Graph Labellings with Variable Weights, a Survey

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Abstract

Graph labellings form an important graph theory model for the channel assignment problem. An optimum labelling usually depends on one or more parameters that ensure minimum separations between frequencies assigned to nearby transmitters. The study of spans and of the structure of optimum labellings as functions of such parameters has attracted substantial attention from researchers, leading to the introduction of real number graph labellings and λ-graphs. We survey recent results obtained in this area.

The concept of real number graph labellings was introduced a few years ago, and in the sequel, a more general concept of λ-graphs appeared. Though the two concepts are quite new, they are so natural that there are already many results on each. In fact, even some older results fall in this area, but their authors used a different mathematical language to state their achievements. Since many of these results are so recent that they are just appearing in various journals, we would like to offer the reader a single reference for the state of art as well as to draw attention to some older results that fall in this area.

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1 Graph labellings and the channel assignment problem

Graph theory models for radio frequency assignment problems can be traced back to the early 1980's in the paper of Hale [27]. Since then, several different forms of graph colorings were developed to model such problems, e.g., the model of T-colorings of graphs, which forbids certain differences between labels at adjacent vertices [42].

Based on a transmitter frequency problem related to him by Lanfear, in 1988 Roberts proposed a new assignment problem with two levels of interference [43] which Griggs adapted to graphs and extended to a more general graph problem of distance-constrained labellings [26]: For nonnegative integers $p_1, \ldots, p_k$, an $L(p_1, \ldots, p_k)$*-labelling* of a graph G is a labelling of its vertices by nonnegative integers such that vertices at distance exactly i receive labels that differ by at least p_i. The maximum label assigned to any vertex is called the *span* of the labelling. The goal of the problem is to construct an $L(p_1, \ldots, p_k)$-labelling of the smallest span. The smallest span of such a labelling is denoted by $\lambda_{p_1,\ldots,p_k}(G)$. Because of practical applications, the distance constraints are often considered to decrease with the distance [5], i.e., $p_1 \geq p_2 \geq \cdots \geq p_k \geq 1$. However, there also appears in practical applications the case $p_1 = 0, p_2 = 1$ [4, 29].

The idea behind this model is the following: radio-transmitters are represented by vertices of a graph and those that are very close are joined by edges. The highest level of interference appears among transmitters represented by adjacent vertices. However, some interference still appears among transmitters represented by vertices at distance two, three, etc. In order to assign frequencies efficiently from the available range, we seek an assignment of frequencies from the shortest possible interval such that the frequencies assigned to very close transmitters differ a lot (in order to avoid interference), while the frequencies assigned to transmitters that are close but not very close differ less. This idea directly leads to the distance-constrained labellings of graphs as introduced in the previous paragraph.

Distance-constrained labellings are closely related to ordinary graph colorings: If $p_1 = \ldots = p_k = 1$, then the problem reduces to the coloring of the k-th power of the graph G. Hence, many results on colorings of graph powers translate to distance-constrained labellings and vice versa. As an example, the reader is referred to the work of Molloy and Reed [40, 41] on colorings of squares of planar graphs.

The fundamental case of distance-constrained labelling (based on the problem of Lanfear and Roberts) is when $p_1 = 2$ and $p_2 = 1$. A major open problem on $L(2,1)$-labellings of graphs is the conjecture of Griggs and Yeh [26] that asserts that every graph G with maximum degree $\Delta \geq 2$ has an $L(2,1)$-labelling of span at most Δ^2. The conjecture is still open almost 15 years after it was published. In a series of papers [26, 12, 35, 19], the original upper bound $\Delta^2 + 2\Delta$ on $\lambda_{2,1}(G)$ of such graphs G has been decreased successively to the current best bound of Gonçalves [19], $\Delta^2+\Delta-2$. The conjecture was verified for several special classes of graphs, including graphs of maximum degree two, chordal graphs ([45], see also [11, 32]), Hamiltonian cubic graphs [30] and planar graphs with maximum degree $\Delta \neq 3$ [3]. Because of practical applications of the distance-constrained labelling, it is not surprising that there is also a growing body of papers on their algorithmic aspects [1, 6, 14, 15, 31, 38].

McDiarmid's survey [37] deals with a general version of the *channel assignment problem*, which is described by a graph G in which each edge e is assigned a positive integer weight (separation) $w(e)$. We consider labellings c of the vertices of G with positive integers such that the labels of adjacent vertices u and v differ by at least $w(uv)$. The *span* of the labelling c is its maximum label. The goal is to find a labelling of minimum span. When the weights of all edges equal one, then the minimum span is just the chromatic number of G.

It is not hard to see that distance-constrained labellings of graphs with constraints at distance at most k can be viewed as a special instance of the channel assignment problem with the underlying graph being the k-th power of the original graph G.

To capture better some of the properties of distance-constrained labellings as a function of the separations p_i, Griggs et al. was led to introduce the more general model of *real number* labellings with distance constraints [22]. An even more general model, λ-graphs, was later proposed by Babilon et al. [2], which is a special case of the channel assignment problem, in which there are only k possible edge weights, except that the labels and weights are real numbers, not integers. Theoretical results for distance-constrained labellings can be extended to λ-graphs, and several open conjectures on distance-constrained labellings were proven in the more general λ-graph context.

We introduce these real number labellings in the next section, and survey the main results about them in the following sections, including the general theoretical results as well as formulas for specific finite and infinite graphs.

The paper concludes with some of the prominent open problems.

2 Labellings with variable weights

Graph labellings with real numbers were first mentioned in the paper of Griggs and Yeh [26]. They observed that the study of $L(2d,d)$-labellings of graphs for a real number d is equivalent to the study of $L(2,1)$-labellings by the Scaling Property that we recall later. Subsequently, people worked out the optimal $L(p,q)$-spans of specific graphs as functions of (integers) p and q. Among there, Georges and Mauro [18] determine spans of optimal $L(p,q)$-labellings of infinite d-regular trees, and van den Heuvel et al. [28] and Leese and Noble [36] provide results for the circular version of the problem. A natural generalization of distance-constrained labellings based on ideas above is the notion of *real number graph labellings* introduced by Griggs and Jin [22].

An $L(p_1,\ldots,p_k)$-labelling of a (possibly infinite) graph G for real numbers $p_1,\ldots,p_k$ is a function $f:V(G)\to\mathbb{R}$ such that $|f(v)-f(w)|\ge p_i$ for any two vertices v and w at distance exactly i in G. The *span* of the $L(p_1,\ldots,p_k)$-labelling f is equal to the difference of the supremum and the infimum of the labels, i.e., $\sup_{v\in V(G)} f(v)-\inf_{v\in V(G)} f(v)$. The infimum of the span of all $L(p_1,\ldots,p_k)$-labellings of G is denoted by $\lambda(G;p_1,\ldots,p_k)$. It can be shown that there always exists an *optimum* $L(p_1,\ldots,p_k)$-labelling, i.e., a labelling of span $\lambda(G;p_1,\ldots,p_k)$. Let us remark that if all the parameters $p_1,\ldots,p_k$ are integers, then $\lambda_{p_1,\ldots,p_k}(G)=\lambda(G;p_1,\ldots,p_k)$ by Theorem 3 below.

