# Toughness threshold for the existence of 2-walks in $K_4$-minor free graphs

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**Abstract**

We show that every $K_4$-minor free graph with toughness greater than $4/7$ has a 2-walk, i.e., a closed walk visiting each vertex at most twice. We also give an example of a $4/7$-tough $K_4$-minor free graph with no 2-walk.

## 1 Introduction

An active area of graph theory is the study of Hamilton cycles [8, 9], in particular, the study of conditions based on different connectivity parameters that guarantees the existence of a Hamilton cycle in a graph. One of the most famous conjectures in this area is Chvátal's conjecture. Its original version asserts that every 2-tough graph $G$ is hamiltonian. Let us recall that a graph $G$ is *hamiltonian* if it contains a cycle passing through all its vertices, and $G$ is *$\alpha$-tough* if the number $\tau(A)$ of components of $G \setminus A$ is at most $\max\{1, |A|/\alpha\}$ for every non-empty set $A$ of the vertices. The original conjecture has been disproved by Bauer et al. [1] who constructed $(9/4-\varepsilon)$-tough graphs which are not hamiltonian but it remains open whether there exists a constant $\alpha_0$ such that every $\alpha_0$-tough graph is hamiltonian.

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Though Chvátal's conjecture remains open in general, it is known to be true for several special classes of graphs. We mention such results on chordal graphs as an example. Recall that a graph is chordal if it does not contain an induced cycle of length four or more. Every 18-tough chordal graph is hamiltonian [5]. It is conjectured that the bound of 18 can be reduced to two [2]. Better bounds are known for several subclasses of chordal graphs: 1-tough interval graphs [12], 3/2-tough split graphs [13] (also see [11]) and $(1+\varepsilon)$-tough planar chordal graphs are hamiltonian. The two former results are known to be the best possible. In the case of planar graphs, the existence of a Tutte cycle implies that every $(3/2+\varepsilon)$-tough planar graph is hamiltonian and Böhme et al. [4] constructed $(3/2-\varepsilon)$-tough planar graphs with no Hamilton cycle.

Another approach to Chvátal's conjecture is to show the existence of weaker substructures than Hamilton cycles. A *$k$-walk* of $G$ is a closed walk that visits each vertex of $G$ at least once and at most $k$ times. Jackson and Wormald [10] conjectured that every 1-tough graph has a 2-walk. The conjecture is still open. The best result in this direction is that every 4-tough graph has a 2-walk [7].

Motivated by these results, we find the toughness threshold for the existence of 2-walks in $K_4$-minor free graphs. Note that the case of Hamilton cycles is rather trivial for $K_4$-minor free graphs: it is easy to show by the induction based on the construction of series-parallel graphs (see Section 2 for details) that 1-tough $K_4$-minor free graphs are hamiltonian. The bound is optimal since any hamiltonian graph is at least 1-tough.

$K_4$-minor free graphs form an important subclass of planar graphs (recall that a graph is planar if and only if it does not contain $K_5$ or $K_{3,3}$ as a minor). An alternative characterization of $K_4$-minor free graphs involves the notion of tree-width, a notion well-studied both in structural graph theory as well as theoretical computer science [3]. A graph $G$ is $K_4$-minor free if and only if its tree-width is at most two. A tree-width of a graph can be described using the notion of tree decompositions that we do not introduce here, or using the notion of $k$-trees. The class of *$k$-trees* can be defined recursively as follows: a complete graph $K_{k+1}$ of order $k+1$ is a $k$-tree, and if $G$ is a $k$-tree, then a graph obtained from $G$ and $K_{k+1}$ by identifying $k$ vertices contained in a complete subgraph of $G$ and $K_{k+1}$ is also a $k$-tree. Hence, 2-trees are obtained from triangles by identifying pairs of edges. A graph $G$ is $K_4$-minor free if and only if it is a subgraph of a 2-tree. In fact, chordal 2-connected $K_4$-minor free graphs are precisely 2-trees. Finally, $K_4$-minor free graphs are also related to *series-parallel graphs* which we introduce in Section 2: every 2-connected $K_4$-minor free graph is series-parallel and every block of a $K_4$-minor free graph is series-parallel.

Our main result is that every $K_4$-minor free graph which is more than 4/7-tough has a 2-walk. On the other hand, we construct a 4/7-tough $K_4$-minor free graph with no 2-walk. The graph that we construct is a 2-tree, i.e., it is also chordal. Hence, our bound is also the best possible for $K_4$-minor free chordal graphs, the class of graphs that coincide with chordal planar graphs $G$ with $\omega(G) \leq 3$. Let us finally remark that it is not hard to generalize our construction to produce an infinite family of 4/7-tough chordal $K_4$-minor free graphs with no 2-walk.

## 2 Series-parallel graphs

In this paper, we deal with $K_4$-minor free graphs which are more than 4/7-tough. Since each 4/7-tough graph is 2-connected, all graphs that we consider are series-parallel graphs. The class of *series-parallel graphs* can be obtained by the following construction based on *blocks with poles.* The simplest series-parallel block is an edge and its two end-vertices are its poles. If $G$ and $H$ are two blocks with poles $v_1$ and $v_2$ and $w_1$ and $w_2$, the graph obtained by identifying the poles $v_2$ and $w_1$, such that $v_1$ and $w_2$ are its new poles, is the block obtained by a *serial join* of $G$ and $H$. The graph obtained from $G$ and $H$ by identifying the the poles $v_1$ and $w_1$ and the poles $v_2$ and $w_2$ is the block obtained by a *parallel join* of $G$ and $H$. All blocks obtained by a series of serial and parallel joins from single edges form the class of series-parallel graphs. In the rest of the paper, we also refer to blocks used in the construction of series-parallel graphs as to *series-parallel blocks* in order to avoid confusion with 2-edge-connected subgraphs that are also called blocks (though we do not use this term in the alternative meaning at all). Vertices of a series-parallel block distinct from the poles are called *inner* vertices.

