Injective colorings of planar graphs with few colors *

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Abstract

An injective coloring of a graph is a vertex coloring where two vertices have distinct colors if a path of length two exists between them. In this paper some results on injective colorings of planar graphs with few colors are presented. We show that all planar graphs of girth ≥ 19 and maximum degree Δ are injectively Δ-colorable. We also show that all planar graphs of girth ≥ 10 are injectively $(\Delta+1)$-colorable, $\Delta+4$ colors are sufficient for planar graphs of girth ≥ 5 if Δ is large enough, and that subcubic planar graphs of girth ≥ 7 are injectively 5-colorable.

1 Introduction

In this paper some results on injective coloring of planar graphs with large girth and few colors are presented. An *injective coloring* of a graph G is

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a mapping $c : V(G) \to \mathscr{C}$ such that $c(v) \neq c(u)$ for each $v, u \in V(G)$, whenever exists a path of length two between v and u. The elements of the set $\mathscr{C}$ are the *colors*. The minimum number of colors that G needs to be colored injectively is the *injective chromatic number* of G, and it is denoted by $\chi_i(G)$. This type of coloring was introduced by Hahn, Kratochvíl, Širáň and Sotteau [8]. They proved the inequality $\Delta \leq \chi_i(G) \leq \Delta^2 - \Delta + 1$, where Δ is the maximum degree of G. They characterize the graphs for which the upper bound is achived in the inequality, these graphs are precisely the incident graphs of projective planes of order $\Delta - 1$. They also characterize the regular graphs for which the lower bound is achieved. In their article also some interesting results on injective colorings of Cartesian graph products, especially on hypercubes, are presented.

In [3] Doyon, Hahn and Raspaud proved a theorem about the dependence between the maximum average degree of graphs and their injective chromatic number. Let G be a graph, the *maximum average degree* of G is denoted by $\mathrm{Mad}(G) = \max\{2|E(H)|/|V(H)|,\ H \subseteq G\}$. Their main result is the following theorem:

Theorem 1 *Let G be a graph of maximum degree Δ. If $\mathrm{Mad}(G) < \frac{14}{5}$ then $\chi_i(G) \leq \Delta + 3$, if $\mathrm{Mad}(G) < 3$ then $\chi_i(G) \leq \Delta + 4$, and if $\mathrm{Mad}(G) < \frac{10}{3}$ then $\chi_i(G) \leq \Delta + 8$.*

Knowing that for planar graphs of girth g holds the inequality $\mathrm{Mad}(G) < \frac{2g}{g-2}$, they obtain the following corollary for planar graphs:

Corollary 2 *Let G be a planar graph of maximum degree Δ. If $g(G) \geq 7$ then $\chi_i(G) \leq \Delta + 3$, if $g(G) \geq 6$ then $\chi_i(G) \leq \Delta + 4$, and if $g(G) \geq 5$ then $\chi_i(G) \leq \Delta + 8$.*

Hahn, Raspaud and Wang [9] proved that the injective chromatic number of every K_4-minor free graph of maximum degree Δ is $\leq \lceil \frac{3}{2}\Delta \rceil$. They also pose the following conjecture:

Conjecture 1 *For each planar graph G, $\chi_i(G) \leq \lceil \frac{3}{2}\Delta \rceil$.*

Injective coloring of a graph G is related to the usual coloring of the square G^2. The inequality $\chi_i(G) \leq \chi(G^2)$ trivially holds. There are some well known results and conjectures about coloring squares of planar graphs [11]. Wegner [15] proved that the squares of cubic planar graphs are 8-colorable. He conjectured that his bound can be improved to 7, and posed the following conjecture:

Conjecture 2 *Let G be a planar graph with maximum degree Δ. The chromatic number of G^2 is at most 7, if $\Delta = 3$, at most $\Delta + 5$, if $4 \le \Delta \le 7$, and at most $\lfloor \frac{3\Delta}{2} \rfloor + 1$, otherwise.*

If the conjecture holds, then the bounds are the best possible. The Conjecture 2 was verified for several special classes of planar graphs, but it remains open for all values of $\Delta \ge 3$. Dvořák et al. [5] have proved that the chromatic number of the square of a planar graph G with sufficiently large maximal degree is $\Delta + 1$ if the girth of G is at least seven and it is bounded by $\Delta + 2$ if the girth of G is six. On the other hand, Molloy and Salavatipour [13] proved the bound $\chi(G^2) \le \lfloor \frac{5\Delta}{3} \rfloor + 78$. They also showed that $\chi(G^2) \le \lfloor \frac{5\Delta}{3} \rfloor + 25$ holds for Δ large enough.

Montassier and Raspaud [14] obtained some results on colorings of the squares of planar subcubic graphs. They proved that $\chi(G^2) \le 5$ if $g(G) \ge 14$ and $\chi(G^2) \le 6$ if $g(G) \ge 10$.

In this paper we focus on planar graphs with specified girth. We show that all planar graphs of girth ≥ 19 are injectively colorable with Δ colors, where $\Delta \ge 3$ is the maximum degree. We also show that all planar graphs with girth ≥ 10 are injectively $(\Delta + 1)$-colorable, $\Delta + 4$ colors are sufficient for planar graphs of girth ≥ 5 if Δ is large enough, and that subcubic planar graphs of girth ≥ 7 are injectively 5-colorable.

For a planar graph G, let G^* be the graph obtained from G by contracting all 2-vertices. To prove the reducibility of configurations we use the *neighboring graph* $G^{(2)}$ defined by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{uv;\ u$ and v have a common neighbor in $G\}$.

In the proofs we use the Discharging method. We assign an initial charge to vertices and faces of a minimal counterexample G in the following way: for every $v \in V(G)$, define the initial vertex charge

$$\mathrm{ch}_0(v) = 2d(v) - 6,$$

where $d(v)$ denotes the degree of v in G. Let $F(G)$ be the set of faces of the graph G. For every face $f \in F(G)$, define the initial face charge

$$\mathrm{ch}_0(f) = r(f) - 6,$$

where $r(f)$ denotes the size of f. By Euler's formula, the total amount of charge is

$$\begin{aligned}\sum_{v\in V(G)} \mathrm{ch}_0(v) + \sum_{f\in F(G)} \mathrm{ch}_0(f) &= (4|E(G)| - 6|V(G)|) + (2|E(G)| - 6|F(G)|) \\ &= 6(|E(G)| - |V(G)| - |F(G)|) \\ &= -12.\end{aligned}$$

We use $\mathrm{ch}^*(x)$ to denote the final charge of a vertex or a face x. It is easy to see that only ≤ 2-vertices and ≤ 5-faces have negative initial charge. Next, we redistribute the initial charge between the vertices and faces such that the total amount of charge does not change. Eventually, the final charge of each vertex and face will be non-negative, thus contradicting the existence of a minimal counterexample, and establishing the theorem in this way.

Through the article we use the following notation. The girth of a graph G is denoted by $g(G)$. A *k-vertex* is a vertex of degree k, a *k-path* is a path of length k, a *k-cycle* is a cycle of length k, and a *k-face* is a face of size k. A *thread* is an induced path in G whose vertices are all of degree 2 in G. A *k-thread* is a thread with k vertices. A *$\geq k$-vertex* is a vertex of degree $\geq k$. On the other hand, a *$\leq k$-vertex* is a vertex of degree $\leq k$. One can similary define *$\geq k$-face*, *$\leq k$-face*, *$\geq k$-path* and *$\leq k$-path*.

We use the term *configuration* for an induced subgraph H of a graph G. We say that a configuration H is *reducible* if it cannot appear in a minimal counterexample G. The proof of the reducibility of the configuration H usually proceeds in the following way. By the minimality of G, $G - H$ can be properly colored. We then show that an arbitrary proper coloring of $G - H$ can be extended properly to H, thus showing that G is injectively colorable, which is a contradiction.

