with lists $L(u) \subseteq V(H)$ of admissible targets for every vertex $u \in V(G)$. The question is if G allows a (locally injective, bijective etc.) homomorphism $f : G \to H$ such that $f(u) \in L(u)$ for every $u \in V(G)$. We refer to these problems as List-H-Hom, List-H-LIHom etc. A deep result of Hell et al. gives a full characterization of the case of general homomorphisms [5, 12]. The problem List-H-Hom is polynomially solvable for so called double circular arc graphs H and NP-complete otherwise.

Setting the lists to the entire vertex set of the target graph, one immediately sees that H-$\mathsf{Hom} \propto \mathsf{List}$-$H$-$\mathsf{Hom}$, H-$\mathsf{LIHom} \propto \mathsf{List}$-$H$-$\mathsf{LIHom}$, H-$\mathsf{LBHom} \propto \mathsf{List}$-$H$-$\mathsf{LBHom}$ and H-$\mathsf{LSHom} \propto \mathsf{List}$-$H$-$\mathsf{LSHom}$. Thus in each case the borderline between polynomial and NP-complete instances (dichotomy assumed) of the List version will lie within the easy instances of the non-List one. This is well seen in the above mentioned case of general homomorphisms and also in the case of the locally surjective ones — List-H-LSHom remains polynomial for graphs H with at most two vertices. The trouble with the locally injective and locally bijective homomorphisms is that the full characterization of the non-List versions is not known. Nevertheless, lists do help! The purpose of this paper is to show that in the case of locally injective homomorphisms, lists guarantee a full dichotomy.

Theorem 1. *The* List-H-LIHom *problem is solvable in linear time if the graph H contains at most one cycle in each connected component, and it is* NP*-complete otherwise.*

The paper is structured as follows. In the next section we quickly describe the connection to the Constraint Satisfaction Problem. In Section 3 we give the argument for the polynomial part of our theorem. The technical reductions for the NP-hardness part are presented in Section 4 and the proof is summarized in Section 5. The final section contains concluding remarks.

2 Locally constrained homomorphisms as CSP

The CSP is parametrized by a fixed template $\mathcal{X} = (X; S_1, \ldots, S_k)$, where S_i's are relations on a finite set X, the arity of S_i being n_i, $i = 1, 2, \ldots, k$. The input of the $\mathcal{X}$-CSP is a structure $\mathcal{U} = (U; R_1, \ldots, R_k)$, where U is a (large) set and R_i is an n_i-ary relation on U, for $i = 1, 2, \ldots, k$. The question is whether there exists a mapping (in fact, a structural homomorphism) $f : U \to X$ such that for every i and every n_i-tuple $(u_1, \ldots, u_{n_i}) \in U^{n_i}$, $(u_1, \ldots, u_{n_i}) \in R_i$ implies $(f(u_1), \ldots, f(u_{n_i})) \in S_i$. Feder and Vardy [6] conjecture that for every template $\mathcal{X}$, this problem is either polynomial time solvable or NP-complete. This is known for binary structures [19], and for many special cases (cf. e.g., Bulatov et al. [4]).

Though in natural structures one tends to overlook unary relations, from the point of view of the formal definition they form a fully coherent part of the picture. And they correspond to lists. A unary relation is just a subset of the ground set. If a vertex $u \in U$ belongs to unary relations $R_{i_1}, \ldots, R_{i_t}$, then the constraints given by the unary relations of $\mathcal{X}$-CSP merely say that $f(u) \in \bigcap_{j=1}^{t} S_{i_j}$, which is equivalent to setting the list of admissible images of u to $L(u) = \bigcap_{j=1}^{t} S_{i_j}$.

The general homomorphism problem H-Hom is obviously a CSP problem — the template is H itself, and the input structure is G (edges of both graphs are considered as symmetric binary relations). We will show that also locally injective and bijective homomorphisms can be expressed as CSP. Given a graph H with h vertices, set

$$D = \{(x, y)|\ x \neq y \in V(H)\},$$

$$D_i = \{(x)|\ \deg_H(x) = i, x \in V(H)\},\ i = 0, 1, \ldots, h-1.$$

Here D is the symmetric binary relation containing all pairs of distinct vertices and D_i's are unary relations controlling the degrees. We derive the following templates from H:

$$\mathcal{L}_H = (V(H); S_1 = E(H), S_2 = D)$$

and

$$\mathcal{B}_H = (V(H); S_1 = E(H), S_2 = D, S_{3+i} = D_i, i = 0, \ldots, h-1).$$