By the Compactness Principle, if the maximum degree of a graph G is bounded, there exists a labelling of finite span. On the other hand, if $k\ge 2$ and G contains vertices of arbitrarily large degree, then G need not have a labelling of finite span. Since the labels of an optimum labelling f can be made nonnegative by subtracting $\inf_{v\in V(G)} f(v)$ from the label of each vertex, there is always an optimum labelling with nonnegative reals. Hence, we can require the labels $f(v)$ of the vertices to be nonnegative reals and define the span of the labelling to be the supremum of the labels used in the labelling. The spans of optimum labellings with nonnegative reals coincide with $\lambda(G;p_1,\ldots,p_k)$.

The values of $\lambda(G;p_1,\ldots,p_k)$ can be viewed as a function of the parameters $p_1,\ldots,p_k$. Griggs and Jin [22] showed that $\lambda(G;p_1,\ldots,p_k)$ is a continuous *piecewise linear* function of its parameters on $[0,\infty)^k$. We say that a function is *piecewise linear*, if there exists a partition of its domain into (possibly infinitely many) measurable parts such that the function is linear on each of them.

The function $\lambda(G; p_1, \ldots, p_k)$ also satisfies the so-called Scaling Property [22], i.e., for every positive real number α:

$$\alpha\lambda(G; p_1, \ldots, p_k) = \lambda(G; \alpha p_1, \ldots, \alpha p_k) \ .$$

In particular, if $k = 2$, the values of $\lambda(G; p_1, p_2)$ are fully determined by the values of the function for $p_2 = 1$. Hence, we often describe only the values of $\lambda(G; x, 1)$ instead of giving the entire description of the function $\lambda(G; x, y)$.

The instances of the channel assignment problem derived from distance-constrained labellings of graphs are of special structure. However, there is no need to restrict ourselves to the channel assignment problems only of this type. Similarly, as real number graph labellings generalize distance-constrained labellings of graphs, we can generalize the notion of the channel assignment problem. A *λ-graph* G is a graph with k types of edges. We allow two vertices to be joined by several edges of different types. A labelling f of the vertices of the λ-graph G with nonnegative real numbers is said to be *proper* with respect to the parameters $x_1, \ldots, x_k$ if the labels of every pair of vertices u and v joined by an edge of the i-th type differ by at least x_i. The *span* of the labelling is the supremum of the labels of the vertices. The infimum of the spans of proper labellings is denoted by $\lambda_G(x_1, \ldots, x_k)$. The values of $\lambda_G(x_1, \ldots, x_k)$ viewed as a k-parameter function form the *λ-function* of G.

It is important to note for the remainder of the paper, we will always implicitly assume that for every choice of the parameters $x_1, \ldots, x_k$, the value of the function $\lambda_G(x_1, \ldots, x_k)$ is finite. As we discuss later, this is equivalent to the statement that G (viewed as an ordinary graph) can be colored (in the ordinary sense) with a finite number of colors.

The results of [22] on real number graph labellings readily translate to the more general setting of λ-graphs:

Theorem 1. *Let G be a (possibly infinite) λ-graph with k types of edges. The λ-function of G is a continuous, non-decreasing and piecewise linear function of $x_1, \ldots, x_k$ on $[0, \infty)^k$.*

It is not hard to see that the λ-function also satisfies the Scaling Property:

Theorem 2 (Scaling Property). *Let G be a (possibly infinite) λ-graph with k types of edges and $\alpha, x_1, \ldots, x_k$ nonnegative reals. It holds that:*

$$\alpha\lambda_G(x_1, \ldots, x_k) = \lambda_G(\alpha x_1, \ldots, \alpha x_k) \ .$$

Because of the Scaling Property, the function $\lambda_G(x_1, \ldots, x_k)$ is finite for all values of $x_1, \ldots, x_k$ if and only if it is finite for one choice of positive values of $x_1, \ldots, x_k$. Since $\lambda_G(1, \ldots, 1)$ is equal to the chromatic number of (the underlying graph of) G decreased by one, our assumption that the function λ_G is well-defined is equivalent to the assumption that G has a finite chromatic number.

The notion of λ-graphs provides us with a more general framework for the study of real number graph labellings. Consider a graph H and reals $p_1, \ldots, p_k$. Let us consider the following λ-graph G with k types of edges: the vertices of G are the same as those of H and two vertices u and v are joined by an edge of the i-th type, $1 \le i \le k$, if their distance in H is exactly i. It is easy to see that the following holds:

$$\lambda_G(x_1, \ldots, x_k) = \lambda(H; x_1, \ldots, x_k) \ .$$

Because of this close connection, we decided to use similar notations for the functions describing spans of optimum labellings of λ-graphs and optimum real number graph labellings.

In the rest of this paper, we survey results obtained on real number graph labellings and λ-graphs. We start with general results obtained in this area and we then focus on results obtained for specific infinite and finite graphs.

3 General structural results

In this section, we survey general results on λ-graphs and real number graph labellings. One of the first questions that comes to mind is whether the sets of labels used in an optimum labelling can be assumed to be of some special form. The answer to this question was provided in the paper [22]. They define the *D-set* $D(x_1, \ldots, x_k)$ of $x_1, \ldots, x_k$ to be the set of all combinations of $x_1, \ldots, x_k$ with non-negative integer coefficients. Let us remark at this point that $\mathbb{N}$ denotes the set of non-negative integers throughout the paper.

$$D(x_1, \ldots, x_k) = \left\{ \sum_{i=1}^{k} \alpha_i x_k \text{ for some } \alpha_i \in \mathbb{N} \right\} \ .$$

The next theorem of [22] was proved for real number graph labellings, but its proof readily translates to the setting of λ-graphs (with finite spans):

Theorem 3 (Griggs and Jin [22]). *Let G be a (possibly infinite) λ-graph with k types of edges. For all nonnegative real numbers $x_1, \ldots, x_k$, there is an optimal labelling f with respect to $x_1, \ldots, x_k$, such that $f(v) = 0$ for some vertex and for all vertices v, $f(v) \in D(x_1, \ldots, x_k)$. In particular, $\lambda_G(x_1, \ldots, x_k) \in D(x_1, \ldots, x_k)$.*

