Description of symmetric and skew–symmetric solution set

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Abstract

We consider a linear system $Ax = b$, where A is varying inside a given interval matrix $\boldsymbol{A}$ and b is varying inside a given interval vector $\boldsymbol{b}$. The solution set of such a system is described by famous Oettli–Prager Theorem. But if we are restricted only on symmetric (or skew–symmetric) matrices $A \in \boldsymbol{A}$, the problem is much more complicated. So far, the symmetric/skew–symmetric solution set description could be obtained only by a lengthy Fourier–Motzkin elimination applied on each orthant. We present a simple explicit description of the symmetric and skew–symmetric solution set by means of nonlinear inequalities the number of which is, however, still exponential.

Keywords: *Linear interval systems, solution set, interval matrix, symmetric matrix.*

1 Introduction

Real-life problems are often subjected to uncertainties in data measurements. Such uncertainties can be dealt with by methods of interval analysis [1]; instead of exact values we compute with compact real intervals. An interval matrix is defined as

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A} \leq \overline{A}$ are fixed matrices (n-dimensional interval vectors can be regarded as interval matrices n-by-1). By

$$A^c \equiv \frac{1}{2}(\underline{A} + \overline{A}), \quad A^\Delta \equiv \frac{1}{2}(\overline{A} - \underline{A})$$

we denote the midpoint and radius of $\boldsymbol{A}$, respectively. Let us consider a linear interval system of equalities

$$\boldsymbol{A}x = \boldsymbol{b}.$$

The solution set

$$\Sigma \equiv \{x \in \mathbb{R}^n \mid Ax = b,\ A \in \boldsymbol{A},\ b \in \boldsymbol{b}\}$$

is described by well-known Oettli–Prager condition [9]

$$x \in \Sigma \quad \Leftrightarrow \quad A^\Delta|x| + b^\Delta \geq |A^c x - b^c|.$$

In interval analysis, we usually suppose, that values varies in given intervals independently. But in some applications, dependencies can occur (cf. [5], [8]). Especially, we will focus on some types of the matrix A. The symmetric solution set

$$\Sigma_{sym} \equiv \{x \in \mathbb{R}^n \mid Ax = b,\ A = A^T,\ A \in \boldsymbol{A},\ b \in \boldsymbol{b}\}$$

and the skew–symmetric solution set

$$\Sigma_{skew} \equiv \{x \in \mathbb{R}^n \mid Ax = b,\ A = -A^T,\ A \in \boldsymbol{A},\ b \in \boldsymbol{b}\}$$

have been exhaustively studied in recent years [2]–[7]. Its description can be obtained by a Fourier–Motzkin elimination applied on each of 2^n orthants. In contrary to Σ, the symmetric solution set Σ_{sym} is not polyhedral, its shape is described by quadrics [3]–[6] and it is not generally convex even if intersected with an orthant. All these properties simply follow from the proposed Theorem 2 and Figures 1 to 2 (in Section 3).

Notation

$\mathbb{IR}^{m\times n}$	the set of all m-by-n interval matrices
$\mathbb{IR}^{n}$	the set of all n-dimensional interval vectors
$\prec_{\text{lex}}$	strict lexicographic ordering of vectors, i.e., $p \prec_{\text{lex}} q$ if for some k we have $p_i = q_i$, $i < k$, and $p_k < q_k$
$\preceq_{\text{lex}}$	lexicographic ordering of vectors, i.e., $p \preceq_{\text{lex}} q$ if $p \prec_{\text{lex}} q$ or $p = q$
$\square S$	interval hull of a set S
$A_{i,\bullet}$	the i-th row of a matrix A
$A_{\bullet,j}$	the j-th column of a matrix A
p^+	positive part of a real number p; i.e., $p^+ = max(0, p)$

2 Linear interval equations with particular dependences

This section aims to give a characterization of the linear interval system equipped with a certain dependency (Theorem 1); the matrix A occurs twice in the system – in (2) and transposed in (3). The description of the symmetric/skew–symetric solution set will be a simple consequence of Theorem 1. Another reason for dealing with such a dependency is that similar relations (occurence of a matrix and its transposition in a system) can appear in some applications, e.g. optimality conditions in linear programming.