Observation 1 *For every graph H, H-*LIHom $\propto$ *$\mathcal{L}_H$-CSP and H-*LBHom $\propto$ *$\mathcal{B}_H$-CSP.*

Proof. For an input graph G, define $\mathcal{U} = (V(G); R_1 = E(G), R_2 = \{(x, y)|; x \neq y \ \wedge \ \exists z : xz, yz \in E(G)\})$. Then U is a feasible instance for $\mathcal{L}_H$-CSP if and only if G allows a locally injective homomorphism into H — the relations R_1 and S_1 control that a candidate vertex mapping is a graph homomorphism, and R_2 with S_2 control the local injectivity. For the locally bijective case, just add $R_{3+i} = \{u \in V(G)|\ \deg_G(u) = i\}$ for each $i = 0, 1, \ldots, h-1$, to ensure that this mapping is locally bijective on every neighborhood.

Observe, however, that this does not mean that the Feder-Vardy conjecture would imply dichotomy for H-LIHom or H-LBHom. Our observation is useful only when the corresponding CSP problem is polynomially solvable, which is unfortunately only seldom (whenever H has at least three vertices, 3-COLORABILITY $\propto$ $\mathcal{L}_H$-CSP $\propto$ $\mathcal{B}_H$-CSP and both problems are NP-complete). The point is that the inputs of $\mathcal{L}_H$-CSP (or $\mathcal{B}_H$-CSP) derived from G are not arbitrary. In view of this becomes the fact that adding lists endorses dichotomy on List-H-LIHom even more interesting.

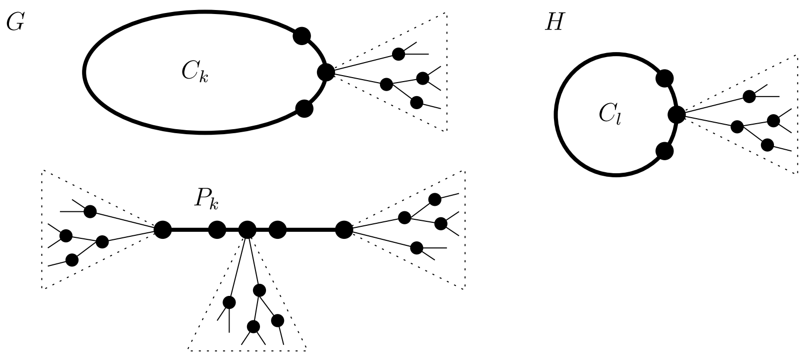

Fig. 1. The polynomial instances for the List-H-LIHom problem. The $2l$ possibilities of of mapping of the emphasized C_k and P_k are afterwards verified for the pending trees.

3 The polynomial case

Lemma 1. *For any connected graph H containing at most one cycle, the* List-H-LIHom *problem is solvable in linear time.*

Proof. Without loss of generality we may assume that the input graph G is connected, since otherwise we can treat each component separately.

If H is a tree, then any connected graph G that allows a locally injective homomorphism $G \to H$ is a subtree of H. Since H is fixed, the graph G itself must have a bounded number of vertices and the problem H-LIHom is solvable in constant time even with lists being incorporated.

Let H have exactly one cycle. We first note that the number of connected graphs G of diameter at most $2\,\mathrm{diam}(H)$ that allow a locally injective homomorphism to H is bounded (each such graph has at most $1 + \Delta^{2\,\mathrm{diam}(H)}$ vertices, where Δ is the maximum vertex degree of H). Hence, for such instances, the List-H-LIHom problem can be decided in constant time.

For the rest of the proof suppose that $\mathrm{diam}(G) > 2\,\mathrm{diam}(H)$. Denote by C_l the unique cycle of H (consult Fig 3). We distinguish two cases:

– G contains a cycle, say C_k: Then this cycle must be mapped onto C_l. This might happen in at most $2l$ ways. Some of these $2l$ ways may be further excluded by the list constraints. If we fix a mapping $C_k \to C_l$, it remains to decide whether this mapping can be extended to the remaining vertices of G. For every vertex u of C_k, we can solve this question in constant time, since this is equivalent to the tree List-H-LIHom problem. (Now the instance is the component of $G \setminus E_{C_k}$ containing u and the parameter is the component of $H \setminus E_{C_l}$ containing the image of u. Both are trees.)