Theorem 3 can also be derived from a later theorem of Babilon et al. [2]: they established an analogue of the classical Gallai-Roy Theorem for the channel assignment problem for an infinite underlying graph with finitely many edge-weights. Recall that the Gallai-Roy Theorem [16, 44] states that the chromatic number of a graph G decreased by one is equal to the length (the number of edges) of a longest oriented path of an orientation of G for which the length of a longest path is minimized. We state the result of [2] using the language of λ-graphs. An orientation of a graph G is said to be *finitary* if there is a constant $K \geq 0$ such that every oriented walk has length at most K. In particular, a finitary orientation is acyclic and does not contain infinite oriented paths. Note that there could be an acyclic orientation without infinite oriented paths that is not finitary. The *weight of a path* is the sum of the weights of its edges. Finally, the *weight of an orientation* is the supremum of the weights of its oriented paths. Note that if the orientation of a λ-graph is finitary, the supremum is always attained and is finite (since G has only finitely many edge types). Moreover, it can be shown that there exists a finitary orientation of minimum weight, and its weight is equal to the span of an optimum labelling:

Theorem 4 (Babilon et al. [2]). *Let G be a (possibly infinite) λ-graph with k types of edges. The optimal span of the labelling of G with respect to $x_1, \ldots, x_k$ is equal to the minimum weight of a finitary orientation of G.*

Let us remark that the proof of Theorem 4 involves the Axiom of Choice.

Given a finitary orientation of the minimum weight, it is easy to construct an optimum labelling of a λ-graph G: let $f(v)$ of a vertex v be the maximum weight of an oriented path that ends at v. It is straightforward to check that the labelling f is a proper labelling of G and that its span is equal to the weight of the orientation. By Theorem 4, the labelling f is optimum.

We already know that the λ-function of any λ-graph is a continuous piecewise linear function. A natural question is whether the λ-function is always comprised of only finitely many linear parts. Griggs and Jin [22] proposed that this is true for real number graph labellings (the *Piecewise Linearity Conjecture*). They verified their conjecture for finite graphs, and got more support for the conjecture by verifying it for infinite graphs with conditions at distance at most two. The proof of the conjecture for λ-graphs with two types of edges is implicitly contained in [2]. The conjecture was eventually proved (for general k) by Král' [33] in the more general setting of λ-graphs. Moreover, he proved it in the following stronger form (we state the result as Theorem 5 later): not only the λ-function of each λ-graph is a piecewise linear function comprised of finitely many parts, but, for every fixed $k \geq 1$ and $\chi \geq 1$, there exists a single finite partition of $[0, \infty)^k$ such that the λ-function of *every* λ-graph with k types of edges and chromatic number at most χ is linear on every part of the partition.

Griggs and Jin [22] also conjectured that Theorem 3 can be refined in the sense that it is enough to consider combinations of the parameters with nonnegative integer coefficients that do not exceed a constant depending only on G and not the separation parameters x_i themselves (the *Coefficient Bound Conjecture*). Let us state their conjecture more formally. Let $D(A; x_1, \ldots, x_k)$ to be the set of all numbers of the form $\sum_{i=1}^{k} \alpha_i x_i$ for some integers α_i, $0 \leq \alpha_i \leq A$. The conjecture asserts that for every graph G and every integer $k \geq 1$, there exists a number A such that for every choice of $x_1, \ldots, x_k$, there is an optimal $L(x_1, \ldots, x_k)$-labelling f of G with labels $f(v) \in D(A; x_1, \ldots, x_k)$. Griggs and Jin [22] made the stronger conjecture that the number A could be chosen depending only on k and the maximum degree Δ of G, not on G itself (the *Delta Bound Conjecture*).

Both the Coefficient Bound Conjecture and the Delta Bound Conjecture were proven in [33] in a stronger form: first, the proof works in the more general setting of λ-graphs, and second, it works more generally for graphs with bounded chromatic number χ, not just for those with bounded maximum degree Δ (note that $\chi \leq \Delta + 1$). Let us now state the result.

Theorem 5 (Král' [33]). *For every $k \geq 1$ and $\chi \geq 1$, there exist constants $A_{k,\chi}$ and $B_{k,\chi}$ such that the space $[0, \infty)^k$ can be partitioned into $B_{k,\chi}$ polyhedral cones, such that the λ-function $\lambda_G(x_1, \ldots, x_p)$ of every λ-graph G (possibly infinite) with k types of edges and chromatic number at most χ is a linear function of $x_1, \ldots, x_p$ on each of the cones.*

Moreover, for each G and each of the cones, there exist linear functions $f_v(x_1, \ldots, x_k)$ such that a vertex labelling of G assigning a vertex v the value of $f_v(x_1, \ldots, x_k)$ is an optimal labelling of G, and $f_v(x_1, \ldots, x_k) \in$

$D(A_{k,\chi}; x_1, \dots, x_k)$.

Even more surprising is the following consequence concerning the number of possible λ-functions of λ-graphs.

Theorem 6 (Král' [33]). *There exist only finitely many (piecewise-linear) functions that can be the λ-function of a λ-graph with at most k types of edges and chromatic number at most χ.*

The key ingredient of the proofs of Theorem 5 and 6 is Theorem 4. It is shown that for every fixed k and χ, the length of the longest oriented path of a minimum-weight finitary orientation of G can be bounded by a constant $C_{k,\chi}$ depending only on k and χ. However, the constant $C_{k,\chi}$ grows enormously in k. In particular, it is not even bounded by a tower function of k. We believe that the bounds obtained in [33] are far from optimal and can be improved. In particular, we think that $C_{k,\chi}$ can be bounded by a function exponential in k and χ and maybe even polynomial in one of the parameters (see Problem 1 in Section 6).

Besides Theorems 5 and 6, the technique used in [33] provides the following generalization of the Compactness Principle to λ-graphs.

Theorem 7 (Král' [34]). *Every λ-graph G with k types of edges and a finite chromatic number contains a finite subgraph H such that $\lambda_G(x_1, \dots, x_k) = \lambda_H(x_1, \dots, x_k)$ for all $(x_1, \dots, x_k) \in [0, \infty)^k$.*

Note that the Compactness Principle implies the existence of such a finite subgraph H for every choice of $(x_1, \dots, x_k)$, but it does not guarantee the existence of such a universal finite subgraph H. Also note Theorem 7 implies the Piecewise Linearity Conjecture of Griggs and Jin.

We mentioned that the constants $A_{k,\chi}$ and $B_{k,\chi}$ in Theorem 5 are really huge. Substantially better bounds than these are known for real number graph labellings with conditions at distance at most two, as well as for λ-graphs with two types of edges. Here is a refinement of Theorem 5 for real number graph labellings from [22].

