Block transitivity and degree matrices ⋆

Jiří Fiala [a], José Soto [b,1],

[a] *KAM, DIMATIA and Institute for Theoretical Computer Science (ITI)
Charles University, Faculty of Mathematics and Physics,
Malostranské nám. 2/25, 118 00 Prague, Czech Republic
Email: fiala@kam.mff.cuni.cz* [2]

[b] *Department of Mathematics, MIT, Cambridge, MA 02139. USA
and Depto. Ing. Matemática, Universidad de Chile
Correo 3, Santiago 170-3, Chile
Email: jsoto@math.mit.edu*

Abstract

We say that a square matrix $\mathbf{M}$ of order r is a degree matrix of a given graph G if there is a so called equitable partition of its vertices into r blocks. This partition satisfies that for any i and j it holds that a vertex from the i-th block of the partition has exactly $m_{i,j}$ neighbors inside the j-th block.

We ask whether for a given degree matrix $\mathbf{M}$, there exists a graph G such that $\mathbf{M}$ is a degree matrix of G, and in addition, for any two edges e, f spanning between the same pair of blocks there exists an automorphism of G that sends e to f. In this work, we affirmatively answer the question for all degree matrices and show a way to construct a graph that witness this fact.

We further explore a case where the automorphism is required to exchange given pair of edges and show some positive and negative results.

Key words: degree matrix, transitive graphs

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1 Introduction

The class of vertex transitive graphs (as well as edge or arc transitive graphs) is an interesting and well studied graph class for its algebraic and topological properties, see e.g. a classical monograph of Biggs [2] or a modern textbook of Godsil and Royle [5].

The defining property of a vertex transitive graph is that for any pair of vertices u and v there exists an automorphism φ that maps u onto v, can be informally rephrased as saying that vertex transitive graphs have very rich structure of automorphism that allows to send an arbitrary vertex anywhere in the given graph. One trivial consequence of the above fact is that any vertex transitive graph must be regular. Our aim is to study graphs with analogously rich structure of automorphism but which need not to be regular.

If we take any automorphism φ of an arbitrary graph, it holds that it must preserve vertex degree, degrees of neighbours, degrees of neighbours of neighbours and so on. This property can be formalized in terms of equitable partition (to our knowledge first defined by Corneil and Gotlieb in [3]) since equitable partitions are exactly those patrtitions where two vertices from the same block cannot be distinguished by counting their neighbors inside any other block. It is worth to note that the partition with the fewest number of blocks can be computed in time $O(m \log(n))$ [1]. Also observe that every automorphism of G must preserve this coarsest equitable partition.

With an equitable partition can be associated so called *degree matrix* in which each row represents a block of the partition, and the entries describes the numbers of their neighbors in different blocks. Kratochvíl, Proskurowski and Telle showed that the test whether a given graph G admits a equitable partition with prescribed matrix is NP-complete [7]. This result was obtained in the setting of locally constrained graph homomorphisms. See also [4] for a comprehensive study of this relationship.

We focus our attention to the question whether for a given degree matrix $\mathbf{M}$ there exist a graph G with an equitable partition corresponding to the matrix $\mathbf{M}$, such that it has as rich structure of automorphism as possible. It is required that for any pair of arcs that are not excluded due to trivial degree reasons there exist an automorphism of G that maps one on the other. (Consequently we can obtain an analogous property for vertices instead of arcs.) We call such graphs block transitive. Ordinary vertex, edge or arc, resp., transitive graphs can be obtained as block transitive graphs with at most two blocks.

In other words, the question in which we are interested can be restated as: Given a degree matrix M, does there exist a block transitive graph G that such M is a degree matrix for G?

We show an algebraic construction providing an affirmative answer for the above problem for all degree matrices. We further explore an even stronger notion of transitive graphs and derive some positive and some negative results.

2 Preliminaries

For a positive integer k the symbol $[k]$ stands for the set $\{1, 2, \ldots, k\}$. Elements of a matrix $\mathbf{M}$ will be denoted by lowercase indexed letters, i.e. $m_{i,j} = (\mathbf{M})_{i,j}$. For a set V we denote by $\binom{V}{2}$ the set of unordered pairs of V.

We consider simple, undirected and (if not stated otherwise) finite graphs. In other words a graph G is a pair (V, E), where $V = V_G$ is a set of vertices and $E = E_G \subseteq \binom{V}{2}$ is a set of edges.

For a graph $G = (V, E)$ and a set $W \subseteq V$ we define the subgraph of G induced by the set W as the graph $(W, E \cap \binom{W}{2})$.

The complete graph K_n is the graph on n nodes in which all vertices are adjacent, i.e. connected by an edge. Formally, $K_n = ([n], \binom{[n]}{2})$. The complete bipartite graph $K_{m,n}$ is the graph with a disjoint union of two sets of sizes m and n as its vertex set, in which only the edges between these two sets are present. Formally, $K_{m,n} = ([m+n], \{uv \mid u \leq m < v\})$.

The complement of a graph $G = (V, E)$ is $\overline{G} = (V, \binom{V}{2} \setminus E)$. For a bipartite graph $G = (V, E)$ with given bipartition $G \subseteq K_{m,n}$ we define its bipartite complement as $\overline{G}' = (V, E_{K_{m,n}} \setminus E)$.

For a vertex $v \in V$ of a graph $G = (V, E)$ we denote by $N_G(u)$ the set of neighbors of u, i.e. $N_G(u) = \{v \mid uv \in E\}$.