– G is a tree: A necessary condition for G to allow a locally injective homomorphism to H is that G contains a path P_k of length $k = \mathrm{diam}(G) - 2\,\mathrm{diam}(H)$ such that the components of $G \setminus E_{P_k}$ stemming from the inner vertices of P_k map locally injectively to the components of $H \setminus E_{C_l}$ and the two trees of diameter $\mathrm{diam}(H)$ hanging on the termini of P_k map locally injectively to H. We first find P_k by $\mathrm{diam}(H)$ many iterations of peeling off vertices of degree one. Then, as in the above case, we try all $2l$ possibilities of a locally injective mapping $P_k \to C_l$ and exclude those which do not satisfy the list constraints. Finally for each such partial mapping, we check if it can be extended to the entire G by solving at most $k+1$ List-H-LIHom problems of constant size.

4 Auxiliary NP-hardness reductions

In this section we show NP-hardness of the List-H-LIHom problem for three basic types of graphs depicted in Fig. 1 (they are informally called the Theta graph, the Flower graph and the Weight graph). The problem we use for our NP-hardness reductions is Edge-precoloring extension. It has been shown NP-complete even when restricted to cubic bipartite graphs [7]. The input of this variant consists of a cubic bipartite graph together with a partial coloring of its edges by three colors — say red, blue and white[1]. The question is whether this partial coloring can be extended to the entire edge set such that each vertex is incident with edges of all three colors.

For positive integers a, b, c, where $b, c \geq 2$, let $\Theta(a, b, c)$ be the graph on $a+b+c-1$ vertices consisting of two vertices of degree three connected by paths of lengths a, b, and c (cf. Fig. 1 top left).

Lemma 2. *For arbitrary positive integers a, b, c, where $b, c \geq 2$, the* List-$\Theta(a, b, c)$-LIHom *problem is* NP*-hard.*

Proof. If $a = b = c \geq 2$, we can reduce the Edge-precoloring extension problem directly. Let G be the instance of the Edge-precoloring extension problem. We replace each edge $e \in E(G)$ by a path P^e of length

[1] The favorite tricolor for Czechs — as well as for several other nations.

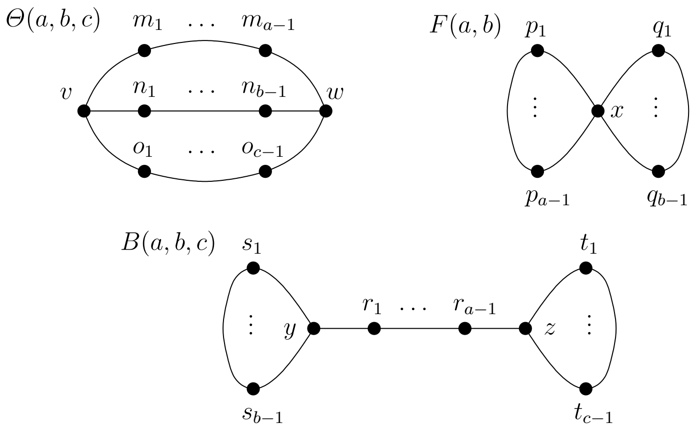

Fig. 2. The basic three types of graphs.

a. We further associate colors red, blue and white with the inner vertices of the three paths of $\Theta(a,a,a)$ in such a way that vertices of the first path represent color red, of the second path color blue, and those of the third one the white color. If the edge e was precolored by a color α, we assign the vertices of P^e lists consisting of vertices of the path in $\Theta(a,a,a)$ representing the color α. For the other vertices we let the lists be the whole vertex set of $\Theta(a,a,a)$.

Straightforwardly, proper 3-edge colorings of G are in one-to-one correspondence with valid locally injective homomorphisms to $\Theta(a,a,a)$. (Such homomorphisms must in fact be locally bijective.) The local conditions on vertices of degree three represent the condition that the three colors on the edges of the original graph must be distinct.

For the case of $a \leq b \leq c$, $a < c$, the construction is slightly more sophisticated. The edge colors will be represented by sequences of images of the vertices along the paths representing the edges. We use vertex lists to enforce that only three feasible patterns may exist.