Theorem 8 (Griggs and Jin [22]). *Let G be a (possibly infinite) graph G with maximum degree Δ. For all nonnegative real numbers $x_1, \dots, x_k$, there is an optimal labelling f with respect to $x_1, \dots, x_k$ such that $f(v) \in D(2\Delta^5; x_1, \dots, x_k)$ for every vertex v of G. In particular, $\lambda_G(x_1, \dots, x_k) \in D(2\Delta^5; x_1, \dots, x_k)$.*

Theorem 5 can be refined for λ-graphs G with two types of edges using the next lemma:

Lemma 9 (Babilon et al. [2]). *Let G be a (possibly infinite) λ-graph G with two types of edges and let Δ be the maximum degree of G. The λ-function is a linear function on the set* $\left[0, \frac{1}{2\Delta^2+2\Delta+1}\right] \times \{1\}$.

By the Scaling Property, the number of linear parts of the λ-function on $[0,\infty)^2$ is equal to the number of its linear parts on $[0,\infty) \times \{1\}$. Fix a λ-graph G with maximum degree Δ, and let $h(x) = \lambda_G(x, 1)$. By Lemma 9, the function h is linear on the intervals $\left[0, \frac{1}{2\Delta^2+2\Delta+1}\right] \times \{1\}$ and $[2\Delta^2+2\Delta+1, \infty) \times \{1\}$. By the Compactness Principle, $h(1) \leq \Delta$. An argument used in the proof of Theorem 4.5 in [2] yields that every linear piece of h starts and ends at a point of the form α/β where $\alpha + \beta \leq \Delta(2\Delta^2 + 2\Delta + 1) = O(\Delta^3)$. Since there are at most $O(\Delta^6)$ such points, the following refinement of Theorem 5 holds:

Theorem 10. *Let $\Delta \geq 1$ be a fixed integer. The space $[0,\infty)^2$ can be partitioned into at most $s = O(\Delta^6)$ parts $S_1, \ldots, S_s$ such the λ-function of every λ-graph G with maximum degree Δ is a linear function of x_1 and x_2 on each S_i, $i = 1, \ldots, s$.*

To conclude, we briefly mention bounds on the number of linear parts of finite λ-graphs in terms of their orders.

Theorem 11 (Babilon et al. [2]). *Let G a λ-graph of order n with two types of edges. The number of linear parts of its λ-function does not exceed $O(n^2)$.*

As we discuss in Section 6 before Problem 5, the functions $\lambda(G; x, 1)$ for most graphs G have the "up-down" behavior. The opposite type of behavior is that the λ-function is either convex or concave. In such a case, the bound of Theorem 11 can be significantly improved.

Theorem 12 (Babilon et al. [2]). *Let G a λ-graph of order n that has two types of edges such that its λ-function is convex. The number of linear parts of λ_G does not exceed $O(n^{2/3})$. Moreover, there exists a λ-graph G of order n such that λ_G is a convex function with $\Omega(n^{2/3})$ linear parts.*

Babilon et al. [2] conjecture that the bound on the number of parts contained in Theorem 11 can be improved to a linear function of n (see Problem 2 in Section 6).

4 Results on specific finite graphs

Determining optimal spans of real number graph labellings for particular graphs is also important. Though determining the spans even for small graphs is quite challenging, there is already a large family of graphs for which spans of their optimal real number labellings are known. For the reader's convenience, we include some of them here. For paths and cycles, Georges and Mauro [17] worked out the values of $\lambda(G; x_1, x_2)$ for integers $x_1 \geq x_2 \geq 1$, which can be extended by the Scaling Property and continuity to give $\lambda(G; x, 1)$ for all reals $x \geq 1$. The results of [17] are extended to $x < 1$ and proven in the real number model in [23]. The new methods could also be helpful for computing the optimal spans of real number graph labellings of other graphs.

Theorem 13 (Griggs and Jin [23], cf. [17]). *The following values are spans of optimal real number graph labellings of the path P_n, $2 \leq n \leq 6$:*

$$\lambda(P_n; x, 1) = \begin{cases} x, & \text{if } n = 2, \\ 1, & \text{if } n = 3 \text{ and } 0 \leq x \leq 1/2, \\ 2x, & \text{if } n = 3 \text{ and } 1/2 \leq x \leq 1, \\ x+1, & \text{if } n = 3 \text{ and } x \geq 1, \\ x+1, & \text{if } n = 4, \\ x+1, & \text{if } n \in \{5,6\} \text{ and } 0 \leq x \leq 1, \\ 2x, & \text{if } n \in \{5,6\} \text{ and } 1 \leq x \leq 2, \text{ and} \\ x+2, & \text{if } n \in \{5,6\} \text{ and } x \geq 2. \end{cases}$$

For paths P_n with $n \geq 7$ vertices, the following values are spans of optimal labellings:

$$\lambda(P_n; x, 1) = \begin{cases} x+1, & \text{if } 0 \leq x \leq 1/2, \\ 3x, & \text{if } 1/2 \leq x \leq 2/3, \\ 2, & \text{if } 2/3 \leq x \leq 1, \\ 2x, & \text{if } 1 \leq x \leq 2, \text{ and} \\ x+2, & \text{otherwise } (x \geq 2). \end{cases}$$

The function $\lambda(P_n; x, 1)$ is depicted in Figure 1.

Similarly as for paths, the values of $\lambda(C_n; x, 1)$ for cycles C_n can be inferred for $x \geq 1$ from the paper of Georges and Mauro [17]. In [23], the results were extended to $x < 1$ and fit in the scenario of the real number graph labellings. Note that unlike in the case of paths, the optimal span of a cycle depends for large cycles also on the congruence class of its length modulo twelve.

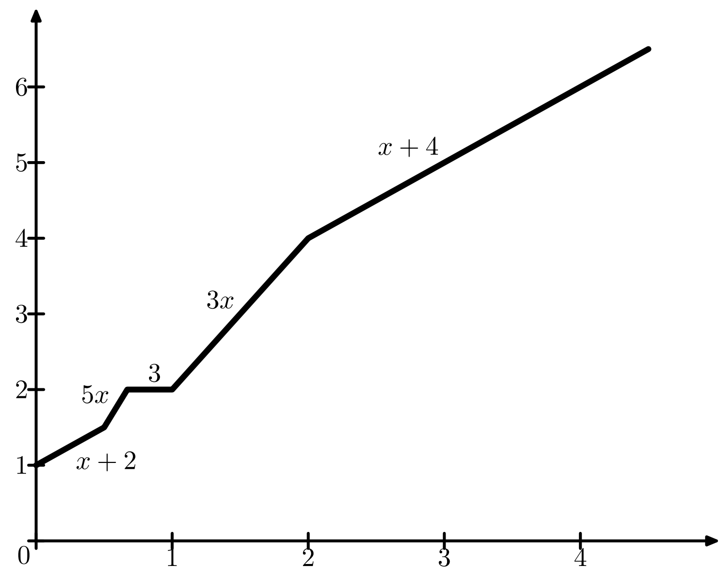

Figure 1: The function $\lambda(P_n; x, 1)$ for paths P_n with at least seven vertices ($n \geq 7$).