Lemma 1. *Let $a, b \in \mathbb{R}^n$, $C, D^1, D^2 \in \mathbb{R}^{n\times n}$ and consider the function*

$$f(w) \equiv a^T w + b^T |w| + \sum_{i,j=1}^{n} c_{ij} |d^1_{ij} w_i + d^2_{ij} w_j|. \tag{1}$$

Checking whether $f(w) \geq 0 \;\forall w \in \mathbb{R}^n$ can be done by the following procedure.

(i) If $n = 1$, then $f(w) = a_1 w + (b_1 + c_{11}|d^1_{11} + d^2_{11}|)|w|$ and relation $f(w) \geq 0$ is true for all $w \in \mathbb{R}$ if and only if the inequality $b_1 + c_{11}|d^1_{11} + d^2_{11}| \geq |a_1|$ holds.

(ii) If $n > 1$, then we will proceed by recursion. $\mathcal{S}$ denotes the system of inequalities. Put $\mathcal{S} = \emptyset$.

(a) For each $i = 1$ to n do the following:
Put $w_i = 0$. Then $f(w)$ is a function of $n-1$ variables. Recursively compute the corresponding inequalities and add them to the system $\mathcal{S}$.

(b) For each $i, j = 1$ to n do the following:
Put $w_i = d^2_{ij} z$ and $w_j = -d^1_{ij} z$. Then $f(w, z)$ is again a function of $n-1$ variables. Recursively compute the corresponding inequalities and add them to the system $\mathcal{S}$.

Now, $f(w) \geq 0$ is true for all $w \in \mathbb{R}^n$ if and only if the system of inequalities $\mathcal{S}$ is satisfied.

Proof. If $f(w) \geq 0$ for all $w \in \mathbb{R}^n$, then the system $\mathcal{S}$ is obviously satisfied. For the proof of the converse implication let us consider a point $w^0 \in \mathbb{R}^n$ for which $f(w^0) < 0$ holds. If any absolute value from (1) is zero, then we can restrict our considerations to the less dimension. If no absolute value from (1) is zero, then the function $f(w)$ is linear on the neighbourhood $N(w^0)$ of w^0. Hence $f(w) = r^T w \ \forall w \in N(w^0)$ for some nonzero $r \in \mathbb{R}^n$. The half-space described by $r^T w < 0$ contains at least one axis, therefore there exists $w^1 \in \mathbb{R}^n$ for which $f(w^1) < 0$ and some absolute value from (1) is zero. We proved the statement by induction, the case $n = 1$ holds simply. □

Theorem 1. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, $\boldsymbol{b} \in \mathbb{IR}^n$, $\boldsymbol{d} \in \mathbb{IR}^n$ Then for certain $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $d \in \boldsymbol{d}$ vectors $x, y \in \mathbb{R}^n$ form a solution of the system*

$$Ax = b, \tag{2}$$

$$A^T y = d \tag{3}$$

if and only if they satisfy the following system of inequalities

$$A^\Delta |x| + b^\Delta \geq |r^1|, \tag{4}$$

$$A^\Delta |y| + d^\Delta \geq |r^2|, \tag{5}$$

$$\sum_{i,j=1}^{n} a^\Delta_{ij} |y_i x_j (p_i - q_j)| + \sum_{i=1}^{n} (b^\Delta_i |y_i p_i| + d^\Delta_i |x_i q_i|) \geq \left| \sum_{i=1}^{n} (r^1_i y_i p_i - r^2_i x_i q_i) \right| \quad \forall p, q \in \{0,1\}^n \setminus \{0^n\}, \tag{6}$$

where $r^1 \equiv -A^c x + b^c$, $r^2 \equiv -(A^c)^T y + d^c$.

Proof. Let $x, y \in \mathbb{R}^n$. Then x, y satisfy (2)–(3) iff for a certain $\alpha \in [-1,1]^{n\times n}$ relations

$$A^c_{i,\bullet}x + \sum_{k=1}^{n} \alpha_{ik} a^{\Delta}_{ik} x_k \in [b^c_i - b^{\Delta}, b^c_i + b^{\Delta}],\ \forall i = 1,\dots,n,$$
$$\left(A^c_{\bullet,j}\right)^T y + \sum_{k=1}^{n} \alpha_{kj} a^{\Delta}_{kj} y_k \in [d^c_j - d^{\Delta}, d^c_j + d^{\Delta}],\ \forall j = 1,\dots,n$$

