Coloring of triangle-free graphs on the double torus

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Abstract

We show that every triangle-free graph on the double torus is 4-colorable. This settles a problem raised by Gimbel and Thomassen [Trans. Amer. Math. Soc. 349 (1997), 4555–4564].

1 Introduction

Colorings of graphs on surfaces permanently attract attention of researchers in graph theory. The most classical result is the Four Color Theorem [1, 13] which asserts that every planar graph is 4-colorable. Another classical result is Grötzsch's theorem [8] which states that every planar graph with no triangles is 3-colorable; also see [15, 16] for short proofs of this result.

Gimbel and Thomassen [7] generalized Grötzsch's theorem to surfaces of higher genera. While the chromatic number of a graph embedded on a surface of genus g is bounded by $O(g^{1/2})$ [9], the chromatic number of a triangle-free graph that can be embedded on a surface of genus g is bounded by $O\left((g/\log g)^{1/3}\right)$ [7]. On the other hand, there exist triangle-free graphs embeddable on a surface of genus g that have chromatic number $\Omega\left(g^{1/3}/\log g\right)$ [7].

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Let us focus on triangle-free graphs on surfaces of small genera. As we have already mentioned, triangle-free plane graphs are 3-colorable. Every triangle-free graph in the projective plane is 3-degenerate and thus 4-colorable. On the other hand, there are non-bipartite triangle-free projective planar quadrangulations and each such quadrangulation is 4-chromatic [18]. Kronk and White [11] established that every triangle-free graph on the torus is 4-colorable; as in the case of the projective plane, the bound cannot be improved since there exist triangle-free graphs on the torus that are not 3-colorable (one example is the Cayley graph for the group Z_{13} with generators 1 and 5 [2]). Gimbel and Thomassen [7] asked whether the result of Kronk and White can be extended to the double torus S_2:

Problem 1 ([7], Problem 9). *Is every graph on S_2 of girth four 4-colorable?*

In the present paper, we answer this question in the affirmative way (see Corollary 23). As there are triangle-free graphs on the torus that are not 3-colorable, the bound cannot be improved.

2 Preliminary observations

In this section, we recall standard graph theory notation related to graph colorings and critical graphs. However, we do not provide any detailed introduction to topological graph theory. We refer the reader to a recent monograph [12] if interested. The only fact that we will need in our further considerations is the following corollary of Euler's formula: the number of edges of a simple triangle-free graph that can be embedded on the surface S_g is at most $2n - 4 + 4g$. In particular, the following holds:

Lemma 1. *The number of edges of an n-vertex triangle-free graph that can be embedded on the double torus is at most $2n + 4$.*

We also assume that the reader is familiar with basic graph theory concepts such as k-colorability or the chromatic number of graphs. In our investigations, the structure of "extremal" non-$(k-1)$-colorable graphs will play a crucial role: a graph G is *k-critical* if it is not $(k-1)$-colorable but each proper subgraph of G is $(k-1)$-colorable. Clearly, the minimum degree of a k-critical graph is at least $k-1$. The vertices of a k-critical graph can be partitioned into two groups: *low-degree* vertices of degree $k-1$ and *high-degree* vertices of degree k or more. The subgraph induced by the low-degree vertices of G is denoted by $L_{k-1}(G)$ and that induced by the high-degree vertices by $H_{k-1}(G)$. One of the first results on critical graphs is the following theorem of Gallai that restricts the possible structure of the components of the low-degree subgraph. Recall that a *Gallai tree* is a graph such that each block (maximal 2-connected subgraph) is an odd cycle or a complete graph.

Theorem 2 (Gallai [6]). *If G is a k-critical graph, then each component of $L_{k-1}(G)$ is a Gallai tree.*

Gallai [6] also studied which colorings of the high-degree subgraph cannot be extended to the components of low-degree subgraphs. In particular, he showed that if each component of $L_{k-1}(G)$ contains a vertex adjacent to two vertices of $H_{k-1}(G)$ of the same color, then the coloring of $H_{k-1}(G)$ can be extended to all the components of $L_{k-1}(G)$. This is in fact a special case of a more general phenomenon studied in list colorings [3, 5, 17]. Results obtained in this area allow us to replace the original condition of Gallai by several others. In particular, the following theorem is true:

Theorem 3. *Let G be a graph and $k \geq 1$ a fixed integer. Any precoloring of $H_{k-1}(G)$ with k colors can be extended to any component of $L_{k-1}(G)$ that has*

- *a vertex adjacent to two vertices of $H_{k-1}(G)$ of the same color, or*
- *two adjacent vertices v_1 and v_2 of degree two such that their neighbors in $H_{k-1}(G)$ have at least three distinct colors.*

We will also use the following result of Stiebitz [14] that allows us to bound the number of components of the high-degree subgraph by the number of components of the low-degree subgraph:

Theorem 4 (Stiebitz [14]). *If G is a k-critical graph that contains a vertex of degree $k-1$, then the number of components of $H_{k-1}(G)$ does not exceed the number of components of $L_{k-1}(G)$.*

Our goal is to show that every triangle-free graph on the double torus is 4-colorable. In order to do so, we show that there are no 5-critical triangle-free graphs that can be embedded on the double torus (Theorem 22). As we have already observed, the minimum degree of a 5-critical graph is four. On the other hand, Lemma 1 implies that the average degree of each triangle-free graph on the double torus is at most slightly above four. Hence the high-degree subgraph of a triangle-free 5-critical graph on the double torus cannot contain too many vertices.