Theorem 14 (Griggs and Jin [23], cf. [17]). *The following values are spans of optimal labellings of the cycle C_3 and C_5:*

$$\lambda(C_n; x, 1) = \begin{cases} 2k, & \text{if } n = 3, \\ 2, & \text{if } n = 5 \text{ and } 0 \le x \le 1/2, \\ 4x, & \text{if } n = 5 \text{ and } 1/2 \le x \le 1, \\ 4, & \text{if } n = 5 \text{ and } 1 \le x \le 2, \text{ and} \\ 2x, & \text{otherwise } (n = 5 \text{ and } x \ge 2). \end{cases}$$

The values of spans of optimal labellings of the cycle C_n for $n = 4$ or $n \ge 6$ are given in the following table:

$\lambda(C_n; x, 1)$	$n \equiv_{12} 0$	$n \equiv_{12} 1, 5, 7, 11$	$n \equiv_{12} 2, 10$	$n \equiv_{12} 3, 9$	$n \equiv_{12} 4, 8$	$n \equiv_{12} 6$
$x \in [0, 1/2]$	$x + 1$	2	2	2	$x + 1$	2
$x \in [1/2, 2/3]$	$3x$	2	2	2	$3x$	2
$x \in [2/3, 1]$	2	$3x$	$3x$	2	$3x$	2
$x \in [1, 2]$	$2x$	$x + 2$	$x + 2$	$2x$	$x + 2$	$2x$
$x \in [2, 3]$	$x + 2$	$2x$	$2x$	$2x$	$x + 2$	$2x$
$x \in [3, \infty)$	$x + 2$	$2x$	$x + 3$	$2x$	$x + 2$	$x + 3$

Another class of graphs for which the optimal spans of real number labellings are known is the class of wheels (a wheel W_n is the graph obtained from the cycle C_n by adding a new vertex adjacent to all the vertices of the cycle). In this case, the values of optimal spans for large wheels depend only on the parity of the base cycle.

Theorem 15 (Griggs and Jin [23]). *The following values are spans of optimal labellings of the wheels W_3 and W_4:*

$$\lambda(W_n; x, 1) = \begin{cases} 3x, & \text{if } n = 3, \\ x + 1, & \text{if } n = 4 \text{ and } 0 \le x \le 1/3, \\ 4x, & \text{if } n = 4 \text{ and } 1/3 \le x \le 1, \text{ and} \\ 2x + 2, & \text{otherwise } (n = 4 \text{ and } x \ge 1). \end{cases}$$

For odd wheels W_n, $n \ge 5$, the spans of optimal labellings are given by the following formula:

$$\lambda(W_n; x, 1) = \begin{cases} \frac{n-1}{2}, & \text{if } 0 \le x \le 1/3, \\ 3x + \frac{n-3}{2}, & \text{if } 1/3 \le x \le 1/2, \\ nx, & \text{if } 1/2 \le x \le 1, \\ x + n - 1, & \text{if } 1 \le x \le \frac{n-1}{2}, \text{ and} \\ 3x, & \text{otherwise } (x \ge \frac{n-1}{2}). \end{cases}$$

Finally, for even wheels W_n, $n \geq 6$, we have:

$$\lambda(W_n; x, 1) = \begin{cases} x + n/2 - 1, & \text{if } 0 \leq x \leq 1/3, \\ 4x + n/2 - 2, & \text{if } 1/3 \leq x \leq 1/2, \\ nx, & \text{if } 1/2 \leq x \leq 1, \\ x + n - 1, & \text{if } 1 \leq x \leq n/2 - 1, \text{ and} \\ 2x + n/2, & \text{otherwise } (x \geq n/2 - 1). \end{cases}$$

Other natural classes of graphs to be considered are complete graphs and complete multipartite graphs. The function $\lambda(K_n; x, 1) = (n-1)x$ for complete graphs is straightforward to determine. A little more complex situation is for complete bipartite graphs:

Theorem 16 (Griggs and Jin [25]). *The following values are spans of optimal labellings of the complete bipartite graph K_{n_1,n_2}, $n_1 \geq n_2$:*

$$\lambda(K_{n_1,n_2}; x, 1) = \begin{cases} \max\{n_1 - 1, n_2 - 1 + x\}, & \text{if } x \in [0, 0.5], \\ (2n_2 - 1)x + \max\{n_1 - n_2 - 1 + x, 0\}, & \text{if } x \in [0.5, 1], \\ x + n_1 + n_2 - 2, & \text{otherwise.} \end{cases}$$

There are many other graphs G for which the values of the function $\lambda(G; x_1, \ldots, x_k)$ are still not determined. It is also of interest to compute "maximal" spans for important classes of graphs, such as planar graphs with bounded degree. More precisely, for a class $\mathcal{G}$ of graphs G, let us define $\lambda(\mathcal{G}; x_1, \ldots, x_k)$ as follows:

$$\lambda(\mathcal{G}; x_1, \ldots, x_k) = \sup_{G \in \mathcal{G}} \lambda(G; x_1, \ldots, x_k) .$$

The problem is then to determine the values of the function $\lambda(\mathcal{G}; x_1, \ldots, x_k)$ for a given class $\mathcal{G}$ of graphs. Unfortunately, it seems that a complete solution even for planar graphs with a fixed bounded degree $\Delta \geq 3$ is completely out of reach of the present methods.

5 Results on specific infinite graphs

In practical applications, infinite graphs often provide a convenient model for the underlying topology of the network. Instead of considering a finite graph, one can model a network as a regular tiling of the plane. Hence, the infinite triangular lattice $\Gamma_\triangle$, the infinite square lattice $\Gamma_\square$ and the infinite hexagonal lattice Γ_H naturally appear in such applications. However, let us start with the "simplest" infinite regular graph, an infinite regular tree. Real number graph labellings of infinite d-regular trees have already been studied in the framework of distance-constrained labellings of graphs: Georges and Mauro [18] determined optimum spans of infinite d-regular trees T_d for $x \geq 1$ and Calamoneri et al. [10] completed the characterization for $x \in [0,1]$. Though infinite trees seem to be very simple graphs, the characterization of their optimum spans, in particular for $x \in (3/2, d-1)$, is very complex.