hold. Equivalently, iff the following problem

$$\max\ 0.\alpha$$

subject to

$$\begin{aligned}
-\sum_{k=1}^{n} \alpha_{ik} a^{\Delta}_{ik} x_k &\le -r^1_i + b^{\Delta},\ \forall i = 1,\dots,n,\\
\sum_{k=1}^{n} \alpha_{ik} a^{\Delta}_{ik} x_k &\le r^1_i + b^{\Delta},\ \forall i = 1,\dots,n,\\
-\sum_{k=1}^{n} \alpha_{kj} a^{\Delta}_{kj} y_k &\le -r^2_j + d^{\Delta},\ \forall j = 1,\dots,n,\\
\sum_{k=1}^{n} \alpha_{kj} a^{\Delta}_{kj} y_k &\le r^2_j + d^{\Delta},\ \forall j = 1,\dots,n,\\
\alpha_{ij} &\le 1,\ \forall i,j = 1,\dots,n,\\
-\alpha_{ij} &\le 1,\ \forall i,j = 1,\dots,n
\end{aligned}$$

has an optimal solution. From duality theory in linear programming this problem has an optimal solution iff the problem

$$\min\ (-r^1 + b^{\Delta})^T w^1 + (r^1 + b^{\Delta})^T w^2 + (-r^2 + d^{\Delta})^T w^3 +$$
$$(r^2 + d^{\Delta})^T w^4 + \sum_{i,j=1}^{n} (w^5_{ij} + w^6_{ij})$$

subject to

$$-a^{\Delta}_{ij} x_j w^1_i + a^{\Delta}_{ij} x_j w^2_i - a^{\Delta}_{ij} y_i w^3_j + a^{\Delta}_{ij} y_i w^4_j + w^5_{ij} - w^6_{ij} = 0,\ \forall i,j = 1,\dots,n,$$
$$w^1, w^2, w^3, w^4, w^5, w^6 \ge 0$$

has an optimal solution. After substitution $u \equiv w^2 - w^1$, $v \equiv w^4 - w^3$ we can this problem rewrite as

$$\min\ (r^1 + b^\Delta)^T u + 2(b^\Delta)^T w^1 + (r^2 + d^\Delta)^T v + 2(d^\Delta)^T w^3 + \sum_{i,j=1}^n (w^5_{ij} + w^6_{ij})$$

subject to

$$\begin{aligned} a^\Delta_{ij} x_j u_i + a^\Delta_{ij} y_i v_j + w^5_{ij} - w^6_{ij} = 0,\ \forall i, j = 1, \ldots, n, \\ w^1 \geq -u,\ w^3 \geq -v,\ w^1, w^3, w^5, w^6 \geq 0. \end{aligned}$$

For optimal w^1, w^3, w^5, w^6 we have $w^5_{ij} + w^6_{ij} = |a^\Delta_{ij} x_j u_i + a^\Delta_{ij} y_i v_j|$ (since one of w^5_{ij},w^6_{ij} is equal to zero), $w^1 = (-u)^+$, $w^3 = (-v)^+$. Hence the problem can be reformulated as an unconstrained optimization problem

$$\begin{aligned} \min_{u,v \in \mathbb{R}^n}\ & (r^1 + b^\Delta)^T u + 2(b^\Delta)^T (-u)^+ + (r^2 + d^\Delta)^T v + \\ & 2(d^\Delta)^T (-v)^+ + \sum_{i,j=1}^n a^\Delta_{ij} |x_j u_i + y_i v_j|. \end{aligned}$$

The positive part of real number p is equal to $p^+ = \frac{1}{2}(p + |p|)$ and the problem comes in the form

$$\min_{u,v \in \mathbb{R}^n}\ (r^1)^T u + (b^\Delta)^T |u| + (r^2)^T v + (d^\Delta)^T |v| + \sum_{i,j=1}^n |a^\Delta_{ij} x_j u_i + a^\Delta_{ij} y_i v_j|.$$

This problem has an optimal solution iff the function

$$f(u, v) \equiv (r^1)^T u + (b^\Delta)^T |u| + (r^2)^T v + (d^\Delta)^T |v| + \sum_{i,j=1}^n a^\Delta_{ij} |x_j u_i + y_i v_j| \tag{7}$$

is nonnegative pro all $u, v \in \mathbb{R}^n$.

