Distance Constrained Labelings of K_4-minor Free Graphs

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Abstract

Motivated by previous results on distance constrained labelings and coloring of squares of K_4-minor free graphs, we show that for every $p \geq q \geq 1$, there exists Δ_0 such that every K_4-minor free graph G with maximum degree $\Delta \geq \Delta_0$ has an $L(p,q)$-labeling of span at most $q\lfloor 3\Delta(G)/2 \rfloor$. The obtained bound is the best possible.

1 Introduction

Distance constrained labelings of graphs form an important graph theoretical model for the channel assignment problem. An $L(p,q)$*-labeling* of a graph G for integers $p \geq q \geq 1$ is a labeling of its vertices by non-negative integers such that the labels of adjacent vertices differ by at least p and those at distance two by at least q. The smallest K for which there exists an $L(p,q)$-labeling with labels $0, \ldots, K$ is called the $L(p,q)$*-span* of G and denoted by $\lambda_{p,q}(G)$. The notion of $L(p,q)$-labeling is closely related to classical graph colorings: the $L(1,1)$-span of a graph G is equal to the chromatic number of G^2 decreased by one.

In this paper, we focus on distance constrained labelings of graphs that do not contain the complete graph of order four as a minor. Such graphs form a subclass of planar graphs which includes all outer-planar

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graphs. Both $L(p,q)$-labelings and colorings of squares of planar graphs were intensively studied, yielding many results: Wegner [30] proved that $\chi(G^2) \le 8$ for planar graphs G with the maximum degree three and conjectured that the bound can be improved to seven. This conjecture has been recently confirmed by Thomassen [28]. For planar graphs with the maximum degree $\Delta \ge 4$, Wegner [30] conjectured that $\chi(G^2) \le \Delta + 5$ for $\Delta \in \{4,5,6,7\}$ and $\chi(G^2) \le \lfloor 3\Delta/2 \rfloor + 1$, otherwise. The best upper bound $\chi(G^2) \le \lfloor 5\Delta/3 \rfloor + 78$ was established in [24, 25] and closely related results on coloring of higher order powers of planar graphs were given in [2, 3]. In the case of $L(p,q)$-labelings of planar graphs, van den Heuvel et al. [15] show that $\lambda_{p,q}(G) \le (4q-2)\Delta + 10p + 38q - 24$, and Borodin et al. [6] provide the bound of $\lambda_{p,q}(G) \le (2q-1)\lceil 9\Delta/5 \rceil + 8p - 8q + 1$ for $\Delta \ge 47$. The best asymptotic result $\lambda_{p,q}(G) \le q\lceil 5\Delta/3 \rceil + 18p + 77q - 18$ is due to Molloy and Salavatipour [24, 25]. Better bounds are known for planar graphs without short cycles [27], e.g. $\lambda_{p,q}(G) \le (2q-1)\Delta + 4p + 4q - 4$ if G is a planar graph of girth at least seven. The bound for planar graphs with girth seven has recently been improved to $2p + q\Delta - 2$ [10] under the assumption that the maximum degree Δ is sufficiently large (this bound is the best possible if $q = 1$ which includes the case of $L(2,1)$-labelings).

For general graphs, the research was focused mainly on $L(2,1)$-labelings because of their practical applications. Another reason is the conjecture of Griggs and Yeh [14] which assumes that $\lambda_{2,1}(G) \le \Delta^2$ for every graph G with maximum degree $\Delta \ge 2$. The conjecture was verified for several special classes of graphs, including graphs of maximum degree two, outer planar graphs [8], planar graphs with maximum degree $\Delta \ne 3$ [4], chordal graphs [26] (see also [7, 21]), hamiltonian cubic graphs [16, 17], direct and strong products of graphs [18], etc. For general graphs, the original bound $\lambda_{2,1}(G) \le \Delta^2 + 2\Delta$ of [14] was improved to $\lambda_{2,1}(G) \le \Delta^2 + \Delta$ in [9]. A more general result contained in [20] yields $\lambda_{2,1}(G) \le \Delta^2 + \Delta - 1$ and the best known bound of $\Delta^2 + \Delta - 2$ was given by Gonçalves [13]. Algorithmic aspects of $L(2,1)$-labelings as well as $L(p,q)$-labelings are also well investigated [1, 5, 11, 12, 19, 23] because of their applications in practice.

Motivated by results of [29], we study $L(p,q)$-labelings of K_4-minor free graphs. This class of graphs includes series-parallel graphs, an interesting class of graphs obtained by two simple operations from single edges as reviewed in Section 2. Wang et al. [29] show that every K_4-minor free graph G with maximum degree Δ has an $L(p,q)$-labeling, $p + q \ge 3$, with span at most $2(2p-1) + (2q-1)\lfloor 3\Delta/2 \rfloor$. This result generalizes the previous result of Lih, Wang and Zhu [22] that the squares of K_4-minor free graphs with maximum degree $\Delta \geq 4$ are $(\lfloor 3\Delta/2 \rfloor + 1)$-colorable, i.e., $\lambda_{1,1}(G) = \lfloor 3\Delta/2 \rfloor$. Since the latter bound is optimal, one may ask whether the bound obtained in [29] is also optimal. We show that this is not the case and the bound for any $p \geq q = 1$ matches the bound for $L(1,1)$-labelings if the maximum degree Δ is sufficiently large. More precisely, we show that for every $p \geq 1$, there exists Δ_0, such that every K_4-minor free graph with maximum degree $\Delta \geq \Delta_0$ has an $L(p,1)$-labeling with span at most $\lfloor 3\Delta/2 \rfloor$. Since this bound matches the optimal bound of [22], it cannot be further decreased. In this paper, we only focus on establishing the existence of Δ_0. Our results also translate to $L(p,q)$-labelings with $q > 1$.

Let us remark that all graphs considered in this paper are simple, i.e., without loops and parallel edges and we use standard graph theory notation which can be found in most textbooks on graph theory.

2 Structure of Series-parallel Graphs

In this section, we introduce notation related to K_4-minor free graphs and series-parallel graphs in particular. *Series-parallel graphs* can be obtained by the following recursive construction based on graphs with two distinguished vertices called *poles*. The simplest series-parallel graph is an edge uv and the two poles of it are its end-vertices. If G_1 and G_2 are series-parallel graphs with poles u_1 and v_1, and u_2 and v_2, respectively, then the graph G obtained by identifying the vertices v_1 and u_2 is also a series-parallel graph and its two poles are the vertices u_1 and v_2. The graph G obtained in this way is called the *serial join* of G_1 and G_2. The *parallel join* of G_1 and G_2 is the graph obtained by identifying the pairs of vertices u_1 and u_2 and v_1 and v_2 with the poles being the identified vertices. The series-parallel graphs are precisely those that can be obtained from edges by a series of serial and parallel joins.

It is well-known that every 2-edge-connected K_4-minor free graph is a series-parallel graph. Let us now state this fact as a separate lemma:

Lemma 1. *Every block of a K_4-minor free graph is a series-parallel graph.*

The construction of a particular series-parallel graph G can be encoded by a rooted tree which is called the *SP-decomposition tree* of G. Each node of the tree corresponds to a subgraph of G obtained at a step of the recursive construction of G. The leaves correspond to simple paths with their end-vertices being poles (such graphs are obtained by successive serial joins from edges) and each inner node of the tree corresponds to either a serial or a parallel join. Based on this, there are two types of inner nodes: *S-nodes* and *P-nodes*. The inner nodes have at least two children: the subgraphs corresponding to their children were joined together by a sequence of serial or parallel joins depending on the type of the node. Since the result of a sequence of serial joins depends on the order in which the serial joins are applied, the children of each inner node are ordered. Without loss of generality, we can assume that the children of a P-node are S-nodes and leaves only, and the children of an S-node are P-nodes and leaves only. We can also assume that no two consecutive children of an S-node are leaves.