Theorem 17 (Calamoneri et al. [10], Georges and Mauro [18]). *The following values are spans of optimal labellings of the infinite d-regular tree T_d, $d \geq 2$:*

$$\lambda(T_d; x, 1) = \begin{cases} x + d - 1, & \text{if } 0 \leq x \leq 1/2, \\ (2d-1)x, & \text{if } 1/2 \leq x \leq d/(2d-1), \\ d, & \text{if } d/(2d-1) \leq x \leq 1, \\ d \cdot x, & \text{if } 1 \leq x \leq d/(d-1), \\ x + d, & \text{if } d/(d-1) \leq x \leq 3/2, \\ 2x + d - 2, & \text{if } d - 1 \leq x \leq d, \text{ and} \\ x + 2d - 2, & \text{if } d \leq x. \end{cases}$$

If $x \in (3/2, \frac{d+1}{2})$ and $x - \lfloor x \rfloor > 1/2$, then the optimal span is given by the following:

$$\lambda(T_d; x, 1) = \begin{cases} (2s+1)(x - \lfloor x \rfloor) + 2x + d - 2 - s, & \text{if } x - \lfloor x \rfloor \leq \frac{s+2}{2s+3}, \text{ and} \\ 2\lfloor x \rfloor + d, & \text{otherwise,} \end{cases}$$

where $s = \left\lfloor \frac{d - \lfloor x \rfloor - 2}{2\lfloor x \rfloor + 1} \right\rfloor$.

Finally, if either $x \in [2, \frac{d+1}{2})$ and $x - \lfloor x \rfloor \leq 1/2$ or $x \in [\frac{d+1}{2}, d-1)$, then the optimal span is given by the following:

$$\lambda(T_d; x, 1) = \begin{cases} \frac{d + \lfloor x \rfloor}{\lfloor x \rfloor} x + \lfloor x \rfloor - 2, & \text{if } x \leq \lfloor x \rfloor \frac{d+1}{d} \text{ and } d \equiv_{\lfloor x \rfloor} 0, \\ \frac{d + \lfloor x \rfloor - 1}{\lfloor x \rfloor} x + \lfloor x \rfloor - 1, & \text{if } x \leq \lfloor x \rfloor \frac{d}{d-1} \text{ and } d \equiv_{\lfloor x \rfloor} 1, \\ \frac{d + 2\lfloor x \rfloor - r}{\lfloor x \rfloor} x + r - 1, & \text{if } x \leq \lfloor x \rfloor \frac{d + \lfloor x \rfloor - r + 1}{d + \lfloor x \rfloor - r}, \ d \equiv_{\lfloor x \rfloor} r \neq 0, 1, \\ x + \lfloor x \rfloor + d - 1, & \text{otherwise.} \end{cases}$$

Since the values of $\lambda(T_d; x, 1)$ can be quite hard to read out from Theorem 17 even for a small fixed integer d, let us state as its corollaries the values of spans of optimal labellings of the infinite path (the infinite 2-regular tree) and the infinite 3-regular and 4-regular trees (the functions are depicted in

Figures 1, 2 and 3). Note that the values of optimal labellings of the infinite path coincide with the values of optimal labellings of long paths (this follows from the Compactness Principle).

Corollary 18. *The following values are spans of optimal labellings of the infinite path T_2:*

$$\lambda(T_2; x, 1) = \begin{cases} x+1, & \text{if } 0 \le x \le 1/2, \\ 3x, & \text{if } 1/2 \le x \le 2/3, \\ 2, & \text{if } 2/3 \le x \le 1, \\ 2x, & \text{if } 1 \le x \le 2, \text{ and} \\ x+2, & \text{otherwise } (x \ge 2). \end{cases}$$

Corollary 19. *The following values are spans of optimal labellings of the infinite 3-regular tree T_3:*

$$\lambda(T_3; x, 1) = \begin{cases} x+2, & \text{if } 0 \le x \le 1/2, \\ 5x, & \text{if } 1/2 \le x \le 3/5, \\ 3, & \text{if } 3/5 \le x \le 1, \\ 3x, & \text{if } 1 \le x \le 3/2, \\ x+3, & \text{if } 3/2 \le x \le 2, \\ 2x+1, & \text{if } 2 \le x \le 3, \text{ and} \\ x+4, & \text{otherwise } (x \ge 3). \end{cases}$$

Corollary 20. *The following values are spans of optimal labellings of the infinite 4-regular tree T_4:*

$$\lambda(T_4; x, 1) = \begin{cases} x+3, & \text{if } 0 \le x \le 1/2, \\ 7x, & \text{if } 1/2 \le x \le 4/7, \\ 4, & \text{if } 4/7 \le x \le 1, \\ 4x, & \text{if } 1 \le x \le 4/3, \\ x+4, & \text{if } 4/3 \le x \le 3/2, \\ 11x/3, & \text{if } 3/2 \le x \le 18/11, \\ 6, & \text{if } 18/11 \le x \le 2, \\ 3x, & \text{if } 2 \le x \le 2.5, \text{ and} \\ x+5, & \text{if } 2.5 \le x \le 3, \text{ and} \\ 2x+2, & \text{if } 3 \le x \le 4, \text{ and} \\ x+6, & \text{otherwise } (x \ge 4). \end{cases}$$

Let us turn our attention to infinite plane lattices. The problem for the triangular lattice $\Gamma_\triangle$ has a rich history. Griggs [21] posed an integer

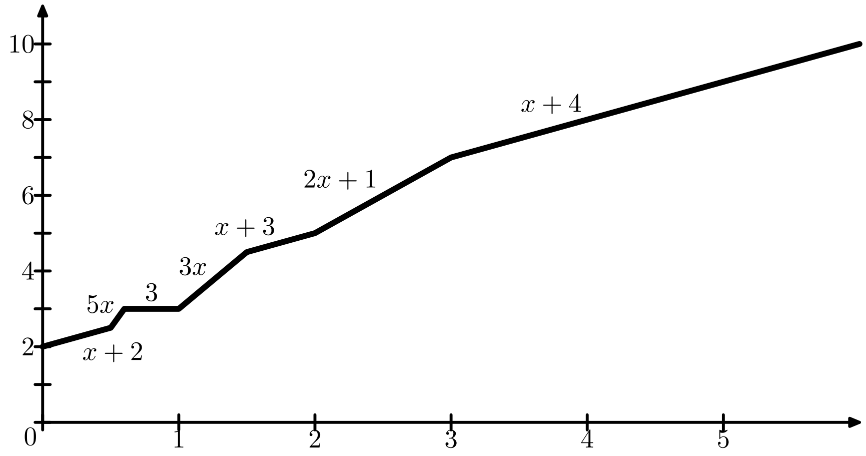

Figure 2: The function $\lambda(T_3; x, 1)$.