Denote first by v, w the two vertices of degree three in $\Theta(a,b,c)$. The inner vertices along the path of length a from v to w will be denoted by $m_1, m_2, \ldots, m_{a-1}$ (this set may be empty), along the path of length b in the same direction by $n_1, \ldots, n_{b-1}$, and along the remaining path by $o_1, \ldots, o_{c-1}$.

We define three sequences of length $a+b+c+1$ by

$$\begin{aligned} R &= (v, m_1, \ldots, m_{a-1}, w, n_{b-1}, \ldots, n_1, v, o_1, \ldots, o_{c-1}, w) \\ B &= (v, n_1, \ldots, n_{b-1}, w, o_{c-1}, \ldots, o_1, v, m_1, \ldots, m_{a-1}, w) \\ W &= (v, o_1, \ldots, o_{c-1}, w, m_{a-1}, \ldots, m_1, v, n_1, \ldots, n_{b-1}, w) \end{aligned}$$

Let G be an instance of the EDGE-PRECOLORING EXTENSION problem and let V^1, V^2 be its two classes of the bipartition. We replace each edge $e \in E_G$ by a path P^e of length $a+b+c$ and call the resulting graph G'.

The vertices of each P^e are denoted by $u_1^e, \ldots, u_{a+b+c+1}^e$ in such a way that the first vertex u_1^e belongs to V^1 and the last vertex $u_{a+b+c+1}^e \in V^2$. We define the lists of the vertices in G' so that for each $e \in E(G)$ and each $i = 1, 2, \ldots, a+b+c+1$, we set

$$L(u_i^e) = \begin{cases} \{R_i, B_i, W_i\} & \text{if } e \text{ is not precolored} \\ \{R_i\} & \text{if } e \text{ is precolored } \textit{red} \\ \{B_i\} & \text{if } e \text{ is precolored } \textit{blue} \\ \{W_i\} & \text{if } e \text{ is precolored } \textit{white}. \end{cases}$$

We claim that G' allows a locally injective homomorphism to $\Theta(a, b, c)$ that respects all list constraints if and only if the edge precoloring of the original graph G can be extended to a proper 3-edge coloring of G.

Suppose first that $f : G' \to \Theta(a, b, c)$ is a locally injective homomorphism. We prove that only the sequences given by R, B and W may appear along any path P^e. The local injectivity constraints then imply that the derived edge-coloring is proper.

Assume first the case $a < b < c$. Let P^e be a path in G'. By list constraints we have $f(u_1^e) = v$ for any such P^e. Assume that $f(u_2^e) = m_1$ (or $f(u_2^e) = w$ if $a = 1$). Then the mapping f is uniquely determined for the next $a-1$ vertices of P^e due to the local constraints. As the subsequence $v, o_{c-1}, \ldots, o_1, w$ starts at a different position in the sequence B it cannot be used as the further extension of f on P^e. In other words the mapping f has to follow the sequence R for the next $b-1$ vertices as well as for the final segment of $c-1$ vertices. For the other two possibilities, i.e., when $f(u_2^e) = n_1$ or o_1, resp., we involve similar arguments to conclude that the only feasible pattern of the mapping f along P^e is given by the sequences B or W, respectively.

It remains to consider the case when $b = a$ or $b = c$ and $a < c$. Without loss of generality assume $a = b$, i.e., the target graph is $\Theta(a, a, c)$. By the arguments presented so far, it might be possible that a mapping of some P^e may follow on the first a vertices the sequence R, while after u^e_{a+1} it continues along B. The crucial observation here is that if $f(u^e_2) = o_1$ then the whole path must be mapped according to the sequence W. Viewed from the other side also if $f(u^e_{a+b+c}) = o_{c-1}$, then the mapping of P^e follows the pattern R. Then the pattern W is used for $|V^1|$ edges, hence giving every vertex of the side V^2 a neighbor mapped onto n_{b-1}. Similarly the pattern R is used for $|V^2| = |V^1|$ edges, giving each vertex on the V^1 side a neighbor mapped onto m_1. For the paths P^e corresponding to the remaining matching in G, the local injectivity constraints imply that $f(u^e_2) = n_1$ and $f(u^e_{a+b+c}) = m_{a-1}$, and hence only the pattern B may be used along these paths.