Lemma 5. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. Let ℓ be the number of vertices of $H_4(G)$ and $d_1, \ldots, d_\ell$ their degrees in G. The sum $\sum_{i=1}^{\ell}(d_i - 4)$ is at most eight. In particular, the number of vertices of $H_4(G)$ is at most eight.*

Lemma 5 easily follows from Lemma 1 and the fact that the minimum degree of a 5-critical graph is four, and we leave its detailed proof to the reader.

We finish this section with two results on the minimum number of vertices of a triangle-free graph that is not 3- or 4-colorable. Let us remark that the bounds given in the following two theorems are best possible.

Theorem 6 (Chvátal [4]). *Every triangle-free graph on at most 10 vertices is 3-colorable.*

Theorem 7 (Jensen and Royle [10]). *Every triangle-free graph on at most 21 vertices is 4-colorable.*

3 Structure of the low-degree subgraph

In this section, we focus on the possible structure of the low-degree subgraph of triangle-free 5-critical graphs on the double torus. We define the *weight* of a component G_0 of $L_4(G)$ to be the number of edges between G_0 and $H_4(G)$. Note that the weight of each component of $L_4(G)$ is even. The next lemma provides us with a simple bound on the total weight of the components of $L_4(G)$ in terms of the number of high-degree vertices and the number of edges of $H_4(G)$.

Lemma 8. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. If $H_4(G)$ has ℓ vertices and m edges, then the total weight of the components of $L_4(G)$ is at most $4\ell + 8 - 2m$.*

Proof. Let $d_1, \ldots, d_\ell$ be the degrees of the vertices of $H_4(G)$. By Lemma 5, the sum of the degrees $\sum_{i=1}^{\ell} d_i$ of the vertices of $H_4(G)$ is at most $4\ell + 8$. If the number of edges of $H_4(G)$ is m, the number of edges between $H_4(G)$ and $L_4(G)$ is at most $4\ell + 8 - 2m$. Hence the total weight of the components of $L_4(G)$ is at most $4\ell + 8 - 2m$. □

By Theorem 2, each component of $L_4(G)$ is a Gallai tree. Let us call a component G_0 of $L_4(G)$ that has no vertices of degree one and that is not an odd cycle a *grunter*. Since each end-block of a grunter must be an odd cycle of length at least five (recall that we are considering triangle-free graphs), the next lemma readily follows from Theorem 2:

Lemma 9. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus and let G_0 be a component of $L_4(G)$. If G_0 does not contain a vertex of degree one, then either G_0 is an odd cycle or it is a grunter. In particular, if G_0 does not contain a vertex of degree one, it contains two adjacent vertices of degree two.*

The simplest components of $L_4(G)$ are trees. A weight of such an n-vertex component is $2n+2$. In the next lemma, we show that if the weight of an n-vertex component is significantly smaller than $2n+2$, then n must be quite large. This will allow us to efficiently analyze the structure of components of $L_4(G)$ based on their weights in our further considerations.

Lemma 10. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. If G' is an n-vertex component of $L_4(G)$, then the weight of G' is equal to $2n+2$ if and only if G' is a tree, and equal to $2n$ if and only if G' is a cycle or a unicyclic graph. Moreover, if the weight of G' is $2n-2\ell$, then $n \geq 5+4\ell$. In particular, if the weight of G' is less than $2n$, the weight of G' is at least 16.*

Specifically, if G' has no vertex of degree one, then its weight is at least $2n \geq 10$, and if G' is a grunter, then its weight is at least 16.

Proof. Observe first that if G' has k blocks that are odd cycles, then the weight of G' is $2n+2-2k$. Moreover, if G' has k blocks that are odd cycles, the number of vertices of G' is at least $4k+1$. Hence if the weight of G' is $2n+2$, then $k=0$ and G' is a tree. If the weight of G' is $2n$, then $k=1$ and G' is a cycle or a unicyclic graph.

Assume now that the weight of G' is $2n-2\ell$. We conclude that G' contains $k=\ell+1$ blocks that are odd cycles and thus it contains at least $4k+1=4\ell+5$ vertices. In particular, if the weight of G' is less than $2n$, i.e., $\ell \geq 1$, then G' has weight at least

$$2 \cdot (4\ell+5) - 2\ell = 6\ell + 10 \geq 16 .$$

It remains to prove the last part of the lemma. If G' has no vertex of degree one, then $k \geq 1$ and the weight of G' is at least

$$2n+2-2k \geq 2(4k+1)+2-2k = 6k+4 \geq 10 .$$

If G' is a grunter, then $k \geq 2$ and consequently its weight is at least $6k+4 \geq 16$. The whole lemma has now been established. □

We now aim to utilize our observations on the structure of the low-degree subgraph of a triangle-free 5-critical graph on the double torus. Let us start by showing that the number of components of $L_4(G)$ must be at least two.