For two sets V and W the symbol $V \times W$ denotes the Cartesian product of these sets. When $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are graphs we mean by $G \times H$ the categorical product of graphs G and H which is defined as follows:

$$V_{G\times H} = V_G \times V_H \qquad E_{G\times H} = \{(u, \bar{u})(v, \bar{v}) \mid uv \in E_G, \bar{u}\bar{v} \in E_H\}$$

Definition 1 *We call a square matrix* $\mathbf{M}$ *of order* r *a degree matrix of a graph* G *if there is a partition of* V_G *into disjoint blocks* $\mathcal{G} = (V_1, \ldots, V_r)$ *such that, for every* i *and every* $u \in V_i$*, we have:*

$$\forall j : |\mathcal{N}_G(u) \cap V_j| = m_{i,j}.$$

Such a partition $\mathcal{G}$ *is called* equitable partition *of* G *(with degree matrix* $\mathbf{M}$*).*

Degree matrices are fully characterized in the following way [4]:

Lemma 2 *A non-negative integer square matrix* $\mathbf{M}$ *of order* r *is a degree matrix if and only if the following conditions are satisfied simultaneously:*

(1) (Plus-symmetry) For every $1 \leq i, j \leq r$, $m_{i,j} = 0 \implies m_{j,i} = 0$.

(2) (Cycle product identity)
For every sequence of indices $i_1, i_2, \ldots, i_k, i_{k+1}$, $k \geq 3$, *such that* $i_{k+1} = i_1$,

$$\prod_{j=1}^{k} m_{i_j,i_{j+1}} = \prod_{j=1}^{k} m_{i_{j+1},i_j}.$$

We denote the set of all degree matrices by $\mathcal{M}$.

We note here that for any $\mathbf{M} \in \mathcal{M}$ the block sizes of the smallest graph G with $\mathbf{M}$ as its degree matrix can be computed as the smallest nontrivial integral solution to a system of linear equations, which can be achieved in time being polynomial in the computational size of the matrix $\mathbf{M}$ [4].

To simplify the following statements we represent the partition $\mathcal{G}$ also by the equivalence relation $\sim_{\mathcal{G}}$ on V_G, where $u \sim_{\mathcal{G}} v$ holds if and only if u and v belong to the same block of the equitable partition $\mathcal{G}$.

Definition 3 *We say that a graph G with degree matrix $\mathbf{M}$ of order r and equitable partition $\mathcal{G}$ is* block transitive, *if for each pair of edges $e = uv$ and $f = \bar{u}\bar{v}$ where $u \sim_{\mathcal{G}} \bar{u}$ and $v \sim_{G} \bar{v}$, there exists an automorphism φ of G that preserves the partition, i.e., $u \sim_{\mathcal{G}} \varphi(u)$, and that* sends e to f, *i.e., $\varphi(u) = \bar{u}$ and $\varphi(v) = \bar{v}$.*

The main result in this work is that the answer for our original problem is always positive. i.e. for every matrix $\mathbf{M} \in \mathcal{M}$ we can construct a block transitive graph that has $\mathbf{M}$ as its degree matrix.

3 Block product

In this section we introduce an useful binary operation between block transitive graphs.

Definition 4 *Let G and H be graphs with equitalbe partitions $\mathcal{G} = (V_1, \ldots, V_r)$ and $\mathcal{H} = (W_1, \ldots, W_r)$ respectively. We construct the* block product graph $G \otimes H$ *according to the partitions $\mathcal{G}$ and $\mathcal{H}$ as follows:*

(1) $V_{G\otimes H} = (V_1 \times W_1) \cup (V_2 \times W_2) \cup \ldots \cup (V_r \times W_r)$.
(2) $E_{G\otimes H} = \{(u,x)(v,y) \,|\, uv \in E_G, xy \in E_H\}$.

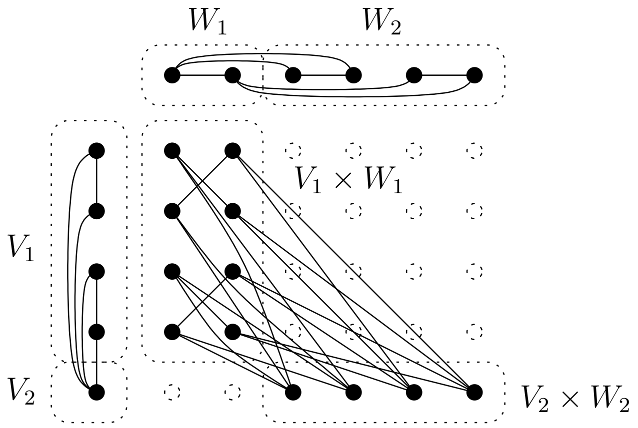
In other words, $G \otimes H$ is the subgraph of the Cartesian product $G \times H$ induced by the vertex set $\bigcup_{i=1}^{r} V_i \times W_i$.