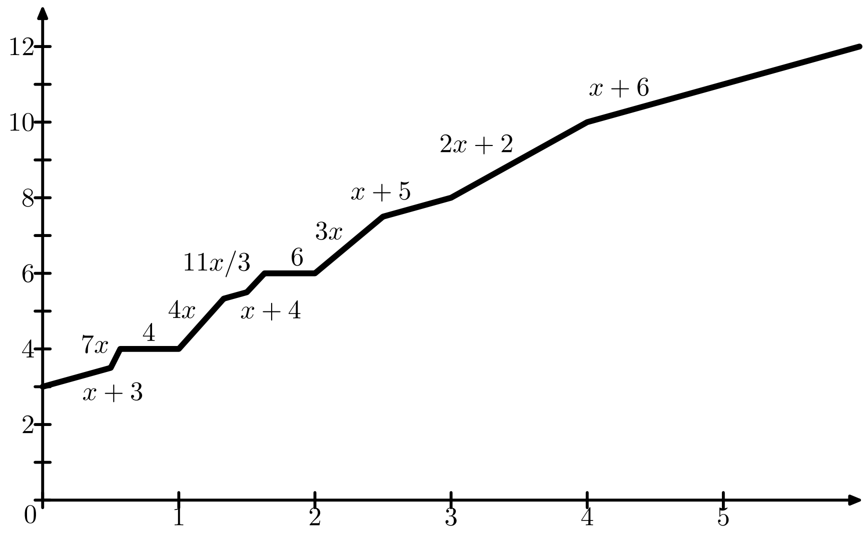

Figure 3: The function $\lambda(T_4; x, 1)$.

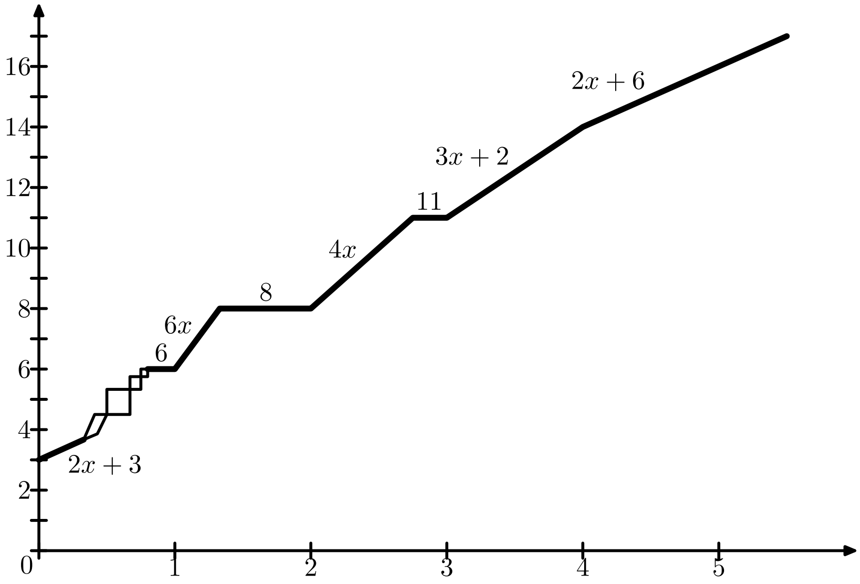

Figure 4: The function $\lambda(\Gamma_\triangle; x, 1)$—the known values are depicted by bold lines and the lower and upper bound by thin lines.

version of the problem in the 2000 International Math Contest in Modeling (MCM). Among 271 teams which participated in the contest, five teams [7, 13, 20, 39, 46] obtained new results for particular choices of parameters. In particular, Goodwin, Johnston and Marcus [20] determined $\lambda(\Gamma_\triangle; x, 1)$ for $x \geq 4$. Several other choices of $x \geq 1$ were later settled by Jin and Yeh [29] and by Zhu and Shi [47]. Calamoneri [9] determined the function for $x \geq 3$ and gave bounds for $x \in [1, 3]$. The function $\lambda(\Gamma_\triangle; x, 1)$ has still not been determined for values $x \in (1/3, 1)$ (see Figure 4 for the known values). The most complete characterization is given in the next theorem.

Theorem 21 (Griggs and Jin [24], cf. Calamoneri [9]). *The following values are spans of optimal labellings of the triangular lattice* $\Gamma_\triangle$ *(or bounds* *on them where they have not been determined):*

$$\lambda(\Gamma_{\triangle}; x, 1) = \begin{cases} 2x+3, & \text{if } 0 \le x \le 1/3, \\ \in [2x+3, 11x], & \text{if } 1/3 \le x \le 9/22, \\ \in [2k+3, 9/2], & \text{if } 9/22 \le x \le 3/7, \\ \in [9k, 9/2], & \text{if } 3/7 \le x \le 1/2, \\ \in [9/2, 16/3], & \text{if } 1/2 \le x \le 2/3, \\ \in [16/3, 23/4], & \text{if } 2/3 \le x \le 3/4, \\ \in [23/4, 6], & \text{if } 3/4 \le x \le 4/5, \\ 6, & \text{if } 4/5 \le x \le 1, \\ 6x, & \text{if } 1 \le x \le 4/3, \\ 8, & \text{if } 4/3 \le x \le 2, \\ 4x, & \text{if } 2 \le x \le 11/4, \\ 11, & \text{if } 11/4 \le x \le 3, \\ 3x+2, & \text{if } 3 \le x \le 4, \text{ and} \\ 2x+6, & \text{otherwise } (x \ge 4). \end{cases}$$

A Manhattan cellular system [5] related to the square lattice finds its applications in the cellular networks in cities. The values of $\lambda(\Gamma_{\square}; x, 1)$ for all $x \in [0, \infty)$ were determined in [24]. Independently, Calamoneri [8, 9] determined the function $\lambda(\Gamma_{\square}; x, 1)$ for $x \ge 3$ (let us remark that some bounds in [8] are not completely correct and were fixed in the journal version [9] of the paper). Let us point out the following interesting fact (that as we will see also holds for the hexagonal lattice): since T_4 is homomorphic to $\Gamma_{\square}$, the function $\lambda(T_4; x, 1)$ is bounded from above by $\lambda(\Gamma_{\square}; x, 1)$. Surprisingly, the values of $\lambda(\Gamma_{\square}; x, 1)$ and $\lambda(T_4; x, 1)$ agree for $x \notin (1.5, 3)$. The reader can compare the functions $\lambda(\Gamma_{\square}; x, 1)$ and $\lambda(T_4; x, 1)$ depicted in Figures 5 and 3.