We claim that it happens iff the conditions (4)–(6) are satisfied. If $f(u, v) \geq 0\ \forall u, v \in \mathbb{R}^n$, then put $u_i = 1$, $u_j = 0\ \forall j \neq i$, and $v = 0$ to obtain the i-th inequality in (4) and similarly for inequalities in (5). Inequalities in (6) follows from putting $u_i = y_i p_i$, $v_i = -x_i q_i$ and from putting $u_i = -y_i p_i$, $v_i = x_i q_i$.

The reverse implication will follow from Lemma 1 applied on the function (7). In each step of the presented procedure one absolute value is put to zero. Let us consider one branch of this procedure, i.e. the sequence of steps resulting in a one-variable function and one inequality (case (i)). Let I_u be the set of all indices $i \in \{1, \ldots, n\}$ such that in some step of the procedure there is put to zero the binomial (case (ii)-(b)) containing the variable u_i, but the new variable z will never be put to zero in this branch of procedure. In the similar way define the index set I_v for variables v_i. If $I_v = \emptyset$, then the resulting inequality is a convex combination of inequalities from (4). If $I_u = \emptyset$, then the resulting inequality is a convex combination of inequalities from (5). Otherwise, $x_i \neq 0$ holds for all $i \in I_v$ and $y_i \neq 0$ holds for all $i \in I_u$ (from definition of $I_u,\ I_v$). Then

$$u_i = y_i z,\ i \in I_u, \text{ and } v_i = -x_i z,\ i \in I_v, \tag{8}$$

which can be easily proven by induction: Putting an absolute value $|x_j u_i + y_i v_j|$ to zero means making a substitution $u_i \equiv y_i z$, $v_j \equiv -x_j z$. In the next steps of the procedure we can put an absolute value $|x_j u_k - y_k x_j z|$ to zero by substitution $u_k \equiv y_k z$, $z \equiv z$. Likewise for $|x_k y_i z + y_i v_k|$.

From (8) it follows that the resulting inequality is contained in the system (6) for $p_i = 1,\ i \in I_u,\ p_i = 0,\ i \notin I_u$ and $q_i = 1,\ i \in I_v,\ q_i = 0,\ i \notin I_v$. □

3 Symmetric solution set

In this section, let us suppose without loss of generality that $\boldsymbol{A} = \boldsymbol{A}^T$, i.e. matrices A^c, A^Δ are symmetric. Otherwise we restrict our considerations on the interval matrix $(\boldsymbol{a}_{ij} \cap \boldsymbol{a}_{ji})_{i,j=1}^n$.

Simple corollary of Theorem 1 enables us to obtain explicit description of the symmetric solution set Σ_{sym}. Nevertheless, the number of inequalities in the description is still exponential. Therefore when checking $x \in \Sigma_{sym}$ for only one vector x, it is better, from the theoretical viewpoint, to use the linear programming problem (from the proof of Theorem 1), which is polynomially solvable. The question whether Σ_{sym} can be described by polynomial number of inequalities is still open.

Theorem 2. *The symmetric solution set Σ_{sym} is described by the following* *system of inequalities*

$$A^{\Delta}|x| + b^{\Delta} \geq |r|, \tag{9}$$

$$\sum_{i,j=1}^{n} a_{ij}^{\Delta}|x_i x_j(p_i - q_j)| + \sum_{i=1}^{n} b_i^{\Delta}|x_i(p_i + q_i)| \geq \left|\sum_{i=1}^{n} r_i x_i(p_i - q_i)\right|$$
$$\forall p, q \in \{0,1\}^n \setminus \{0^n, 1^n\}, p \prec_{\text{lex}} q, \tag{10}$$

where $r \equiv -A^c x + b^c$*.*

Proof. Σ_{sym} can be equivalently described as the set of all $x \in \mathbb{R}^n$ satisfying

$$Ax = b^1, \tag{11}$$
$$A^T x = b^2 \tag{12}$$

for some $A \in \boldsymbol{A}$, $b^1, b^2 \in \boldsymbol{b}$ (since $\frac{1}{2}(A + A^T) \in \boldsymbol{A}$ is symmetric matrix and $\frac{1}{2}(b^1 + b^2) \in \boldsymbol{b}$). Put $y \equiv x$, $\boldsymbol{d} \equiv \boldsymbol{b}$ and apply Theorem 1 on the system (11)–(12). We obtain that Σ_{sym} is described by (9)–(10). To reduce the number of inequalities in (10), it is sufficient due to symmetry to consider only vectors $p, q \in \{0,1\}^n \setminus \{0^n, 1^n\}$ for which $p \prec_{\text{lex}} q$. □