An example of the construction of a block product graph is depicted in Fig. 1

We denote by $\mathcal{G} \otimes \mathcal{H}$ the natural partition of $V_{G\otimes H}$ induced by this product, i.e. $\mathcal{G} \otimes \mathcal{H} = (V_1 \times W_1, V_2 \times W_2, \ldots, V_r \times W_r)$. Note that for any node (u, x) the number of its neighbors in $V_i \times W_i$ is exactly the product of the number of neighbors of u in V_i with the number of neighbors of x in W_i. Consequently, $\mathcal{G} \otimes \mathcal{H}$ is a equitable partition and the following observation holds:

Claim 5 *Let G and H be graphs with degree matrices $\mathbf{M}$ and $\mathbf{N}$, respectively, of order r associated to equitable partitions $\mathcal{G}$ and $\mathcal{H}$, resp. Then $\mathcal{G} \otimes \mathcal{H}$ is an equitable partition of $G \otimes H$, and the degree matrix associated to this partition is the* coordinate product $\mathbf{M} \otimes \mathbf{N}$, *defined as:*

$$\forall i,j : \;\; (\mathbf{M} \otimes \mathbf{N})_{i,j} = m_{i,j} n_{i,j}$$



$$\begin{pmatrix}1 & 1\\4 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 2\\1 & 1\end{pmatrix} = \begin{pmatrix}1 & 2\\4 & 0\end{pmatrix}$$

Fig. 1. Example of a block product graph and the corresponding degree matrices

We show that this product behaves well with respect to block transitivity.

Theorem 6 *Let G and H be two block transitive graphs with equitable partitions of size r, and degree matrices $\mathbf{M}$ and $\mathbf{N}$, then $G \otimes H$ is also a block transitive graph with degree matrix $\mathbf{M} \otimes \mathbf{N}$.*

PROOF. Let $\mathcal{G}$ and $\mathcal{H}$ be the equitable partitions associated to the graphs G and H, respectively, with degree matrices $\mathbf{M}$ and $\mathbf{N}$, resp.

Let $e = (u, x)(v, y)$ and $f = (\bar{u}, \bar{x})(\bar{v}, \bar{y})$ be two edges of $G \otimes H$ between the same pair of blocks of the partition, i.e., $(u, x) \sim_{\mathcal{G}\otimes\mathcal{H}} (\bar{u}, \bar{x})$ and $(v, y) \sim_{\mathcal{G}\otimes\mathcal{H}} (\bar{v}, \bar{y})$. Since G is block transitive, there exists an automorphism $\varphi \in \mathrm{Aut}(G)$ that sends uv to $\bar{u}\bar{v}$. Similarly there is $\psi \in \mathrm{Aut}(H)$ that sends xy to $\bar{x}\bar{y}$. Consider the mapping $\varphi \otimes \psi$ on the set $V_{G\otimes H}$ defined by:

$$(\varphi \otimes \psi)(w, z) = (\varphi(w), \psi(z))$$

It is straightforward to verify that $\varphi \otimes \psi$ is an automorphism of $G \otimes H$: if $(w, z)(w', z') \in E_{G\otimes H}$ then $ww' \in E_G$ and $zz' \in E_H$.

As φ and ψ are automorphisms we know that $\varphi(w)\varphi(w') \in E_G$ and $\psi(z)\psi(z') \in E_H$. Finally, we get that

$$(\varphi(w), \psi(z))\,(\varphi(w'), \psi(z')) = (\varphi\otimes\psi)(w, z)\,(\varphi\otimes\psi)(w', z') \in E_{G\otimes H}$$

As by the construction, the mapping $\varphi \otimes \psi$ sends e to f, the statement of the theorem holds.

4 Construction of block transitive graphs

Lemma 7 *Let* $\mathbf{M}$ *be a degree matrix with 0's outside the diagonal. Then all 0 entries outside the diagonal can be replaced with appropriate positive numbers in a way that the resulting matrix* M' *is a degree matrix.*

PROOF. For the degree matrix $\mathbf{M}$ take G be the smallest graph G with an equitable partition $\mathcal{G}$ corersponding to $\mathbf{M}$. Let $S = (s_1, s_2, \ldots, s_r)$ the sizes of blocks on $\mathcal{G}$. (This vector can be computed as the minimal solution for the block sizes problem associated to $\mathbf{M}$, see [4].) For each pair of different blocks V_i and V_j that are not connected (i.e., that $m_{i,j} = m_{j,i} = 0$) we insert $h_{i,j} = \gcd(s_i, s_j)$ disjoint copies of the complete bipartite graph $K_{(s_i/h_{i,j}),(s_j/h_{i,j})}$ using the vertices of V_i and V_j. The resulting graph will have the following matrix $\mathbf{M}'$:

$$m'_{i,j} = \begin{cases} m_{i,j} & \text{if } m_{i,j} \neq 0 \text{ or } i = j. \\ s_j/h_{i,j} & \text{if } m_{i,j} = 0 \text{ and } i \neq j. \end{cases}$$

We now introduce simple matrices that we will use to factorize degree matrices. If not stated otherwise all matrices are of order r.

Definition 8 *For a nonnegative integer* m *and indices* $i, j \in [r]$ *we define the symmetric matrix* $\mathbf{S}_{i,j}(m)$ *of order* r *as follows:*

$$(\mathbf{S}_{i,j}(m))_{k,l} = \begin{cases} 0 & \textit{if } k = l. \\ m & \textit{if } \{k, l\} = \{i, j\}. \\ 1 & \textit{in other case.} \end{cases}$$

For a positive integer m and a set of indices $I \subseteq [r]$ we define the matrix $\mathbf{A}_I(m)$ of order r as follows:

$$(\mathbf{A}_I(m))_{k,l} = \begin{cases} 0 & \textit{if } k = l. \\ m & \textit{if } k \notin I, l \in I. \\ 1 & \textit{in other case.} \end{cases}$$

It is straightforward to verify that all $\mathbf{S}_{i,j}(m)$ and also all $\mathbf{A}_I(m)$ are indeed degree matrices. To support this fact we note here that a block transitive graph for any such a matrix will be constructed later in this section.