2 Injective 3-coloring of subcubic planar graphs

In this section we show a result on injective 3-colorings of subcubic planar graphs. We prove that every subcubic planar graph with girth ≥ 19 can be colored in such a way. Moreover, we present a subcubic planar graph with girth 10 that is not injectively 3-colorable.

Theorem 3 *Every subcubic planar graph G with girth ≥ 19 is injectively 3-colorable.*

Proof. Suppose that the theorem is false. Let G be a counterexample to the theorem with $|V(G)| + |E(G)|$ as small as possible. Thus, G is a planar graph of girth ≥ 19 and it is not injectively 3-colorable. Moreover, every proper subgraph of G is injectively 3-colorable.

Reducible configurations. Let us first pose some reducible configurations for injective 3-colorings. Some of them will be used later in other results, where we use four or more colors.

Lemma 4 *A* 1*-vertex and a* 4*-thread are reducible configurations.*

Proof. Let u be a 1-vertex in G. The unique neighbor of a 1-vertex is of degree ≤ 3, so u has at most two neighbors at distance two. Thus, it has at least one available color to extend c.

Let $uvwz$ be a 4-thread. After coloring the rest of the graph, at least one free color remains for the vertices u and z, and two free colors for v and w. Color each of u and z by its free color. Since these two vertices are at distance three, they can be colored with a same color. Afterwards, for each of vertices v and w remains at least one color. Notice that they can be colored with a same color, if necessary, so we can extend the coloring to them as well. □

Lemma 5 *The configurations of Fig. 1 are reducible.*

Proof. We consider separately every configuration H of Fig. 1. Suppose that H is a subgraph of G. By the minimality of G, the graph $G - H$ has a proper injective 3-coloring c. We prove that c can be extended properly to G and obtain a contradiction, which proves the reducibility of H.

(a) In this case H is a 20-cycle, and we use the labeling of its vertices as described in Fig. 1(a). After coloring the graph $G - H$, every vertex a_i has at least one available color, every vertex b_i has two, and each c_i has three available colors.

The proper vertex coloring of the graph $H^{(2)}$ is precisely the proper injective coloring of H. Graph $H^{(2)}$ has two components, first one is the 10-cycle $a_1c_1a_2 \dots a_5c_5a_1$ and the second one is the 10-cycle $b_1b_2 \dots b_{10}b_1$. Since each a_i has an available color, we can color them first. Afterwards, at least one available color is left for each c_i, and so we color them as well. In such a way the first component is colored. In

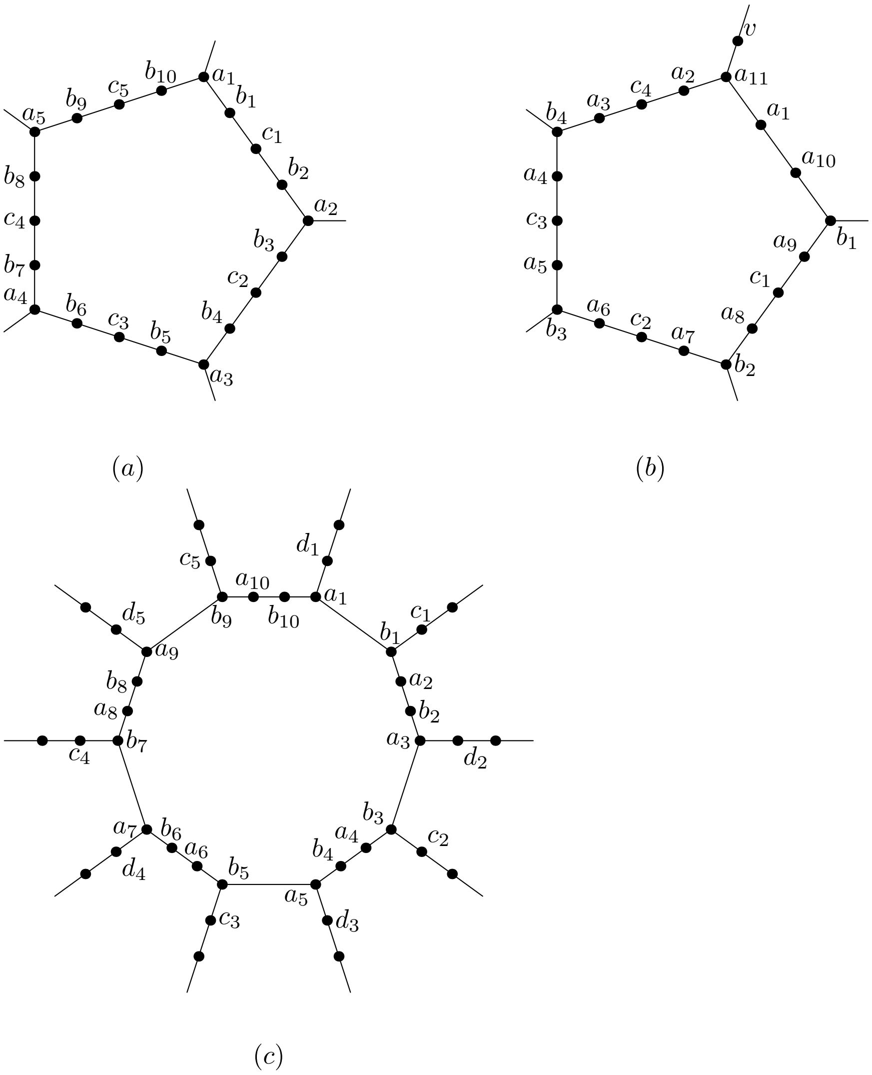

Figure 1: Reducible configurations for injective 3-colorings

the second component, every vertex has two available colors but since it is an even cycle, it can be easily colored. Thus, we obtain a proper injective 3-coloring of G, and so it follows that the configuration H is reducible.

(b) In this case H is a 19-cycle with one pendant vertex, we use the labeling as in Fig. 1(b). Notice that v is a 2-vertex in G. By the minimality of G it follows that there exists a proper injective coloring c of $G-(H-v)$. Afterwards, note that each a_i has two available colors, each c_i has three, and each b_i has at least one free color, by which we color b_i.

To prove the reducibility of H we use again the neighboring graph $H^{(2)}$. Since each b_i is already colored, in $H^{(2)}$ are only left non-colored all vertices a_i and c_i. The non-colored vertices form four components in $H^{(2)}$. Three of them are just isolated vertices c_1, c_2 and c_3. Each of them has three available colors, but after coloring their neighbors in $H^{(2)} - \{b_1, b_2, b_3, b_4\}$, they have at least one free color left, which is enough. The last component is the path $a_1 a_2 \dots a_{10} a_{11} c_4$. The vertices of the path have two free colors, so the graph $H^{(2)}$ can be colored, and therefore the injective 3-coloring c can be extended on G.

(c) The configuration H is a 20-cycle C plus some pendant 2-threads as in Fig. 1(c), we use the same labeling. We extend easily the coloring c also to the vertices at distance two from the cycle C. Only C and the vertices at distance one from C remain non-colored. We denote that graph as $K = C \cup \{c_1, d_1, \dots, c_5, d_5\}$. The neighboring graph $K^{(2)}$ has two isomorphic components. We use the Alon-Tarsi Theorem [1] to prove that each of them is colorable with the given list of colors.

First, let us define Eulerian subgraphs. A subdigraph H of a directed graph D is called *Eulerian* if the indegree $d_H^-(v)$ of every vertex v of H is equal to its outdegree $d_H^+(v)$. The graph H is *even* if it has an even number of edges, otherwise, it is *odd*. Let $E^e(D)$ and $E^o(D)$ be the numbers of even and odd Eulerian subgraphs of D, respectively.