The number of inequalities in (10) is exponential, namely $\frac{1}{2}(2^n - 2)(2^n - 3) = \mathcal{O}(4^n)$, but not as tremendous as by using Fourier–Motzkin elimination. For $n = 2$ we have only one additional inequality, for $n = 3$ the number rise up to 15 (cf. [3], [4], [6], Fourier–Motzkin elimination leads to 44 inequalities for only one orthant).

Example 1. For two dimensional case, the system (10) comes into only one inequality

$$a_{11}^{\Delta}x_1^2 + a_{22}^{\Delta}x_2^2 + b_1^{\Delta}|x_1| + b_2^{\Delta}|x_2| \geq |r_1 x_1 - r_2 x_2|,$$

or, equivalently

$$a_{11}^{\Delta}x_1^2 + a_{22}^{\Delta}x_2^2 + b_1^{\Delta}|x_1| + b_2^{\Delta}|x_2| \geq | - a_{11}^{c}x_1^2 + a_{22}^{c}x_2^2 + b_1^{c}x_1 - b_2^{c}x_2|.$$

In the list bellow we mention some particular examples. Figures 1 to 2 illustrate solution set (light grey color) and symmetric solution set (grey color):

1. (Figure 1) $\boldsymbol{A} = \begin{pmatrix} [1,2] & [0,a] \\ [0,a] & -1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$; here interval hull $\square\Sigma$ can be arbitrarily larger than $\square\Sigma_{sym}$, depending on the real parameter $a > 0$.

2. (Figure 2) $\boldsymbol{A} = \begin{pmatrix} -1 & [-5,5] \\ [-5,5] & 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ [1,3] \end{pmatrix}$; here $\Sigma = \mathbb{R}^2$ is unbounded, but Σ_{sym} is bounded.

3. For $\boldsymbol{A} = \begin{pmatrix} [0,1] & [1,2] \\ [1,2] & [-1,0] \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}$ we have $\Sigma = \Sigma_{sym}$ and both are bounded.

4. For $\boldsymbol{A} = \begin{pmatrix} [-1,1] & [0,2] \\ [0,2] & [-1,1] \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} [0,1] \\ [0,1] \end{pmatrix}$ we have $\Sigma = \Sigma_{sym}$ and both are unbounded.

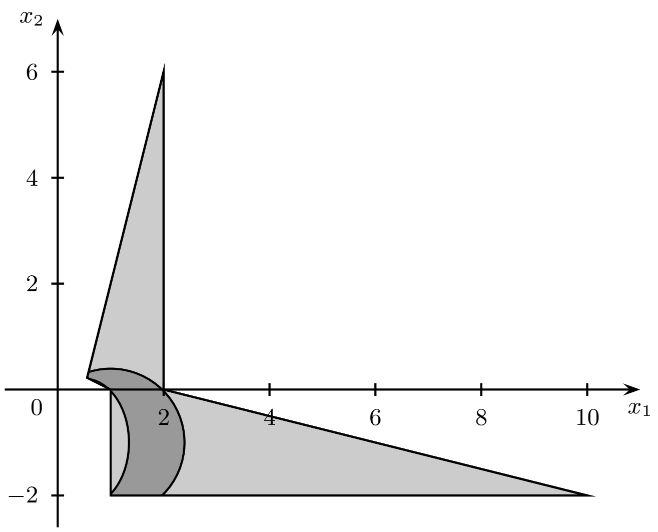

Figure 1: Solution set arbitrarily larger than symmetric solution set, $a = 4$.