An SP-decomposition tree corresponding to a series-parallel graph G is not unique. In fact, there is a lot of freedom in its choice as can be seen in the following well-known result:

Lemma 2. *Let G be a series-parallel graph and v a vertex of G. There exists an SP-decomposition tree such that v is one of the poles of the graph corresponding to the root of the SP-decomposition tree.*

In the proof of our main result, we show that a minimal possible counter-example does not contain certain subgraphs. Their structure is based on the subtrees corresponding to them in the SP-decomposition tree. A subgraph of G corresponding to a leaf of the tree, i.e., a path consisting of 2-vertices , is called an *ℓ-subgraph* of G (ℓ stands for leaf). A subgraph obtained by a parallel join of A_1-subgraph, A_2-subgraph, ..., A_k-subgraph, is a $P(A_1, \ldots, A_k)$-subgraph and a subgraph obtained by a serial join of such subgraphs is an $S(A_1, \ldots, A_k)$-subgraph. For instance, a $P(\ell, \ell, \ell)$-subgraph is a subgraph of G that corresponds to a P-node with three leaves. Since the result of a serial join depends on the order in which the subgraphs are joined, we require the sequence $A_1, \ldots, A_k$ to respect this order. Subgraphs obtained by a parallel join of several A-subgraphs are called $P(A*)$-subgraphs and those obtained by a serial join $S(A*)$-subgraphs. $P(\ell*)$-subgraphs are called P-subgraphs for short. An example of this notation can be found in Figure 2.

Finally, we introduce a special name for particular P-subgraphs of a series-parallel graph G. A P-subgraph of G obtained by a parallel join of several two-edge paths and possibly an edge is called a *crystal*. Its vertices distinct from its poles are said to be its *inner* vertices. The *size* of a crystal is the number of edges incident with each of its poles, i.e., if the poles are adjacent, the size of a crystal is the number of the paths forming it increased by one. If A is a crystal, $\mathrm{Inner}(A)$ denotes its set of inner vertices and size(A) denotes its size.

3 Labelings of K_4-minor Free Graphs

In this section, we state and prove our results on $L(p,q)$-labelings of K_4-minor free graphs. For integers $p \geq 1$ and $D \geq 1$, a graph G is said to be (D,p)-*bad* if G is K_4-minor free, has maximum degree is at most D and has no $L(p,1)$-labeling with span at $\lfloor 3D/2 \rfloor$. A graph G is (D,p)-*minimal* if it is (D,p)-bad and there is no (D,p)-bad graph of smaller order. Finally, a function $c : V(G) \to \{0,1,\ldots,k\}$ is an $L_D(p,1)$-labeling of G if it is an $L(p,1)$-*labeling* of G and its span is at most $\lfloor 3D/2 \rfloor$. Clearly, a graph G is (D,p)-bad if it is K_4-minor free and there is no $L_D(p,1)$-labeling of G.

The following theorem shows that (for a fixed positive integer p) there are only finitely many (D,p)-minimal graphs.

Theorem 3. *For every positive integer p, there exists an integer D_0 such that there is no (D,p)-bad graph for any $D \geq D_0$.*

Before we present the proof of Theorem 3, let us state the following immediate corollary of it which gives a clearer statement of the result.

Corollary 4. *For every positive integers $p \geq q$, there exists Δ_0 such that every K_4-minor free graph with maximum degree $\Delta \geq \Delta_0$ has an $L(p,q)$-labeling with span at most $q\lfloor 3\Delta/2 \rfloor$.*

Proof. Fix integers p and q, and set Δ_0 to be the constant D_0 from Theorem 3 for $p' = \lceil p/q \rceil$. To see that Δ_0 has the required properties, fix a K_4-minor free graph G such that $\Delta(G) \geq \Delta_0$. By Theorem 3 (for $D = \Delta(G)$), there exists an $L(\lceil p/q \rceil, 1)$-labeling c of G of span at most $\lfloor 3\Delta(G)/2 \rfloor$. Set $c'(v) = qc(v)$ for each $v \in V(G)$. Since the differences of the labels assigned to neighboring vertices by c' are at least q and the differences of the labels of vertices at distance two are at least $q\lceil p/q \rceil \geq p$, c' is an $L(p,q)$-labeling of G and since its span is at most $q\lfloor 3\Delta(G)/2 \rfloor$, the statement of the corollary follows. □

In a series of lemmas, we show that if D is sufficiently large (in terms of p), then certain subgraphs cannot appear in a (D,p)-minimal graph and we eventually conclude that there is no (D,p)-minimal graph. The main idea of each of the proofs is to modify the given (D,p)-minimal graph G to a smaller one which has an $L_D(p,1)$-labeling by the (D,p)-minimality of G

and use that labeling to obtain an $L_D(p,1)$-labeling of G contradicting the assumption that G is (D,p)-bad.

3.1 Overture

Clearly, every (D,p)-minimal graph is connected. In the following lemma, we show that it cannot contain vertices of degree one or two adjacent vertices of degree two.

Lemma 5. *For every positive integer p, there exists a constant D_5 such that no (D,p)-minimal graph, $D \geq D_5$, contains a vertex of degree at most one or two adjacent vertices of degree two.*

Proof. We prove the lemma for $D_5 = 8p-4$. Let us fix a (D,p)-minimal graph G, $D \geq D_5$. First, consider the case that there is a vertex v in G of degree one. Remove the vertex and find an $L_D(p,1)$-labeling c of $G \setminus v$ (such a labeling exists by the minimality of G). We now aim to extend c to an $L_D(p,1)$-labeling of the entire graph G. To show that there is a suitable label for v, we count the number of labels in the set $\{0,\ldots,\lfloor 3D/2\rfloor\}$ which cannot be used on v without violating the constraints of $L(p,1)$-labelings. We say that those labels are *forbidden* for v. In particular, we show that the number of labels forbidden for v is at most $\lfloor 3D/2\rfloor$, and thus there is at least one label available for v. The label of the only neighbor w of v forbids at most $2p-1$ labels to be assigned to v and the neighbors of w forbid at most additional $D-1$ labels, hence the total number of labels which cannot be assigned to v is at most $2p-1+D-1 = D+2p-2 \leq D+4p-2 = D+\lfloor D_5(p)/2\rfloor \leq \lfloor 3D/2\rfloor$. In particular, c can be extended to an $L_D(p,1)$-labeling of G, i.e., G is not (D,p)-bad—a contradiction.

Next, we show that there are no two adjacent vertices u and v of degree two. Remove u and v from G and find an $L_D(p,1)$-labeling c of $G \setminus \{u,v\}$. Let x be the neighbor of u different from v and y the neighbor of v different from u, i.e., G contains a path $xuvy$. We first find a label for u: there are at most $2p-1$ labels forbidden by x, at most $D-1$ forbidden by the neighbors of x and at most one label forbidden by y. Together, there are at most $2p-1+D-1+1 = D+2p-1 \leq \lfloor 3D/2\rfloor$ labels forbidden for u and therefore, we can label u properly. The case of v is analogous, except that there are at most $2p-1$ additional labels forbidden by u. We conclude that c can be extended to the entire graph G which contradicts the (D,p)-minimality of G. $\square$

In the rest of the proof, we choose one of the end-blocks of the block-decomposition of a K_4-minor free graph G (we choose the entire G if G is 2-connected) and show that it cannot contain certain types of subgraphs. We refer to the chosen end-block as to the *final block*, and write $G^\star$ for it. By Lemma 1, the final block is a series-parallel graph and, by Lemma 2, we may assume that one of the poles of the graph corresponding to the root of its SP-decomposition is its cut-vertex. In case that G is 2-vertex-connected, we consider an arbitrary SP-decomposition of G. One such (fixed) decomposition of $G^\star$ will be denoted by $T^\star$. Notice that since $G^\star$ is 2-connected, the root of $T^\star$ is a P-node.