An important notion used in our proofs is a notion of an $A$-bridge. If $A \subseteq V(G)$, then an *$A$-bridge* of $G$ is a maximal subgraph of $G$ such that any two vertices of it are joined by a path with all inner vertices distinct from those contained in $A$. The vertices of an $A$-bridge contained in the set $A$ are called *attachments* and its other vertices are *inner vertices.* A simplest $A$-bridge is an edge with both end-vertices contained in $A$; an $A$-bridge with internal vertices is said to be *non-trivial.* Hence, $\tau(A)$ is equal to the number of non-trivial $A$-bridges.

Let us now state a simple structural lemma on series-parallel blocks:

**Lemma 1.** *Let $G$ be a series-parallel block with poles $v_1$ and $v_2$. If $G$ is not a single edge, then there exists an inner vertex $v_0$ such that each $\{v_1, v_2, v_0\}$-bridge has exactly two attachments, and there are a $\{v_1, v_2, v_0\}$-bridge with the attachments $v_1$ and $v_0$ and a $\{v_1, v_2, v_0\}$-bridge with the attachments $v_2$ and $v_0$.*

*Proof.* We proceed by induction on the number of inner vertices of a series-parallel block. If $G$ is obtained by a serial join of two blocks, set $v_0$ to be the pole of the two blocks that was identified. If $G$ is obtained by a parallel join of two blocks, at least one of the two blocks is not a single edge (we deal with simple graphs only) and this block contains a vertex $v_0$ with the properties described in the statement of the lemma. Since the other block used in the parallel join is a $\{v_1, v_2, v_0\}$-bridge with attachments $v_1$ and $v_2$, all the $\{v_1, v_2, v_0\}$-bridges have two attachments. □

We finish this section with introducing a notion of proper series-parallel blocks. Let $G$ is a series-parallel graph, $H$ is one of the blocks obtained in the construction of $G$, and $v_1$ and $v_2$ are the poles $H$. We say that $H$ is a *proper* block if $H$ has only one $\{v_1, v_2\}$-bridge but $G$ has at least one non-trivial $\{v_1, v_2\}$-bridge different from $H$. Note that being a proper block is a property that depends not only on the block $H$ but also on $G$.

# 3 Notation used in the proof

In this section, we introduce notation that we use in the proof of our main result. We show that a proper block of a 4/7-tough series-parallel graph contains 2-walks of certain specific types unless it contains one of the obvious obstacles for their existence. The considered types of 2-walks are called green, red, blue, black and grey. Similarly, the obstacles are called green, red and blue.

We start with introducing the types of 2-walks. Let $G$ be a proper series-parallel block of a 4/7-tough series-parallel graph and let $v_1$ and $v_2$ be its poles. Examples of all the types of walks that we introduce can be found in Figure 1. A *green walk from $v_i$* is an open walk that starts and ends at $v_i$, visits each inner vertex of $G$ once or twice and does not visit any of the poles except at the beginning and the end of the walk. A *red walk* is an open walk that starts at $v_1$, ends at $v_2$, visits each inner vertex of $G$ once

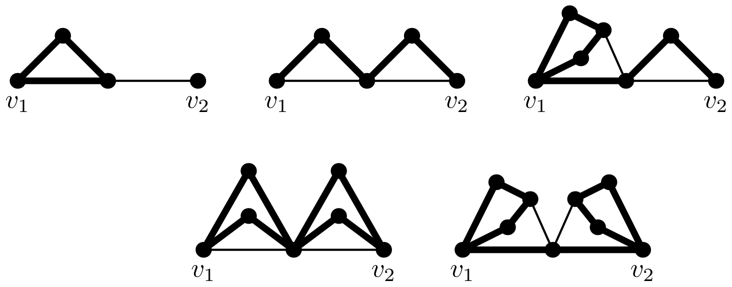


Figure 1: Examples of green, red, blue, black and grey walks (in this order). The green and blue walks are from the vertex $v_1$.

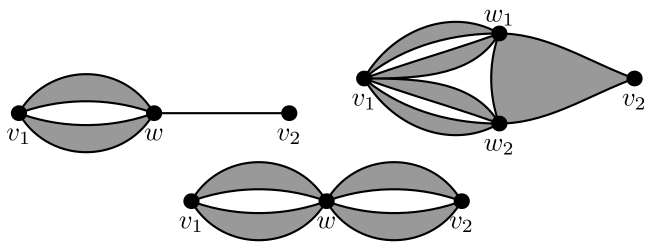


Figure 2: Green and blue obstacles at the vertex $v_1$ and a red obstacle (depicted in this order).

or twice and does not visit any of the poles except at the beginning and the end of the walk. A *blue walk from* $v_i$ is an open walk that starts at $v_i$, ends at the other pole of $G$, visits each inner vertex of $G$ once or twice, visits $v_i$ at most twice but it visits the other pole only at the end of the walk. A *black walk* is a closed walk that visits both $v_1$ and $v_2$ once and each inner vertex of $G$ once or twice. Note that a black walk can also be viewed as a collection of two open walks each starting at $v_1$ and ending at $v_2$ that visit together all the inner vertices of $G$ once or twice. Finally, a *grey walk* is an open walk that starts at $v_1$, ends at $v_2$ and visits each vertex of $G$ once or twice.

We now describe some obvious obstacles for the existence of the described types of walks. It turns out that these obstacles, under the assumption that $G$ is more than $4/7$-tough, are the only ones that can exclude the existence of a particular type of a walk. We say that $G$ contains a *green obstacle at* $v_i$ if there exists an inner vertex $w$ such that there are two non-trivial $\{v_1, v_2, w\}$-bridges with the attachments $v_i$ and $w$ (see Figure 2). Clearly, if $G$ contains a green obstacle at $v_i$, it cannot contain a green walk from the other pole: indeed, such a walk must enter each of the two bridges from $w$ since it must avoid the vertex $v_i$ and thus the vertex $w$ would be visited three times—for the first time before tracing the first of the bridges, for the second time after tracing the first and before tracing the second bridge, and for the third time after tracing the second bridge.