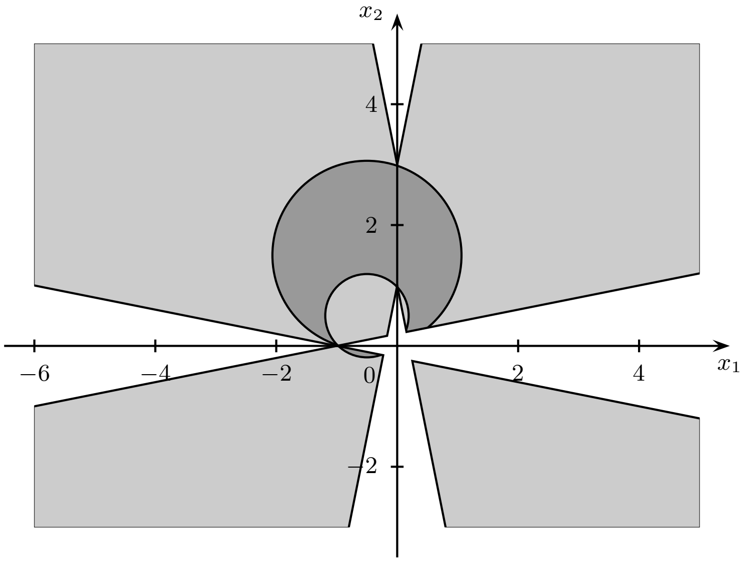

Figure 2: Unbounded solution set and bounded symmetric solution set.

4 Skew–symmetric solution set

In this section, let us suppose without loss of generality that $\boldsymbol{A} = -\boldsymbol{A}^T$ and the diagonal of $\boldsymbol{A}$ is zero. Therefore A^c is skew–symmetric and A^Δ is a symmetric matrix. The description of the skew–symmetric solution set Σ_{skew} follows from Theorem 1. The resulting number of inequalities in the description is again exponential. But in comparison with the upper bound $8\left(\frac{3}{2}\right)^{2^{\kappa+1}}$, $\kappa = \frac{1}{2}n(n+1)$, for the final number of inequalities obtained by Fourier–Motzkin elimination (see [4]), the improvement is significant.

Theorem 3. *The skew–symmetric solution set Σ_{skew} is described by the* *following system of inequalities*

$$A^\Delta|x| + b^\Delta \geq |r|, \tag{13}$$

$$\sum_{i,j=1}^{n} a_{ij}^\Delta |x_i x_j (p_i - q_j)| + \sum_{i=1}^{n} b_i^\Delta |x_i(p_i + q_i)| \geq \left|\sum_{i=1}^{n} r_i x_i (p_i + q_i)\right|$$
$$\forall p, q \in \{0,1\}^n \setminus \{0^n\}, p \preceq_{\text{lex}} q, \tag{14}$$

where $r \equiv -A^c x + b^c$.

Proof. Σ_{skew} can be equivalently described as the set of all $x \in \mathbb{R}^n$ satisfying

$$Ax = b^1, \tag{15}$$

$$A^T(-x) = b^2 \tag{16}$$

for some $A \in \boldsymbol{A}$, $b^1, b^2 \in \boldsymbol{b}$ (since $\frac{1}{2}(A - A^T) \in \boldsymbol{A}$ is a skew–symmetric matrix and $\frac{1}{2}(b^1 + b^2) \in \boldsymbol{b}$). Put $y \equiv -x$, $\boldsymbol{d} \equiv \boldsymbol{b}$. Then

$$r^1 = -A^c x + b^c = -(-A^c)(-x) + b^c = -(A^c)^T y + d^c = r^2 \equiv r.$$

Apply Theorem 1 on the system (15)–(16). We obtain that Σ_{skew} is described by (13)–(14). To reduce the number of inequalities in (14), it is sufficient due to symmetry to consider only vectors $p, q \in \{0,1\}^n \setminus \{0^n\}$ for which $p \preceq_{\text{lex}} q$. $\square$

Remark 1. The number of inequalities in (14) is $2^{n-1}(2^n - 1)$ and can be furthermore decreased to the number $2^n - n - 1$. We claim that it is sufficient to consider only such inequalities for which $p = q$ holds. For given vectors p, q define $I_p = \{i \mid p_i = 1\}$, $I_q = \{i \mid q_i = 1\}$ and denote the inequality (14) corresponding to p, q by $Ineq(I_p, I_q)$. Let p, q be fixed and define $I = I_p \cap I_q$, $J = I_p \cup I_q$. We prove that $Ineq(I_p, I_q)$ is a consequence of