We adopt the notation of A-subgraphs introduced in Section 2, and we say that an A-subgraph is *contained* in $G^\star$, if there is a subtree T_A of the form described by A with root r in $T^\star$ such that there is no descendant w of r in $T^\star$ whose depth (measured from r) is greater than depth of every descendant of r in T_A (in other words, we allow the subtree of the node r to be more complex than just A-subgraph, but we do not want it to be significantly more complex).

An immediate consequence of Lemma 5 is that all P-subgraphs contained in the final block are crystals. In fact, crystals are the "building blocks" of many of the reducible subgraphs we deal with later in the proof. The following lemma gives two useful estimates on the size of crystals in (D,p)-minimal graphs.

Lemma 6. *For every positive integer p, there exist a constant K such that no (D,p)-minimal graph G, contains a crystal of size greater than $\lceil D/2\rceil + K$. Moreover, if C_1 and C_2 are two crystals in G sharing a vertex v such that v is incident to no vertex of G except for the vertices of C_1 and C_2 and C_2 contains at least one inner vertex, then the size of C_1 is at least $\lfloor D/2\rfloor - K$.*

Proof. We prove the lemma for $K = 4p-4$. Let us fix a (D,p)-minimal graph G. To see the first claim, suppose that there is a crystal C with poles u and v of size $k \geq \lceil D/2\rceil + 4p - 3$ and let w be an inner vertex of C. Remove w and find a proper $L_D(p,1)$-labeling c of $G \setminus w$. It is now possible to extend c to an $L_D(p,1)$-labeling of G because there are at most $\lfloor 3D/2\rfloor$ labels which cannot be assigned to w: the labels of vertices u and v forbid at most $2p-1$ labels each, at most $k-1$ labels are forbidden by the labels of the remaining inner vertices of C, at most $D-k$ labels are forbidden by neighbors of u outside C and other at most $D-k$ labels are forbidden by the labels of the neighbors of v outside C. Altogether, there are at most

$2D + 4p - 3 - k \leq \lfloor 3D/2 \rfloor$ forbidden labels, hence there is still at least one label available for w, thus G is not (D, p)-bad—a contradiction.

To prove the second claim, suppose there are two crystals C_1 and C_2 with the common pole v which is connected only to the vertices of C_1 and C_2, and C_2 contains an inner vertex w. Let u be the pole of C_2 different from v, k be the number of inner vertices of C_2 and $\ell \leq \lfloor D/2 \rfloor - 4p + 3$ be the size of C_1. As in the first part of the proof, we remove w from G and find an $L_D(p,1)$-labeling of $G \setminus w$ which we extend to an $L_D(p,1)$-labeling of G. The number of labels forbidden for w is again at most $\lfloor 3D/2 \rfloor$: vertices u and v forbid at most $2p-1$ labels each, at most $k-1$ labels are forbidden by the inner vertices of C_2, at most ℓ labels are forbidden by the vertices in C_1 neighboring with v, and at most $D-k$ labels are forbidden by neighbors of u outside of C_2; altogether $4p - 3 + D + \ell \leq \lfloor 3D/2 \rfloor$ forbidden labels, thus there is again at least one label available for w. Hence, there exists an $L_D(p,1)$-labeling of G which contradicts the (D,p)-minimality of G. □

Let us turn our attention back to the SP-decomposition $T^\star$. We already know that the deepest inner nodes are P-nodes and that they correspond to crystals; to proceed with the proof, we investigate the neighborhood of those crystals. There are two possibilities—either the P-node is the entire decomposition $T^\star$ or it has an S-node parent S. Let us start with the former case:

Lemma 7. *For every positive integer p, there exists a constant D_7 such that the SP-decomposition of the final block of every (D,p)-minimal graph, $D \geq D_7$, has at least two inner nodes.*

Proof. We prove the lemma for $D_7 = \max\{8p - 6, D_5\}$, where D_5 is the constant from Lemma 5. For the sake of contradiction, assume that there is a (D,p)-minimal graph G (for some $D \geq D_7$) whose final block $G^\star$ violates the statement. By Lemma 5, the entire decomposition cannot be just a leaf. Hence, we may assume that the decomposition consists of a single P-node with several leaves. In other words, $G^\star$ is a single crystal with poles u and v, which is possibly connected to the rest of G through the pole v. Let w be one of the inner vertices of the crystal. Remove w, find an $L_D(p,1)$-labeling c of $G \setminus w$ and then extend the labeling to w. The number of labels forbidden for w is at most $4p - 3 + D \leq \lfloor 3D/2 \rfloor$: at most $2(2p-1)$ because of u and v, at most $k-1$ because of the inner vertices of $G^\star$, and at most $D-k$ because of the neighbors of v outside $G^\star$. Hence, we can extend c to an $L_D(p,1)$-labeling of G. Therefore, G is not (D,p)-bad. □

Figure 1: A $S(P,\ell)$-subgraph and the corresponding subtree.

3.2 Allegro

By Lemma 7, if D is large enough, we know that every bottommost P-node P_0 in $G^\star$ has an S-node parent S_0. Let us investigate the other children of S_0: in the next two lemmas, we show that S_0 must have exactly two children, both being P-subgraphs.

Lemma 8. *For every positive integer p, there exists a constant D_8 such that there is no (D,p)-minimal graph, $D \geq D_8$, whose final block contains an $S(P,\ell)$-subgraph.*

Proof. We prove the lemma for $D_8 = 8p - 4$. Let us fix a (D,p)-minimal G (for some $D \geq D_8$). Suppose that $G^\star$ contains an $S(P,\ell)$-subgraph. In other words, there is a crystal A of size $k \geq 2$ with poles u and v connected to an edge vx (see Figure 1). Since the size of A is at least two, it contains an inner vertex w. Remove w and find an $L_D(p,1)$-labeling c of $G \setminus w$. Then, extend the labeling to w. The number of labels forbidden for w is at most $D + 4p - 2 \leq \lfloor 3D/2 \rfloor$: at most $2(2p-1)$ because of u and v, at most $k-1$ because of the inner vertices of A, at most $D-k$ because of neighbors of u outside A, and 1 because of the vertex x. Hence, there is at least one label available for w, and therefore c can be extended to G. This implies that G is not (D,p)-bad—a contradiction. $\square$

The lemma we just proved shows that every bottommost P-subtree has an S-node parent S_0 whose children are only P-subgraphs. The following lemma yields that there exactly two such children, i.e., the subtree of S_0 is an $S(P,P)$-subgraph as claimed before.

Lemma 9. *For every positive integer p, there exists a constant D_9 such there is no (D,p)-minimal graph, $D \geq D_9$, whose final block contains an $S(P,P,P)$-subgraph.*

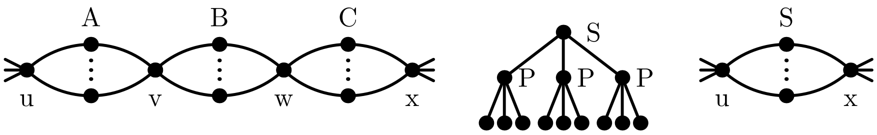

Figure 2: An $S(P,P,P)$-subgraph, the subtree corresponding to it and its reduction.

