High Girth Cubic Graphs Map to the Clebsch Graph

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Abstract

We give a (computer assisted) proof that the edges of every graph with maximum degree 3 and girth at least 17 may be 5-colored (possibly improperly) so that the complement of each color class is bipartite. Equivalently, every such graph admits a homomorphism to the Clebsch graph (Fig. 1).

Hopkins and Staton [8] and Bondy and Locke [2] proved that every (sub)cubic graph of girth at least 4 has an edge-cut containing at least $\frac{4}{5}$ of the edges. The existence of such an edge-cut follows immediately from the existence of a 5-edge-coloring as described above, so our theorem may be viewed as a kind of coloring extension of their result (under a stronger girth assumption).

Every graph which has a homomorphism to a cycle of length five has an above-described 5-edge-coloring; hence our theorem may also be viewed as a weak version of Nešetřil's Pentagon Problem: Every cubic graph of sufficiently high girth maps to C_5.

1 Introduction

Throughout the paper all graphs are assumed to be finite, undirected and simple. For any positive integer n, we let C_n denote the cycle of length

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n, and K_n denote the complete graph on n vertices. If G is a graph and $U \subseteq V(G)$, we put $\delta(U) = \{uv \in E(G) : u \in U \text{ and } v \notin U\}$, and we call any subset of edges of this form a *cut*. The maximum size cut of G, denoted $\mathrm{MAXCUT}(G) = \max_{U \subseteq V} |\delta(U)|$ is a parameter which has received great attention. Next, we normalize and define

$$b(G) = \frac{\mathrm{MAXCUT}(G)}{|E(G)|}.$$

Determining $b(G)$ (or equivalently $\mathrm{MAXCUT}(G)$) for a given graph G is known to be NP-complete, so it is natural to seek lower bounds. It is an easy exercise to show that $b(G) \geq 1/2$ for any graph G and $b(G) \geq 2/3$ whenever G is cubic (that is 3-regular). The former inequality is almost attained by a large complete graph, the latter is attained for $G = K_4$: any triangle contains at most two edges from any bipartite subgraph, and each edge of K_4 is in the same number of triangles (two). This suggests that triangles play a special role, and raises the question of improving this bound for cubic graphs with higher girth. In the 1980's, several authors independently considered this problem [2, 8, 17], the strongest results being

- $b(G) \geq 4/5$ for G with maximum degree 3 and no triangle [2]
- $b(G) \geq 6/7 - o(1)$ for cubic G with girth tending to infinity [17]

On the other hand, cubic graphs exist with arbitrarily high girth and satisfying $b(G) < 0.94$ [10]. As far as we know, this result did not appear in print; it is, however, relatively straightforward to prove it by considering random cubic graphs, see [16] for a nice survey.

Define a set of edges C from a graph G to be a *cut complement* if $C = E(G) \setminus \delta(U)$ for some $U \subseteq V(G)$. Then the problem of finding a cut of maximum size is exactly equivalent to that of finding a cut complement of minimum size. A natural relative of this is the problem of finding many disjoint cut complements. Indeed, packing cut complements may be viewed as a kind of coloring version of the maximum cut problem.

There are a variety of interesting properties which are equivalent to the existence of $2k+1$ disjoint cut complements, so after a handful of definitions we will state a proposition which reveals some of these equivalences. This proposition is well known, but we have provided a proof of it in Section 3 for the sake of completeness. For every positive integer n, we let Q_n denote the n-dimensional cube, so the vertex set of Q_n is the set of all binary vectors of length n, and two such vertices are adjacent if they differ in a single coordinate. The n-dimensional projective cube,[1] denoted PQ_n, is the simple graph obtained from the $(n+1)$-dimensional cube Q_{n+1} by identifying pairs of antipodal vertices (vertices that differ in all coordinates). Equivalently, the projective cube PQ_n can be described as a Cayley graph, see Section 3. If G, H are graphs, a *homomorphism* from G to H is a mapping $f : V(G) \to V(H)$ with the property that $f(u)f(v)$ is an edge of H whenever uv is an edge of G. A mapping $g : E(G) \to E(H)$ is *cut-continuous* if the preimage of every cut is a cut. Now we are ready to state the relevant equivalences.

Proposition 1.1 *For every graph G and nonnegative integer k, the following properties are equivalent.*

(1) There exist $2k$ pairwise disjoint cut complements.

(2) There exist $2k+1$ pairwise disjoint cut complements with union $E(G)$.

(3) G has a homomorphism to PQ_{2k}.

(4) G has a cut-continuous mapping to C_{2k+1}.

Perhaps the most interesting conjecture concerning the packing of cut complements—or equivalently homomorphisms to projective cubes—is the following conjectured generalization of the Four Color Theorem. Although not immediately obvious, this is equivalent to Seymour's conjecture on edge-coloring of planar r-graphs for odd values of r.

Conjecture 1.2 (Seymour) *Every planar graph with all odd cycles of length greater than $2k$ has a homomorphism to PQ_{2k}.*

Since the graph PQ_2 is isomorphic to K_4, the $k = 1$ case of this conjecture is equivalent to the Four Color Theorem. The $k = 2$ case of this conjecture concerns homomorphisms to the graph PQ_4 which is also known as the Clebsch graph (see Figure 1). This case was resolved in the affirmative by Naserasr [11] who deduced it from a theorem of Guenin [4]. The following theorem is the main result of this paper; it shows that graphs of maximum degree three without short cycles also have homomorphisms to PQ_4. The *girth* of a graph is the length of its shortest cycle, or ∞ if none exists.