$$\frac{1}{2}\left(Ineq(I, I) + Ineq(J, J)\right). \tag{17}$$

The right-hand side of $Ineq(I_p, I_q)$ is $\left|\sum_{i \in I_p} r_i x_i + \sum_{i \in I_q} r_i x_i\right| = \left|\sum_{i \in I} r_i x_i + \sum_{i \in J} r_i x_i\right|$, which is not greater than $\left|\sum_{i \in I} r_i x_i\right| + \left|\sum_{i \in J} r_i x_i\right|$, the right-hand side of (17). The second sum in $Ineq(I_p, I_q)$ is equal to $\sum_{i \in I_p} b_i^\Delta |x_i| + \sum_{i \in I_q} b_i^\Delta |x_i|$, which is equal to $\sum_{i \in I} b_i^\Delta |x_i| + \sum_{i \in J} b_i^\Delta |x_i|$, the second sum in (17). To prove the remaining relations, it is sufficient to consider any indices $i_1 \neq i_2$ and the only four cases:

1. Case $i_1 \in I_p \setminus I_q, i_2 \notin I_p \cup I_q$. The corresponding terms in $Ineq(I_p, I_q)$ are $a^{\Delta}_{i_1 i_1}|x_{i_1} x_{i_1}| + a^{\Delta}_{i_1 i_2}|x_{i_1} x_{i_2}|$, while the corresponding term in (17) is only $a^{\Delta}_{i_1 i_2}|x_{i_1} x_{i_2}|$.
2. Case $i_1, i_2 \in I_p \setminus I_q$. The corresponding terms in $Ineq(I_p, I_q)$ are $a^{\Delta}_{i_1 i_1}|x_{i_1} x_{i_1}| + 2a^{\Delta}_{i_1 i_2}|x_{i_1} x_{i_2}| + a^{\Delta}_{i_2 i_2}|x_{i_2} x_{i_2}|$, while the corresponding term in (17) is empty.
3. Case $i_1 \in I_p \setminus I_q, i_2 \in I_q \setminus I_p$. The corresponding terms in $Ineq(I_p, I_q)$ are $a^{\Delta}_{i_1 i_1}|x_{i_1} x_{i_1}| + a^{\Delta}_{i_2 i_2}|x_{i_2} x_{i_2}|$, while the corresponding term in (17) is empty.
4. Case $i_1 \in I_p \cap I_q, i_2 \in I_p \setminus I_q$. The corresponding terms in $Ineq(I_p, I_q)$ are $a^{\Delta}_{i_1 i_2}|x_{i_1} x_{i_2}| + a^{\Delta}_{i_2 i_2}|x_{i_2} x_{i_2}|$, while the corresponding term in (17) is only $a^{\Delta}_{i_1 i_2}|x_{i_1} x_{i_2}|$.

Hence the system of inequalities (14) is equivalent to the system

$$\sum_{i<j} a^{\Delta}_{ij}|x_i x_j (p_i - p_j)| + \sum_{i=1}^{n} b^{\Delta}_i |x_i p_i| \geq \left| \sum_{i=1}^{n} r_i x_i p_i \right|, \ \forall p \in \{0,1\}^n \setminus \{0^n\}. \tag{18}$$

The last reduction follows from the fact that for each unit vector $p \equiv e_i$ the corresponding inequality in (18) represents a $|x_i|$-multiple of the i-th Oettli–Prager inequality (13).

Example 2. For $n = 2$, the system (18) is composed of only one inequality

$$b^{\Delta}_1 |x_1| + b^{\Delta}_2 |x_2| \geq |r_1 x_1 + r_2 x_2|,$$

or, equivalently

$$b^{\Delta}_1 |x_1| + b^{\Delta}_2 |x_2| \geq |b^c_1 x_1 + b^c_2 x_2|.$$

In this two dimensional case the set Σ_{skew} represents a polyhedral set which is convex in each orthant (cf. [4]). Some particular examples:

1. For $\boldsymbol{A} = \begin{pmatrix} 0 & [1,2] \\ [-2,-1] & 0 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} [0,2] \\ [-2,2] \end{pmatrix}$ we have $\Sigma = \Sigma_{skew}$ and both are bounded.
2. For $\boldsymbol{A} = \begin{pmatrix} 0 & [-1,1] \\ [-1,1] & 0 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} [0,2] \\ [-2,2] \end{pmatrix}$ we have $\Sigma = \Sigma_{skew}$ and both are unbounded.

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