Proof. We prove the lemma for $D_9 = 4K+16p-4$, where K is the constant from Lemma 6. Let G be a (D,p)-minimal graph for some $D \geq D_9$ whose final block contains an $S(P,P,P)$-subgraph. In other words, there exist three crystals A, B, and C with poles u and v, v and w, and w and x, respectively. The configuration is depicted in Figure 2. By Lemma 6, we know that the size of each of the crystals is between $\lfloor D/2 \rfloor - K$ and $\lceil D/2 \rceil + K$. By symmetry, we may assume that the size of A is smaller than or equal to the size of C.

Construct an auxiliary graph G' as follows: remove the crystals A, B and C from G and connect u and x by ℓ paths of length two, where ℓ is the size of A. This newly created crystal is denoted by S. Since the order of G' is smaller than the order of G, there exists an $L_D(p,1)$-labeling c of G'. We now extend c to the original graph G. First, the vertices u, x, and all the vertices outside S get the same label as they are assigned by c. We use the labels assigned by c to inner vertices of S to label all vertices in the crystal A and size(A) vertices in the crystal C (note that since the distance of an inner vertex of A from an inner vertex of C is at least three, the labels of those vertices are not in a conflict). After this operation, all inner vertices of A are properly labeled and there are at most $2K+1$ vertices in C without a label.

Next, we find a label for the vertex v avoiding the conflicting labels except for the labels of inner vertices of A. The number of forbidden labels for v is at most $D+2p-1 \leq \lfloor 3D/2 \rfloor$: at most $2p-1$ because of u, at most ℓ because of the inner vertices of C, and at most $D-\ell$ because of neighbors of u outside A. To resolve possible conflicts with the labels in A, unlabel the inner vertices of A in conflict. Notice that at most $2p-1$ vertices can be unlabeled. Use a similar approach to label w—but this time, the roles of A and C are interchanged, i.e., we avoid the labels of inner vertices of A and if there is a conflict with an inner vertex of C, we unlabel the conflicting vertex. The number of labels forbidden for w is at most $D+4p-2 \leq \lfloor 3D/2 \rfloor$, since we have to avoid the label of v as well.

When v and w are labeled, we can finish labeling the inner vertices of A and C (those which did not get label yet or have been unlabeled). Let k be the number of inner vertices of A. The number of forbidden labels of an inner vertex of A (similarly for C) is at most $D+4p-2 \leq \lfloor 3D/2 \rfloor$: at most $2(2p-1)$ because of u and v, at most 1 because of w, at most $k-1$ because of the inner vertices of A, and at most $D-k$ because of the neighbors of u outside A. The final step is labeling of the inner vertices of B. Notice that the inner vertices of A and B use at most size$(A)+2K+4p-1$ distinct labels: at most size(A) for the labels taken from S, at most $2K+1$ for vertices which did not get the initial labels, and at most $2(2p-1)$ new labels of the unlabeled vertices. Therefore, the number of labels forbidden for an inner vertex of B is at most $D+2K+8p-2 \leq \lfloor 3D/2 \rfloor$: at most $2(2p-1)$ because of v and w, at most $D-\text{size}(A)-1$ because of the other inner vertices of B, at most size(A) $+ 2K + 4p - 1$ because of the inner vertices of A and C, and at most 2 because of u and x. We infer from the preceding calculations that c can be extended to G. Hence, G is not (D,p)-bad. □

Lemma 9 provides a nice characterization of possible configurations of S-nodes of the largest depth. It shows that those nodes are roots of $S(P,P)$-subgraphs in $G^\star$. Since the root of the decomposition $T^\star$ must be a P-node, every $S(P,P)$-subgraph must have a P-node parent. The following lemma shows that no such P-node has two or more $S(P,P)$-children. In particular, this shows that every $S(P,P)$-subgraph of the largest depth in $T^\star$ is contained in a $P(S(P,P),l*)$-subgraph.

Lemma 10. *For every positive integer p, there exists a constant D_{10} such that there is no (D,p)-minimal graph, $D \geq D_{10}$, whose final block contains an $P(S(P,P),\ S(P,P))$-subgraph.*

Proof. We prove the lemma for $D_{10} = 16p+8K-4$, where K is the constant from Lemma 6. Fix a (D,p)-minimal graph G, $D \geq D_{10}$, such that its final block $G^\star$ contains a $P(S(P,P),S(P,P))$-subgraph. In particular, there are two vertices u and v connected by two crystals A_L and B_L with the common pole x and by another two crystals A_R and B_R with the common pole y (see Figure 3). The entire subgraph (the four vertices and four crystals) is denoted by R. By Lemma 6, the size of each of the crystals

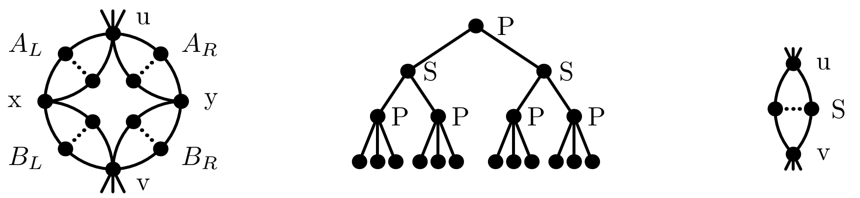

Figure 3: A $P(S(P,P),S(P,P))$-subgraph, the subtree corresponding to it and its reduction.

is at least $\lfloor D/2 \rfloor - K$, so the number neighbors of u (or v) outside R is small (at most $2K+1$). We construct an auxiliary graph G' as follows: remove the interior of R (leave only u and v) and join u and v by ℓ paths where $\ell = \min\{\text{size}(A_L)+\text{size}(A_R), \text{size}(B_L)+\text{size}(B_R))\}$. The newly added crystal is denoted by S. Since G' has less vertices than G and its maximum degree is at most D, there exists a proper $L_D(p,1)$-labeling c of G'. We extend c to a proper $L_D(p,1)$-labeling of G in the following way. First, split the labels of the inner vertices of S into two sets X and Y, such that each set contains at least $\lfloor D/2 \rfloor - K$ labels. The elements of X are used to label as many inner vertices of A_L and B_R as possible. The elements of Y are used to label the inner vertices of A_R and B_L in a similar way. Note that after this step, at most $4K+2$ inner vertices of crystals in R are not labeled. Next, label vertices x and y. As in Lemma 9, unlabel some neighboring inner vertices if there is a conflict. The number of forbidden labels for x (analogously for y) is at most $D + 4p + 2K \leq \lfloor 3D/2 \rfloor$: at most $2(2p-1)$ because of u and v, at most ℓ because of the inner vertices of A_R and B_R, at most $2K+1$ because of the neighbors of u outside R, at most $D-\ell$ because of neighbors of v outside R, and at most 1 because of the vertex y. Since the number of forbidden labels is at most $\lfloor 3D/2 \rfloor$, it is possible to label both the vertices x and y. Note that the number of unlabeled inner vertices is bounded by $2(2p-1)$. Finally, we label the remaining inner vertices (those which were unlabeled or were not labeled yet). Since there are at most $2(2p-1)+4K+2$ such vertices, the number of labels forbidden for an inner vertex of A_L is bounded by $D + 8p + 4K - 2 \leq \lfloor 3D/2 \rfloor$: $2(2p-1)$ because of x and v, $D-\ell$ because of the neighbors of v outside R, 1 because of u, and $\ell + 2(2p-1) + 4K + 1$ because of the labels of the inner vertices of A_L, B_L, and B_R. The cases of the inner vertices of the remaining three crystals are analogous. Therefore, G can be properly $L_D(p,1)$-labeled—a contradiction. □

3.3 Intermezzo

Before we continue with the proof, let us establish the following technical lemma. Before stating it, we need some additional notation: if p, t, and K are non-negative integers, then $B_K(t,p)$ denotes the set of integers x such that $0 \le x \le K$ and $|t-x| < p$. Notice that if u and v are two adjacent vertices of G and u is labeled with t, then $B_K(t,p)$ is precisely the set of labels which cannot be used to label v in any proper $L(p,1)$-labeling of G with span K.