[1] sometimes called folded cube

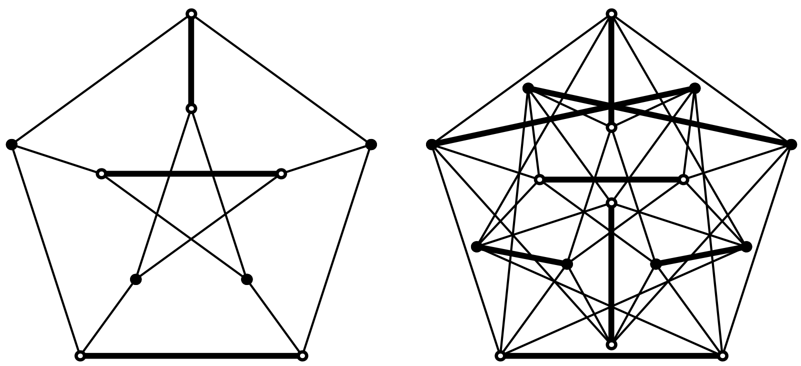

Figure 1: Petersen and Clebsch graph with one cut complement emphasized, the respective bipartition of the vertex set is depicted, too. The other four cut complements are obtained by a rotation.

Theorem 1.3 *Every graph of maximum degree 3 and girth at least* 17 *has a homomorphism to PQ_4 (also known as the Clebsch graph), or equivalently has 5 disjoint cut complements. Furthermore, there is a linear time algorithm which computes the homomorphism and the cut complements.*

Clearly no graph with a triangle can map homomorphically to the triangle-free Clebsch graph (equivalently, have 5 disjoint cut complements), but we believe this to be the only obstruction for cubic graphs. We highlight this and one other question we have been unable to resolve below.

Conjecture 1.4 (Šámal) *Every triangle-free cubic graph has a homomorphism to PQ_4.*

Problem 1.5 *What is the largest integer k with the property that all cubic graphs of sufficiently high girth have a homomorphism to PQ_{2k}?*

As we mentioned before, there are high-girth cubic graphs with $b(G) < 0.94$. Such graphs do not admit a homomorphism to PQ_{2k} for any $k \geq 8$, so there is indeed some largest integer k in the above problem. At present, we know only that $2 \leq k \leq 7$.

Another topic of interest for cubic graphs of high girth is circular chromatic number, a parameter we now pause to define. For any graph G, we let

$G^{\geq k}$ denote the simple graph with vertex set $V(G)$ and two nodes adjacent if they have distance at least k in G. The *circular chromatic number* of G, is $\chi_c(G) = \inf\{\frac{n}{k} : G \text{ has a homomorphism to } C_n^{\geq k}\}$. Every graph satisfies $\lceil \chi_c(G) \rceil = \chi(G)$ so the circular chromatic number is a refinement of the usual notion of chromatic number. The following curious conjecture asserts that cubic graphs of sufficiently high girth have circular chromatic number $\leq \frac{5}{2}$ (since $C_{2k+1} \cong C_{2k+1}^{\geq k}$).

Conjecture 1.6 (Nešetřil's Pentagon Conjecture [12]) *If G is a cubic graph of sufficiently high girth then there is a homomorphism from G to C_5.*

It is an easy consequence of Brook's Theorem that the above conjecture holds with C_3 in place of C_5 (every cubic graph of girth at least 4 is 3-colorable). On the other hand, it is known that the conjecture is false if we replace C_5 by C_{11} [9], consequently it is false if we replace C_5 by any C_n for odd $n \geq 11$. Later, it was shown that the conjecture is false also for C_9 [15] and C_7 [5] in place of C_5.

An important extension of Conjecture 1.6 is the problem to determine the infimum of real numbers r with the property that every cubic graph of sufficiently high girth has circular chromatic number $\leq r$. The above results show that this infimum must lie in the interval $[\frac{7}{3}, 3]$, but this is the extent of our knowledge. It is tempting to try to use the fact that girth ≥ 17 cubic graphs map to the Clebsch graph and girth ≥ 4 cubic graphs map to C_3 to improve the upper bound, but the circular chromatic numbers of C_3, the Clebsch graph, and their direct product are all at least three,[2] so no such improvement can be made. Neither were we able to use our result to improve upper bounds on fractional chromatic numbers of cubic graphs. This is conjectured to be at most $14/5$ for triangle-free cubic graphs (Heckmann and Thomas [7]), and proved to be at most $3 - 3/64$ (Hatami and Zhu [6]).

It is easy to prove directly that Conjecture 1.6, if true, implies Theorem 1.3 (perhaps with a stronger assumption on the girth). This follows from part (4) of Proposition 1.1 and the following easy observation.

Observation 1.7 *If there is a homomorphism from G to H, then there is a cut-continuous mapping from G to H.*

[2] The only nontrivial case is the product $PQ_4 \times K_3$. By a theorem of [3] this graph is uniquely 3-colorable; consequently $\chi_c(PQ_4 \times K_3) = 3$.

Proof Let $f : V(G) \to V(H)$ be a homomorphism and define the mapping $f^\sharp : E(G) \to E(H)$ by the rule $f^\sharp(uv) = f(u)f(v)$. If $S = \delta(U)$ is a cut in H, then $(f^\sharp)^{-1}(S) = \delta(f^{-1}(U))$, which is also a cut. $\square$

The relationship between homomorphisms and cut-continuous maps is studied in greater detail in [14] and [13] where it is shown that, perhaps surprisingly, existence of a cut-continuous mapping from G to H frequently implies the existence of a homomorphism from G to H. Unfortunately, it does not appear likely that these techniques can be used to extend the main theorem of this paper to attain Conjecture 1.6.

We finish the introduction with another conjecture due to Nešetřil (personal communication) concerning the existence of homomorphisms for cubic graphs of high girth.