For each $v \in V(D)$, let $L(v)$ be a set of $d_D^+(v)+1$ distinct colors, where $d_D^+(v)$ is the outdegree of v. The Alon-Tarsi Theorem states that in a directed graph D there is a proper vertex-coloring $c : V(D) \to \mathscr{C}$ such that $c(v) \in L(v)$ for all $v \in V$, if $E^e(D) \neq E^o(D)$.

Since the components of $K^{(2)}$ are isomorphic, we only prove the colorability of the component C_1 induced by vertices $\{a_1, c_1, a_2, a_3, c_2, a_4, a_5, c_3, \dots, a_9, c_5, a_{10}\}$. Observe that vertices $a_2, a_4, a_6, a_8, a_{10}$ have three available colors, the remaining vertices have only two free colors. We notice that there are five 3-faces in C_1. Make the edges in each of them directed in sense that the orientation of the 3-face $a_1c_1a_2$ is counter-clockwise. Orient the other 3-faces also counter-clockwise. So, every vertex have one outgoing edge and one ingoing edge. Only the edges connecting 3-faces remained undirected. We orient them in a clockwise direction.

In the component C_1 is now sixteen odd Eulerian graphs and seventeen even, the combinations of triangles, 10-cycle $a_1a_2 \dots a_{10}a_1$ and the empty graph. The vertices with three available colors have two outgoing edges, others have only one. The assumptions of Alon-Tarsi Theorem are fulfilled, and so we can color C_1. This implies that H is reducible.

Discharging rule. We apply the Discharging method on G^* using only the following rule. Notice that the rule assumes that vertices a, b, c, d of Fig. 2 are of degree 3 and f_2 corresponds to a 19-face in G.

Rule R1: Let f_1 be a ≥ 7-face and f_2 a 5-face in G^* such that they correspond in G as it is presented in Fig. 2, so the only edge of f_2 which is subdivided by two vertices is adjacent to f_1. Then, f_1 sends 1 to f_2.

Final charge. Observe that the graph G^* is cubic. Since the total charge of G^* is -12, there must be at least one ≤ 5-face in G^*. However, G^* cannot contain a ≤ 4-face, otherwise G contains a 4-thread in order to satisfy the girth assumption. Notice that by Lemma 5, the 4-threads are reducible. Considering the 5-faces in G^*, we notice that they correspond to three different configurations in G. The first and the second configuration are isomorphic to the reducible configurations (a) and (b) of Fig. 1, respectively. The third one is presented in Fig. 2, where the vertices a and d are of degree three.

Now, considering that $g(G) \geq 19$ and G does not contain a 4-thread, one obtain that the face f_1 must be of size ≥ 7 in G^*. Hence, every 5-face has an adjacent ≥ 7-face at the edge that is subdivided by two vertices, as the edge bc in Fig. 2. Thus, it receives 1 by rule R1, so it has a non-negative

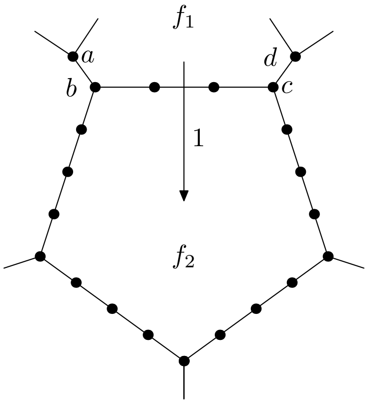

Figure 2: Rule R1

final charge.

Now, we show that faces which are sending charge by rule R1 have non-negative final charge. We consider several cases regarding the size. First, observe that every k-face in G^*, $k \geq 7$, have at most $\lfloor \frac{k}{2} \rfloor$ adjacent 5-faces to which it sends charge. This holds since the edges ab and cd of Fig. 2 are not subdivided in G. It can be easily seen that the final charge of an $\geq$11-face f is

$$\mathrm{ch}^*(f) = ch_0(f) - \left\lfloor \frac{r(f)}{2} \right\rfloor = \left\lceil \frac{r(f)}{2} \right\rceil - 6 \geq 0.$$

Now, let us consider the faces of size between 7 and 10. Notice that a 7-face has enough charge to send only to one 5-face. Let us assume that there exists a 7-face f_1 in G^* such that it sends charge to two 5-faces. Then, f_1 corresponds to a $\leq$17-cycle in G, which contradicts the girth assumption. Similary, 8-faces in G^* could have at most two and 9-faces at most three adjacent 5-faces to which they send charge. However, their final charge will remain non-negative.

A 10-face, which sends charge to at most four 5-faces has non-negative final charge. It remains only to consider a 10-cycle with five adjacent 5-faces. This configuration is reducible in G due to Lemma 5, since it corresponds to the configuration of Fig. 1(c).

We have shown that all faces in G^* have non-negative charge and since G^* is cubic, also its vertices have non-negative final charge. Thus, we obtain a contraction which establish the theorem. $\square$

Not every subcubic planar graph is injectively 3-colorable. We pose such a graph of girth 10.

Proposition 6 *The planar subcubic graph on the left side of Fig. 3 is not injectively* 3*-colorable.*

Proof. We try to injectively 3-color the graph H on the left side of Fig. 3. Consider the neighboring graph $H^{(2)}$. It has two components. The vertices of the first component are drawn as squares, and the vertices of the second component are drawn as circles in the same figure.

Observe that it is equivalent to 3-color properly the second component of $H^{(2)}$ and to 3-edge-color the right graph of Fig. 3, since the first graph (the second component of $H^{(2)}$) is the line graph of the second (the right graph of Fig. 3). The second graph is the Dodecahedron with one edge subdivided. Use now the well known fact that a cubic graph with one edge subdivided is not 3-edge-colorable. $\square$

3 Injective 4-coloring of subcubic planar graphs

Here we decrease the girth bound to 10 of Theorem 3 by using one extra color.

Theorem 7 *Every subcubic planar graph G with girth ≥ 10 is injectively* 4*-colorable.*

Proof. Suppose that the theorem is false and suppose that a subcubic planar graph G with girth ≥ 10 is a minimal counterexample. We use again the Discharging method on G^* to obtain a contradiction.

Reducible configurations. First we pose few reducible configurations.

Lemma 8 *The graph G neither contains a* 1*-vertex nor a* 2*-thread.*

Proof. Since a 1-vertex is reducible for injective 3-colorings, it is also reducible for injective 4-colorings.

In a 2-thread uv, each vertex has at most three neighbors in $G - u - v$ and therefore at least one available color. However, vertices u and v have no common neighbor, and so they could be same colored. Therefore the configuration is reducible. $\square$

To prove the theorem of this section we explore again different configurations in G that correspond to a 5-face in G^*. The first observation is this lemma:

Lemma 9 *The configurations of Fig. 4 are reducible.*

Proof. We prove the reducibility of each configuration separately.

(a) Let H be the configuration of Fig. 4(a), and let H be a subgraph of G. By the minimality, it follows that there exists an injective 4-coloring c of $G - H$. We extend c to H. First, notice that all vertices of H have at least two free colors, moreover each of a_1, b_1, b_5 has three free colors.

As before, we want to color $H^{(2)}$ properly. The graph $H^{(2)}$ has two components, the first one is the 5-cycle $a_1a_2 \dots a_5a_1$ and the second one is the 6-cycle $b_1b_2 \dots b_5vb_1$ plus the diagonal b_1b_5. In the first component, the vertex a_1 has three available colors, others have at least two. It is easy to color such a 5-cycle.