Lemma 11. *Let p be a non-negative integer, G a graph with no adjacent 2-vertices, c its partial $L(p,1)$-labeling of span at most $K \ge |V(G)|-1$ such that every vertex which is not labeled by c is a 2-vertex, and every label is used at most once in c. Further, let $P = \{v_1,\dots,v_k\}$ be the set of all vertices whose degree is different from 2. If every label in the set $\bigcup_{i=1}^{k} B_K(c(v_i),p)$ is used on some vertex v in $V(G)$, then c can be extended to an $L(p,1)$-labeling of the entire graph G with span at most K.*

Proof. In order to extend c to the entire G, we assign the unused labels (from the set $\{0,\dots,K\}$) arbitrarily to the 2-vertices which are not labeled by c, in such a way that each label is used at most once. It is now routine to check that this extension of c is an $L(p,1)$-labeling. The condition for vertices at distance two is clearly satisfied as no two vertices get the same label. If u and v are neighboring vertices, we know that at least one of them, say u, has degree different from 2 and therefore, is labeled by c. If v is not labeled by c, the distance of the labels of u and v must be at least p because all the labels conflicting with $c(u)$ are used somewhere else in the prelabeling c. If v is labeled by c, then the proper difference of labels is guaranteed by the fact that c is a partial $L(p,1)$-labeling. □

The main benefit of the lemma is that we do not have to specify the assignment of all the labels, but only the labels of vertices with degrees different from two and the labels which are "close" (in terms of p) to those labels. This will be quite useful in the proofs of the next few lemmas which involve constructions of $L_D(p,1)$-labelings of potentially large graphs with only a few vertices which are not 2-vertices.

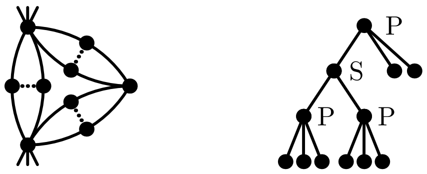

Figure 4: A $P(S(P,P),\ell*)$-subgraph and the corresponding subtree.

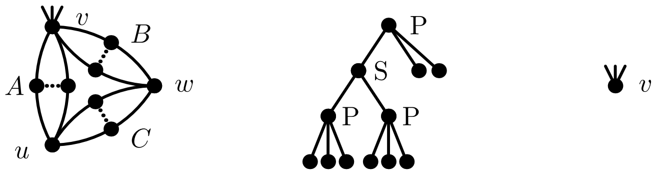

Figure 5: A $P(S(P,P),\ell*)$-subgraph being the entire final block, the corresponding subtree, and its reduction.

3.4 Largo

By Lemmas 7–10, if D is large enought, the final block of a (D,p)-minimal graph contains a $P(S(P,P),\ell*)$-subgraph (see Figure 4). This subgraph is either the entire final block or the root of the subtree corresponding to it has a parent (which is an S-node and must have another parent which is a P-node). First, we deal with the former case.

Lemma 12. *For every positive integer p, there exists a constant D_{12} such that there is no (D,p)-minimal graph, $D \geq D_{12}$, whose entire final block is a $P(S(P,P),\ell*)$-subgraph.*

Proof. We prove the lemma for

$$D_{12} = \max\left\{\frac{2}{3}(10p + 4Kp - 2K - 3), 6K + 4p + 4\right\}$$

where K is the constant from Lemma 6. For the sake of contradiction, assume that there is a (D,p)-minimal graph G, $D \geq D_{12}$, whose final block $G^\star$ is a $P(S(P,P),\ell*)$-subgraph. In particular, $G^\star$ consists of three vertices

u, v, w and three crystals A, B, and C such that the poles of the crystal A are u and v, the poles of the crystal B are v and w, and the poles of the crystal C are u and w. If $G^\star$ is not the entire graph G, then v is the cut-vertex separating $G^\star$ from the rest of G. Let N_v be the set of neighbors of v outside $G^\star$. The configuration is depicted in Figure 5. In order to produce an $L_D(p,1)$-labeling of G, we construct an auxiliary graph G' from G by replacing $G^\star$ by a single vertex v and eventually find an $L_D(p,1)$-labeling c of G'.

In the rest of the proof, we aim to extend c to an $L_D(p,1)$-labeling c' of the entire graph G. By Lemma 6, the sizes of A, B, and C are at least $\lfloor D/2 \rfloor - K$, and thus $|N_v| \leq 2K+1$. We start with the vertices u, v and w. Their labels should satisfy the following: they differ by at least $2p$ from each other and and each of them differs by at least p from all the labels of the vertices in N_v. Since this is satisfied for v, the vertex v can keep its original label and we only have to label u and w. Calculating the number of labels forbidden for u (similarly for w), we get that there are at most $2(4p-1)+(2K+1)(2p-1) \leq \lfloor 3D/2 \rfloor$ such labels. In particular, there exist suitable labels for u and w.

To finish the prelabeling, we assign the labels of vertices in N_v to some inner vertices of C and we assign any unused labels in $B_{\lfloor 3D/2 \rfloor}(c'(v),p)$ to some inner vertices in the crystal C. Since $\lfloor D/2 \rfloor - K > (2K+1) + 2p - 2 + 1$, the crystal C always contains enough inner vertices for the assignment and moreover, there will remain at least one inner vertex of C without an assigned label. The existence of such a vertex will be important in the final part of the proof. Similarly, we assign any unused elements of $B_{\lfloor 3D/2 \rfloor}(c'(u),p)$ to some inner vertices of B and the unused elements of $B_{\lfloor 3D/2 \rfloor}(c'(w),p)$ to some inner vertices of A. Notice that the resulting labeling is a valid partial $L_D(p,1)$-labeling of $G^\star$.

Next, we would like to estimate the size of $G^\star$. By the degree condition for vertices u, v, and w, we obtain the following inequalities:

$$\begin{aligned} \text{size}(A) + \text{size}(C) + |N_v| &\leq D \\ \text{size}(A) + \text{size}(B) &\leq D \\ \text{size}(A) + \text{size}(C) &\leq D \end{aligned}$$

Summing these values up, we get that

$$\begin{aligned} |V(G^\star)| &= |\text{Inner}(A)| + |\text{Inner}(B)| + |\text{Inner}(C)| + 3 \\ &\leq \text{size}(A) + \text{size}(B) + \text{size}(C) + 3 \leq \lfloor 3D/2 \rfloor + 3. \end{aligned}$$

However, the statement of Lemma 11 requires $|V(G^\star)| \leq \lfloor 3D/2 \rfloor + 1$. To overcome this problem, we consider the following cases.

Case 1: *None of A, B, and C contains an edge joining the poles of the crystal.* In this case, we can remove one unlabeled inner vertex w' from A and one unlabeled inner vertex u' from B. Let G' be the resulting graph. Since two vertices are removed in the construction of G', $|V(G')| \leq \lfloor 3D/2 \rfloor + 1$ as required. By Lemma 11, we obtain an $L_D(p,1)$-labeling $c^\star$ of G'. The labeling $c^\star$ is eventually extended to the entire $G^\star$ by setting $c^\star(w') = c^\star(w)$ and $c^\star(u') = c^\star(u)$.

Case 2: *Exactly one of A, B, and C contains an edge joining the poles of the crystal.* By the symmetry, we can assume that A contains an edge joining the end-vertices of A, i.e., $\text{size}(A) = |\text{Inner}(A)| + 1$. Remove an unlabeled inner vertex w' from A and let G' be the resulting graph. Again, $|V(G')| \leq \lfloor 3D/2 \rfloor + 1$ as required. By Lemma 11, there exists an $L_D(p,1)$-labeling $c^\star$ of G' which can be extended to an $L_D(p,1)$-labeling of $G^\star$ by setting $c^\star(w') = c^\star(w)$.