Lemma 11. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. The subgraph $L_4(G)$ contains at least two components.*

Proof. By Theorem 4, $L_4(G)$ contains at least one component. Assume for the sake of contradiction that $L_4(G)$ has a single component G'. Since G' is a Gallai tree it contains a vertex v of degree at most two. Let w_1 and w_2 be two neighbors of v in $H_4(G)$. Note that the vertices w_1 and w_2 are not adjacent since G is triangle-free. Consider a coloring of $H_4(G)$ with three colors, which exists by Lemma 5 and Theorem 6, and recolor the vertices w_1 and w_2 with the fourth (unused) color. By Theorem 3, the precoloring of the vertices of $H_4(G)$ can be extended to a coloring of G with four colors—a contradiction. □

We finish this section with three lemmas which bound the number of components of $L_4(G)$ under certain assumptions on the total number of vertices and the total weight of the components of $L_4(G)$.

Lemma 12. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. If the total weight of the components of $L_4(G)$ is at most 26 and $L_4(G)$ has at least 15 vertices, then the number of components of $L_4(G)$ does not exceed one.*

Proof. Assume for the sake of contradiction that $L_4(G)$ has at least two components. Since $26 < 2 \cdot 15$, $L_4(G)$ must contain an n_1-vertex component G_1 of weight at most $2n_1 - 2$.

If the weight of G_1 is exactly $2n_1 - 2$, $L_4(G)$ must contain another n_2-vertex component G_2 of weight less than $2n_2$. By Lemma 8, the weight of each of G_1 and G_2 is at least 16. Since the total weight of the components of $L_4(G)$ is at most 26, the component G_2 could not exist. We conclude that the weight of G_1 is at most $2n_1 - 4$.

By Lemma 8, the weight of G_1 is at least 22. Since the total weight of the components of $L_4(G)$ is at most 26, the weight of G_1 is 22 and the weight of the other component of $L_4(G)$ is four. We conclude that G_1 has 13 vertices and the other component of $L_4(G)$ is comprised of a single vertex. Hence

$L_4(G)$ has $n_1 + 1 = 13 + 1 = 14$ vertices which contradicts our assumption that the number of vertices of $L_4(G)$ is at least 15. $\square$

In the next lemma, we show that the number of components of $L_4(G)$ does not exceed two under some weaker assumptions.

Lemma 13. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. If the total weight of the components of $L_4(G)$ is at most 28 and $L_4(G)$ has at least 14 vertices, then the number of components of $L_4(G)$ does not exceed two.*

Proof. Assume for the sake of contradiction that $L_4(G)$ is comprised of three or more components. Since $28/3 < 10$, $L_4(G)$ contains a component that is a tree by Lemma 10. Let G_1 be such a component and n_1 its number of vertices. By Lemma 10, the weight of G_1 is $2n_1 + 2$.

Since the number of vertices of $L_4(G)$ is at least 14 and the total weight of its components is 28, $L_4(G)$ must contain an n_2-vertex component G_2 of weight at most $2n_2 - 2$. Since the weight of G_2 is at most $28 - 2 \cdot 4 = 20$, the weight of G_2 is $2n_2 - 2$ by Lemma 10. On the other hand, since the sum of the weights of G_1 and G_2 is at least $4 + 16 = 20$, any component of $L_4(G)$ distinct from G_1 and G_2 is a tree by Lemma 10. Hence G_2 is the only component of $L_4(G)$ which is not a tree. We infer from Lemma 10 that the total weight of the components of $L_4(G)$ is at least $2 \cdot 14 + 2 = 30$ which is impossible. This completes the proof. $\square$

Finally, under even weaker assumptions, the number of components of $L_4(G)$ is at most three.

Lemma 14. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. If the total weight of the components of $L_4(G)$ is at most 30 and $L_4(G)$ has at least 14 vertices, then the number of components of $L_4(G)$ does not exceed three.*

Proof. Assume for the sake of contradiction that $L_4(G)$ contains four or more components. Since the average weight of the components of $L_4(G)$ is at most $30/4 < 10$, one of the components of $L_4(G)$ is a tree, say G_1. Since the average weight of the components of $L_4(G)$ distinct from G_1 is at most $(30 - 4)/3 < 10$, $L_4(G)$ contains another component G_2 that is a tree. Let n_1 and n_2 be the number of vertices of G_1 and G_2, respectively. By Lemma 10, the sum of the weights of G_1 and G_2 is $2(n_1 + n_2) + 4$.