We say that $G$ contains a *blue obstacle at* $v_i$ (see Figure 2) if there exist inner vertices $w_1$ and $w_2$ such that there are two non-trivial $\{v_1, v_2, w_1, w_2\}$-bridges with the attachments $v_i$ and $w_1$, two non-trivial $\{v_1, v_2, w_1, w_2\}$-bridges with the attachments $v_i$ and $w_2$, and a non-trivial $\{v_1, v_2, w_1, w_2\}$-bridge with the attachments $v_{3-i}$, $w_1$ and $w_2$. If $G$ contains a blue obstacle at $v_i$, then $G$ cannot contain a blue walk from $v_{3-i}$: such a blue walk must enter or exit one of the two $\{v_1, v_2, w_1, w_2\}$-bridges with the attachments $v_i$ and $w_1$ through $v_i$ (otherwise, $w_1$ would be visited three times), and similarly one of the bridges with the attachments $v_i$ and $w_2$ must enter or exit through $v_i$ (otherwise, $w_2$ would be visited three times). Hence, the vertex $v_i$ would be visited twice and thus there is no blue walk from $v_{3-i}$. Note that we do not need the $\{v_1, v_2, w_1, w_2\}$-bridge with the attachments $v_{3-i}$, $w_1$ and $w_2$ to be non-trivial in order to prevent the existence of a blue walk, however, in our considerations, the bridge will always be non-trivial.

Finally, we say that $G$ contains a *red obstacle* if there exists an inner vertex $w$ such that there are two $\{v_1, v_2, w\}$-bridges with the attachments $v_1$ and $w$, and two $\{v_1, v_2, w\}$-bridges with the attachments $v_2$ and $w$ (see Figure 2). If $G$ contains a red obstacle, then it cannot contain a red walk—indeed, such a walk can enter only one of the two $\{v_1, v_2, w\}$-bridges with the attachments $v_1$ and $w$ from the vertex $v_1$, and thus it must enter and exit the other bridge through $w$. Similarly, one of the two $\{v_1, v_2, w\}$-bridges with the attachments $v_2$ and $w$ is entered and exited through $w$. Then, $w$ is visited three times—we conclude that there is no red walk. Similarly, the presence of a blue obstacle at any of the two poles prevents the existence of a red walk.

We now state five lemmas on the existence of each type of a walk. These lemmas will be proven in the next section.

**Lemma 2.** *Let $G$ be a proper series-parallel block of a $4/7$-tough series-*

*parallel graph and let $v_1$ and $v_2$ be its poles. If $G$ does not contain a green obstacle at the pole $v_2$, then $G$ contains a green walk from $v_1$. Analogously, if $G$ does not contain a green obstacle at the pole $v_1$, then $G$ contains a green walk from $v_2$.*

**Lemma 3.** *Let $G$ be a proper series-parallel block of a $4/7$-tough series-parallel graph and let $v_1$ and $v_2$ be its poles. If $G$ contains neither a blue obstacle at the pole $v_1$ or $v_2$, nor a red obstacle, then $G$ contains a red walk.*

**Lemma 4.** *Let $G$ be a proper series-parallel block of a $4/7$-tough series-parallel graph and let $v_1$ and $v_2$ be its poles. If $G$ does not contain a blue obstacle at the pole $v_1$, then $G$ contains a blue walk from $v_2$. Analogously, if $G$ does not contain a blue obstacle at the pole $v_2$, then $G$ contains a blue walk from $v_1$.*

**Lemma 5.** *Every proper series-parallel block $G$ of a $4/7$-tough series-parallel $G$ contains a black walk.*

**Lemma 6.** *Every proper series-parallel block $G$ of a $4/7$-tough series-parallel $G$ contains a grey walk.*

# 4 Main result

Before we proceed with proving Lemmas 2–6, let us derive the main result assuming we have already proven the lemmas.

**Theorem 7.** *If $G$ is a $K_4$-free minor graph that is more than $4/7$-tough, then $G$ has a 2-walk.*

*Proof.* Since $G$ is more than $1/2$-tough, it is 2-connected and thus series-parallel. If $G$ has less than four vertices, then it is either a single vertex, an edge or a triangle and the statement of the theorem readily follows. We assume in the rest that $G$ has at least four vertices. Since $G$ is 2-connected, it is obtained by a parallel join of series-parallel blocks $B_1, \ldots, B_k$ with poles $v_1$ and $v_2$. Without loss of generality, we can assume that each $B_i$ is either an edge or a series-parallel block obtained by a serial join. Since $G$ is $4/7$-tough, at most three of the blocks $B_1, \ldots, B_k$ are non-trivial.

If there are three non-trivial blocks $B_1$, $B_2$ and $B_3$, then neither of them contains a red or a blue obstacle at $v_1$ or $v_2$. If $B_1$ contained a red obstacle (with a vertex $w$ as in the definition), then $G$ would have six non-trivial $\{v_1, v_2, w\}$-bridges which is impossible because of the toughness assumption

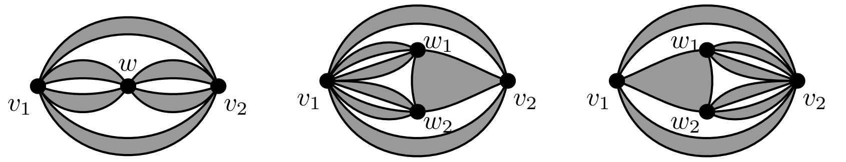


Figure 3: Configurations in the proof of Theorem 7 in case of three non-trivial series-parallel blocks.

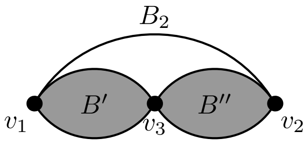


Figure 4: Notation used in the proof of Theorem 7 in the case of a single non-trivial series-parallel block.