Case 3: *At least two crystals contain an edge connecting the poles.* Then,

$$\begin{aligned}
|\text{Inner}(A)| + |\text{Inner}(C)| + |N_v| + 1 &\leq D, \\
|\text{Inner}(A)| + |\text{Inner}(B)| + 1 &\leq D, \\
|\text{Inner}(A)| + |\text{Inner}(C)| + 1 &\leq D,
\end{aligned}$$

and at least one of those inequalities is strict. Therefore, we get $|V(G^\star)| \leq \lfloor 3D/2 \rfloor + 1$ and Lemma 11 can be applied to $G^\star$ directly, yielding an $L_D(p,1)$-labeling $c^\star$ of $G^\star$.

Based on the discussion above, we can find an $L_D(p,1)$-labeling $c^\star$ of $G^\star$ consistent with the prelabeling. It is routine to check that(because of the construction of the prelabeling) $c^\star$ combined with c is a proper $L_D(p,1)$-labeling of the G, hence G is not (D,p)-bad—a contradiction. $\square$

By Lemmas 7–12, the final block of a (D,p)-minimal graph contains an $S(P(S(P,P),\ell*),\ell)$-subgraph, an $S(P(S(P,P),\ell*),P)$-subgraph, or an $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph. In the next three lemmas, we show that none of these cases actually applies if D is large enough.

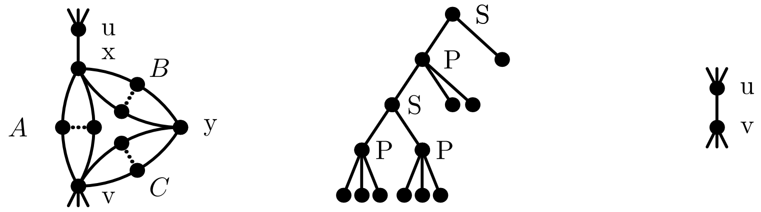

Figure 6: An $S(P(S(P,P),\ell*),\ell)$-subgraph, the subtree corresponding to it and its reduction.

Lemma 13. *For every positive integer p, there exists a constant D_{13} such that there is no (D,p)-minimal graph, $D \geq D_{13}$, whose final block contains an $S(P(S(P,P),\ell*),\ell)$-subgraph.*

Proof. We prove the statement of the lemma with

$$D_{13} = \max\{4K + 2L + 8p, (4p-2)(K+L+1) + 24p - 8\}$$

where K is the constant from Lemma 6 and $L = 4p - 3$. Let G be a (D,p)-minimal graph for some $D \geq D_{13}$ whose final block $G^\star$ contains an $S(P(S(P,P),\ell*),\ell)$-subgraph. In particular, $G^\star$ contains four vertices u, v, x and y, three crystals A, B, and C such that the poles of crystal A are x and v, the poles of crystal B are x and y, the poles of crystal C are y and v, and there is an edge joining u with x. The entire subgraph (the four vertices and the three crystals) is denoted by R and is depicted in Figure 6. Finally, let N_u and N_v be the set of the neighbors of u and v that are not contained in R. Moreover, if there is an edge uv in $G^\star$ which is not contained in R, then we also set $v \in N_u$ and $u \in N_v$. Both N_u and N_v are nonempty, otherwise x is a cut-vertex and $G^\star$ is not 2-connected.

By Lemma 6, the sizes of both B and C are at least $\lfloor D/2 \rfloor - K$. Next, we show that the size of A is at least $\lfloor D/2 \rfloor - L$. Assume the contrary, i.e., $\mathrm{size}(A) \leq \lfloor D/2 \rfloor - L - 1$. Let us remove an inner vertex w from B and find an $L_D(p,1)$-labeling c of $G \setminus w$. Since the number of labels forbidden for w is at most $D-1+\lfloor D/2 \rfloor - L - 1 + 1 + 2(2p-1) = \lfloor 3D/2 \rfloor - L + 4p - 3 = \lfloor 3D/2 \rfloor$, c can be extended to G. However, this is impossible by the (D,p)-minimality of G. Hence, the size of A must be at least $\lfloor D/2 \rfloor - L$. Combined with the lower bound on the size of C, we obtain that $|N_v| \leq K + L + 1$.

In order to prove the statement of the lemma, we construct a new graph G' from G by replacing R with an edge uv and find an $L_D(p,1)$-labeling c of G' which we eventually extend to an $L_D(p,1)$-labeling of G. The proof proceeds similarly to the proof of Lemma 12. First, we find the prelabeling: the labels of the vertices x and y are chosen in such a way that they differ by at least $2p$ from the labels of both u and v, by at least p from the labels of the vertices in N_v, by at least one from the labels of the vertices in N_u, and by at least $2p$ from each other. The number of forbidden labels is bounded by $2(4p-1)+(K+L+1)(2p-1)+D-1+4p-1 \le \lfloor 3D/2 \rfloor$.

Next, the labels of vertices in $N_v \setminus u$ are used to label some inner vertices of B, and the unused labels in $B_{\lfloor 3D/2 \rfloor}(c(v),p) \cup B_{\lfloor 3D/2 \rfloor}(c(u),p)$ are used to label some inner vertices of B. Since $\lfloor D/2 \rfloor - K > (K+L+1)+2(2p-2)+1$, the number of inner vertices in the crystal B is sufficient so that all the labels described above can be used on some vertices of B. Moreover, there always remains at least one inner vertex of B without a label. Finally, the unused labels in $B_{\lfloor 3D/2 \rfloor}(c(y),p)$ are used to label some inner vertices of A and the unused labels in $B_{\lfloor 3D/2 \rfloor}(c(x),p)$ are used to label some inner vertices of C.

It is straightforward to verify that this partial labeling satisfies the conditions on the prelabeling given in Lemma 11 for the subgraph R and span $\lfloor 3D/2 \rfloor$ with a possible exception for the condition that $|V(R)| \le \lfloor 3D/2 \rfloor$. Since the degree of each of v, x, and y is at most D, we obtain the following inequalities:

$$\begin{aligned} \mathrm{size}(A) + \mathrm{size}(B) + 1 &\le D, \\ \mathrm{size}(B) + \mathrm{size}(C) &\le D, \\ \mathrm{size}(A) + \mathrm{size}(C) + |N_v| &\le D. \end{aligned}$$

Summing these inequalities up and using $|N_v| \ge 1$, we conclude that

$$2(\mathrm{size}(A) + \mathrm{size}(B) + \mathrm{size}(C)) \le 3D - 2.$$

Thus,

$$\begin{aligned} |V(R)| &= 4 + |\mathrm{Inner}(A)| + |\mathrm{Inner}(B)| + |\mathrm{Inner}(C)| \\ &\le 4 + \lfloor (3D-2)/2 \rfloor = \lfloor 3D/2 \rfloor + 3. \end{aligned}$$

As in Lemma 12, we cannot apply Lemma 11 directly to R in general, because the number of vertices could be greater than $\lfloor 3D/2 \rfloor + 1$. However, by considering the same three cases as in Lemma 12, we conclude that the prelabeling can always be extended to an $L_D(p,1)$-labeling c_R of R. By

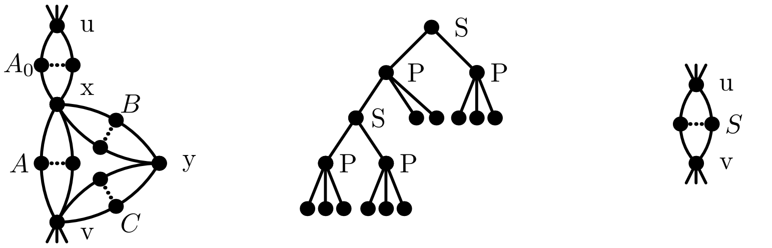

Figure 7: An $S(P(S(P,P),\ell*),P)$-subgraph, the subtree corresponding to it and its reduction.