Since $L_4(G)$ has 14 vertices and the total sum of the weights of the components of $L_4(G)$ is 30, G must contain an n_3-vertex component G_3 of weight at most $2n_3 - 2$. Since the weight of G_3 is at most $30 - 3 \cdot 4 = 18$, the weight of G_3 is $2n_3 - 2$ by Lemma 10. Note that the weight of G_3 must also be at least 16. Because the total weight of G_1, G_2 and G_3 is at least 24, $L_4(G)$ contains four components and the component G_4 is a tree. Let n_4 be the number of vertices of G_4. By Lemma 10, the total weight of the components of $L_4(G)$ is $2(n_1+n_2+n_3+n_4)+4$. Since $n_1+n_2+n_3+n_4 \geq 14$, the total weight of the components of $L_4(G)$ must be at least 32 which is impossible. □

4 Number of edges in the high-degree subgraph

In this section we focus on estimates on the number of edges that could be contained in the high-degree subgraph ($H_4(G)$) of a triangle-free 5-critical graph on the double torus. We start by showing a lower bound on the number of such edges.

Lemma 15. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. The subgraph $H_4(G)$ contains at least three edges.*

Proof. Assume to the contrary that there exists such a graph G in which $H_4(G)$ contains at most two edges. If $H_4(G)$ has at most one edge or two edges sharing a vertex, we can color the vertices of $H_4(G)$ in such a way that all the vertices of $H_4(G)$ have the same color except for a single vertex with a different color. Let G' be a component of $L_4(G)$. If G' contains a vertex v of degree one, then v is adjacent to two vertices of $H_4(G)$ of the same color. If G' contains no vertices of degree one, then G' contains two adjacent vertices v_1 and v_2 of degree two by Lemma 9. Since G is triangle-free, the vertices v_1 and v_2 are adjacent to four different vertices of $H_4(G)$. Hence one of them is adjacent to two vertices of the same color. We conclude that each component of $L_4(G)$ has a vertex adjacent to two vertices of $H_4(G)$ of the same color. Theorem 3 implies that G is 4-colorable which contradicts our assumption that G is 5-critical.

It remains to consider the case when $H_4(G)$ is formed by two disjoint edges, say x_1x_2 and y_1y_2, and ℓ isolated vertices, $0 \leq \ell \leq 4$. Let c_{ij}, $i, j = 1, 2$, be the coloring of $H_4(G)$ that assigns the vertices x_i and y_j the same color and all the remaining vertices of $H_4(G)$ another color. Let us consider a component G' of $L_4(G)$. Assume that G' has no vertices of degree one and each vertex of G' is adjacent to vertices of $H_4(G)$ with mutually distinct colors. Since $H_4(G)$ is colored with only two colors, G' has no vertices of degree one. Moreover, each vertex of G' of degree two is adjacent to either x_i or y_j. Since two vertices adjacent in G' cannot both be neighbors of x_i or of y_j, G' is not an odd cycle. Hence Lemma 9 yields that G' is a grunter. Moreover, each vertex of degree two contained in an end-block of the grunter is adjacent to either x_i or y_j (note that the vertices of degree two in an end-block must be adjacent alternately to x_i and y_j).

Let us now distinguish two cases based on the number of grunter components of $L_4(G)$. The number of grunter components of $L_4(G)$ is at most two since each grunter has weight at least 16 and the total weight of the components of $L_4(G)$ is at most $40 - 4 = 36$ by Lemma 8.

If $L_4(G)$ has a single grunter component G_1 and all the vertices of degree two of G_1 are adjacent to two vertices of the same color for both the colorings c_{11} and c_{12}, then all the vertices of G_1 of degree two in its end-blocks are adjacent to x_1 and y_1 (because of the coloring c_{11}) and to x_1 and y_2 alternately (because of the coloring c_{12}). Hence all such vertices adjacent to y_1 are also adjacent to y_2 and vice versa. We conclude that G contains a triangle since the vertices y_1 and y_2 are adjacent.

It remains to analyze the case when $L_4(G)$ has two components that are grunters. Let G_1 and G_2 be these two components. Assume that for each of the four colorings c_{ij}, all the vertices of G_1 or all the vertices G_2 of degree two are adjacent to vertices of $H_4(G)$ with two different colors. An argument analogous to that used at the end of the previous paragraph yields that there is a vertex of degree two in G_1 adjacent to two vertices of the same color in the coloring c_{11} or c_{12}. Hence the "bad" colorings for G_1 are c_{11} and c_{22} and the "bad" colorings for G_2 are c_{12} and c_{21}. In particular, each vertex of degree two lying in an end-block of G_1 is either adjacent to x_1 and y_2 or to x_2 and y_1. Note that G_1 has at least four vertices of the former kind and at least four vertices of the latter kind. Similarly, each vertex of degree two lying in an end-block of G_2 is either adjacent to x_1 and y_1 or to x_2 and y_2, and the number of vertices of each of the two kinds is at least four. We conclude that all the vertices x_1, x_2, y_1 and y_2 have eight neighbors in G_1 and G_2. Hence the degree of each of them in G is at least nine which is impossible by Lemma 5.

We infer from our discussion that for at least one of the colorings c_{11}, c_{12}, c_{21}, c_{22} each component of $L_4(G)$ has a vertex adjacent to two vertices of $H_4(G)$ of the same color: indeed, we have argued that for the components

G_1 and G_2, and other components contain vertices of degree one. They are adjacent to vertices of $H_4(G)$ of the same color since the vertices of $H_4(G)$ are 2-colored. We can now infer from Theorem 3 that G is 4-colorable which contradicts our assumption that G is 5-critical. □

We have just established a lower bound on the number of edges of $H_4(G)$. Our next aim is to find an upper bound.