For the reverse implication, assume $E(G)$ is properly colored. We define the mapping $f : G' \to \Theta(a, b, c)$ such that for every $e \in E(G)$ and all $i = 1, 2, \ldots, a + b + c + 1$, $u^e_i = R_i$ if e is red, and analogously with B_i representing blue and with W_i representing white colors. On every vertex of G' the mapping preserves the list constraints and is locally injective: on the inner vertices of any P^e since the patterns R, B, W respect local injectivity constraints, and on the vertices of degree three since the coloring is proper.

For integers $a, b \geq 3$, let $F(a, b)$ be the graph on $a + b - 1$ vertices consisting of two cycles C_a and C_b sharing exactly one vertex (cf. Fig. 1 top right).

Lemma 3. *For arbitrary integers $a, b \geq 3$, the* List-$F_{a,b}$-LIHom *problem is* NP*-hard.*

Proof. Assume x is the vertex of $F(a, b)$ of degree four and that the inner vertices of the two cycles are $p_1, \ldots, p_{a-1}$ and $q_1, \ldots, q_{b-1}$.

We extend the proof of the previous lemma for the NP-hardness of the List-$\Theta(a, a, b)$-LIHom problem. Let G' be the graph constructed from an instance G of the Edge-precoloring extension problem. Define lists L' such that for every $u \in V_{G'}$ (see Fig. 3):

$$\begin{aligned} x \in L'(u) &\iff v \in L(u) \ \vee \ w \in L(u) \\ p_i \in L'(u) &\iff m_i \in L(u) \ \vee \ n_{a-i} \in L(u) \\ q_i \in L'(u) &\iff o_i \in L(u). \end{aligned}$$

We claim that G' allows a locally injective homomorphism $f : G' \to \Theta(a, a, b)$ respecting the list constraints L if and only if there exists a locally injective $g : G' \to F(a, b)$ respecting the list constraints L'.

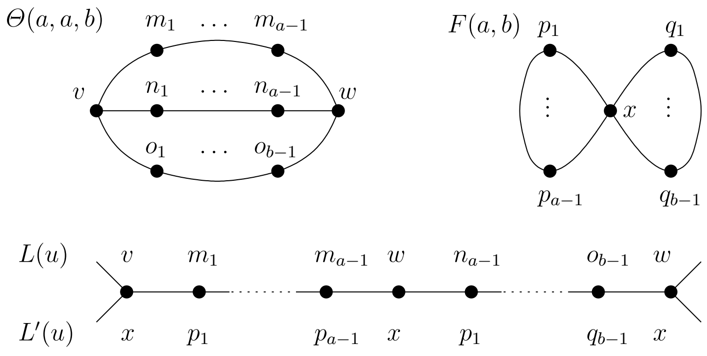

Fig. 3. The reduction for List-$F(a,b)$-LIHom.
The lower part shows the new lists along a path P^e (for e precolored red).

As $\Theta(a,a,b)$ can be mapped locally injectively onto $F(a,b)$, the "only if" implication is straightforward (taking into account also the way how the new lists L' are constructed from L). For the opposite direction, note that the lists L' defined above together with local constraints assure that along each P^e the vertex x will be involved exactly four times (twice on the end). (This follows by a simple case analysis when $b \neq 2a$, and a little more subtle argument analogous to the case $a = b < c$ of the proof of Lemma 2 works for $b = 2a$. The latter is omitted because of space limitations.) We modify a feasible $g : G' \to F(a,b)$ by a series of substitutions. All odd occurrences of x along every P^e will be replaced by v and even occurrences by w. We further replace each pattern $v, p_1, \ldots, p_{a-1}, w$ by the pattern $v, m_1, \ldots, m_{a-1}, w$. Similarly, we replace patterns $v, p_{a-1}, \ldots, p_1, w$ by $v, n_1, \ldots, n_{a-1}, w$ and also all patterns $v, q_1, \ldots, q_{b-1}, w$ by $v, o_1, \ldots, o_{b-1}, w$. In this way we obtain a feasible locally injective homomorphism $f : G' \to \Theta(a,a,b)$.

For positive integers a, b, c, where $b, c \geq 3$, let $B(a,b,c)$ be the only graph on $a+b+c-1$ vertices consisting of two disjoint cycles C_b and C_c connected by a path of length a.