(see Figure 3). If $B_2$ contained a blue obstacle at $v_1$ (with vertices $w_1$ and $w_2$ as in the definition), then $G$ would have seven non-trivial $\{v_1, v_2, w_1, w_2\}$-bridges which is also impossible because of the toughness assumption. The case that $B_2$ contained a blue obstacle at $v_2$ is symmetric. We conclude that each $B_i$, $i = 1, 2, 3$, contains a red walk by Lemma 3 (note that all the blocks $B_i$ are proper). The red walks of $B_1$ and $B_2$ and the black walk of $B_3$ (which exists by Lemma 5) combine to a 2-walk of $G$.

If there are exactly two non-trivial blocks $B_1$ and $B_2$, then each of them is proper and thus contains a black walk by Lemma 5. The two black walks combine to a 2-walk of $G$.

The last case is that there is a single non-trivial block $B_1$. Note we cannot apply Lemma 5 since $B_1$ is not a proper block. In this case, $k = 2$ and $B_2$ is a single edge. The block $B_1$ was obtained by a serial join of two blocks $B'$ and $B''$ (see Figure 4). Since $G$ has at least four vertices, one of the blocks $B'$ and $B''$ is non-trivial, say $B'$ is a non-trivial series-parallel block. Let $v_3$ be the common pole of $B'$ and $B''$. Observe now that the graph $G$ can also be obtained in the following way: perform the serial join of $B''$ and $B_2$ identifying the vertex $v_2$ and let $B_0$ be the obtained block with poles $v_1$ and $v_3$. $G$ is then obtained by the parallel join of $B_0$ and $B'$. Since both $B_0$ and $B'$ are non-trivial, we can now proceed as in the case of two or three non-trivial blocks which we have analyzed before and conclude that $G$ has a 2-walk. $\square$

We prove Lemmas 2–6 together by induction on the number of their vertices. In the proof, we use the induction assumption that all the five lemmas have been established for all proper blocks with fewer vertices.

*Proof of Lemmas 2–6.* If $G$ is a single edge or a two-edge path, the statements of all the lemmas clearly hold. In the rest, we assume that $G$ contains at least two inner vertices. Let $v_0$ be the vertex of $G$ as described in Lemma 1. Let $A_1, \ldots, A_k$ be the $\{v_1, v_2, v_0\}$-bridges with the attachments $v_1$ and $v_0$, and $B_1, \ldots, B_\ell$ the $\{v_1, v_2, v_0\}$-bridges with the attachments $v_2$ and $v_0$. Note that since $G$ is proper, any bridge with the attachments $v_1$ and $v_2$ must be a single edge and a walk does not have to trace it (it does not have any inner vertices). Hence, we can assume that there is no such bridge at all.

At most two of the bridges $A_1, \ldots, A_k$ are non-trivial: otherwise, the original graph contains four non-trivial $\{v_1, v_0\}$-bridges (at least three bridges $A_i$ and a bridge containing $v_2$) and the entire graph is at most 1/2-tough contradicting the assumption. If $k \geq 2$, then we can assume that all the bridges $A_1, \ldots, A_k$ are non-trivial, since the trivial bridges do not have to be traced by a walk and we can remove them from the list. Hence, it is enough to consider the following three cases:

- $k = 1$ and $A_1$ is a bridge formed by a single edge,
- $k = 1$ and $A_1$ is a non-trivial bridge, and
- $k = 2$ and both $A_1$ and $A_2$ are non-trivial bridges.

Similarly, only the following three cases need to be considered regarding the bridges $B_1, \ldots, B_\ell$:

- $\ell = 1$ and $B_1$ is a bridge formed by a single edge,
- $\ell = 1$ and $B_1$ is a non-trivial bridge, and
- $\ell = 2$ and both $B_1$ and $B_2$ are non-trivial bridges.

Also note that each non-trivial bridge $A_i$ is a non-trivial series-parallel block with the poles $v_1$ and $v_0$, and each non-trivial bridge $B_i$ is a non-trivial series-parallel block with the poles $v_2$ and $v_0$.

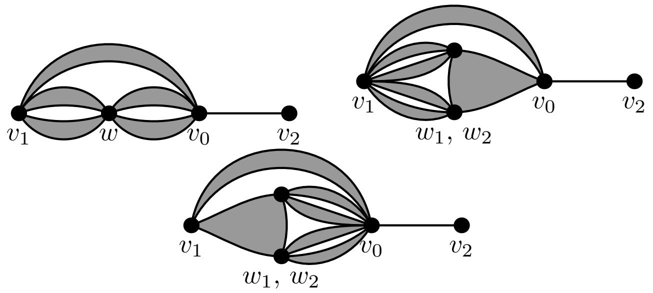


Figure 5: Possible configurations if $k = 2$ and $A_1$ does not contain a red walk.

We now prove several technical claims on the existence of certain walks in the blocks $A_i$ and $B_i$ that we later use to construct the desired walks in $G$.

**Claim 1.** *If $k = 2$, then each of the blocks $A_1$ and $A_2$ has a red walk. Analogously, if $\ell = 2$, then each of the blocks $B_1$ and $B_2$ has a red walk.*

By symmetry, we can only focus on the case that $k = 2$ and show that $A_1$ has a red walk. By the induction assumption, it is enough to show that $A_1$ does not contain a red obstacle or a blue obstacle at $v_1$ or $v_0$: if $A_1$ contained a red obstacle, then the entire graph would contain six non-trivial $\{v_1, v_0, w\}$-bridges, contradicting the assumption that the graph is more than 4/7-tough (see Figure 5). If $A_1$ contained a blue obstacle, then then the entire graph would contain seven non-trivial $\{v_1, v_0, w_1, w_2\}$-bridges, also contradicting the assumption that the graph is more than 4/7-tough.