Lemma 16. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. The subgraph $H_4(G)$ contains at most six edges.*

Proof. Assume for the sake of contradiction that $H_4(G)$ has seven or more edges. By Lemma 5 and Theorem 7, $L_4(G)$ has at least 14 vertices, and, by Lemma 11, $L_4(G)$ is comprised of at least two components. By Lemma 8, the total weight of the components of $L_4(G)$ cannot exceed $40 - 14 = 26$. Since $26 < 2 \cdot 14$, $L_4(G)$ has an n_1-vertex component G_1 of weight less than $2n_1$. On the other hand, the weight of G_1 is at least 16 by Lemma 10. Hence the weight of any other component G_2 of $L_4(G)$ is at most 10. If G_2 were a tree, then $L_4(G)$ would have to contain another component of weight less than twice the number of its vertices which is impossible. We conclude that G_2 is a cycle of length 5 and its weight is exactly 10. Consequently, the weight of G_1 is 16 and G_1 is a double-5-cycle, i.e., two 5-cycles sharing a vertex. Since the total weight of G_1 and G_2 is 26, the number of vertices of $H_4(G)$ is eight and it contains exactly seven edges. Moreover, the degree of each vertex of $H_4(G)$ in G is five.

Since $H_4(G)$ has eight vertices and seven edges, $H_4(G)$ contains at least four vertices v of degree at most two. Since G is triangle-free, the vertices of G_2 are adjacent to at least five different vertices of $H_4(G)$ and thus there is a vertex v of $H_4(G)$ of degree at most two adjacent to a vertex of G_2. Since each vertex of $H_4(G)$ can be adjacent to at most two vertices of G (otherwise G would contain a triangle) and the degree of every vertex of $H_4(G)$ in G is five, there exists a vertex v_0 of $H_4(G)$ adjacent to both a vertex of G_1 and a vertex of G_2. Let v_i be any of the neighbors of v_0 in G_i, $i = 1, 2$. Note that the degree of v_i in G_i is two. Since G_1 is a double-5-cycle, the vertex v_1 has a neighbor v'_1 of degree two in G_1. Similarly, v_2 has a neighbor v'_2 of degree two in G_2.

Now color the vertices of $H_4(G)$ with three colors (this is possible by Theorem 6) and recolor the vertex v_0 with the fourth (unused) color. If the vertex v'_1 is not adjacent to two vertices of $H_4(G)$ of the same color, then the neighbors of v_1 and v'_1 in $H_4(G)$ must have at least three distinct colors (two colors appear because of the neighbors of v_1' and the third color appears because of the vertex v_0; note that v_0 is not a neighbor of v_1' since G is triangle-free). Similarly, if v_2' is not adjacent to two vertices of $H_4(G)$ of the same color, then the neighbors of v_2 and v_2' have at least three distinct colors. We infer from Theorem 3 that the precoloring of the vertices of $H_4(G)$ can be extended to a coloring of all the vertices of G with four colors—a contradiction. $\square$

Lemmas 15 and 16 now imply the following:

Lemma 17. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. The number of edges of $H_4(G)$ is between three and six.*

5 Structure of the high-degree subgraph

In this section, we further refine our knowledge about the possible structure of high-degree subgraphs of triangle-free 5-critical graphs G on the double torus. Let us start by showing that the number of vertices of $H_4(G)$ must be eight for every such graph G.

Lemma 18. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. The number of vertices of $H_4(G)$ is eight. In particular, the degree of each vertex of $H_4(G)$ is five.*

Proof. Assume first that $H_4(G)$ has at most six vertices. By Lemma 17, $H_4(G)$ contains at least three edges. Hence the total weight of the components of $L_4(G)$ is at most $32 - 6 = 26$ by Lemma 8. By Lemma 11, $L_4(G)$ is comprised of at least two components, and by Theorem 7, $L_4(G)$ has at least $22 - 6 = 16$ vertices which is impossible by Lemma 12.

It remains to exclude the case when the number of vertices of $H_4(G)$ is seven. By Lemma 17, the number of edges of $H_4(G)$ is between three and six. We first exclude the case when $H_4(G)$ has only three edges. Since $H_4(G)$ has seven vertices, $H_4(G)$ is comprised of at least four components. Hence the number of components of $L_4(G)$ is at least four by Theorem 4. By Lemma 8, the total weight of the components of $L_4(G)$ is $36 - 6 = 30$, and by Theorem 7, the number of vertices of $L_4(G)$ is at least 15. However, such a graph G cannot exist by Lemma 14.

We conclude that $H_4(G)$ has at least four edges. Recall that the number of vertices of $L_4(G)$ is at least 15 by Theorem 7. If $H_4(G)$ has four edges, then Lemma 8 yields that the total weight of the components of $L_4(G)$ is at most $36 - 8 = 28$ and Theorem 4 yields that the number of components of $L_4(G)$ is at least three. Such a graph G cannot exist by Lemma 13. If $H_4(G)$ has five or more edges, then by Lemma 8 the total weight of the components of $L_4(G)$ is at most $36 - 10 = 26$ and Theorem 4 yields that the number of components of $L_4(G)$ is at least two. However, Lemma 12 excludes the existence of such a graph G. We conclude that $H_4(G)$ must have eight vertices. The rest follows from Lemma 5. □

Another fact that we establish in this section is that $H_4(G)$ must be bipartite.