Conjecture 1.8 *For every integer k there is a graph H of girth at least k and an integer N, such that for every cubic graph G with girth at least N we have*

$$G \xrightarrow{hom} H\,.$$

Our theorem shows this conjecture to be true for $k \le 5$, but the other cases remain open.

2 The Proof

The goal of this section is to prove the main theorem. We begin with a lemma which reduces our task to cubic graphs.

Lemma 2.1 *If Theorem 1.3 holds for every cubic G then it holds for every subcubic G, too.*

Proof Let G be a subcubic graph of girth at least 17. We will find a cubic graph G' such that girth of G' is at least 17 and $G' \supseteq G$. The lemma then follows, as restriction of any homomorphism $G' \xrightarrow{hom} PQ_4$ to $V(G)$ is the desired homomorphism $G \xrightarrow{hom} PQ_4$.

To construct G', put $r = \sum_{v \in V(G)} (3 - \deg(v))$. Let H be an r-regular graph of girth at least 17 (it is well-known that such graphs exists, see e.g. [1] for a nice survey). We take $|V(H)|$ copies of G. For every edge uv of H we choose two vertices of degree less than 3, one from a copy of G corresponding to each of u and v; then we connect these by an edge. Clearly, this process will lead to a cubic graph containing G and with girth at most equal to the minimum of girths of G and H. $\square$

To show that cubic graphs of girth ≥ 17 have homomorphism to the Clebsch graph, we shall use property (1) from Proposition 1.1. Accordingly, we define a *labeling* of a graph G to be a four-tuple $X = (X_1, X_2, X_3, X_4)$ so that each X_i is a subset of $E(G)$. We call a labeling X a *cut labeling* if every X_i is a cut, and a *cut complement labeling* if every X_i is a cut complement. If $X_i \cap X_j = \emptyset$ whenever $1 \leq i < j \leq 4$ we say that the labeling is *wonderful.*

Define function $a : \{0, 1, \ldots, 4\} \to \mathbb{Z}$ by $a(0) = 0$, $a(1) = 1$, $a(2) = 10$, $a(3) = 40$, and $a(4) = 1000$. Now, for any labeling X, we define the *mark* of an edge e (with respect to X) to be $m_X(e) = \{i \in \{1, 2, 3, 4\} : e \in X_i\}$, the *weight* of e to be $w_X(e) = |m_X(e)|$, and the *cost* of e to be $\mathrm{cost}_X(e) = a(w_X(e))$. Finally, we define the *cost* of X to be $\mathrm{cost}(X) = \sum_{e \in E(G)} \mathrm{cost}_X(e)$.

The structure of our proof is quite simple: we prove that any cut complement labeling of minimum cost in a cubic graph of girth ≥ 17 is wonderful. To show that such a labeling is wonderful, we shall assume it is not, and then make a small local change to improve the cost—thus obtaining a contradiction. The observation below will be used to make our local changes. For any sets A, B we let $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$ be the symmetric difference. If $X = (X_1, \ldots, X_4)$ and $Y = (Y_1, \ldots, Y_4)$ are labelings, then we let $X \,\Delta\, Y = (X_1 \,\Delta\, Y_1, \ldots, X_4 \,\Delta\, Y_4)$. We say we obtain $X \,\Delta\, Y$ from X by *switching* Y.

Observation 2.2 *If C is a cut and D is a cut complement, then $C \,\Delta\, D$ is a cut complement. Similarly, if X is a cut complement labeling and Y is a cut labeling, then $X \,\Delta\, Y$ is a cut complement labeling.*

Proof Let $C = \delta(U)$ and $D = E(G) \setminus \delta(V)$. Then $C \,\Delta\, D = E(G) \setminus (\delta(U) \,\Delta\, \delta(V)) = E(G) \setminus \delta(U \,\Delta\, V)$ so it is a cut complement. For labelings we consider each coordinate separately. □

The graphs we consider will have high girth, so they will look like trees locally. Our proof will exploit this by using the above observation to make changes on a tree. To state our method precisely, we now introduce a family of rooted trees. Let T_i denote a rooted tree of "depth i" in which all vertices have degrees 1 and 3, and the root vertex, denoted r, has degree 1. Explicitly, we let T_1 be an edge (with one end being the root). Having defined T_i, we form T_{i+1} by joining two copies of T_i by identifying their root vertices and then connecting this common vertex to a new vertex, which will be the new root. The unique edge incident with the root we shall call the *root edge.* We let $2T_i$ denote the tree obtained from two copies of T_i by identifying their root edges in the opposite direction (the resulting edge will be called the *central* edge of $2T_i$). A vertex of T_i or $2T_i$ is *interior* if either it has degree 3, or it is the root of T_i. A cut C (cut labeling X) of T_i or $2T_i$ is called *internal* if $C = \delta(Z)$ ($X = (\delta(Z_1), \ldots, \delta(Z_4))$) for some set Z (sets $Z_1, \ldots, Z_4$) of interior vertices. Now we are ready to state and prove a lemma that forms the first step of the proof: it will be used to show that any cut complement labeling of minimum cost has no edges of weight > 2.

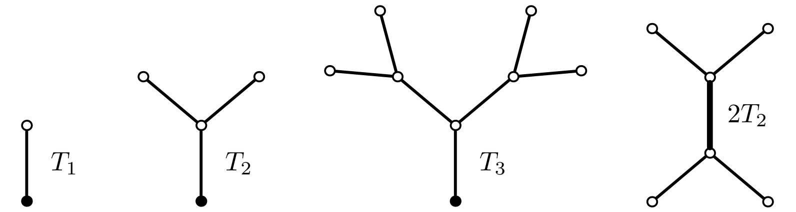

Figure 2: Illustration of definitions, root vertex/central edge are emphasized.