The second component is easily colored by the list version of Brooks Theorem, which states that a connected graph is degree-choosable unless it is a Gallai tree, that is each of its blocks is either complete graph or an odd cycle [2, 7].

(b) Let H be the configuration of Fig. 4(b), which is a subgraph of G. By the minimality of G, there exists a proper injective 4-coloring c of the graph $G - H$. Afterwards, the vertex b_1 has all four colors available, a_1, b_2, a_5 and b_5 have three available colors, and all the remaining vertices have at least two.

The vertices of H comprise two components in $H^{(2)}$. The first is the 8-cycle $a_1a_2 \dots a_8a_1$ plus the edge a_1a_5 and the second is comprised of the cycles $b_1b_2 \dots b_5b_1$ and $b_1b_6 \dots b_9b_1$ together with the edges b_2b_9 and b_5b_6. Both components can be easily colored using the list version of Brooks Theorem. $\square$

Discharging rule. In order to redistribute the initial charge, we use the following rule:

Rule R1: Every $\geq$7-face in G^* sends $\frac{1}{5}$ to every adjacent 5-face.

Final charge. Note that a 5-face is the smallest face in G^* due to the girth assumption of G and the fact that a 2-thread is reducible.

Graph G^* is cubic, so there is no vertex with negative charge. Therefore, G^* contains a ≤ 5-face due to the negative total charge. Next, we show that after applying the discharging rule R1 to G^*, each face has a non-negative charge, therefore a contradiction is obtained.

Two adjacent 5-faces in G^* contain in G a configuration isomorphic to the one of Fig. 4(a), which is reducible by Lemma 9. Such a configuration cannot occur in G^* due to the minimality of G. A 5-face adjacent to a 6-face in G^* induces in G a configuration isomorphic to the configuration of Fig. 4(b). Again it cannot happen. Knowing that 5-faces have five adjacent faces of size ≥ 7, after applying R1, they have a non-negative final charge. Faces of size ≥ 8 have enough charge even if all their neighbors are 5-faces. So we have to consider 7-faces. However, knowing that two 5-faces cannot be adjacent, it follows that each 7-face has at most three adjacent 5-faces, and so it has non-negative final charge. Each vertex and face of G^* has a non-negative charge, therefore we obtain a contradiction. $\Box$

Notice that there exist graphs which are not injective 4-colorable. Such a graph is presented in [9].

4 Injective 5-coloring of subcubic planar graphs

In this section we prove a theorem about injective 5-colorings of subcubic planar graphs.

Theorem 10 *Every subcubic planar graph G with girth ≥ 7 is injectively 5-colorable.*

Proof. Suppose that the theorem is false. Let a subcubic planar graph G with girth ≥ 7 be a minimal counterexample. We will obtain a contradiction.

Reducible configurations. ¿From the previous section we know that a 1-vertex and a 2-thread are reducible for 4-colorings, therefore they are also reducible for 5-colorings. We use two more configurations:

Lemma 11 *The configurations (a) and (b) of Fig. 5 are reducible.*

Proof. (a) Let H be the configuration of Fig. 5(a) that is a subgraph of G. Then, by the minimality, it follows that there exists an injective 5-coloring c of $G - H$. Extending c on G would prove the reducibility of H. The vertex v has at least one free color, vertices u and w have at least two. We color first v with its available color. Afterwards, we color differently the remaining two vertices, which still have two available colors.

(b) Let H be the configuration of Fig. 5(b), and let H be a subgraph of G. By the minimality it follows that there exists an injective 5-coloring of the graph $G - H$. Now, the vertex a_5 has at least one free color, vertices a_4 and a_6 have three, and vertices a_1, a_2, a_3 and a_7 have at least two free colors. Let us color the graph $H^{(2)}$. It consists of a 7-cycle $a_1a_2a_3a_4a_5a_6a_7$. First, color the vertex a_5. That reduces the number of free colors of its neighbors a_4 and a_6, they have now two free colors. What remains is a 6-path with each vertex having at least two available colors. Such a path is easily colored, thus a contradiction is obtained. □

The graph G^* is again cubic and as such contains at least one 5-face. It cannot contain smaller faces, due to the reducibility and the girth assumption in graph G. Afterwards, note that each 5-face in G^* is reducible in G, since at least two edges have to be subdivided by one vertex, due to the girth assumption, and such a configuration is reducible by Lemma 11. □

5 Injective Δ-coloring of the planar graphs

The results for subcubic graphs are in the following few sections generalized to the graphs with higher maximum degree.

Theorem 12 *Every planar graph G with maximum degree $\Delta \geq 4$ and girth ≥ 19 is injectively Δ-colorable.*

Proof. Suppose that the theorem is false. Let a planar graph G with maximum degree $\Delta \geq 4$ and girth ≥ 19 be a minimal counterexample for injective Δ-coloring. We will obtain a contradiction.

Reducible configurations. We have proved that a 1-vertex and a 4-thread are reducible for injective 3-colorings in subcubic graphs. These configurations are reducible also for injective Δ-colorings, from very similar reasons.

Lemma 13 *The configurations of Fig. 6 are reducible, where in configuration* (b) *one of the vertices* a_1*,* a_2 *is of degree* $\leq \Delta - 1$*.*

Proof. (a) Let H be the configuration of Fig. 6(a), and let H be a subgraph of G. By the minimality of G, there exists an injective coloring c of the graph $(G-H) \cup \{a_1, a_2, \ldots, a_5\}$. Now, the vertices $b_1, b_2, \ldots, b_{10}$ have at least two free colors, and the vertices $c_1, c_2, \ldots, c_5$ have $\Delta - 2$ of them.

Let $K^{(2)}$ be the graph obtained from $H^{(2)}$ by removing the vertices $a_1, a_2, \ldots, a_5$. A proper vertex coloring of $K^{(2)}$, with the number of free colors for each vertex as defined above, is a proper injective coloring of $H - \{a_1, a_2, \ldots, a_5\}$ and also gives the extension of c to G. The graph $K^{(2)}$ has six components. Five of them are trivial - including just one vertex with $\Delta - 2$ available colors. The last component is the 10-cycle $b_1 b_2 \ldots b_{10}$ where each vertex has two available colors. So, they can be colored. It follows that the configuration H is reducible.

(b) The proof of reducibility of the configuration of Fig. 6(b) is similar as for the previous one. Without loss of generality, we assume that vertex a_1 is of degree $\Delta - 1$. Let H be the configuration of Fig. 6(b), and let H be a subgraph of graph G. By the minimality of G, there exists a coloring c of the graph $(G - H) \cup \{a_1, a_2, \ldots, a_5\}$.

The non-colored vertices are $b_1, b_2, \ldots, b_{10}$ and c_1, c_2, c_3, c_4. Vertex b_{10} has at least one available color, b_2 has three, every other vertex b_i, $i \in \{1, 3, 4, \ldots, 9\}$, has two, and vertices $c_1, \ldots, c_4$ have $\Delta - 2$ free colors. Let us define the graph $K^{(2)} := H^{(2)} - \{a_1, a_2, \ldots, a_5\}$ and color it. The graph $K^{(2)}$ has five components, four trivial with just one vertex, and a 9-path. First, we color the trivial components. Those are vertices c_i, $i \in \{1, 2, 3, 4\}$, which have $\Delta - 2$ free colors, therefore they are easily colored. It remains to color the path $b_1 b_2 \ldots b_{10}$. We start at b_{10} and then continue easily with b_9, b_8 to the last one, which is b_1. The graph $K^{(2)}$ is colored, so the coloring c can be extended to the graph G, and the configuration H is reducible. □

Discharging rule. We use the following rule to redistribute the initial charge of G and establish a contradiction by obtaining positive final charge:

Rule R1: Every k-vertex in G^*, $k \geq 4$, sends $\frac{1}{5}$ to each adjacent 5-face.