Lemma 19. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. The subgraph $H_4(G)$ is bipartite.*

Proof. Assume for the sake of contradiction that $H_4(G)$ is not bipartite. By Lemma 18, the number of vertices of $H_4(G)$ is eight. Since the number of edges of $H_4(G)$ does not exceed six by Lemma 17 and G is triangle-free, $H_4(G)$ contains five or six edges. In particular, $H_4(G)$ contains a cycle of length five and possibly one more edge. In the rest, we distinguish two cases based on the number of edges of $H_4(G)$ and eventually obtain contradiction in each of them.

Assume first that $H_4(G)$ contains exactly five edges, i.e., $H_4(G)$ is comprised of a cycle of length five and three isolated vertices. In particular, the number of components of $L_4(G)$ is at least four by Theorem 4. By Lemma 8, the total weight of the components of $L_4(G)$ is at most $40 - 2 \cdot 5 = 30$, and by Theorem 7, the number of vertices of $L_4(G)$ is at least 14. Lemma 14 now excludes the existence of G.

We have shown that $H_4(G)$ must contain six edges. Consequently, the number of components of $H_4(G)$ is three. By Lemma 8, the total weight of the components of $L_4(G)$ is $40 - 2 \cdot 6 = 28$, and by Theorem 7, the number of vertices of $L_4(G)$ is at least 14. Theorem 4 and Lemma 13 now exclude the existence of such a graph G. □

6 Bipartite high-degree subgraph with eight vertices

In this section, we utilize our observations to exclude the existence of a 5-chromatic triangle-free graph that can be embedded on the double torus.

Let us start by observing that if the high-degree subgraph is bipartite, then at least one of the components of the low-degree subgraph is an odd cycle or a grunter.

Lemma 20. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. If $H_4(G)$ is bipartite, then $L_4(G)$ has a component that is either an odd cycle or a grunter.*

Proof. Assume the contrary, i.e., that each component of $L_4(G)$ has a vertex of degree one by Theorem 4, and consider any 2-coloring of the vertices of $H_4(G)$. Since each component of $L_4(G)$ has a vertex adjacent to two vertices of the same color, G is 4-colorable by Theorem 3 contrary to our assumption that G is 5-critical. □

In the proof of the main theorem, we consider 3-colorings of $H_4(G)$ such that one of the colors is assigned to a single vertex. We claim that all such colorings can be extended to all components of $L_4(G)$ of weight at most eight.

Lemma 21. *Let G be a triangle-free 5-critical graph that can be embedded on the double torus. If the vertices of $H_4(G)$ are precolored with three colors in such a way that one of the three colors is assigned to a single vertex of G, then the precoloring of $H_4(G)$ can be extended to any component of $L_4(G)$ of weight at most 8.*

Proof. Let G' be a component of $L_4(G)$ of weight at most 8. By Lemma 8, G' is a tree. Hence G' is either a single vertex, an edge, or a path comprised of two edges. If G' contains a vertex adjacent to two vertices of $H_4(G)$ of the same color, then the precoloring can be extended to G' by Theorem 3. Hence if the precoloring of $H_4(G)$ cannot be extended to G', then G' is a path $v_1v_2v_3$ and the vertices v_1 and v_3 are adjacent to vertices of three distinct colors. Since $H_4(G)$ is precolored with three distinct colors, the vertices v_1 and v_3 can be colored with the fourth color. Since the vertex v_2 has degree four, it is adjacent to vertices of at most three distinct colors and the coloring can also be extended to v_2 as desired. □

We are now ready to prove our main result.

Theorem 22. *There is no triangle-free 5-critical graph G that can be embedded on the double torus.*

Proof. By Lemmas 18 and 19, $H_4(G)$ is a bipartite graph with eight vertices. By Lemma 17, the number of edges of $H_4(G)$ is between three and six. Hence the total weight of the components of $L_4(G)$ is at most 34 by Lemma 8.

In the rest of the proof, we establish a series of claims that eventually yield the proof of the theorem.

Claim 1. *If $L_4(G)$ contains a component G_1 whose weight is less than twice the number of the vertices of G_1, then $L_4(G)$ contains another component of weight ten or more.*

Let n_1 be the number of vertices of G_1. Since G_1 is the only component of $L_4(G)$ of weight ten or more, the remaining components of $L_4(G)$ are trees of weight at most eight by Lemma 10. By Lemmas 10 and 20, G_1 is a grunter (G_1 cannot be a cycle since its weight is less than n_1). Consider any 2-coloring c of $H_4(G)$. If the precoloring c cannot be extended to $L_4(G)$, then each vertex of G_1 of degree two is adjacent to two vertices of $H_4(G)$ of two distinct colors. Let v_1 and v_2 be any two adjacent vertices of degree two in G_1. Now recolor any vertex of $H_4(G)$ adjacent to v_1 with the third (unused) color. G_1 now contains two adjacent vertices of degree two, namely v_1 and v_2, such that their neighbors in $H_4(G)$ are colored with three distinct colors. In particular, the precoloring of $H_4(G)$ can be extended to G_1 by Theorem 3. Since the precoloring of $H_4(G)$ can be extended to the remaining components of $L_4(G)$ by Lemma 21, G is 4-colorable which is impossible.