Lemma 2.3 *Let X be a labeling of the tree $2T_2$ and assume that the weight of the central edge is > 2. Then there exists an internal cut labeling Y of $2T_2$ so that* $\mathrm{cost}(X \,\Delta\, Y) < \mathrm{cost}(X)$.

Proof Let e be the central edge, let x be a vertex incident with e, let f, g be the other edges incident with x, and let $A = m_X(e)$, $B = m_X(f)$, and $C = m_X(g)$. We will construct a cut labeling $Y = (\delta(Z_1), \ldots, \delta(Z_4))$ (where each Z_i is either $\emptyset$ or $\{x\}$) so that $\mathrm{cost}(X \,\Delta\, Y) < \mathrm{cost}(X)$. For convenience, we shall say that we *switch* a set $I \subseteq \{1, 2, 3, 4\}$ if we set $Z_i = \{x\}$ if $i \in I$ and $Z_i = \emptyset$ otherwise.

If $S = A \cap B \cap C$ is nonempty then we may switch S, thereby reducing the cost of each of e, f, g. Hence we may suppose S is empty.

Case 1. $|A| = 4$**:** If $B = C = \emptyset$ then we switch $\{1\}$ decreasing the cost from $a(4)$ to $a(3) + 2a(1)$. Otherwise we switch $B \cup C$; this leads to a mark $\{1, 2, 3, 4\} \setminus (B \cup C)$ on e, C on f and B on g, reducing the cost again.

Case 2. $|A| = 3$**:** We may suppose $A = \{1, 2, 3\}$ and $|A| \geq |B| \geq |C|$ Moreover, $|C| < 3$ for otherwise $A \cap B \cap C$ is nonempty. If B and A have a common element, then we switch it. This changes the weights of edges in T from 3, $|B|$, $|C|$ to 2, $|B| - 1$, $|C| + 1$ and as $|C| < 3$, this is an improvement in the total cost. It remains to consider the cases when both B and C are subsets of $\{4\}$. In each of these cases we switch $\{1\}$, this reduces the cost from at least $a(3)$ to at most $3a(2)$. $\square$

The next lemma, which provides the second step of the proof, is analogous to the previous one, but is considerably more complicated to prove.

Lemma 2.4 *Let X be a labeling of the tree $2T_9$ and assume that every edge has weight ≤ 2 and that the central edge has weight exactly 2. Then there exists an internal cut labeling Y of $2T_9$ so that* $\mathrm{cost}(X \,\Delta\, Y) < \mathrm{cost}(X)$.

Before discussing the proof of this lemma we shall use it to prove the main theorem.

Proof It follows from Lemma 2.1 and Proposition 1.1 that it suffices to prove that all cubic graphs with girth at least 17 have wonderful cut complement labelings. Let G be such graph and let X be a cut complement labeling of G of minimum cost. It follows immediately from Lemma 2.3 that every edge of G has weight ≤ 2. Suppose there is an edge e of weight 2. Then it follows from our assumption on the girth that G contains a subgraph isomorphic to $2T_9$ (possibly with some of the leaf vertices identified) where e is the central edge. Now Lemma 2.4 gives us an *internal* cut labeling Y of $2T_9$ (hence a cut labeling of G) such that $\mathrm{cost}(X \,\Delta\, Y) < \mathrm{cost}(X)$. This contradiction shows that X is wonderful, and completes the proof.

Next we give a short description of a linear-time algorithm that finds the partition. We start with a cut complement labeling $(E(G), E(G), E(G), E(G))$. Then we repeatedly pick a bad edge e—that is an edge for which $w(e) > 1$. By Lemma 2.3 and 2.4 we can decrease the total cost by switching a cut labeling that contains only edges at distance at most 8 from e. We can therefore find the cut labeling in constant time (e.g. by brute force if we do not try to minimize the constant)—we only have to use efficient representation of the graph, namely a list of edges, list of vertices, and pointers between the adjacent objects. As the cost of the starting coloring is $a(4) \cdot |E(G)|$ and at each step the decrease is at least by 1, it remains to handle the operation "pick a bad edge" in constant time. For this, we maintain a linked list of bad edges, for each element of the list there is a pointer from and to the corresponding edge in the main list of edges. This allows us to change the list of bad edges after each switch in constant time (although, we repeat, the constant is impractically large). $\square$

It remains to prove Lemma 2.4, and our proof of this requires a computer. Unfortunately, both the number of labelings and the number of possible cuts is far too large for a brute-force approach: There are $2(2^9 - 1) - 1$ edges of $2T_9$, which means more than 11^{1000} labelings, even if we use Lemma 2.3 to eliminate labeling with edges of weight 3 or 4. Moreover, there are roughly $(2^{2\cdot 2^8})^4$ internal cut labelings in $2T_9$, hence we cannot use brute-force even for one labeling. To overcome the second problem we shall recursively compute all of the necessary information, called a "menu" on the subtrees, leading to an efficient algorithm for a given labeling. To solve the first problem, instead of enumerating all labelings of $2T_9$ and computing the menu for them, we will iteratively find all menus corresponding to all labelings of $T_1, T_2, \ldots, T_8$. This way we avoid considering the same "partial labeling" several times. To further reduce the computational load, we will consider only "worst possible menus" in each T_i. Now, to the details.