Final charge. Due to the reducibility of a 4-thread and the girth assumption, G^* has only ≥ 5-faces. We will redistribute the charge in such a way that every vertex and every face in G^* will have positive final charge.

Using the Lemma 13 we see that each vertex of the 5-face in G^* is of degree $\Delta \geq 4$ in G and thus also in G^*. Therefore, when applying rule R1 to G^*, each 5-face f receives $5 \cdot \frac{1}{5}$ of charge, and its final charge is non-negative.

Now, we only have to show that vertices does not have negative charge after applying rule R1. The initial charge for vertices is non-negative, since the minimal degree of vertices in G^* is 3. After applying R1, the final charge of a ≥ 4-vertex v is at least

$$\mathrm{ch}_0(v) - \frac{1}{5}d(v) = 2d(v) - 6 - \frac{1}{5}d(v) = \frac{9}{5}d(v) - 6 > 0.$$

All the vertices and faces have non-negative charge, therefore we obtain a contradiction which establishes the theorem. □

6 Injective (Δ+1)-coloring of the planar graphs

In this section the result on coloring graphs with $(\Delta + 1)$ colors is presented.

Theorem 14 *Every planar graph G with girth ≥ 10 and maximum degree $\Delta \geq 4$ is injectively $(\Delta + 1)$-colorable.*

Proof. Suppose that the theorem is false. Let a planar graph G, with $\Delta \geq 4$ and girth ≥ 10, be a counterexample with the smallest number of vertices and edges. We will obtain a contradiction.

Reducible configurations. We start again at reducible configurations.

Lemma 15 *The following configurations are reducible:*

(a) a 1-vertex and a 2-thread;

(b) The configuration of Fig. 7 if one of the vertices u_1, u_2, u_3, and u_4 has degree $\leq \Delta - 1$.

Proof. (a) A 1-vertex is reducible for injective Δ-colorings, therefore it is also for injective $(\Delta + 1)$-colorings.

Each vertex in a 2-thread has at most Δ neighbors at distance two, so exactly one available color. Vertices of a 2-thread have no common neighbor, therefore they can be colored with the same color, if necessary. So, every coloring can be extended to this configuration, thus it is reducible.

(b) Let H be a configuration of Fig. 7, and let it be a subgraph of G. Proving the reducibility of H, we distinguish two cases: either one of the vertices u_1, u_4 is of degree $< \Delta$ or one of the vertices u_2, u_3 is of degree $< \Delta$. In both cases by the minimality of G, there exists an injective $(\Delta+1)$-coloring c of the graph $(G - H) \cup \{v, u_1, u_2, u_3, u_4\}$. So, in both cases it remains only to color vertices a_i, $i \in \{1, 2, 3, 4\}$, and b_j, $j \in \{1, 2, \dots, 5\}$.

Suppose first u_1 is of degree $< \Delta$ (for u_4 it is symmetric). Notice that b_1 and b_5 have at least two free colors, and b_2, b_3 and b_4 have at least $\Delta - 1$ available colors. Now, we define a graph $K^{(2)} := H^{(2)} - \{v, u_1, \dots, u_4\}$. Graph $K^{(2)}$ has two components. First is the 5-cycle $b_1 b_2 \dots b_5$ which is trivially colorable, since two vertices have at least two available colors, and three vertices have at least $\Delta - 1 \geq 3$ free colors. The second component is the 4-path $a_1 a_2 a_3 a_4$ where the vertex a_4 has only one free color, and each of the remaining vertices has at least two available colors. Such a path is colorable, so we can extend the coloring c to G.

Considering the second case, suppose u_2 is of degree $< \Delta$ (for u_3 it is symmetric). We define the graph $K^{(2)}$ as in the previous case. Each of the vertices b_j has at least two available colors, so the 5-cycle $b_1 b_2 \dots b_5$ is colorable. The vertices a_1 and a_4 have only one available color, a_2 has three free colors, enabling the path to be properly colored. The $K^{(2)}$ can be colored, therefore the configuration H is reducible. $\square$

Discharging rules. We redistribute the initial charge of the vertices and faces of G^* using the following two rules:

Rule R1: Every ≥ 4-vertex of graph G^* sends $\frac{1}{2}$ of charge to every adjacent 5-face.

Rule R2: Let v be the vertex of degree Δ and u its neighbor in G of degree 3. If u lies on a 5-face f such that v and f are not incident, then v sends $\frac{1}{4}$ to f.

Final charge. We show that the total charge of G^* is non-negative after applying the discharging rules. Let us start with faces again. The inital charge is negative only for 5-faces. Every 5-face in G^* contains either two ≥ 4-vertices or at least four 3-vertices with all neighbors u_i of degree Δ, where each u_i corresponds to a vertex v of rule R2, otherwise a reducible configuration in G is encountered. In the former case, each of ≥ 4-vertices sends $\frac{1}{2}$ by rule R1, so the 5-face receives 1. In the latter case, every vertex u_i sends $\frac{1}{4}$ by rule R2. We have at least four such vertices, thus the 5-face receives 1 of charge, and it is non-negative again.

It remains to show that all vertices have non-negative charge. The 3-vertices have non-negative charge, since the rules do not affect them.

Let v be a k-vertex, $4 \leq k < \Delta$ in G^*. It sends charge only by rule R1. The k-vertex v has at most k adjacent 5-faces, therefore it sends at most $\frac{k}{2}$ of charge, and its final charge is non-negative.

Now, only the Δ-vertices remain to be considered. They send charge by rules R1 and R2. Suppose the Δ-vertex v has k adjacent 5-faces. Observe that if there exists a 3-vertex u in G adjacent to a Δ-vertex v, then the edge uv is not adjacent to a 5-face in G^* due to the girth assumption. Therefore, rule R2 is at most $\Delta - 2k$ times applied at v. So, the final charge is

$$\mathrm{ch}^*(v) \geq \mathrm{ch}_0(v) - \frac{1}{2}k - \frac{1}{4}(\Delta - 2k) = \frac{7}{4}\Delta - 6 > 0.$$

All charges are non-negative, a contradiction. □

7 Injective $(\Delta + 4)$-coloring

In this section we show that $\Delta + 4$ colors are sufficient to injectively color a planar graph G with girth > 4 and large enough maximum degree Δ. Notice that girth > 4 is necessary, since there are graphs of girth four that have precisely $\frac{3}{2}\Delta$ colors. For example, take a Δ-regular Shanon's triangle with each edge subdivided and Δ even.

Theorem 16 *Every planar graph G with girth ≥ 5 and maximum degree $\Delta \geq 139$ can be injectively colored with $\Delta + 4$ colors.*

Proof. Suppose that the theorem is false. Let a graph G be a minimal counterexample. We will obtain a contradiction.

Let $\epsilon \leq \frac{1}{5}$, $\epsilon \in \mathbb{R}^+$, and $b = \lceil \frac{6}{\epsilon} \rceil \geq 30$. If a vertex has degree $\geq b$, it is called a *big* vertex. Vertices which are not big are *small*.

Reducible configurations. In the previous sections, we proved that a 1-vertex and a 2-thread are reducible for the injective $(\Delta+1)$-coloring, thus they are also for the injective $(\Delta+4)$-coloring. In the proof we also use the reducible configurations of Fig. 8. Small vertices are drawn as squares and circled vertices have degree as it is depicted or denoted. We want to emphasize that in Fig. 8 drawings are not fixed. Only the types of the neighborhoods of the vertex v are prescribed. For example, in (C_4) the vertices w_1 and w_2 can be consecutive around v in the embedding of G.