Lemma 4. *For arbitrary positive integers a, b, c, where $b, c \geq 3$, the* List-$B(a,b,c)$-LIHom *problem is* NP*-hard.*

Proof. Let y, z be the two vertices of $B(a,b,c)$ of degree three. Denote the inner vertices of the path P_a by $r_1, \ldots, r_{a-1}$ (this set may be empty), and

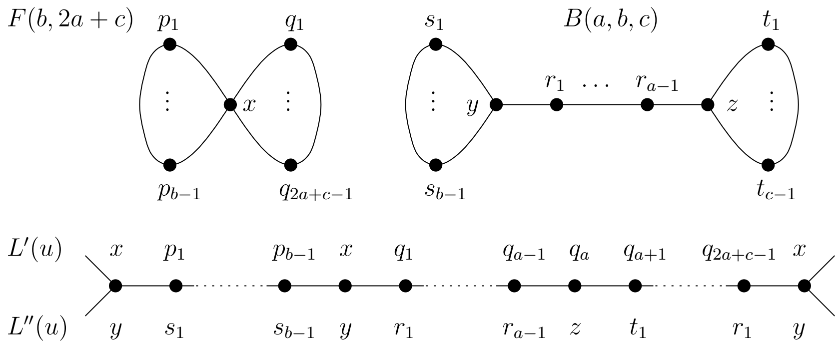

Fig. 4. The reduction for List-$B(a,b,c)$-LIHom.
The lower part shows the new lists along a path P^e (for e precolored red).

let $s_1, \ldots, s_{b-1}$ be the inner vertices of C_b and $t_1, \ldots, t_{c-1}$ the inner vertices of C_c.

We extend the proof of the previous lemma for the NP-hardness of the List-$F(b,2a+c)$-LIHom. Let G' be the graph constructed from an instance G of the Edge-precoloring extension problem. Define lists L'' such that for every $u \in V_{G'}$ (see Fig. 4):

$$\begin{aligned}
y \in L''(u) &\iff x \in L'(u)\\
s_i \in L''(u) &\iff p_i \in L'(u)\\
r_i \in L''(u) &\iff q_i \in L'(u) \ \vee \ q_{2a+c-i} \in L'(u)\\
z \in L''(u) &\iff q_a \in L'(u) \ \vee \ q_{a+c} \in L'(u)\\
t_i \in L''(u) &\iff q_{a+i} \in L'(u)
\end{aligned}$$

We claim that G' allows a locally injective homomorphism $g : G' \to F(b, 2a+c)$ respecting the list constraints L' if and only if there exist a locally injective homomorphism $h : G' \to B(a,b,c)$ with list constraints L''.

The core argument is that for any homomorphism g feasible for the instance (G', L'), every vertex mapped onto x has at most one neighbor mapped onto one of q_1, q_{2a+c-1}, and so the feasible homomorphisms $g : C' \to F(b, 2a+c)$ are in a one-to-one correspondence with locally injective homomorphisms $h : G' \to B(a,b,c)$ obeying the list constraints L''.

5 Proof of Theorem 1

Observation 2 *If a graph H is an induced subgraph of a graph H', then* List-H-$\mathsf{LIHom} \propto \mathsf{List}$-$H'$-$\mathsf{LIHom}$.

Proof. Given an input (G, L) of List-H-LIHom, it suffices to use it as an instance of List-H'-LIHom. Since the lists $L(u), u \in V(G)$ do not contain vertices of $H' \setminus H$, any feasible homomorphism $G \to H'$ uses only vertices of H, and since H is an induced subgraph of H', such a mapping is a homomorphism into H. In the opposite direction, any feasible (locally injective) homomorphism $G \to H$ is clearly a (locally injective) homomorphism to H'.

Proof of Theorem 1:

The polynomial-time algorithm Assume each component H_j of H contains at most one cycle. Then for each component G_i of G and every H_j we test whether G_i allows a locally injective homomorphism (we may restrict lists of the vertices in G_i to subsets of H_j). Each such a subproblem can be decided independently. The overall problem has an affirmative answer if and only if for every G_i, there exists at least one H_j allowing a locally constrained homomorphism. The overall computational complexity remains linear.

The NP-complete part The membership of the List-H-LIHom problem in NP is obvious. Assume H contains a component with at least two cycles. Let H' be a smallest induced subgraph of H containing at least two cycles.