If $S \subseteq \{1,2,3,4\}$, we define an internal cut labeling Y of T_i to be an *internal S-swap* if $Y = (\delta(Z_1), \ldots, \delta(Z_4))$ where every Z_i is a set of interior nodes (note that the root is interior) and $S = \{i \in \{1,2,3,4\} : r \in Z_i\}$. Informally, an internal S-swap 'switches S between the root and the leaves'. A *menu* is a mapping $M : \mathcal{P}([4]) \to \mathbb{Z}$. If T_i is a copy of a rooted tree with root r and X is a labeling of T_i then the *menu corresponding to* X is defined as follows

$$M_X(S) = \min\{\mathrm{cost}(X \,\Delta\, Y) - \mathrm{cost}(X) : Y \text{ is an internal } S\text{-swap}\}\,. \qquad (1)$$

Thus, the menu M_X associated with X is a function which tells us for each subset $S \subseteq \{1,2,3,4\}$ the minimum cost of making an internal S-swap. This is enough information to check whether we can decrease the cost of a given labeling: if T_1, T_2, T_3 are trees meeting at a vertex and X_i is the restriction of a labeling X to T_i, then we can decrease the cost by a local swap if we have $M_{X_1}(S) + M_{X_2}(S) + M_{X_3}(S) < 0$ for some $S \in \mathcal{P}([4])$.

For menus M, N and a set $R \subseteq \{1,2,3,4\}$ we define $\mathrm{Parent\_menu}(M,N,R) : \mathcal{P}([4]) \to \mathbb{Z}$ to be mapping given by the following rule:

$$\mathrm{Parent\_menu}(M,N,R)(S) = \min_{Q\in\mathcal{P}([4])}\Big(M(Q)+N(Q)+a(|R\Delta S\Delta Q|)-a(|R|)\Big)\,. \qquad (2)$$

The motivation for this definition is the following observation, which is the key to our recursive computation.

Observation 2.5 *Let X be a labeling of the tree T_i where $i \geq 2$. Assume that T_i is composed of the root edge e and two copies of T_{i-1} denoted T' and T'', and let X' and X'' be the restrictions of the labeling X to the trees T' and T''. Then*

$$M_X = \mathrm{Parent\_menu}(M_{X'}, M_{X''}, m_X(e))\,.$$

Proof Let v be the end of the edge e which is distinct from r and pick $S, Q \in \mathcal{P}([4])$. Now choose a cut labeling $Y = (\delta(Z_1), \ldots, \delta(Z_4))$ so that $\text{cost}(X \,\Delta\, Y) - \text{cost}(X)$ is minimal subject to the following constraints

(i) Z_i is internal for $1 \le i \le 4$,

(ii) $S = \{i \in \{1,2,3,4\} : r \in Z_i\}$, and

(iii) $Q = \{i \in \{1,2,3,4\} : v \in Z_i\}$.

Then $m_{X\Delta Y}(e) = m_X(e) \,\Delta\, S \,\Delta\, Q$ and we find that

$$\text{cost}(X\Delta Y) - \text{cost}(X) = M_{X'}(Q) + M_{X''}(Q) + a(|m_X(e)\Delta S \Delta Q|) - a(|m_X(e)|)\,.$$

It follows from this that $M_X = \text{Parent\_menu}(M_{X'}, M_{X''}, m_X(e))$ as desired. $\square$

Using the above observation, it is relatively fast to compute the menu associated with a fixed labeling of a tree T_i. However, for our problem, we need to consider all possible labelings of T_i. Accordingly, we now define a few collections of menus which contain all of the information we need to compute to resolve Lemma 2.4. Prior to defining these collections, we need to introduce the following partial order on menus: if M_1 and M_2 are menus, we write $M_1 \preccurlyeq M_2$ if $M_1(s) \le M_2(s)$ for every $s \in \mathcal{P}([4])$.

We let $\mathcal{M}_i$ be the set of all M_X, where X is a labeling of T_i, and every $e \in E(T_i)$ satisfies $w_X(e) \le 2$. We let $\mathcal{W}_i$ denote the set of maximal ('worst') elements (with respect to $\preccurlyeq$) of $\mathcal{M}_i$. Further, we define two subsets of these sets: $\mathcal{M}'_i$ denotes the set of menus corresponding to those labelings X of T_i where each edge is of weight at most 2 and where the root edge is marked by $\{1,2\}$. Finally, $\mathcal{W}'_i$ is the set of maximal elements of $\mathcal{M}'_i$. The following observation collects the important properties of these sets.

Observation 2.6 *For every $i \ge 2$ we have*

(1) $\mathcal{M}_i = \big\{\text{Parent\_menu}(M, N, s) \mid M, N \in \mathcal{M}_{i-1}, s \in \mathcal{P}([4]), |s| \le 2\big\}$

(2) $\mathcal{W}_i = \max\limits_{in\, \preccurlyeq}\big\{\text{Parent\_menu}(M, N, s) \mid M, N \in \mathcal{W}_{i-1}, s \in \mathcal{P}([4]), |s| \le 2\big\}$

(3) $\mathcal{W}'_i = \max\limits_{in\, \preccurlyeq}\big\{\text{Parent\_menu}(M, N, \{1,2\}) \mid M, N \in \mathcal{W}_{i-1}\big\}$

Proof Part (1) follows immediately from Observation 2.5. The second part follows from this and from the fact that the mapping Parent_menu is monotone with respect to the order $\preccurlyeq$ on menus. Part 3 follows by a similar argument. $\square$

Next we state the key claim proved by our computer check.

Claim 2.7 (verified by computer) *For every $W_1 \in \mathcal{W}_9'$, and $W_2, W_3 \in \mathcal{W}_8$ there exists $S \in \mathcal{P}([4])$ such that $W_1(S) + W_2(S) + W_3(S) < 0$.*

We use the Observation 2.6 to give a practical scheme for computing the collections $\mathcal{W}_8$ and $\mathcal{W}_9'$ followed by a simple test for each possible triple. Further details are described in the Code Listing. With this, we are finally ready to give a proof of Lemma 2.4.