**Claim 2.** *If $k = 2$ and $G$ does not contain a blue obstacle at $v_1$, then $A_1$ or $A_2$ contains a green walk from $v_0$. Analogously, if $\ell = 2$ and $G$ does not contain a blue obstacle at $v_2$, then $B_1$ or $B_2$ contains a green walk from $v_0$.*

If both $A_1$ and $A_2$ contained green obstacles at $v_1$ with vertices $w_1$ and $w_2$, then $G$ would contain a blue obstacle at $v_1$ with $w_1$ and $w_2$. Note that the $\{v_1, v_2, w_1, w_2\}$-bridge with the attachments $v_2$, $w_1$ and $w_2$ is non-trivial since it contains $v_0$ (see Figure 6).

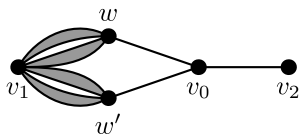


Figure 6: The configuration if $k = 2$ and both $A_1$ and $A_2$ contain green obstacles at $v_1$.

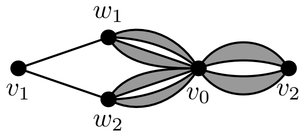


Figure 7: The configuration if $k = \ell = 2$ and both $A_1$ and $A_2$ contain green obstacles at $v_0$.

**Claim 3.** *If $k = \ell = 2$, then $A_1$ or $A_2$ contains a green walk from $v_1$. Analogously, if $k = \ell = 2$, then $B_1$ or $B_2$ contains a green walk from $v_2$.*

If both $A_1$ and $A_2$ contained a green obstacle at $v_0$, say with vertices $w_1$ and $w_2$, then $G$ would contain seven non-trivial $\{v_0, v_2, w_1, w_2\}$-bridges contradicting our assumption that $G$ is more than 4/7-tough (see Figure 7). The claim now follows from the induction assumption. Analogously, $B_1$ or $B_2$ contains a green walk from $v_0$.

**Claim 4.** *$A_1$ or $B_1$ contains neither a blue obstacle at $v_0$ nor a red obstacle.*

If both $A_1$ and $B_1$ contained blue obstacles at $v_0$ with vertices $w_1$ and $w_2$, and $w_1'$ and $w_2'$, respectively, then there would be nine non-trivial $\{v_0, w_1, w_2, w_1', w_2'\}$-bridges (see Figure 8). Hence, the graph would be at most 5/9-tough contradicting the assumption that it is more than 4/7-tough.

If $A_1$ contained a blue obstacle at $v_0$ (with vertices $w_1$ and $w_2$) and $B_1$ a red obstacle (with a vertex $w$), then the whole graph would have nine non-trivial $\{v_0, v_2, w, w_1, w_2\}$-bridges. Again, this is excluded by the assumption. The case that $A_1$ contained a red obstacle and $B_1$ a blue one is symmetric.

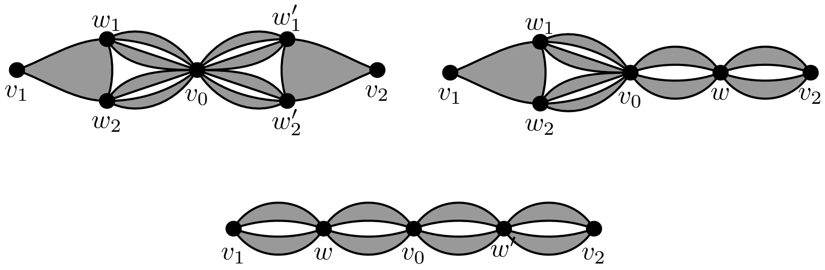


Figure 8: Possible configurations if both $A_1$ and $B_1$ contain a blue obstacle at $v_0$ or a red obstacle.

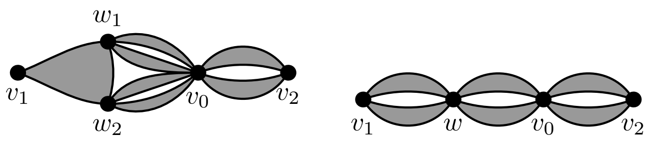


Figure 9: Possible configurations if $A_1$ contains a blue obstacle at $v_0$ or a red obstacle, and $\ell = 2$.

If $A_1$ contained a red obstacle (with a vertex $w$) and $B_1$ also contained a red obstacle (with a vertex $w'$), then the whole graph would have nine non-trivial $\{v_0, v_1, v_2, w, w'\}$-bridges which is impossible by the assumption. The statement of the claim now readily follows.

**Claim 5.** *If $\ell = 2$, then $A_1$ contains neither a blue obstacle at $v_0$ nor a red obstacle. Analogously, if $k = 2$, then $B_1$ contains neither a blue obstacle at $v_0$ nor a red obstacle.*

If $A_1$ contained a blue obstacle at $v_0$, then there would be seven non-trivial $\{w_1, w_2, v_0, v_2\}$-bridges contradicting the assumption that the graph is more than 4/7-tough (also see Figure 9). If $A_1$ contained a red obstacle, then there would be seven non-trivial $\{v_1, w, v_0, v_2\}$-bridges. The case of $k = 2$ is symmetric.

We are now ready to construct the desired types of walks in $G$.

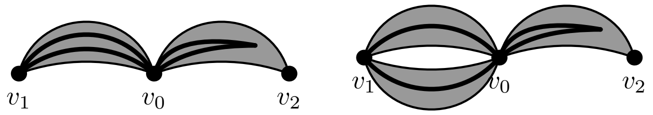


Figure 10: Green walks (drawn in bold) constructed in the proof of Lemma 2.

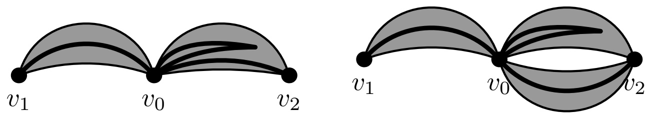


Figure 11: Red walks (drawn in bold) constructed in the proof of Lemma 3.