Lemma 9 *Every degree matrix $\mathbf{M}$ with zeros on the diagonal can be decomposed into coordinate product of finitely many matrices $\mathbf{S}_{i,j}(m)$ and $\mathbf{A}_I(m)$ with suitable parameters i, j, m and I.*

PROOF. According to Lemma 7 we construct matrix $\mathbf{M}'$ and write:

$$\mathbf{M} = \mathbf{M}' \otimes \bigotimes_{i<j,\, m_{i,j}=0} \mathbf{S}_{i,j}(0).$$

In addition, we divide symmetric entries of $\mathbf{M}'$ by the greatest common divisor and obtain a matrix $\mathbf{M}''$, where the symmetric entries are relative primes. It holds that:

$$\mathbf{M}' = \mathbf{M}'' \otimes \bigotimes_{i<j,\, \gcd(m'_{i,j}, m'_{j,i})>1} \mathbf{S}_{i,j}(\gcd(m'_{i,j}, m'_{j,i})).$$

Let $P = \{p_1, \ldots, p_k\}$ be the set of prime divisors of elements of $\mathbf{M}''$. If P is empty then $\mathbf{M}''$ is the identity matrix and the Lemma is proved. In the other case, for each $p \in P$ we define the matrix $\mathbf{M}^p$ elementwise such that for all $k \in [r] : m^p_{k,k} = 0$ and for $l \in [r], l \neq k$ the entry $m^p_{k,l}$ is the greatest power of p dividing $m''_{k,l}$.

Every matrix $\mathbf{M}^p$ is a degree matrix, because it satisfies the cycle product identity. Any product along a cycle of indices is equal to the greatest power of p dividing the corresponding product in the original matrix $\mathbf{M}''$.

Now we can further decompose matrix $\mathbf{M}''$ as follows:

$$\mathbf{M}'' = \bigotimes_{p \in P} \mathbf{M}^p.$$

It remains to disassemble each matrix $\mathbf{M}^p$ into coordinate product of matrices of form $\mathbf{A}_I(p)$. For that we iteratively repeat the following procedure:

1. Select I to be the set of indices of rows of $\mathbf{M}^p$ with all ones and one zero.
2. If $I = [r]$ then stop, otherwise divide coordinatewise $\mathbf{M}^p$ by $\mathbf{A}_I(p)$ and continue by step 1.

To argue correctness we need an insight into block sizes $|V_1|, \ldots, |V_r|$ of a graph G with degree matrix M^p. If $m^p_{k,l} = m^p_{l,k} = 1$ then both blocks V_k and V_l are of the same size. In the other case holds either that $m^p_{k,l} = 1$ and $p|m^p_{l,k}$ or vice versa. Assume w.l.o.g. the first case, which means that $|V_k| = m^p_{l,k}|V_l|$. As the matrix $\mathbf{M}^p$ is not the identity matrix, we know that this case at least once appears, i.e., G has at least two blocks of different size.

The choice of the index set I corresponds to the selection of indices of blocks that are of the maximum size. A vertex from a smaller block $V_l, l \notin I$ must have $m^p_{k,l} \geq p$ neighbors in any $V_k, k \in I$. In other words, whenever $k \in I, l \notin I$ we know that $p|m^p_{k,l}$. Hence, the division in step 2 always provides an integral result.

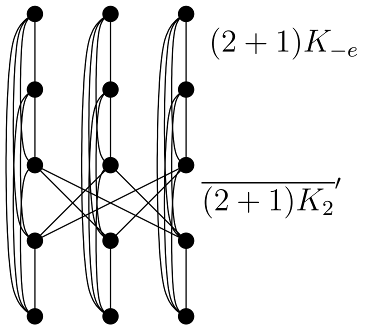
Observe also that the products along a cycle of indices in the former and in the modified matrix $\mathbf{M}^p$ differ only in a factor of p^t. This t is the number of times the cycle traverses between the set I and its complement and it is independent on the direction of the cycle. Therefore the modified matrix $\mathbf{M}^p$ obtained in step 2 satisfies cycle product identity, i.e., it is a degree matrix, and the procedure can be iterated.

The number of rounds of the procedure is equal to the greatest power of p in $\mathbf{M}^p$. Hence, it is finite and the statement of the Lemma holds.

We continue with a construction of a block transitive graph with degree matrix $\mathbf{S}_{i,j}(m)$. We take $V = [m+1] \times [r]$ and

$$E = \{(a,k)(a,l) \mid \{k,l\} \neq \{i,j\}\} \cup \{(a,i)(b,j) \mid a \neq b\}.$$

In explanation the first set of edges is a disjoint union of $m+1$ copies of the graph $K_r - e$ (a complete graph with one edge removed) joined by the second set of edges that define a bipartite



$$\mathbf{S}_{3,4}(2) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Fig. 2. Example of a block transitive graph $G_{\mathbf{S}}$ for the matrix $\mathbf{S}_{3,4}(2)$, $r = 5$.
(Rows in the drawing of G_S form blocks of the corresponding equitable partition.)

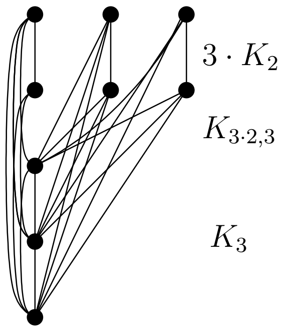
complement of a $(m+1)$-matching between the set of vertices with second coordinate i and the one with second coordinate j. See Fig. 2 for an example.