Proof (of Lemma 2.4) Let X be an edge labeling of $2T_9$ as in the lemma; we may suppose the central edge uv is labeled by $\{1, 2\}$. Let T^1, T^2, T^3 be the three distinct maximal subtrees of $2T_9$ which have v as a leaf, and assume that T^1 contains the central edge. Let X_j denote the restriction of X to T^j, and write $M_j = M_{X_j}$. Choose $W_1 \in \mathcal{W}_9'$, $W_2, W_3 \in \mathcal{W}_8$ so that $M_j \preccurlyeq W_j$ holds for each j. By Claim 2.7, we may choose $S \in \mathcal{P}([4])$ for which $W_1(S) + W_2(S) + W_3(S) < 0$ and by definition of $\preccurlyeq$ we have $M_1(S) + M_2(S) + M_3(S) < 0$, too. Let X_j be the internal S-swap for which the minimum in the definition of M_j (equation (1)) is attained. Then $Y = X_1 \,\Delta\, X_2 \,\Delta\, X_3$ is an internal cut labeling of $2T_9$ and $\mathrm{cost}(X \,\Delta\, Y) - \mathrm{cost}(X) = M_1(S) + M_2(S) + M_3(S) < 0$. This completes the proof. $\square$

Remark 2.8 *In the definition of cost of a coloring, the values of parameters $a(i)$ can be chosen in a variety of ways—provided we do penalize edges of weight 1. Perhaps it seems more natural to have $a(1) = 0$ but this straightforward approach does not work. Consider the edge labeling of $2T_4$ the upper part of which is depicted in Figure 3. It is rather easy to verify, that switching any local cut labeling does not get rid of edge of weight 2. Moreover, this labeling can be extended to arbitrary $2T_n$ by the 'growing rules' depicted in the figure (a, b, c, d stand for $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ in any order). On the other hand, by switching $\{2\}$ and $\{3\}$ on the cuts depicted in the figure, we decrease the cost of the coloring by $a(1)$.*

Remark 2.9 *We note that we could prove Lemma 2.3 by the same method as Lemma 2.4; in fact a simple modification of the code verifies both of these Lemmas at the same time. The reason we put Lemma 2.3 separately is that it allows for an easy proof by hand, and this hopefully makes the proof easier to understand.*

Another remark is that an easy modification of our method of verifying Claim 2.7 could decrease the running time by 30%. We did not want to *obscure the main proof for this relatively small saving, but we wish to mention the trick here. In the process of enumerating the sets $\mathcal{W}_i$, we can throw away all menus M that satisfy $M(\emptyset) < 0$. It is not hard to show that we still consider all 'hard cases'.*

Remark 2.10 *The necessity to use computer for huge amount of checking is not entirely satisfying (although this point of view may be rather historically conditioned aesthetic criterion). It would be interesting to find a proof of Lemma 2.4 without extensive case-checking, perhaps by a careful inspection of the sets $\mathcal{W}_i$.*

3 Some Equivalences

The goal of this section is to prove Proposition 1.1 from the Introduction (restated here for convenience as Proposition 3.2), which gives several graph properties equivalent to the existence of a homomorphism to a projective cube PQ_{2k}. To prove this, it is convenient to first introduce another family of graphs. For every positive integer n, let H_n denote the graph with all binary vectors of length n forming the vertex set and with two vertices being adjacent if they agree in exactly one coordinate (note that H_n is a Cayley graph on $\mathbb{Z}_2^n$).

For odd n, the graph H_n has exactly two components, one containing all vertices with an even number of 1's, and the other all vertices with an odd number of 1's; we call the components H_n^e and H_n^o, respectively.

Observation 3.1 *For every $k \geq 1$ we have $H_{2k+1}^e \cong H_{2k+1}^o \cong PQ_{2k}$.*

Proof The mapping that sends each binary vector to its complementary vector gives an isomorphism between H_{2k+1}^o and H_{2k+1}^e. Thus, the simple graph obtained from H_{2k+1} by identifying complementary vectors is isomorphic to H_{2k+1}^e (and to H_{2k+1}^o). However, this graph is also isomorphic to PQ_{2k}, since viewing the vertices of each as a pair of complementary vectors, we see that u and v will be adjacent if and only if one vector associated with u and one vector associated with v differ in exactly 1 coordinate. □

Now we are ready to prove the proposition.

Proposition 3.2 *For every graph G and nonnegative integer k, the following properties are equivalent.*

(1) There exist $2k$ pairwise disjoint cut complements.

(2) There exist $2k+1$ pairwise disjoint cut complements with union $E(G)$.

(3) G has a homomorphism to PQ_{2k}.

(4) G has a cut-continuous mapping to C_{2k+1}.

Proof We shall show (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1).

To see that (1) $\implies$ (2), let S_1, S_2, …, S_{2k} be pairwise disjoint cut complements, and for every $1 \leq i \leq 2k$ let $W_i = E(G) \setminus S_i$. Now setting $S_{2k+1} = E(G) \setminus \cup_{1 \leq i \leq 2k} S_i = E(G) \setminus \Delta_{1 \leq i \leq 2k} W_i$ we have (2).

Next we shall show that (2) $\implies$ (3). Let S_1,S_2,…,S_{2k+1} be $2k+1$ disjoint cut complements with union $E(G)$ and for every $1 \leq i \leq 2k+1$ choose $U_i \subseteq V(G)$ so that $S_i = E(G) \setminus \delta(U_i)$. Now assign to each vertex v a binary vector x^v of length $2k+1$ by the rule $x^v_i = 1$ if $x \in U_i$ and $x^v_i = 0$ otherwise. This mapping gives a homomorphism from G to H_{2k+1}, so by Observation 3.1 we conclude that G has a homomorphism to PQ_{2k}.