Lemma 17 *The configurations of Fig. 8 are reducible.*

Proof. We prove the reducibility of each configuration separetly. In each proof we suppose that the configuration is contained in the counterexample G. By minimality of G, we assume that there exists a coloring c of the graph G/uv, where uv is an edge in each configuration of Fig. 8. We expand back the edge uv and extend the coloring c to G by recoloring vertices u and v. In this way, we establish the reducibility of the configuration.

For each non-colored vertex $x \in \{u, v\}$, we define $L(x)$ as the list of its free colors. Let $l(x) = |L(x)|$ be the number of available colors for vertex x. Notice that the number of available colors is obtained by counting all the possible neighbors at distance two. We subtract this number from $\Delta + 4$ in order to obtain a lower bound of $l(x)$.

Since we use the same procedure in all proofs, as it was described above, we list just the lower bounds of $l(u)$ and $l(v)$ for all configurations of Fig. 8. Since u and v have no common neighbor, it is sufficient for each of them to preserve one available color.

(C_1) $l(v) \geq 1$ and $l(u) \geq 3$;

(C_2) $l(v) \geq \Delta - 2b + 7$ and $l(u) \geq 3$;

(C_3) $l(v) \geq \Delta - 5b + 12$ and $l(u) \geq \Delta - 2b + 3$;

(C_4) $l(v) \geq 2$ and $l(u) \geq 2$;

(C_5) $l(v) \geq \Delta - 2b + 6$ and $l(u) \geq 2$;

(C_6) $l(v) \geq 1$ and $l(u) \geq 1$;

(C_7) $l(v) \geq \Delta - 2b + 5$ and $l(u) \geq 1$;

(C_8) $l(v) \geq \Delta - 2$ and $l(u) \geq \Delta - b + 1$;

(C_9) $l(v) \geq \Delta - b + 1$ and $l(u) \geq \Delta - b + 1$;

(C_{10}) $l(v) \geq \Delta - 2b + 4$ and $l(u) \geq \Delta - b + 1$;

(C_{11}) $l(v) \geq \Delta - 5$ and $l(u) \geq \Delta - b - 2$;

(C_{12}) $l(v) \geq \Delta - b - 1$ and $l(u) \geq \Delta - b - 1$;

(C_{13}) $l(v) \geq \Delta - 2b + 3$ and $l(u) \geq \Delta - b$.

Notice that vertices u and v in all configurations have enough available colors for $\Delta > 5b - 12 \geq 138$. $\square$

Initial charge. We assign charge to vertices and faces of G. For every $v \in V(G)$, we assign an initial vertex charge $\mathrm{ch}_0(v) = \frac{9}{5}d(v) - 6$, and for every face $f \in F(G)$, we assign an initial face charge $\mathrm{ch}_0(f) = \frac{6}{5}r(f) - 6$. Using Euler's formula, in a similar way as in the introduction, one can easily show that the total amount of charge is -12.

Discharging rules. We use the following discharging rules to make the final charge of all faces and vertices positive.

Rule R1: A $\geq$3-vertex sends $\frac{6}{5}$ to every adjacent 2-vertex.

Rule R2: A big vertex v sends $\frac{9}{5} - \epsilon$ to each adjacent 3-vertex w, if w has an adjacent 2-vertex. If other two neighbors of w are of degree ≥ 3, then v sends $\frac{3}{5}$ to w.

Rule R3: A small vertex v of degree ≥ 5 sends ϵ to each adjacent 3-vertex w, if the other two neighbors of w are a 2-vertex and a big vertex.

Rule R4: Suppose a big vertex v has an adjacent vertex w of degree 3, 4, 5 or 6. Then, v sends $\frac{1}{5}$ to each neighbor z of w that has a degree 3 and that has the other two neighbors small.

Rule R5: A small vertex of degree ≥ 7 sends $\frac{1}{5}$ to an adjacent 3-vertex w if the other two neighbors of w are small and of degree ≥ 3.

Rule R6: A big vertex sends $\frac{6}{5}$ to an adjacent 4-vertex, which has at least two adjacent 2-vertices.

Rule R7: A big vertex sends $\frac{3}{5} + 2\epsilon$ to an adjacent 5-vertex.

Rule R8: A big vertex sends 2ϵ to an adjacent 6-vertex.

Rule R9: A big vertex sends $\frac{6}{5}$ to an adjacent 6-vertex w, if all other neighbors of w are 2-vertices.

Rule R10: Suppose that a big vertex v has an adjacent 2-vertex w. Then, v sends $\frac{2}{5}$ to the other neighbor of w if it is of degree 6, 7, 8 or 9.

Rule R11: A big vertex sends $\frac{4}{5}$ to each adjacent 7-vertex.

Final charge. Let v be a d-vertex of G. We consider several cases regarding d:

- *v is a 2-vertex.* It cannot have an adjacent 2-vertex, since the 2-threads are reducible. Therefore, v does not send any charge. It receives $\frac{6}{5}$ of charge from each neighbor by rule R1, so its final charge is

$$\mathrm{ch}^*(v) = -\frac{12}{5} + 2 \cdot \frac{6}{5} = 0.$$

- *v is a 3-vertex.* The initial charge of a 3-vertex v is $-\frac{3}{5}$. To prove that its final charge is non-negative, we consider a few subcases:

  (*i*) *v has more than one adjacent 2-vertex.* Then the reducible configuration (C_1) occurs.

  (*ii*) *v has only one adjacent 2-vertex.* Then, v sends $\frac{6}{5}$ to it by R1. We denote the other two neighbors of v by x and y. Consider three possibilities. If $d(x) \leq 4$ or $d(y) \leq 4$, we obtain the reducible configuration (C_1). In case that x and y are both small, we obtain the reducible configuration (C_2). Otherwise, one of them is big and the other one has degree >4. Then, the discharging rules R2 and eventually R3 are used. So, the final charge is

$$\mathrm{ch}^*(v) \geq -\frac{3}{5} - \frac{6}{5} + \left(\frac{9}{5} - \epsilon\right) + \epsilon = 0.$$

(*iii*) *v has no adjacent 2-vertex.* If v is adjacent with some big vertex, then the rule R2 is applied, hence v receives $\frac{3}{5}$ and it has non-negative final charge. If v has no big neighbor, then rule R4 or R5 applies, since the configuration (C_3) is reducible. From discharging rules, we obtain that each neighbor sends or it is sent through it exactly $\frac{1}{5}$ of charge, so in total $\frac{3}{5}$, which is sufficient for v to have non-negative charge.

- *v is a 4-vertex.* It has $\frac{6}{5}$ of initial charge. We consider few subcases again:

  (*i*) *v has at least three adjacent 2-vertices.* Then, we obtain the reducible configuration (C_4).

  (*ii*) *v has exactly two adjacent 2-vertices.* If it has also an adjacent big vertex, then the rule R6 is used. Thus, the big vertex sends $\frac{6}{5}$ of charge and v gives away $\frac{12}{5}$, therefore its final charge is 0. If there is no adjacent big vertex, we obtain (C_5).

  (*iii*) *v has at most one adjacent 2-vertex.* In this case, it has enough charge, since it sends at most $\frac{6}{5}$.

- *v is a 5-vertex.* It has initial charge $\frac{15}{5}$ and it sends $\frac{6}{5}$ of charge to every adjacent 2-vertex by R1, and ϵ to every adjacent 3-vertex by R3. We consider three subcases:

  (*i*) *v has at least four adjacent 2-vertices.* Then, the configuration (C_6) occurs.