Lemma 10 *The graph* $G_{\mathbf{S}} = (V, E)$ *is block transitive and* $\mathbf{S}_{i,j}(m)$ *is its degree matrix.*

PROOF. Note first that the partition $\mathcal{G} = \{V_1, \ldots, V_r\}$ defined by $V_i = [m+1] \times \{i\}$ is an equitable partition of $G_{\mathbf{S}}$ with the degree matrix $\mathbf{S}_{i,j}(m)$.

To argue that $G_{\mathbf{S}}$ is block transitive we distinguish two cases:

(1) The given edges $e = (a,k)(a,l)$ and $f = (b,k)(b,l)$ span between V_k and V_l, where $\{k,l\} \neq \{i,j\}$. Then let π be the transposition of $[m+1]$ that swaps a with b, i.e., $\pi = (a\ b)$ written in cycle notation.
(2) The edges $e = (a,i)(b,j)$ and $f = (c,i)(d,j)$ span between V_i and V_j. By the construction of $G_{\mathbf{S}}$ we know that $a \neq b$ and $c \neq d$. Now let π be any permutation of $[m+1]$ such



$$\mathbf{A}_{\{1,2\}}(3) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 3 & 3 & 0 & 1 & 1 \\ 3 & 3 & 1 & 0 & 1 \\ 3 & 3 & 1 & 1 & 0 \end{pmatrix}$$

Fig. 3. Example of $G_{\mathbf{A}}$ for the matrix $\mathbf{A}_{\{1,2\}}(3)$, $r = 5$.

that $\pi(a) = c$ and $\pi(b) = d$.

$$\pi = \begin{cases} (a\ c)(b\ d) & \text{if all } a, b, c, d \text{ are different.} \\ (b\ d) & \text{if } a = c \text{ and } b \neq d. \\ (a\ c) & \text{if } b = d \text{ and } a \neq c. \\ (a\ b\ d) & \text{if } b = c \text{ and } a \neq d. \\ (a\ b\ c) & \text{if } a = d \text{ and } b \neq c. \\ (a\ b) & \text{if } (a, b) = (d, c). \end{cases}$$

In both cases we obtain the desired automorphism φ of $G_{\mathbf{S}}$ by applying π on the first coordinate, i.e., it is the mapping $\varphi(x, y) = (\pi(x), y)$.

Now we construct a block transitive graph with degree matrix $\mathbf{A}_I(m)$. Without loss of generality we may assume, that the set I is of size s and that it contains the first s natural numbers, i.e. $I = [s]$. A graph with such particular degree matrix has as a degree matrix matrix $\mathbf{A}_J(m)$ for any set $J \subseteq [r]$ of size s, since it can be obtained only by a suitable reordering of blocks in the equitable partition.

We take $V = [1] \times [r] \ \cup \ ([m] \setminus \{1\}) \times [s]$ and

$$E = \{(a, i)(a, j)\} \ \cup \ \{(a, i)(1, j) \ | \ i \leq s, j > s\}.$$

In other words, this graph consists of m disjoint complete graphs K_s joined to a single K_{r-s} by a complete bipartite graph $K_{m\cdot s, r-s}$. See Fig. 3 for an example.

Lemma 11 *The graph $G_{\mathbf{A}} = (V, E)$ is block transitive and $\mathbf{A}_I(m)$ is its degree matrix.*

PROOF. As above the equitable partition according to the second coordinate witness that $\mathbf{A}_I(m)$ is a degree matrix of $G_{\mathbf{A}}$.

We further discuss the property of block transitivity in two cases:

(1) The given edges $e = (a,i)(a,j)$ and $f = (b,i)(b,j)$ span between V_i and V_j, where $i,j \le s$.

(2) The edges $e = (a,i)(1,j)$ and $f = (b,i)(1,j)$ span between V_i and V_j, where $i \le s < j$.

In both cases let π be the transposition of $[m]$ that swaps a with b, i.e., $\pi = (a\ b)$. As in the previous lemma we apply π on the first coordinate to get the desired automorphism $\varphi(x,y) = (\pi(x), y)$.

5 Main Theorem

Theorem 12 *For every degree matrix* $\mathbf{M} \in \mathcal{M}$ *there exists a finite block transitive graph* G *with degree matrix* $\mathbf{M}$*.*

PROOF. We first transform the given matrix $\mathbf{M}$ into a matrix $\mathbf{M}'$ such that $\mathbf{M}'$ contains all non-diagonal entries of $\mathbf{M}$.

As $\mathbf{M}'$ has zeros on the diagonal, we now decompose the matrix $\mathbf{M}'$ into coordinate product of matrices of form $\mathbf{S}_{i,j}(m)$ and $\mathbf{A}_I(m)$ due to Lemma 9. By Theorem 6 we construct a block transitive graph G' with degree matrix $\mathbf{M}'$ according to this decomposition.

It remains to further modify G' to incorporate all nonzero diagonal entries of $\mathbf{M}$. Without loss of generality assume that $m_{1,1}, m_{2,2}, \ldots, m_{k,k}$ are all nonzero diagonal entries of $\mathbf{M}$, and put $z = (m_{1,1}+1)(m_{2,2}+1)\ldots(m_{k,k}+1)$. We take z copies of the graph G' and distinguish the z copies $u_{(i_1,i_2,\ldots,i_k)}$ of every vertex u by indices $(i_1, i_2, \ldots, i_k)$ taking range $[m_{1,1}+1] \times [m_{2,2}+1] \times \cdots \times [m_{k,k}+1]$.