Claim 2. *The subgraph $L_4(G)$ contains no component of weight less than twice the number of its vertices.*

Assume the contrary, i.e., that $L_4(G)$ contains a component G_1 with n_1 vertices of weight at most $2n_1 - 2$. Claim 1 yields that $L_4(G)$ contains another component of weight ten or more. Let G_2 be this component of $L_4(G)$ and n_2 the number of its vertices. Since the weight of G_1 is at most $2n_1 - 2$, its weight is at least 16 by Lemma 10. Since the total weight of G_1 and G_2 is at least $16 + 10 = 26$, the weight of any component of $L_4(G)$ distinct from G_1 and G_2 is at most eight. Assume that G_1 contains a vertex w of degree one. Hence G_2 is a grunter or a cycle by Lemma 20. In particular, G_2 contains two adjacent vertices of degree two, say v_1 and v_2. Since G is triangle-free, v_1 and v_2 have four distinct neighbors in H_4, say $u_1, \ldots, u_4$. By symmetry, we can assume that u_1 is not a neighbor of w.

Now consider a coloring of the vertices of $H_4(G)$ with two colors. If G_2 contains a vertex of degree two adjacent to two vertices of $H_4(G)$ of the same color, then the precoloring of the vertices of $H_4(G)$ can be extended to all the components of $L_4(G)$ by Theorem 3 since each component of $L_4(G)$ has a vertex adjacent to two vertices of $H_4(G)$ of the same color. Otherwise, each vertex of G_2 of degree two is adjacent to two vertices of $H_4(G)$ of distinct colors. Now recolor the vertex u_1 with the third (unused) color. The precoloring of $H_4(G)$ can be extended to G_1 by Theorem 3 since the vertex w is adjacent to two vertices of $H_4(G)$ of the same color, to G_2 since the vertices v_1 and v_2 are adjacent to vertices of $H_4(G)$ of three distinct colors, and to the remaining components of $L_4(G)$ by Lemma 21.

We conclude that G_1 has no vertex of degree one. An analogous argument yields that G_2 also has no vertex of degree one. Since the weight of G_1 is at most $2n_1 - 2$, G_1 is a grunter. If G_2 were also a grunter, then the sum of the weights of G_1 and G_2 would be at least $2 \cdot 16 = 32$ and G_1 and G_2 would be the only components of $L_4(G)$. By Lemma 8, the number of edges of $H_4(G)$ would be at most four and thus the number of components of $H_4(G)$ would be at least four which is impossible by Theorem 4. Hence G_2 is an odd cycle.

Let v_1 and v_2 be two adjacent vertices of degree two in G_1 and let A_1 be the set of their four neighbors in $H_4(G)$. Let A_2 be the set of the neighbors of the vertices of G_2 in $H_4(G)$. Since G is triangle-free, A_2 contains at least five distinct vertices. Hence there is a vertex u contained in both A_1 and A_2. By symmetry, we can assume that u is a neighbor of v_1. Let w_1 be a neighbor of u in G_2 and w_2 a vertex of G_2 adjacent to w_1.

Now consider a coloring of $H_4(G)$ with three distinct colors such that all the vertices except u are colored with only two colors and u is colored with the third color. By Lemma 21, the coloring can be extended to all the components of $L_4(G)$ with a possible exception of G_1 and G_2. If the vertex v_2 is adjacent to two vertices of $H_4(G)$ of the same color, the precoloring can be extended to G_1 by Theorem 3. Otherwise, the vertices v_1 and v_2 are adjacent to vertices of $H_4(G)$ with three distinct colors and the precoloring can be extended to G_1 again by Theorem 3. An analogous argument yields that the precoloring can be extended to G_2. Hence the graph G is 4-colorable which is impossible. The proof of Claim 2 is now complete.

We have established that the weight of each component of $L_4(G)$ is at least twice the number of its vertices. In particular, all the components of $L_4(G)$ are trees, cycles and unicyclic graphs. By Lemma 20, at least one of the components is an odd cycle. Let G_1 be this component and n_1 the number of its vertices. In addition, let A_1 be the neighbors of G_1 in $H_4(G)$.

Note that $|A_1| \geq 5$.

Claim 3. *The subgraph $L_4(G)$ has at least three components of weight ten or more.*

First assume that G_1 is the only component of $L_4(G)$ of weight ten or more. Now choose a vertex $w \in A_1$ and color the vertices of $H_4(G)$ with three colors in such a way that all the vertices of $H_4(G)$ except w are assigned only two colors. The precoloring can be extended to $L_4(G)$ by Theorem 3 and Lemma 21 which is impossible.