the construction of the prelabeling, c_R can be combined with c to yield an $L_D(p,1)$-labeling of the entire graph G. Hence, G is not (D,p)-bad, a contradiction. □

Lemma 14. *For every positive integer p, there exists a constant D_{14} such that there is no (D,p)-minimal graph, $D \geq D_{14}$, whose final block contains an $S(P(S(P,P),\ell*),P)$-subgraph.*

Proof. We prove this lemma for $D_{14} = 10p(K + 6p + 6)$ where K is the constant from Lemma 6. For the sake of contradiction, assume that there exists a (D,p)-minimal graph G, $D \geq D_{13}$, whose final block $G^\star$ contains an $S(P(S(P,P),\ell*),P)$-subgraph. In particular, $G^\star$ contains four vertices u, v, x, and y and four crystals A_0, A, B, and C, such that the poles of A_0 are u and x, the poles of A are x and v, the poles of B are x and y, and the poles of C are v and y. The entire subgraph (the four vertices and the four crystals) is denoted by R and is depicted in Figure 7. Let N_u and N_v be the sets of neighbors of u and v outside R and set $M = 5p$.

By Lemma 6, the sizes of the crystals B and C are at least $\lfloor D/2 \rfloor - K$. Consequently, the sizes of A_0 and A sum to at most $\lceil D/2 \rceil + K$. By an argument analogous to that used to prove Lemma 6, we show that the sum of the sizes of A and A_0 is at least $\lfloor D/2 \rfloor - L$, where $L = 4p-4$. Assume the contrary, i.e., $\text{size}(A) + \text{size}(A_0) \leq \lfloor D/2 \rfloor - L - 1$. Then, remove an inner vertex w from B and find an $L_D(p,1)$-labeling c of $G \setminus w$. Since the number of labels forbidden for w is at most $D - 1 + \lfloor D/2 \rfloor - L - 1 + 2(2p-1) = \lfloor 3D/2 \rfloor - L + 4p - 4 = \lfloor 3D/2 \rfloor$, c can be extended to G. However, this is impossible by the (D,p)-minimality of G.

Let $M = 5p$. We distinguish two cases: $\mathrm{size}(A_0) \geq \frac{D}{2M}$ and $\mathrm{size}(A_0) < \frac{D}{2M}$.

Case $\mathrm{size}(A_0) \geq \frac{D}{2M}$**:** Consider the graph G' obtained from G by contracting the subgraph induced by A, B, and C into the vertex v. In particular, R is transformed to a crystal S with poles u and v. Next, we find an $L_D(p,1)$-labeling c of G' and extend it to an $L_D(p,1)$-labeling of G as described in the following. The labels assigned to the inner vertices of S are used to label the inner vertices of A_0 and as many inner vertices of C as possible. Then, suitable labels for x and y are found (possibly by unlabeling some vertices in A_0 and C). The number of forbidden labels for x and y is at most $D - \frac{D}{2M} + \lceil D/2 \rceil + K + 2(2p-1) + 1 \leq \lfloor 3D/2 \rfloor$. The inner vertices of A and the remaining vertices of A_0 are labeled next. The number of forbidden labels is at most $D - 1 + 2p - 1 + 1 + 2(2p-1) \leq \lfloor 3D/2 \rfloor$ for the inner vertices of A and at most $D - 1 + \lceil \frac{D}{2} \rceil + K + 1 - \frac{1}{2M}D + 1 + 2(2p-1) \leq \lfloor 3D/2 \rfloor$ for the inner vertices of A_0.

The next step is labeling of the remaining inner vertices of C. The number of forbidden labels for those vertices is bounded by $D - 1 + 2(2p-1) + 1 \leq \lfloor 3D/2 \rfloor$. Finally, label the inner vertices of B (the number of forbidden labels for these vertices is at most $D - 1 + \lceil D/2 \rceil + K - \frac{D}{2M} + 2p - 1 + 2(2p-1) \leq \lfloor 3D/2 \rfloor$). We conclude that c can be extended to an $L_D(p,1)$-labeling of the entire G—a contradiction.

Case $\mathrm{size}(A_0) < \frac{D}{2M}$**:** Since $\mathrm{size}(A) + \mathrm{size}(A_0) \geq \lfloor D/2 \rfloor - L$, $\mathrm{size}(A) \geq \frac{(M-1)D}{2M} - L$. Therefore, $|N_v| \leq \frac{D}{2M} + K + L \leq \frac{D}{M}$. By the 2-connectivity of $G^\star$, $|N_v| \geq 1$ (otherwise, x would be a cut-vertex). We proceed analogously to the proof of Lemma 13. Transform G to G' by replacing R with a single edge uv, and find an $L_D(p,1)$-labeling c of G'. In the rest of the proof, we demonstrate how to extend c to an $L_D(p,1)$-labeling of the entire graph G.

First, we find labels for x and y that differ from the labels of u and v by at least $2p$, from the labels of the vertices in N_v by at least p, from the labels of the vertices in N_u by at least one, and from each other by at least $2p$. This is always possible since the number of labels forbidden for x and y is at most $D + \frac{D}{M}(2p-1) + 3(4p-1) \leq \lfloor 3D/2 \rfloor$. Next, labels for the inner vertices of A_0 are found in such a way that the difference of these labels from the labels of u, v, x, and y is at least p and they are different from the labels of all the vertices in $N_u \cup N_v$. Note that the number of forbidden labels for each vertex of A_0 is bounded by $D + 4(2p-1) + \frac{D}{2M} + K + L \leq \lfloor 3D/2 \rfloor$.

Now, assign all the labels of the inner vertices of A_0 to some inner vertices of C and assign the labels of the vertices in N_v to some inner vertices of

B (omit the label $c(u)$ if ux is an edge). Next, construct an auxiliary graph G_0 by taking the subgraph of G induced by the set $\{v, x, y\} \cup A \cup B \cup C$. If ux is an edge in G, add u and the edge ux to G_0 as well. Let c_0 be the obtained prelabeling of G_0. To meet the conditions of Lemma 11, we extend c_0 as follows: the unused labels in $B_{\lfloor 3D/2 \rfloor}(c_0(u), p)$ and $B_{\lfloor 3D/2 \rfloor}(c_0(v), p)$ are assigned to some vertices in B and the unused labels in $B_{\lfloor 3D/2 \rfloor}(c_0(x), p)$ and $B_{\lfloor 3D/2 \rfloor}(c_0(y), p)$ are assigned to some inner vertices of C and A, respectively. Notice that since $\lfloor D/2 \rfloor - K > \frac{D}{M} + 2(2p-2) + 1$ and $\frac{(M-1)D}{2M} - L > 2(2p-2)+1$, every crystal contains enough inner vertices for the assignment of the labels and every crystal will always contain at least one inner vertex without a label.