If H' contains exactly two cycles, these two cycles must be edge-disjoint (the symmetric difference of the two cycles would yield a new cycle otherwise). If the two cycles are further vertex-disjoint, then H' is isomorphic to some $B_{a,b,c}$. If the cycles share one vertex, then H' is isomorphic to some $F_{a,b}$.

Otherwise H' has two intersecting cycles. If the graph is not isomorphic to some $\Theta(a, b, c)$, then between some of the three paths of $\Theta(a, b, c)$ there exists an induced edge. Then either H' contains a smaller $\Theta(a', b', c')$ (contradicting the minimality of H') or H' is isomorphic to K_4.

We have proved in Section 4 that for each of the graphs $B(a, b, c)$, $F(a, b)$, and $\Theta(a, b, c)$, the List-H-LIHom problem is NP-hard. For the case of $H' = K_4$, the NP-hardness follows from the trivial reduction H-LIHom $\propto$ List-H-LIHom and the fact that K_4-LIHom is NP-complete [8]. The conclusion then follows by Observation 2.

6 Concluding remarks

6.1 It is interesting that the case of locally surjective homomorphisms does not seem expressible as a CSP problem. Yet a full dichotomy holds both for List and non-List versions.

6.2 An irritating open problem is an analog of Observation 2 for locally bijective homomorphisms. A direct proof would hinge on an involved garbage collection.

6.3 For many instances of the H-LIHom problem for $H = B(a, b, c)$, $F(a, b)$, and $\Theta(a, b, c)$, we can prove NP-hardness also for the non-List version. However, a full characterization even for these simple graphs is not known.

6.4 Recently Daniel Marx [18] showed that the EDGE-PRECOLORING EXTENSION problem remains NP-complete for planar bipartite 3-regular graphs. Our reductions in the proofs of Lemmas 2-4 preserve planarity. D. Marx suggested [personal communication] a simple planarity preserving reduction for List-K_4-LIHom. The gadget is depicted in Fig. 5.

Here, each edge of a graph G, the instance of EDGE-PRECOLORING EXTENSION is replaced by the graph on 12 vertices, consisting of a cycle on six vertices with attached leaves. All leaves are assigned lists with a single element (e.g. $\{4\}$), hence their mapping onto K_4 is fixed. Two opposite leaves are identified with the two original vertices of the original edge. A simple case study yields, that the opposite vertices of the C_6 must be mapped onto the same vertex in K_4. As there are only three vertices available, the images of the neighbors of the original vertices may represent colors of the original edges. The local constrains assure that any locally injective homomorphism from the modified graph G' to K_4 corresponds to a valid edge-3-coloring. The precoloring of some edges can be embedded in a straightforward way.

From the above, we can conclude that whenever a connected graph H contains at least two cycles, the List-H-LIHom problem is NP-complete even for planar inputs. However, the complexity of planar K_4-LIHom and planar K_4-LBHom (without lists) is still an interesting open problem.

7 Acknowledgment

We would like to thank P. Hell for tempting us to consider the List versions of the locally constrained homomorphism problems.

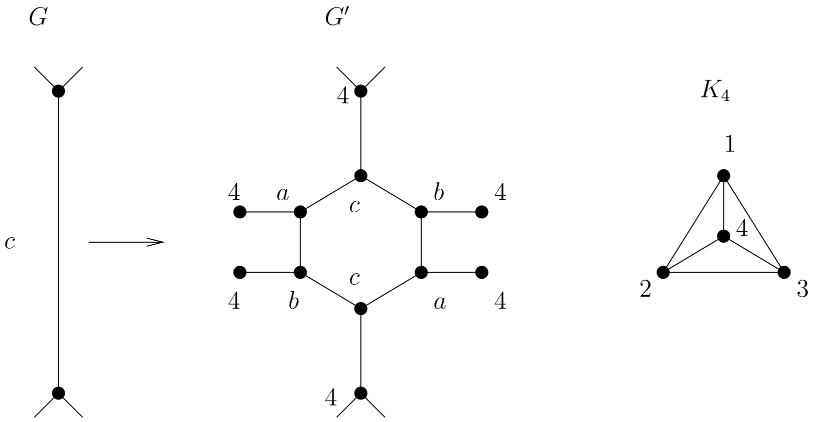

Fig. 5. Edge replacement gadget for the reduction of the planar Edge-precoloring extension problem to the planar List-H-LIHom problem.

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