Next we prove that (3) $\implies$ (4). Since the composition of two cut-continuous mappings is cut-continuous, it follows from Observation 1.7 and Observation 3.1 that it suffices to find a cut-continuous mapping from H_{2k+1} to C_{2k+1}. To construct this, let $E(C_{2k+1}) = \{e_1, e_2, \ldots, e_{2k+1}\}$ and define a mapping $g : E(H_{2k+1}) \to E(C_{2k+1})$ by the rule that $g(uv) = e_i$ if u and v agree exactly in coordinate i. We claim that g is a cut-continuous mapping. To see this, let R be a cut of C_{2k+1}, let $J = \{i \in \{1, 2, \ldots, 2k+1\} : e_i \in R\}$, and note that $|J|$ is even. Now let X be the set of all binary vectors with the property that there are an even number of 1's in the coordinates specified by J. Then $g^{-1}(R) = \delta(X)$ so our mapping is cut-continuous as required.

To see that (4) $\implies$ (1), simply note that the preimage of any edge of C_{2k+1} is a cut complement, so the preimages of the $2k+1$ edges are $2k+1$ disjoint cut complements. □

We can extract the key idea of the above proof as follows. Let $E_i \subseteq E(H_{2k+1})$ be the set of edges uv such that u and v agree in exactly the i-th coordinate.[3] The sets E_1, …, E_{2k+1} form a partition of $E(H_{2k+1})$ into disjoint cut complements.

[3]If you think of H_n as of a Cayley graph, then E_i consists of edges corresponding to the i-th element of the generating set. We thank to Reza Naserasr for this comment.

4 Code Listing

In this section we present the code used to verify Claim 2.7. The code is written in C; it can be downloaded at `http://kam.mff.cuni.cz/~samal/papers/clebsch/`. It runs about 30 minutes on a 2 GHz processor. We have tested it with compilers gcc (version 3.0, 3.3), Intel C, and Borland C++ on several computers to minimize the possibility of error in the proof due to wrong computer hardware/software.

We use Observation 2.6 to iteratively compute $\mathcal{W}_{i+1}$ from $\mathcal{W}_i$, this is accomplished by function W_update. By the same function we compute $\mathcal{W}'_9$ from $\mathcal{W}_8$. Finally, we use final_test to check whether all triples of menus satisfy the inequality of Claim 2.7. To simplify and speed up the code, we use static data structures for $\mathcal{W}_i$'s—that is, the elements of the set $\mathcal{W}_i$ are stored as W[i][j]— with a limit MAX=20000 on the number of elements, if this number turned out to be too small, the program would output an error message (this does not happen).

Marks of edges, that is elements of $\mathcal{P}([4])$ are represented as integers from 0 up to 15. For convenience variables that hold marks have type mark (which is a new name for **short**). Symmetric difference of marks corresponds to bitwise xor—"^". Cost of edges are stored in variables of type cost (a new name for **int**). From equation 2 it is easy to deduce that Parent_menu$(M, N, R)(S) \le M(S) + N(S)$. Consequently, the largest coordinate of an element of $\mathcal{W}_i$ is at most $2^{i-1}a(4)$, and as we only use sets $\mathcal{W}_i$ for $i \le 9$, we will not have to store larger numbers than an **int** can hold. Other new data types are **menu** (array of 16 cost's used to represent a menu), and **comparison**—variables of that type are assigned values -1, 0, 1, or INCOMP=2 if the result of a corresponding comparison (of two menus) is $\prec$, $=$, $\succ$ or incomparable.