Now for every index $l = 1, \ldots, k$ we join vertices inside block V_l by $\frac{z}{m_{l,l}+1}$ cliques $K_{m_{l,l}+1}$ in the way that two vertices become connected if only if they are copies of the same vertex and their indices differ only in the l-th coordinate.

Straightforwardly, if we unify the z copies of each block of the equitable partition of G' into one block, we obtain a partition of V_G into r sets, witnessing that $\mathbf{M}$ is a degree matrix of the graph G.

It remains to show that G is block transitive. We distinguish two cases:

(1) The given edges $e = u_{(i_1,i_2,\ldots,i_k)}v_{(i_1,i_2,\ldots,i_k)}$ and $f = \bar{u}_{(j_1,j_2,\ldots,j_k)}\bar{v}_{(j_1,j_2,\ldots,j_k)}$ span between different blocks. In this case first find an automorphism ψ of G' with $\psi(u) = \bar{u}$ and $\psi(v) = \bar{v}$, whose existence is assured by the block transitivity of G'. Then we define k permutations $\pi_1, \ldots, \pi_k$ such that for each $l = 1, \ldots, k$, the permutation $\pi_l = (i_l, j_l)$ is the transposition of $[m_{l,l} + 1]$ that swaps i_l with j_l. We combine these mappings into the desired automorphism $\varphi(u_{(i_1,i_2,\ldots,i_k)}) = \psi(u)_{(\pi_1(i_1),\pi_2(i_2),\ldots,\pi_k(i_k))}$.

(2) The edges $e = u_{(i_1,i_2,\ldots,i_k)}u_{(i'_1,i'_2,\ldots,i'_k)}$ and $f = \bar{u}_{(j_1,j_2,\ldots,j_k)}\bar{u}_{(j'_1,j'_2,\ldots,j'_k)}$ join vertices from the same block. Then we know that indices $i_1, i_2, \ldots, i_k$ differ from $i'_1, i'_2, \ldots, i'_k$ only in one, say t-th, coordinate $i_t \neq i'_t$. Consequently, $j_1 = j'_1, j_2 = j'_2, \ldots, j_k = j'_k$ except for $j_t \neq j'_t$. As above, we first determine an automorphism ψ of G' with $\psi(u) = \bar{u}$. Then we take k permutations $\pi_1, \ldots, \pi_k$ such that for $l = 1, \ldots, k$ the permutation π_l acts on $[m_{l,l}+1]$ and sends i_l onto j_l, and, in addition, the permutation π_t further satisfies $\pi_t(i'_t) = j'_t$. Together, these mappings provide us the desired automorphism φ of G by $\varphi(u_{(i_1,i_2,\ldots,i_k)}) = \psi(u)_{(\pi_1(i_1),\pi_2(i_2),\ldots,\pi_k(i_k))}$.

    Observe that images of two adjacent vertices differ only in one coordinate and, therefore, are connected by the construction of G.

6 Strongly block transitive graphs

We continue our study with a stronger notion of a block transitive graph. From now, the desired automorphism will be required not only to send given edge e to f, but in addition, also f to e.

Definition 13 *We call a graph G with degree matrix $\mathbf{M}$ of order r and equitable partition $\mathcal{G}$ is* strongly block transitive*, if for each* *pair of edges* $e = uv$ *and* $f = \bar{u}\bar{v}$ *where* $u \sim_{\mathcal{G}} \bar{u}$, $v \sim_{\mathcal{G}} \bar{v}$ *and where* $\bar{u} = v$ *if* $\bar{v} = u$ *there exists an automorphism* φ *of* G *preserving the partition that* swaps e with f, *i.e.,* $\varphi : u \leftrightarrow \bar{u}, v \leftrightarrow \bar{v}$.

The special assumption in the definition excludes the case of $e = uv$, $f = wu$, where u should be swapped simultaneously with v and w, which is impossible.

The following observation gives us a insight into the structure of a strongly block transitive graph:

Observation 14 *Let* G *be a strongly transitive graph with respect to equitable partition* $\mathcal{G} = \{V_1, \ldots, V_k\}$ *with degree matrix* $\mathbf{M}$. *Then the subgraph of* G' *induced by a single block* V_i *or by the edges stemming between two blocks* V_i *and* V_j *does not contain a path on four vertices* P_4 *as an induced subgraph.*

PROOF. Assume that G' contains a path (u, v, w, x). We choose $e = uv$, $f = wx$. If φ was the desired automorphism we get that $\varphi(vw) = \varphi(v)\varphi(w) = xu \in E_{G'}$, i.e. (u, v, w, x) cannot be an induced subgraph of G'.

As complete bipartite graphs are the only bipartite graphs of diameter two, we immediately obtain the following corollary:

Corollary 15 *Let* G *be as in the above observation, then its subgraph given by the edges stemming between two blocks* V_i *and* V_j *is a disjoint union of complete bipartite graphs* $K_{m_{i,j},m_{j,i}}$.

In consequence we see, that we cannot directly import all constructions from the above section, since the construction for the symmetric matrix $\mathbf{S}_{i,j}(m)$ (Lemma 10) involved a complement of a matching as the graph between blocks V_i and V_j, but this graph is not complete bipartite.

On the other hand, the notion of strongly block transitive graphs preserves all the remaining constructions.

Theorem 16 *Let* G *and* H *be two strongly block transitive graphs with equitable partitions of size* r, *and degree matrices* $\mathbf{M}$ *and* $\mathbf{N}$, *then* $G \otimes H$ *is also a strongly block transitive graph with degree matrix* $\mathbf{M} \otimes \mathbf{N}$.