  (*ii*) *v has exactly three adjacent 2-vertices.* If it has no adjacent big vertex, then it is reducible by (C_7). And, if there is a big neighbor, v receives $\frac{3}{5} + 2\epsilon$ by R7, so the final charge of v is

$$\mathrm{ch}^*(v) \geq \frac{15}{5} - 3 \cdot \frac{6}{5} - \epsilon + \left(\frac{3}{5} + 2\epsilon\right) = 0.$$

(*iii*) *v has at most two 2-neighbors.* Then, it has final charge

$$\mathrm{ch}^*(v) \geq \frac{15}{5} - 2 \cdot \frac{6}{5} - 3\epsilon \geq 0.$$

- *v is a 6-vertex.* It has initial charge $\mathrm{ch}_0(v) = \frac{24}{5}$. The vertex v may send $\frac{6}{5}$ to adjacent 2-vertices by rule R1, and ϵ to adjacent 3-vertices by rule R3. We consider four subcases:

  (*i*) *v has six adjacent 2-vertices.* If there is a small vertex at distance two, the configuration is reducible by (C_8). On the other hand, if there are only big vertices at distance two, each of them sends $\frac{2}{5}$ to v by R10, and so the final charge of v is

$$\mathrm{ch}^*(v) = \frac{24}{5} - 6 \cdot \frac{6}{5} + 6 \cdot \frac{2}{5} = 0.$$

  (*ii*) *v has five adjacent 2-vertices.* If v has a big neighbor, then v receives $\frac{6}{5}$ by R9. Thus the final charge is

$$\mathrm{ch}^*(v) = \frac{24}{5} - 5 \cdot \frac{6}{5} + \frac{6}{5} = 0.$$

  Now, we assume that v has no big neighbor. If v has a small vertex at distance two with a 2-vertex as a common neighbor, then we obtain (C_9). Otherwise, we obtain a configuration with five big vertices at distance two, whose common neighbors with v are the five 2-vertices. Each of them sends $\frac{2}{5}$ to v by R10. The vertex v may also send charge by R3 to the neighbor of degree ≥ 3. So, we infer

$$\mathrm{ch}^*(v) = \frac{24}{5} - 5 \cdot \frac{6}{5} + 5 \cdot \frac{2}{5} - \epsilon > 0.$$

  (*iii*) *v has four adjacent 2-vertices.* If v has an adjacent big vertex, then its final charge is

$$\mathrm{ch}^*(v) \geq \frac{24}{5} - 4 \cdot \frac{6}{5} - 2\epsilon + 2\epsilon = 0.$$

  If v has no adjacent big vertex, then it has a big vertex u at distance two with a 2-vertex as a common neighbor, otherwise we get the reducible configuration (C_{10}). The vertex u sends $\frac{2}{5}$ of charge by R10, so the final charge of v is

$$\mathrm{ch}^*(v) \geq \frac{24}{5} - 4 \cdot \frac{6}{5} - 2\epsilon + \frac{2}{5} \geq 0.$$

(iv) *v has at most three adjacent 2-vertices.* Then, its final charge is

$$\mathrm{ch}^*(v) = \frac{24}{5} - 3 \cdot \frac{6}{5} - 3\epsilon \geq 0.$$

- *v is a d-vertex with $d \in \{7, 8, 9\}$.* In this case, the vertex v may also send charge to other vertices by R1, R3 or R5. Let d_2 be the number of adjacent 2-vertices and d_3 the number of adjacent 3-vertices of v. Since $d \geq d_2 + d_3$, the final charge of v is

$$\begin{aligned} \mathrm{ch}^*(v) &\geq \frac{9}{5}d - 6 - \frac{6}{5}d_2 - \frac{1}{5}d_3 \\ &\geq \frac{9}{5}d - 6 - \frac{6}{5}d_2 - \frac{1}{5}\big(d - d_2\big) \\ &\geq \frac{8}{5}d - 6 - d_2. \end{aligned}$$

Thus, the vertex v has non-negative final charge, if $d_2 \leq \frac{8}{5}d - 6$. Now, it remains to consider only the possibilities $d = 7$ with $d_2 \in \{6, 7\}$, $d = 8$ with $d_2 \in \{7, 8\}$, and $d = 9$ with $d_2 = 9$.

(a) Suppose that $d = 7$, 8 or 9 and $d_2 = d$. If v has an adjacent small vertex at distance two, we have a reducible configuration by (C_{11}). Otherwise, v gets $\frac{2}{5}d$ of charge by the rule R10, and so it has enough charge to send to all adjacent 2-vertices.

(b) Suppose that $d = 7$ or 8 and $d_2 = d - 1$. In this case, v has an adjacent $\geq$3-vertex w. We denote vertices at distance two that are not adjacent to w by $w_1, w_2, \ldots, w_{d-1}$. If w and some vertex w_i are small, then the reducible configuration (C_{12}) is obtained. In case that all vertices w_i are big, we use the rule R10 to obtain enough charge. Finally, if w is a big vertex, then the final charge of v is

$$\mathrm{ch}^*(v) = \frac{9}{5}d - \frac{6}{5}\big(d - 1\big) + \frac{4}{5} \geq 0.$$

- *v is a small vertex of degree* ≥ 10. It sends at most $\frac{6}{5}$ along each edge, thus its final charge is at least

$$\mathrm{ch}^*(v) \geq \frac{9}{5}d - 6 - \frac{6}{5}d = \frac{3}{5}d - 6 \geq 0.$$

- *v is a big vertex.* It sends at most $\frac{9}{5} - \epsilon$ along each edge, thus it sends at most $(\frac{9}{5} - \epsilon)d$ of charge. So, after appying the discharging rules its final charge is

$$\mathrm{ch}^*(v) \geq \frac{9}{5}d - 6 - \left(\frac{9}{5} - \epsilon\right)d = \epsilon\, d - 6.$$

  Considering that v has degree $d \geq \left\lceil \frac{6}{\epsilon} \right\rceil$, we infer that their final charge is non-negative.

  This establishes the theorem. $\square$

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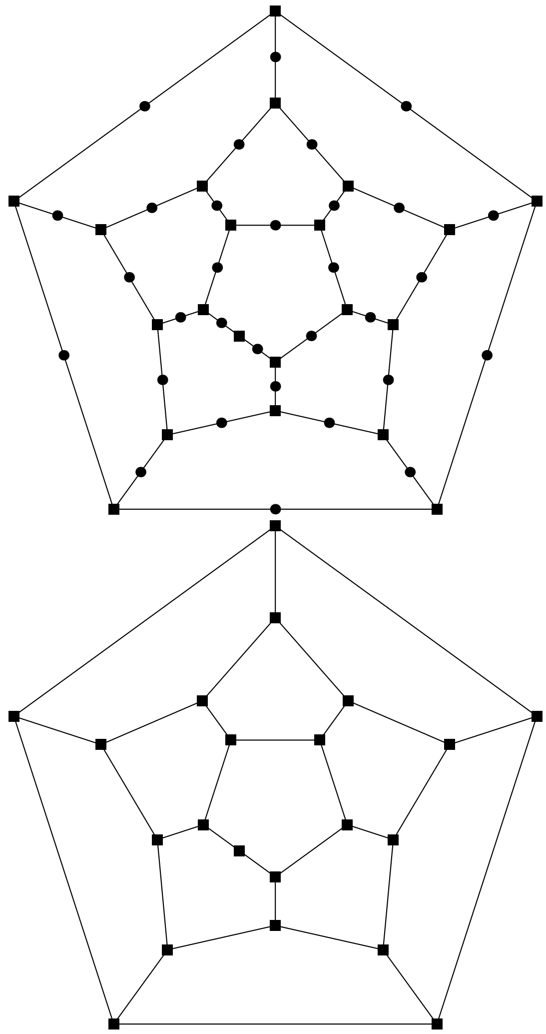


Figure 3: A subcubic planar graph with girth 10 and injective chromatic number 4, and a subcubic planar graph with chromatic index 4