When we need to compute $M =$ Parent_menu(M_1, M_2, c), this is implemented as add_menus(M_1,M_2,child); p_menu(child, parent, M). Here child corresponds to the sum $M_1 + M_2$, parent is a menu corresponding to coloring of the single edge of T_1 by color c. Then we insert the menu in the set $\mathcal{W}_i$ (array W[i]) by calling insert_menu. Note that if did implement the deletion of 'small' menus in this function in a more straightforward manner ('move everything left'), the running time would approximately double.

```
#include <stdio.h>
#define MAX 20000      // limit on size of the sets W_i

typedef short mark;
typedef int cost;
typedef cost menu[16];

cost a[5]={0,1,10,40,1000};
cost markcost[16]; // cost of edge marked by each
                   // possible mark
menu one_mark[1]; // W'_1, i.e. one_mark[0] corresponds
                  // to T1 marked by {1,2}
menu W[9][MAX];
menu Wprime[MAX]; // W'_9
int Wsize[10]; // Wsize[i] is the number of elements of W[i]
int Wprimesize; // the number of elements of Wprime

typedef short comparison;
comparison INCOMP = 2;

void menu_from_mark(mark Q, menu M) {
// M will be the menu corresponding to T1 marked by Q
  mark s;
  for(s=0; s<16; s++)
    M[s] = markcost[Q ^ s] - markcost[Q];
}

void init_variables() {
  mark s;
  for (s=0; s<16; s++)
     markcost[s] = a[(s&1) + ((s>>1) & 1) +
                   ((s>>2) & 1) + ((s>>3)&1)];
// the right hand side is a[n], where n is the number
// of ones in binary representation of s

  menu_from_mark(3,one_mark[0]);

  Wsize[1]=0;
  for(s=0; s<16; s++)
     if (markcost[s] < a[3])
        menu_from_mark(s,W[1][++Wsize[1]]);
}
```

```
void add_menus(menu M1, menu M2, menu sum) {
  mark s;
  for (s=0; s<16; s++)
    sum[s] = M1[s]+M2[s];
}

comparison sign(int n) {
  if (n > 0) return 1;
  if (n < 0) return -1;
  return 0;
}

comparison compare_menus(menu M1, menu M2) {
// returns -1, 0, 1, INCOMP, depending on
// whether M1<M2, M1=M2, M1 > M2, or they are incomparable
  mark s;
  comparison t, current=0;

  for (s=0; s < 16; s++) {
    t = sign (M1[s] - M2[s]);
    if ((t != 0) && (t == -current)) return INCOMP;
    if (current == 0) current = t;
  }
  return current;
}

void p_menu(menu child, menu parent, menu output) {
// child is the sum of two childs,
// parent corresponds to the new edge
  mark s, q;
  cost new, current_best;

  for (s=0; s<16; s++) {
    current_best = a[4];
    for (q = 0; q < 16; q++) {
      new = child[q] + parent[s ^ q];
      if (new < current_best) current_best = new;
    }
    output[s] = current_best;
  }
}
```

```
void insert_menu(menu *book, int *booksize, menu M) {
// book is an array of menus, booksize is the number
// of elements of book, we are inserting M
  int i;
  mark s;
  comparison t=0;

  for (i=0; i < *booksize; i++) {
    t = compare_menus(M, book[i]);
    if (t <= 0) return;   // we will not insert small menu
    if (t == 1) break;    // we will delete book[i]
  }

// we will delete all elements of book that are <= M
  if (t==1)   // i.e. M > book[i]
    for ( ; i < *booksize; i++) {
      while (i < *booksize &&
                  compare_menus(M, book[i])==INCOMP)
        i++;
      while (*booksize > i &&
              compare_menus(book[*booksize-1],M) <= 0 )
        (*booksize)--; // we abandon small menus at the end
      if (*booksize <= i) break;
    // and move big menu from end to the place of book[i]:
      (*booksize)--;
      for (s = 0; s<16; s++)
        book[i][s] = book[*booksize][s];
    }

// we insert M as the last element of book
  if (*booksize == MAX) printf("too short array!\n");
  else {
    for (s = 0; s<16; s++)
      book[*booksize][s] = M[s];
    (*booksize)++;
  }
}

void W_update(menu *oldW, int oldsize, menu *root_edge,
              int rootsize, menu *newW, int *newsize) {
  menu N, child;
```

```
  int i, j, k;

  *newsize = 0;
  for (i=0; i < oldsize; i++)
    for (j=i; j < oldsize; j++) {
      add_menus(oldW[i],oldW[j],child);
      for (k=0; k < rootsize; k++) {
        p_menu(child,root_edge[k],N);
        insert_menu(newW, newsize, N);
      }
    }
}


int final_test(menu *C, int Csize, menu *P, int Psize) {
  int i, j, k;
  mark s;
  int counter=0;
  menu child;

  for (i=0; i < Csize; i++)
    for (j=i; j < Csize; j++) {
      add_menus(C[i],C[j],child);
      for (k=0; k < Psize; k++) {
        counter ++;
        for (s=0; s<16; s++)
          if (child[s]+P[k][s] < 0) { counter--; break;}
        // Claim 2.7 holds for C[i],C[j],P[k]
        // we proceed by testing another triple
      }
    }
  return counter;
}

int main() {
  int i;
  init_variables();

  for (i=1; i<8; i++) {
    W_update(W[i],Wsize[i],W[1],Wsize[1],W[i+1],&Wsize[i+1]);
    printf("The size of W%d is: %d\n",i+1,Wsize[i+1]);
  }
```

```
  W_update(W[8], Wsize[8], one_mark, 1, Wprime, &Wprimesize);

  if (final_test(W[8], Wsize[8], Wprime, Wprimesize) == 0)
    printf("\nProof is finished.\n\n");

  return 0;
}
```


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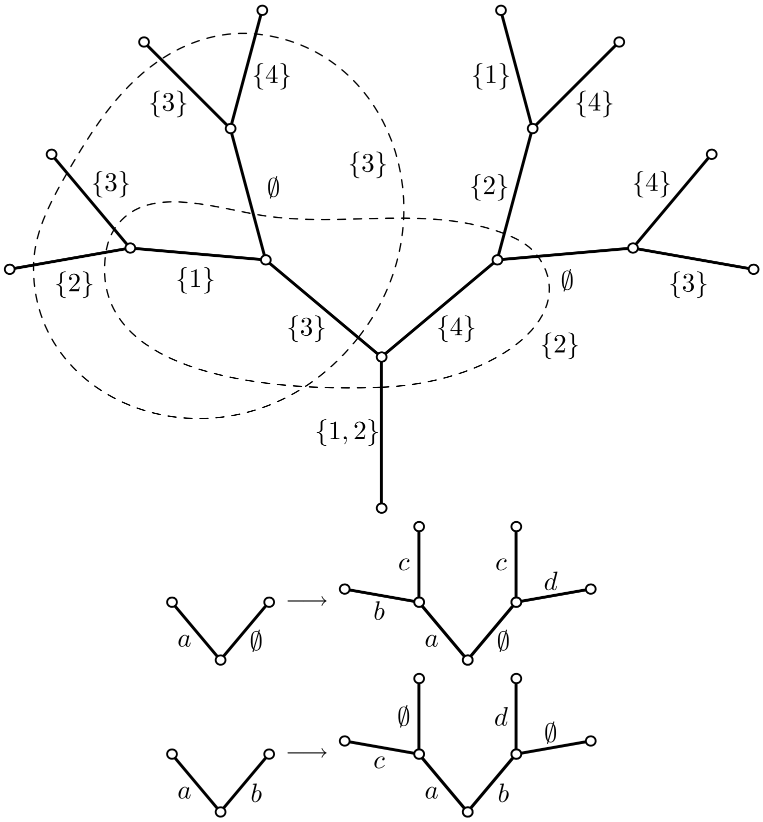

Figure 3: A difficult labeling of T_4.