Lemma 17 *The graph* $G_{\mathbf{A}} = (V, E)$ *is strongly block transitive and* $\mathbf{A}_I(m)$ *is its degree matrix.*

We omit formal proofs of these two statements as the proofs are exactly the same only notion "block transitive graph" should be strengthened by "strongly block transitive graph" and accordingly "automorphism φ that sends u to v" with "automorphism φ that swaps u with v", etc.

We claim also that the construction shown in Theorem 12, which was used to comprehend degree matrices with some positive entries on the diagonal, could be used for strongly block transitive graphs without any change.

Hence, we can focus our attention only to *symmetric* degree matrices. We present a necessary condition for a symmetric nonnegative matrix to be a degree matrix of a strongly block transitive graph:

Definition 18 *We say that a degree matrix* $\mathbf{M}$ *satisfies the* tri-angle inequality property *if for every sequence of indices* $i_1, \ldots, i_k$, $k \geq 3$ *holds that*

$$\prod_{j=1}^{k-1} m_{i_j,i_{j+1}} \neq 0 \quad \Longrightarrow \quad \prod_{j=1}^{k-1} m_{i_j,i_{j+1}} > m_{i_1,i_k}$$

Theorem 19 *Every degree matrix of a strongly transitive graph satisfies the triangle inequality property.*

In the sequel we use the following notation: Assume that the equitable partition $V_1, \ldots, V_r$ of a graph G is fixed. For a vertex $u \in V_1$ and $k = 2, \ldots, r$ we denote by $N_k(u)$ the set of its neighbors in V_k. Now consider all paths on k vertices from u that traverse blocks $V_2, \ldots, V_k$ (i.e. all paths $u, u_2, \ldots, u_k$ such that $u_j \in V_j$ for all $j = 2, \ldots, k$). Let $N'_k(u)$ be the set of final vertices of the above paths (it may be empty).

The proof of Theorem 19 is a direct consequence of the following lemma:

Lemma 20 *Let G be a graph and $V_1, \ldots, V_k$ be a sequence of blocks of one of its equitable partition such that for some $u \in V_1$ it holds that $|N'_k(u)| < |N_k(u)|$. Then the graph is not strongly block transitive for that partition.*

PROOF. We consider two cases:

- $N_k(u) \cap N'_k(u) \neq \emptyset$. We choose two vertices $v, v' \in N_k(u)$ such that $v \in N'_k(u)$ and $v' \notin N'_k(u)$. But no blocks preserving

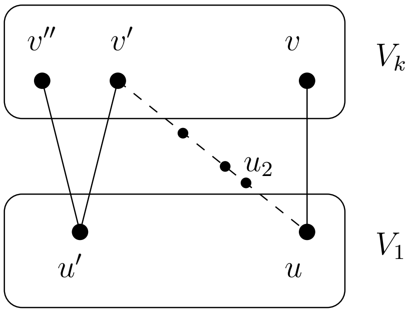


Fig. 4. Illustration for the proof of Theorem 19.

automorphism φ can send the edge uv onto uv', because the image of the path $(u, u_2, \ldots, u_k = v)$ would confirm that v' also belongs to the set $N'_k(u)$.

- $N_k(u) \cap N'_k(u) = \emptyset$. We take any $v \in N_k(u), v' \in N'_k(u)$. Also choose any $u' \in V_{i_k}$ such that $(u', v') \in E_G$. Now $v' \in N_k(u')$ and as $|N_k(u')| = |N_k(u)| = m_{1,k}$ we may choose $v'' \in N_k(u')$ such that $v'' \notin N'_k(u)$ (see Fig. 4).

  If there were blocks preserving automorphisms φ that swaps uv with $u'v'$ and φ' that swaps uv with $u'v''$ then in their composition $\psi = \varphi \circ \varphi'$ the vertex u remains fixed, while v' is swapped with v''. As in the above case, the image of the path $(u, u_2, \ldots, u_k = v')$ under ψ would contradict the choice $v'' \notin N'_k(u)$.

PROOF. [of Theorem 19] Take any graph G with degree matrix $\mathbf{M}$ which does not satisfy the triangle inequality property. Without loss of generality we may assume that witnessing this assumption the blocks are arranged such that their indices are $i_1, \ldots, i_k = 1, \ldots, k$.

By our assumptions we have

$$|N'_k(u)| \leq \prod_{j=1}^{k-1} m_{j,j+1} < m_{1,k} = |N_k(u)|$$

and hence G is not strongly block transitive by Lemma 20.

Theorem 19 breaks our hope into use of factorization method into symmetric matrices $\mathbf{S}_{i,j}(m)$ and asymmetric matrices $\mathbf{A}_I(m)$, as the following corollary shows:

Corollary 21 *No strongly block transitive graph exists with degree matrix $\mathbf{S}_{i,j}(m)$ of order $r \geq 3$ with $m \geq 2$.*

We have shown that the triangle inequality property is a necessary condition for a degree matrix to allow a strongly block transitive graph. We show that, unfortunately, it is not sufficient. For that we further explore the structure of strongly block transitive graphs.

Definition 22 *Let G be a graph with equitable partition $V_1, \ldots, V_r$. We say that $B \subseteq V_i$ is an (i,j)*-block *if $B = V_i \cap N(u)$ for some $u \in V_j$. In such a case we also say that u* generates *the (i,j)-block B.*

Clearly, any (i,j)-block B satisfies $|B| = m_{j,i}$.