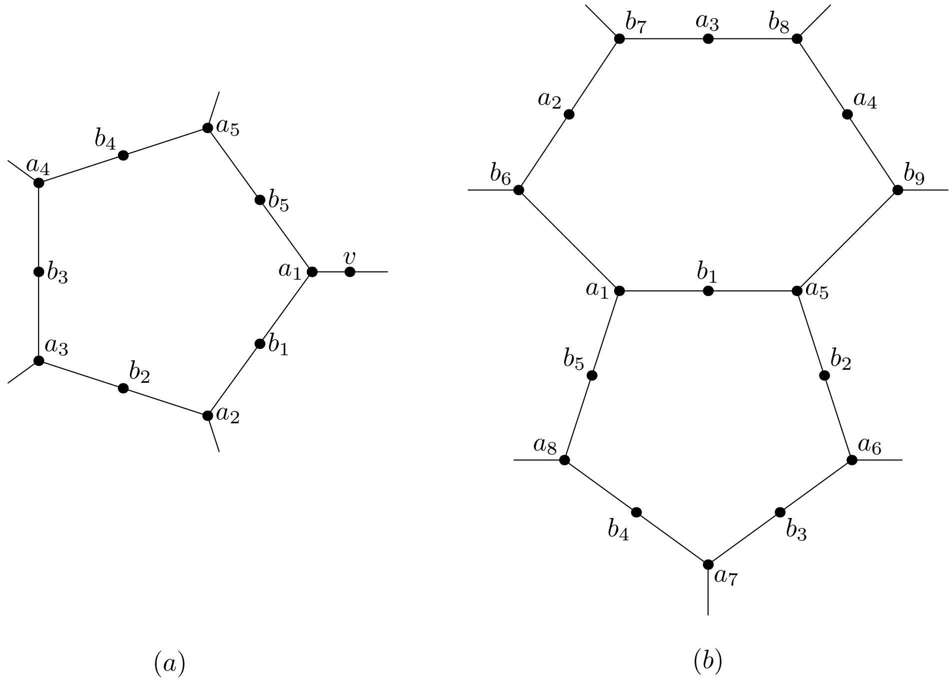


Figure 4: Reducible configurations for injective 4-colorings

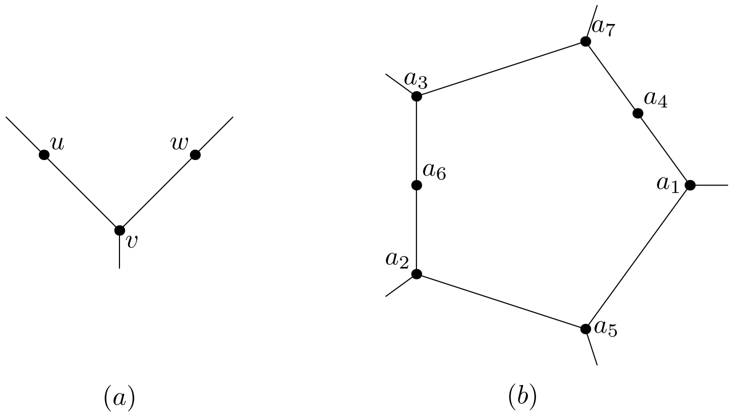


Figure 5: Reducible configurations for injective 5-colorings

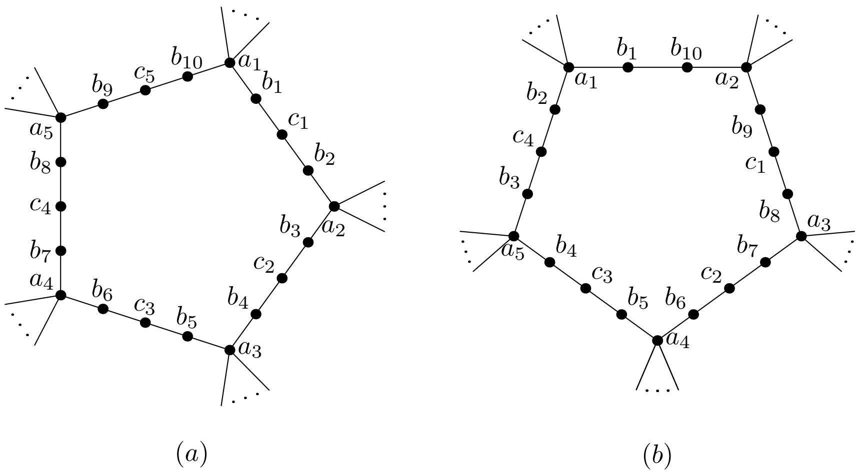


Figure 6:Reducible configurations for injective Δ-colorings

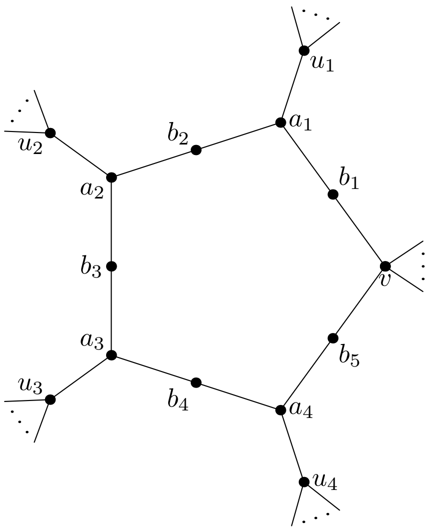
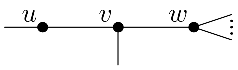
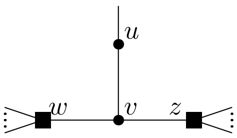


Figure 7: Reducible configuration for injective $(\Delta + 1)$-colorings

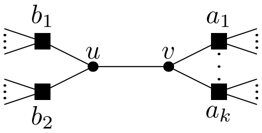


$d(w) \leq 4$

(C_1)

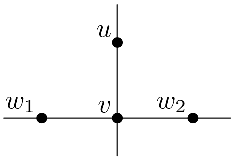


(C_2)

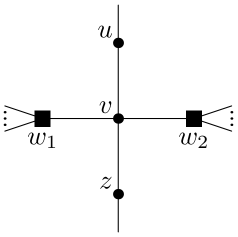


$d(v) \leq 6$

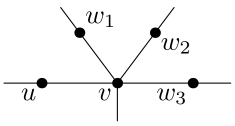
(C_3)



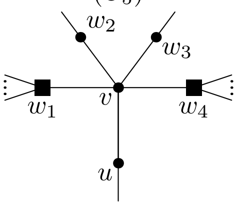
(C_4)



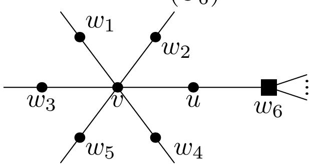
(C_5)



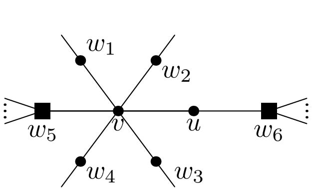
(C_6)

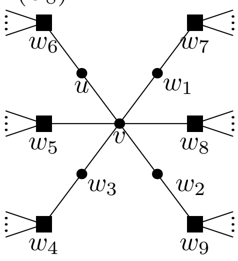


(C_7)



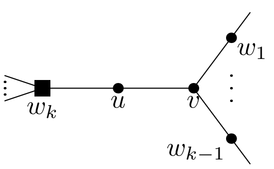
(C_8)



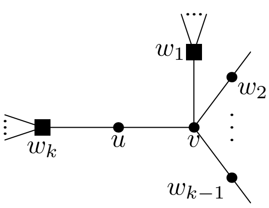


(C_9)

(C_{10})



$7 \leq d(v) \leq 9$



$7 \leq d(v) \leq 8$