Assume next that $L_4(G)$ has two components of weight ten or more; let G_2 be such a component distinct from G_1 and let n_2 be the number of its vertices. If G_2 has a vertex v of degree one, choose $w \in A_1$ that is not a neighbor of v and proceed as in the previous paragraph. Otherwise, G_2 must be an odd cycle. Since the vertices of G_2 have at least five neighbors in $H_4(G)$, there exists a vertex $w \in A_1$ adjacent to both G_1 and G_2. Analogously to the preceding cases, the precoloring assigning the vertices of $H_4(G)$ except w two colors and w the third color can be extended to each component of $L_4(G)$.

Claim 4. *There is no triangle-free 5-critical graph that can be embedded on the double torus.*

Claim 3 implies that there are at least three components, say G_1, G_2 and G_3, that have weight ten or more. If $L_4(G)$ had four components, their total weight would be at least $3 \cdot 10 + 4 = 34$. Hence $H_4(G)$ would have only three edges by Lemma 8. However, $H_4(G)$ would have five components which is impossible by Theorem 4. We conclude that G_1, G_2 and G_3 are the only components of $L_4(G)$.

Since $L_4(G)$ is comprised of only three components, $H_4(G)$ has at least five edges by Theorem 4. Hence the total weight of the components of $L_4(G)$ is at most $40 - 5 \cdot 2 = 30$ by Lemma 8. Consequently, the weight of each G_i, $i = 1, 2, 3$, is 10 and $H_4(G)$ is a forest with exactly five edges.

By our assumptions G_1 is a cycle of length five. If both G_2 and G_3 were trees, the number of vertices of $L_4(G)$ would be $5 + 4 + 4 = 13$ which is impossible by Theorem 7. We conclude that G_2 is also a cycle of length five. Let A_i, $i = 1, 2$, be the set of vertices of $H_4(G)$ that are adjacent to a vertex of G_i. Note that $|A_i| \geq 5$ for $i = 1, 2$. Moreover, if $|A_i| = 5$, then each vertex of A_i is adjacent to exactly two vertices of G_i.

Now assume that G_3 is a tree and let v_1 and v_2 be two of its leaves. If $|A_1| > 5$ or $|A_2| > 5$, there exist three vertices u_1, u_2 and u_3 of $H_4(G)$

adjacent to vertices of both G_1 and G_2. Now consider a 2-coloring of the vertices of $H_4(G)$. If there is a vertex u_i, $i = 1, 2, 3$, not adjacent to v_1, recolor u_i with the third (unused) color. The coloring of the vertices of $H_4(G)$ can now be extended to G_1, G_2 and G_3 by Theorem 3. Hence the vertices u_1, u_2 and u_3 are neighbors of v_1. By symmetry, we can assume that the colors of u_1 and u_2 are the same. Let us now recolor the vertex u_3 with the third (unused) color. Again, Theorem 3 yields that the precoloring can be extended to the vertices of $L_4(G)$ which is impossible since G is not 4-colorable.

It remains to consider the case when $|A_1| = |A_2| = 5$. Note that each vertex of A_i is adjacent to exactly two vertices of G_i since G is triangle-free. Since $H_4(G)$ has eight vertices, there exist two vertices u_1 and u_2 adjacent to vertices of both G_1 and G_2. If u_i is not adjacent to v_1 or v_2, we consider a coloring of the vertices of $H_4(G)$ with three colors that assigns the vertex u_i a color different from all the other vertices of $H_4(G)$, and proceed as in the previous paragraph. Hence both u_1 and u_2 are adjacent to v_1 and v_2. However, this implies that the degree of u_1 is six which contradicts our previous deduction that all the vertices of $H_4(G)$ have degree five (see Lemma 18).

We conclude that the three components G_1, G_2 and G_3 are cycles of length five. As in the previous cases, let A_i be the set of neighbors of G_i in $H_4(G)$. If there exists a vertex $u \in A_1 \cap A_2 \cap A_3$, then we color the vertices of $H_4(G)$ with two colors and recolor u with the third (unused) color. We infer from Theorem 3 that the precoloring can be extended to $L_4(G)$ which contradicts our assumption that G is not 4-colorable.

It remains to consider the case when $A_1 \cap A_2 \cap A_3 = \emptyset$. Let u_{12} be a vertex of $A_1 \cap A_2$ and u_{23} a vertex of $A_2 \cap A_3$. Note that such vertices exist since $|A_i| \geq 5$ for every $i = 1, 2, 3$. Now color the vertices of $H_4(G)$ with two colors, recolor u_{12} with the third (unused) color and u_{23} with the fourth (unused) color. It is straightforward to check that each of the cycles G_1, G_2 and G_3 either contains a vertex adjacent to two vertices of the same color or its vertices are adjacent to vertices with at least three distinct colors and thus it has two adjacent vertices of degree two adjacent to vertices of at least three distinct colors in $H_4(G)$. Theorem 3 now yields that the precoloring can be extended to $L_4(G)$ which is impossible since G is not 4-colorable. $\square$

As a corollary of Theorem 22, we can now settle Problem 1:

Corollary 23. *Every graph on S_2 of girth four is 4-colorable.*

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