**Proof of Lemma 2.** *If $G$ does not contain a green obstacle at $v_2$, then it has a green walk from $v_1$.*

Note that $\ell = 1$, otherwise, $G$ would contain a green obstacle with $w = v_0$. In addition, $B_1$ does not contain a green obstacle at $v_2$ since such an obstacle would also be a green obstacle of $G$. Hence, $B_1$ has a green walk from $v_0$ by the induction. If $k = 1$, the green walk of $B_1$ can be combined with a black walk of $A_1$ to a green walk of $G$. If $k = 2$, the green walk of $B_1$ can be combined with two red walks of $A_1$ and $A_2$ (which exist by Claim 1) to a green walk of $G$. We conclude that $G$ has a green walk from $v_1$ unless it has a green obstacle at $v_2$. The reader can check Figure 10 for the illustration of the proof of this claim.

**Proof of Lemma 3.** *If $G$ contains neither a blue obstacle at $v_1$ or $v_2$ nor a red obstacle, then $G$ has a red walk.*

Since $G$ does not have a red obstacle, $k = 1$ or $\ell = 1$. By symmetry, we can assume that $k = 1$. Let us first consider the case $\ell = 1$. By Claim 4, the assumptions of the claim and the induction, $A_1$ or $B_1$ contains a red walk. By symmetry, let us say that $A_1$ has a red walk. Since $G$ does not contain a blue obstacle at $v_2$, $B_1$ does not contain it either and thus $B_1$ has a blue walk from $v_0$. The red walk of $A_1$ and the blue walk of $B_1$ combine to a red walk of $G$ (see Figure 11).

If $\ell = 2$, then $B_1$ or $B_2$ contains a green walk from $v_0$ by Claim 2. By symmetry, we assume that $B_1$ has a green walk from $v_0$. By Claim 5 and

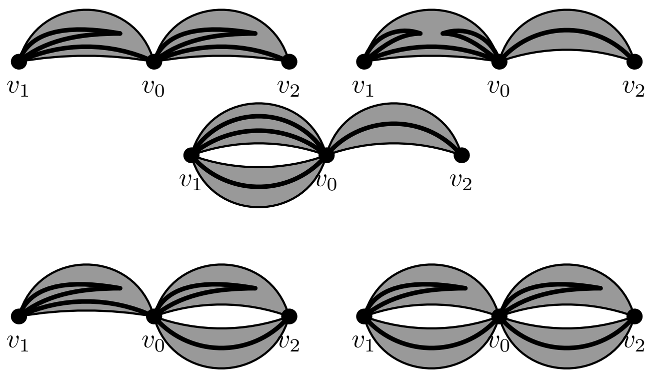


Figure 12: Blue walks (drawn in bold) constructed in the proof of Lemma 4.

the induction, $A_1$ has a red walk. The red walk of $A_1$, the green walk of $B_1$ and a red walk of $B_2$ (which exists by Claim 1) can be combined to a red walk of $G$ (also see Figure 11).

**Proof of Lemma 4.** *If $G$ does not contain a blue obstacle at $v_2$, then it has a blue walk from $v_1$.*

Assume first that $k = \ell = 1$. By Claim 4, $A_1$ or $B_1$ contains neither a red obstacle nor a blue obstacle at $v_0$. If $A_1$ has this property, then $A_1$ contains a blue walk from $v_1$ by the induction and $B_1$ has a blue walk from $v_0$ (otherwise, a blue obstacle at $v_2$ of $B_1$ would also be a blue obstacle at $v_2$ of $G$). The two blue walks combine to a blue walk of $G$ from $v_1$. If $B_1$ contains neither a red obstacle nor a blue obstacle at $v_0$, $B_1$ has a red walk by the induction. In addition, $A_1$ has a grey walk by the induction. The grey and the red walks combine to a blue walk from $v_1$ (see Figure 12).

If $k = 2$ and $\ell = 1$, then $B_1$ has a red walk by Claim 5 and the induction. By Claim 1, both $A_1$ and $A_2$ have red walks, and by the induction, they have black walks, too. The black walk of $A_1$ and the red walks of $A_2$ and $B_1$ combine to a blue walk of $G$ from $v_1$.

If $\ell = 2$, then $B_1$ or $B_2$ contain a green walk from $v_0$ by Claim 2. Assume that $B_1$ does (the other case is symmetric). By Claim 1, $B_2$ contains a red walk. If $k = 1$, a blue walk of $A_1$ from $v_1$ which exists by the induction and Claim 5, combines with the green walk of $B_1$ and the red walk of $B_2$ to a blue walk of $G$ from $v_1$. If $k = 2$, $A_1$ or $A_2$ has a green walk from $v_1$

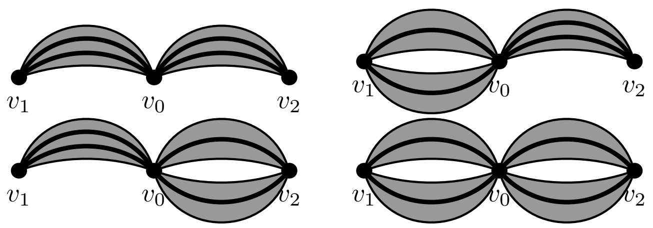


Figure 13: Black walks (drawn in bold) constructed in the proof of Lemma 5.

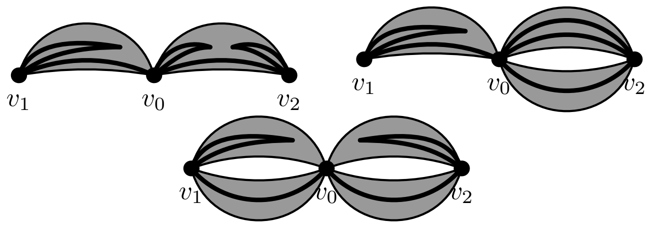


Figure 14: Grey walks (drawn in bold) constructed in the proof of Lemma 6.

by Claim 3. By the symmetry, we can assume that $A_1$ has a green walk from $v_1$. Since $A_2$ has a red walk by Claim 1, the green walks of $A_1$ and $B_1$ and the red walks of $A_2$ and $B_2$ combine to a blue walk of $G$ from $v_1$ (see Figure 12).