Finally, we have to show that $|V(G_0)| \leq \lfloor 3D/2 \rfloor + 1$. Since the degree of each of the vertices x, y, and v is bounded by D, we obtain that

$$\begin{aligned} |\mathrm{size}(A)| + |\mathrm{size}(B)| + \mathrm{size}(A_0) &\leq D \\ |\mathrm{size}(B)| + |\mathrm{size}(C)| &\leq D \\ |\mathrm{size}(A)| + |\mathrm{size}(C)| + |N_v| &\leq D \end{aligned}$$

Summing these inequalities up and using $|N_v| \geq 1$ and $\mathrm{size}(A_0) \geq 2$, we conclude that

$$2(\mathrm{size}(A) + \mathrm{size}(B) + \mathrm{size}(C)) \leq 3D - 3.$$

Thus,

$$\begin{aligned} |V(G_0)| &= 4 + |\mathrm{Inner}(A)| + |\mathrm{Inner}(B)| + |\mathrm{Inner}(C)| \\ &\leq 4 + \lfloor (3D-3)/2 \rfloor = \lfloor (3D-1)/2 \rfloor + 3. \end{aligned}$$

As in the previous two proofs, Lemma 11 cannot be applied to G_0 directly. However, considering the same cases and analyzing them as in Lemma 12, we conclude that G is not (D, p)-bad. □

Lemma 15. *For every positive integer p, there exists a constant D_{15} such that there is no (D, p)-minimal graph, $D \geq D_{15}$, whose final block contains an $S(P(S(P, P), \ell*),\ P(S(P, P), \ell*))$-subgraph.*

Proof. We prove the lemma for $D_{15} = 24p + 28K + 4$ where K is the constant from Lemma 6. For the sake of contradiction, fix G to be a (D, p)-minimal graph for some $D \geq D_{15}$ whose final block $G^\star$ contains an $S(P(S(P, P), \ell*),$

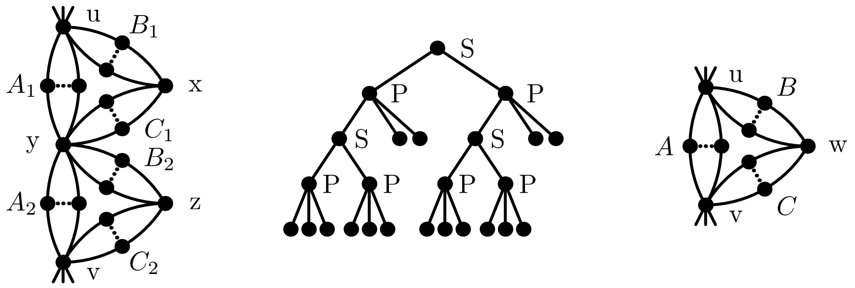

Figure 8: An $S(P(S(P,P),\ell*),P(S(P,P),\ell*))$-subgraph, the subtree corresponding to it and its reduction.

$P(S(P,P),\ell*))$-subgraph. In particular, $G^\star$ contains five vertices u, v, x, y, and z and six crystals A_1, B_1, C_1, A_2, B_2 and C_2 such that the poles of the crystal A_1 are u and y, the poles of the crystal B_1 are u and x, the poles of the crystal C_1 are x and y, the poles of the crystal A_2 are y and v, the poles of the crystal B_2 are y and z, and the poles of the crystal C_2 are v and z. The entire subgraph (the five vertices and the six crystals) is denoted by R and is depicted in Figure 8. Let N_u and N_v be the sets of the neighbors of u and v outside R.

By Lemma 6, the sizes of the crystals B_1, C_1, B_2, and C_2 are at least $\lfloor D/2 \rfloor - K$. Hence, the sizes of A_1 and A_2 sum to at most $2K+1$. An $L_D(p,1)$-labeling of G is obtained as follows: construct a new graph G' from G by contracting the subgraph induced by A_1, B_1, and C_1 into the vertex u. Since the degree of the vertex u could be greater than D after the contraction, it might be necessary to remove several (at most $2K+1$) vertices from the crystal corresponding to B_2. Let A, B and C be the “new” crystals and let w be the common pole of B and C. Find an $L_D(p,1)$-labeling c of G'. We will extend c to an $L_D(p,1)$-labeling of G in what follows.

First, use the labels assigned to the inner vertices of B to label as many inner vertices of B_1 as possible and use the labels assigned to the inner vertices of C to label as many inner vertices of C_2 as possible. The vertex y is assigned the label of an arbitrarily chosen inner vertex of A. Next, find suitable labels for x and z and unlabel some vertices in B_1 or C_2 if required. The number of forbidden labels for each of the vertices x and z is bounded by $D + 2(2p-1) + 2$. The inner vertices of A_1 and A_2 are labeled next. The number of forbidden labels for those vertices is at most $D + 2(2p-1) + 2K \le \lfloor 3D/2 \rfloor$.

It remains to label the inner vertices of B_2 and C_1, and the remaining inner vertices of B_1 and C_2. We start with the remaining vertices of B_1 and C_2. The number of forbidden labels is bounded by $D - 1 + 2(2p-1) + 1 \le \lfloor 3D/2 \rfloor$. Next, label as many inner vertices of C_1 as possible using labels of inner vertices of C and as many inner vertices of B_2 using labels of inner vertices of B. Notice that there are at most $2(2K + 1 + 2(2p-1))$ labels which are used on inner vertices of B_1 but not on inner vertices of B_2, and the same relation holds between C_2 and C_1. Finally, label the remaining vertices of C_1 and B_2. The number of forbidden labels for those vertices is at most

$$D - 1 + 2(2K + 1 + 2(2p-1)) + 2(2p-1) \le \lfloor 3D/2 \rfloor.$$

We infer from the above that c can be extended to the entire graph G, which contradicts its (D,p)-minimality. $\square$

3.5 Finale

Proof of Theorem 3. Fix p and set $D_0 = \max\{D_7, \ldots, D_{15}\}$ where $D_7, \ldots, D_{15}$ are the constants from Lemmas 7–15. For the sake of contradiction, let us assume that there exists a (D,p)-bad graph G, $D \ge D_0$. Since the empty graph is clearly not (D,p)-bad, there must exist a (D,p)-minimal graph G'. Further, let $G^\star$ be the final block of G' and $T^\star$ be its SP-decomposition tree such that if $G^\star$ contains a cut-vertex v of G', then v is one of the poles of the root node of $T^\star$.

By Lemmas 7–9, the final block $G^\star$ contains an $S(P,P)$-subgraph. Consider an $S(P,P)$-subgraph G_0 whose depth in $T^\star$ is the largest among all the $S(P,P)$-subgraphs. By Lemma 10, there is no $P(S(P,P),S(P,P))$-subgraph. So, G_0 must be contained in a $P(S(P,P),l*)$-subgraph G_1. Lemma 12 yields that G_1 cannot be the entire subgraph $G^\star$. In particular, the P-node corresponding to G_1 must have an S-node parent in $T^\star$ which corresponds to an $S(\ldots)$-subgraph G_2. However, Lemmas 13–15 imply that no such G_2 exist. We infer from the above arguments that no (D,p)-bad graph exists. $\square$

4 Direction for Future Research

Corollary 4 yields that the upper bound on the $L(p, 1)$-span of K_4-minor free graphs of the maximum degree Δ matches the corresponding upper bound on the chromatic number of the square if Δ is large enough. Analogous results are known for some other graph classes as well. For instance, the bounds on the $L(p, 1)$-span and the $L(1, 1)$-span of planar graphs of maximum degree Δ obtained by Molloy and Salavatipour [24, 25] differ only by an additive term which (linearly) depends on p. We suspect that this is not a mere coincicence and believe that the following more general statement actually holds.

Conjecture 1. *Let H be a graph and let $f_p^H(\Delta)$ be the maximum $L(p, 1)$-span of an H-minor free graph of the maximum degree Δ. For every positive integer p, there exist two constants Δ_0 and K such that $f_p^H(\Delta) \le f_1^H(\Delta) + K$ for every $\Delta \ge \Delta_0$.*

For the case of K_4-minor free graphs, i.e., $H = K_4$, we have established the above conjecture with $K = 0$ (Corollary 4). It could turn out that this is just an exposure of a more general fact, i.e., Conjecture 1 is true with $K = 0$ for all graphs H.

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