Lemma 23 *Let V_1, V_2 and V_3 be three blocks of some strongly block transitive graph G such that $m_{1,2}m_{2,3} = m_{1,3}$. Then the following holds:*

(1) Every $(2,1)$-block intersects every $(2,3)$-block in at most one vertex.
(2) Each $(3,1)$-block is a disjoint union of $(3,2)$-blocks.

PROOF. Assume for the contrary that B is a $(2,1)$-block that intersects some $(2,3)$-block B' in at least two vertices. Take $u \in V_1$ that generates B and $w \in V_3$ that generates B'. Let $N'_3(u)$ be the set of vertices in V_3 that have in the set V_2 a common neighbor with u.

Clearly $w \in N'_3(u)$. Since u and w have $|B \cap B'| \geq 2$ common neighbors in V_2, we have that $N'_3(u) < m_{1,2}m_{2,3} = m_{1,3} = N_3(u)$. Now, by Lemma 20 the graph G is not strongly block transitive, a contradiction.

For the other claim take a $(3,1)$-block B generated by some vertex $u \in V_1$. Take an arbitrary $v' \in N'_3(u)$ and its neighbor $u' \in V_1$. Now for any $v \in B$ there exists an automorphism φ that swaps uv with $u'v'$ and consequently $v \in N'_3(u')$, i.e. $B \subseteq N'_3(u')$. (Consult Fig 4 with $k = 3$.)

As the set $N'_3(u')$ may have at most $m_{1,2}m_{2,3} = m_{1,3} = |B|$ many vertices, we have that $N'_3(u') = B$. Consequently B must be the disjoint union of the $(2,3)$-blocks generated by the neighbors of u' in V_2.

Now we are ready to show the counterexample.

Example 24 *For the symmetric degree matrix*

$$\mathbf{M} = \begin{pmatrix} 0 & 4 & 2 & 2 & 2 & 2 \\ 4 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 4 & 4 & 4 \\ 2 & 2 & 4 & 0 & 4 & 4 \\ 2 & 2 & 4 & 4 & 0 & 4 \\ 2 & 2 & 4 & 4 & 4 & 0 \end{pmatrix}$$

no strongly block transitive graph exists although $\mathbf{M}$ *satisfies the triangle inequality property.*

PROOF. Take an arbitrary $(2,1)$-block B of a hypothetic strongly block transitive graph G. We apply Lemma 23 (2) for the triples of blocks (V_1, V_i, V_2) with $i = 3, \ldots, 6$. For every i we get that B is a disjoint union of $(2,i)$-blocks. As the four vertices of B can be partitioned into pairs in only three different ways, we get that for some distinct i and j, the $(2,i)$ and $(2,j)$-blocks forming B coincide. But this is in contradiction with Lemma 23 (1) for the triple (V_i, V_2, V_j).

7 Conclusion

We have defined the new notion of a block transitive graph that extends the notion of a transitive graph beyond the class regular graphs. We have shown that for every degree matrix $\mathbf{M}$ always a block transitive graph G with degree matrix $\mathbf{M}$ exists. We have also shown that for yet more restrictive notion of strongly transitive graphs this does not hold in general which yields an open question of classification of degree matrices that allow a strongly block transitive graph.

Finally, we would like mention a folklore algebraic construction of bipartite edge transitive graphs (see e.g. [6]). Take a group Γ such that it has two subgroups Γ_1 and Γ_2. Consider the sets of their cosets $V_i = \{x\Gamma_i \mid x \in \Gamma\}$ for $i = 1, 2$. The intersection graph $G = (V_1 \cup V_2, \{AB \mid A \cap B \neq \emptyset\})$ is bipartite as cosets in both V_1 and V_2 are disjoint. Vertices from V_1 have $\frac{|\Gamma_1|}{|\Gamma_1 \cap \Gamma_2|}$ neighbors and vertices in V_2 have degree $\frac{|\Gamma_2|}{|\Gamma_1 \cap \Gamma_2|}$, so $\begin{pmatrix} 0 & \frac{|\Gamma_1|}{|\Gamma_1 \cap \Gamma_2|} \\ \frac{|\Gamma_2|}{|\Gamma_1 \cap \Gamma_2|} & 0 \end{pmatrix}$ is a degree matrix of G. Moreover, G is arc-transitive as the action $x \to ba^{-1}x$ sends the edge A_1B_1 onto A_2B_2 if we take some $a \in A_1 \cap B_1$ and $b \in A_2 \cap B_2$.

Existence of a suitable group Γ and its subgroups is essentially equivalent to the existence of an bipartite edge transitive graph. From such a G we take $\Gamma = \mathrm{Aut}(G)$ to be the automorphism group of G and for an edge $e = uv$ we take Γ_1 and Γ_2, respectively, to be the set of automorphisms that fix u and v, resp. A coset of Γ_1 consists of all automorphisms of G that send u to a fixed vertex, and analogously for Γ_2.

As G is edge transitive then for any edge $u'v' \in E_G$ there exist an automorphism $\varphi \in \Gamma$ such that $\varphi(u) = u'$ and $\varphi(v) = v'$, i.e. the cosets $\varphi\Gamma_1$ and $\varphi\Gamma_2$ representing u' and v' intersect. Consequently, the graph obtained by the above construction from the group $\Gamma = \mathrm{Aut}(G)$ and the two subgroups Γ_1, Γ_2 is isomorphic to the original graph G.

From our Theorem 12 follows that we can find a group Γ and its subgroups of prescribed relative sizes of their intersections, such that the intersection graph of their cosets yields a block transitive graph with given degree matrix. We pose a question whether such suitable group and its subgroups can be constructed directly in a purely algebraic way.

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