**Proof of Lemma 5.** *$G$ has a black walk.*

The black walk of $G$ is comprised of a black walk of $A_1$ if $k = 1$ or two red walks of $A_1$ and $A_2$ if $k = 2$ (such red walks exist by Claim 1), and of a black walk of $B_1$ if $\ell = 1$ or two red walks of $B_1$ and $B_2$ if $\ell = 2$ (see Figure 13).

**Proof of Lemma 6.** *$G$ has a grey walk.*

Assume first that $k = 1$ and $\ell = 1$. By Claim 4, $A_1$ or $B_1$ does not contain a blue obstacle at $v_0$, say $A_1$ does not. By the induction, $A_1$ has a blue walk from $v_1$ and $B_1$ has a grey walk. The two walks combine to a grey walk of $G$ (see Figure 14).

Next, we consider the case $k = 1$ and $\ell = 2$. By Claim 5, $A_1$ does not contain a blue obstacle at $v_0$. Hence, it has a blue walk from $v_1$ by

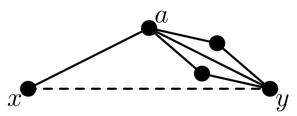

Figure 15: A series-parallel block with poles $x$ and $y$ such that any 2-walk must contain an inner edge incident with $y$.

the induction. Both $B_1$ and $B_2$ have black walks by the induction and red walks by Claim 1. The blue walk of $A_1$ from $v_1$, a black walk of $B_1$ and a red walk of $B_2$ combine to a grey walk of $G$. The case $k = 2$ and $\ell = 1$ is symmetric.

The final case that we need to consider is that $k = \ell = 2$. By Claim 1, $A_1$, $A_2$, $B_1$ and $B_2$ have red walks. By Claim 3, $A_1$ or $A_2$ has a green walk from $v_1$, say $A_1$ does. Similarly, we can suppose that $B_1$ has a green walk from $v_2$. The green walks of $A_1$ and $B_1$ and the red walks of $A_2$ and $B_2$ combine to a grey walk of $G$. □

# 5 A $4/7$-tough graph with no $2$-walk

In this section, we construct a 4/7-tough 2-tree with no 2-walk. We start with introducing two series-parallel blocks which are depicted in Figures 15 and 16. The two blocks have a common property that any 2-walk tracing them must contain an inner edge incident with $y$ as stated in the next two lemmas (an edge of a block is *inner* if it does not join its poles).

**Lemma 8.** *Let $G$ be the series-parallel block with poles $x$ and $y$ depicted in Figure 15. Any $2$-walk contains at least one inner edge of $G$ incident with $y$.*

*Proof.* If no inner edge of $G$ incident with $y$ is contained in a 2-walk, then the 2-walk must come to $a$ from $x$, then visit both the common neighbors of $a$ and $y$ and return to $x$. However, $a$ would be visited three times in this way. The statement of the lemma now follows. □

**Lemma 9.** *Let $G$ be the series-parallel block with poles $x$ and $y$ depicted in Figure 16. Any $2$-walk contains at least one inner edge of $G$ incident with $y$.*

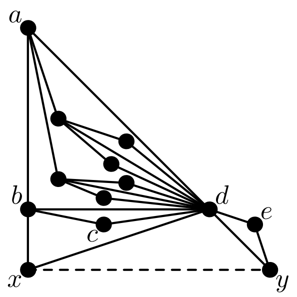


Figure 16: A series-parallel block with poles $x$ and $y$ such that any 2-walk must contain an inner edge incident with $y$.

*Proof.* Let us consider a 2-walk that contains neither the edge $dy$ nor the edge $ey$. Hence, the 2-walk comes to and leaves the vertex $e$ through the edge $de$. By Lemma 8, $d$ is incident with at least one edge contained in each of the two copies of the block depicted in Figure 15 pasted along the edge $ad$. Since the 2-walk visits $d$ at most twice, the 2-walk cannot use the edge $bd$ or the edge $cd$. Hence, the 2-walk enters the block through the vertex $x$, it comes from $x$ to $b$, visits $c$, continues to $a$ (in order to reach $d$), and eventually returns from $a$ to $b$ and leaves the block through $x$. However, in this way, the 2-walk visits $b$ three times which is impossible. We conclude that every 2-walk contains at least one inner edge of $G$ incident with $y$. ◻

The graph that we present as an example of a 4/7-tough series-parallel graph with no 2-walk is depicted in Figure 17. It is easy to verify that the graph is not only series-parallel, but it is in fact a 2-tree. Also note that the graph contains three copies of the block from Figure 15 and two copies of the block from Figure 16. Let us first argue that it has no 2-walk.

**Lemma 10.** *The* 2*-tree depicted in Figure 17 has no* 2*-walk.*

*Proof.* By Lemmas 8 and 9, every 2-walk of the graph contains five edges incident with the vertex $a$. However, such a 2-walk must visit $a$ at least three times which is impossible. ◻

Next, we argue that the graph depicted in Figure 17 is 4/7-tough.

**Theorem 11.** *The* 2*-tree depicted in Figure 17 is an example of a* 4/7*-tough* 2*-tree with no* 2*-walk.*

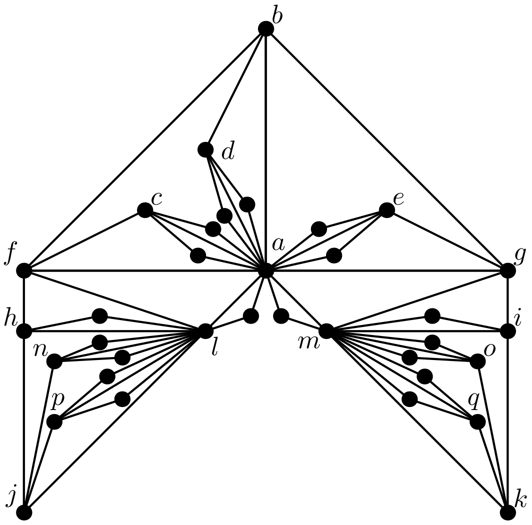


Figure 17: A 4/7-tough 2-tree with no 2-walk.

*Proof.* By Lemma 10, it is enough to show that the graph is 4/7-tough. The number of non-trivial $A$-bridges for $A = \{h, l, n, p\}$ is seven and thus the graph is at most 4/7-tough. In the rest, we show that the graph is 4/7-tough.

Assume that $G$ is less than 4/7-tough, and let $A$ be a non-empty inclusion-wise minimal set of vertices such that $|A|/\tau(A) < 4/7$. For any proper subset $B$ of $A$, we can infer the following from the choice of $A$:

$$\begin{aligned}
\frac{|A|}{\tau(A)} &< \frac{|A| - |B|}{\tau(A \setminus B)} \\
\tau(A)|B| &< |A| \left(\tau(A) - \tau(A \setminus B)\right) \\
\frac{\tau(A)}{|A|} |B| &< \tau(A) - \tau(A \setminus B) \\
\frac{7|B|}{4} &< \tau(A) - \tau(A \setminus B)
\end{aligned}$$

Hence, if $|B| = 1$, $\tau(A) \geq \tau(A \setminus B) + 2$. In particular, each vertex of $A$ is an attachment of at least three non-trivial $A$-bridges and thus $A$ contains no vertices of degree two. Similarly, every pair of vertices of $A$ is incident with five non-trivial $A$-bridges (unless $|A| = 2$), every triple with seven such bridges (unless $|A| = 3$) and every quadruple with nine such bridges (unless $|A| = 4$). In our further considerations, we will argue that $A$ does not contain certain subsets $B$ based on the number of $A$-bridges incident with the vertices $B$ and implicitly assume that $B$ is a proper subset of $A$; the cases that $B = A$ will not be explicitly analyzed and the reader is asked to check that our arguments extend to such cases, too.

Let $B = A \cap \{j, h, l, n, p\}$. If $j \in B$, then neither $h$, $n$ nor $p$ can be contained in $B$ (there would not be three non-trivial $A$-bridges incident with them). On the other hand, $l$ is contained in $B$, since otherwise $j$ would not be incident with at least three non-trivial $A$-bridges. However, the pair $j$ and $l$ is now incident with at most four non-trivial $A$-bridges: those containing $h$, $n$, $p$ and the common neighbor of $a$ and $l$. Since this is impossible, we infer that $j \notin B$.

Assume that $l \in B$. If $B = \{h, l, n, p\}$, then the quadruple $h$, $l$, $n$ and $p$ is incident with at most eight non-trivial $A$-bridges which is impossible. If $B = \{h, l, n\}$, then the triple $h$, $l$ and $n$ is incident with at most six non-trivial $A$-bridges which is also impossible. Similarly, $B \neq \{h, l, p\}$. If $B = \{l, n, p\}$, the triple $l$, $n$ and $p$ is incident with at most six non-trivial $A$-bridges which is impossible. If $B = \{l, n\}$, the pair $l$ and $n$ is incident with at most four non-trivial $A$-bridges which is also impossible. Similarly, $B \neq \{l, p\}$. If $B = \{h, l\}$, then the pair $h$ and $l$ is incident with at most four non-trivial $A$-bridges which is impossible, too. Hence, $B = \{l\}$ and $l$ is incident with at most two non-trivial $A$-bridges which is impossible as well. We eventually conclude that $l \notin B$. Hence, neither $n$ nor $p$ are contained in $A$ (they cannot be incident with three non-trivial $A$-bridges if $l \notin B$). The only two cases that remain are $B = \{h\}$ and $B = \emptyset$. Since the former case is excluded ($h$ would be incident with a single non-trivial $A$-bridge), we infer that $B = \emptyset$. Analogously, it holds that $A \cap \{i, k, m, o, q\} = \emptyset$ and thus that $A \subseteq \{a, b, c, d, e, f, g\}$.

Assume first that $b \in A$. Since $b$ must be incident with at least three non-trivial $A$-bridges, $a$ must also be contained in $A$ (otherwise, $f$ and $g$ cannot be in different $A$-bridges), and $f \notin A$, $g \notin A$, and $d \notin A$. Let $\alpha = |A \cap \{c, e\}|$. There are $3 + 2\alpha$ non-trivial $A$-bridges, and $|A|/\tau(A) = (2 + \alpha)/(3 + 2\alpha) \geq 4/7$. We conclude that $b \notin A$.

Assume now that $f \in A$. Since $f$ must be incident with at least three non-trivial $A$-bridges, $a \in A$ and $c \notin A$. If $g$ is also contained in $A$, let $\alpha = |A \cap \{d\}|$ (note that $e \notin A$ in this case). It is easy to derive that

$|A|/\tau(A) = (3+\alpha)/(5+2\alpha) \geq 4/7$. If $g \notin A$, let $\alpha = |A \cap \{d, e\}|$. We derive that $|A|/\tau(A) = (2+\alpha)/(3+2\alpha) \geq 4/7$. We eventually conclude that $f \notin A$. By symmetry, $g \notin A$. We can now conclude that $A \subseteq \{a, c, d, e\}$.

If $a \notin A$, then none of the vertices $c$, $d$ or $e$ can be incident with three non-trivial $A$-bridges. Hence, $a \in A$. Let $\alpha = |A \cap \{c, d, e\}|$. Since there are $1+2\alpha$ non-trivial $A$-bridges, we have that $|A|/\tau(A) = (1+\alpha)/(1+2\alpha) \geq 4/7$. We can now conclude that there is no set $A$ with $|A|/\tau(A) < 4/7$ and the graph is $4/7$-tough. □

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