Theorem 22 (Griggs and Jin [24], cf. Calamoneri [9]). *The following*

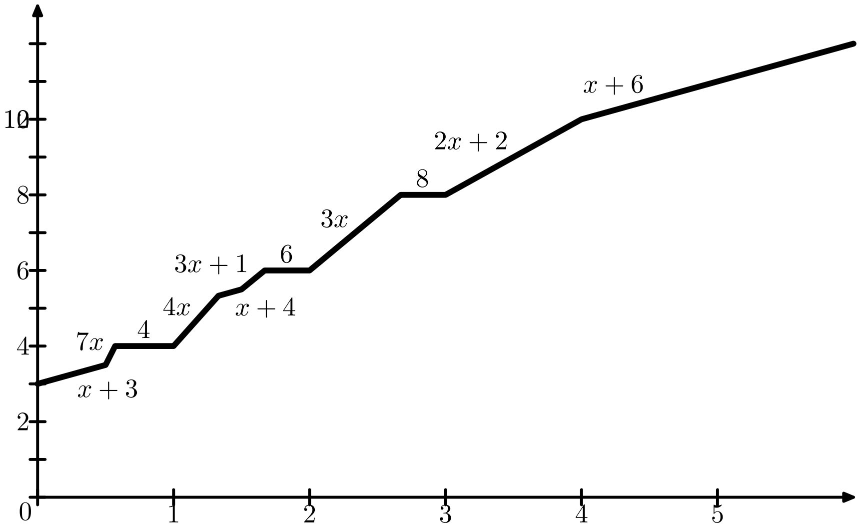

Figure 5: The function $\lambda(\Gamma_\square; x, 1)$.

values are spans of optimal labellings of the square lattice $\Gamma_\square$*:*

$$\lambda(\Gamma_\square; x, 1) = \begin{cases} 2x+3, & \text{if } 0 \le x \le 1/2, \\ 7x, & \text{if } 1/2 \le x \le 4/7, \\ 4, & \text{if } 4/7 \le x \le 1, \\ 4x, & \text{if } 1 \le x \le 4/3, \\ x+4, & \text{if } 4/3 \le x \le 3/2, \\ 3x+1, & \text{if } 3/2 \le x \le 5/3, \\ 6, & \text{if } 5/3 \le x \le 2, \\ 3x, & \text{if } 2 \le x \le 8/3, \\ 8, & \text{if } 8/3 \le x \le 3, \\ 2x+2, & \text{if } 3 \le x \le 4, \text{ and} \\ x+6, & \text{otherwise } (x \ge 4). \end{cases}$$

Besides the triangular and square lattices, the plane can also be tiled by hexagons. Calamoneri [9] determined the values of $\lambda(\Gamma_H; x, 1)$ for $x \in [2, \infty)$ and provided lower and upper bounds for $x \in [1, 2]$. The function $\lambda(\Gamma_H; x, 1)$ was completely determined in [24]. As in the case of the square lattice, the functions $\lambda(\Gamma_H; x, 1)$ and $\lambda(T_3; x, 1)$ agree for most of the values of x—see

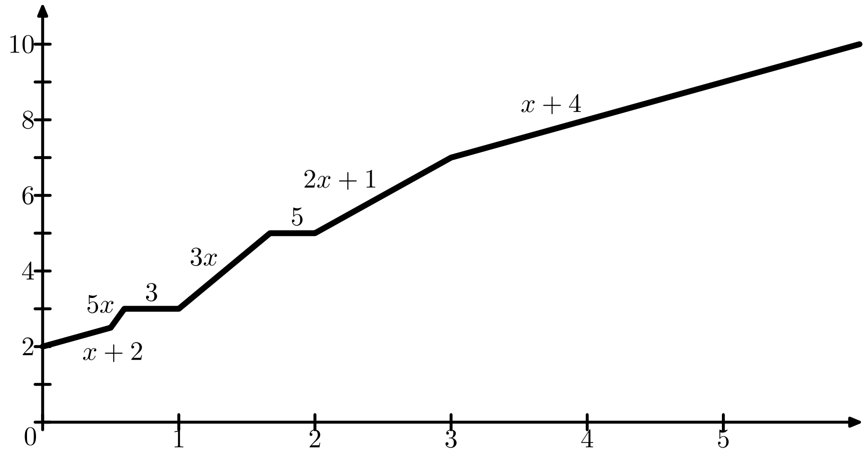

Figure 6: The function $\lambda(\Gamma_H; x, 1)$.

Figures 6 and 2.

Theorem 23 (Griggs and Jin [24], cf. Calamoneri [9]). *The following values are spans of optimal labellings of the hexagonal lattice* Γ_H*:*

$$\lambda(\Gamma_H; x, 1) = \begin{cases} x+2, & \text{if } 0 \leq x \leq 1/2, \\ 5x, & \text{if } 1/2 \leq x \leq 3/5, \\ 3, & \text{if } 3/5 \leq x \leq 1, \\ 3x, & \text{if } 1 \leq x \leq 5/3, \\ 5, & \text{if } 5/3 \leq x \leq 2, \\ 2x+1, & \text{if } 2 \leq x \leq 3, \text{ and} \\ x+4, & \text{otherwise } (x \geq 3). \end{cases}$$

In Theorems 22 and 23, we have seen that the values of $\lambda(\Gamma_\Box; x, 1)$ and $\lambda(\Gamma_H; x, 1)$ coincide with the values of of $\lambda(T_d; x, 1)$ for most values of x where d is the common degree of the vertices of the lattice. We wonder what is the reason for this behavior and what general property caused it.

6 Open problems

We conclude the paper with suggesting several problems for further research on real number graph labellings. The proof of Theorem 5 is based on an inductive argument that yields enormous bounds on the values $A_{k,\chi}$ and $B_{k,\chi}$. It seems that such huge bounds are not necessary and it could be possible to establish better bounds on $A_{k,\chi}$ and $B_{k,\chi}$.

Problem 1. *Determine whether the constants $A_{k,\chi}$ and $B_{k,\chi}$ from Theorem 5 can be bounded by a function exponential in k and χ, or even by a function polynomial in one of the parameters k and χ.*

Another problem is to provide a bound on the number of linear parts of the λ-function of small finite λ-graphs G. It was conjectured [2] that the quadratic bound provided in Theorem 11 can be decreased to a linear one.

Problem 2. *Prove that the number of linear parts of the λ-function of a finite λ-graph G of order n is at most $O(n)$.*

The construction of optimal labellings of infinite regular lattices, for instance in [9, 24], are based on repeating the same pattern of the labels throughout the lattice. It seems natural to ask whether all optimal labellings of regular lattices must be of such a type:

Problem 3. *Investigate the structure, in particular the symmetry properties, of optimal labellings of the infinite regular lattices $G_\triangle$, $G_\square$ and G_H.*

Research of the dependence of the circular analogue of the channel assignment problem on its parameters preceded the real number graph labellings, see, e.g., [28, 36]. It seems natural to ask whether Theorem 5 in particular can be proven in the setting of circular labellings.

Problem 4. *Explore circular labelling analogues of the real number graph labellings and determine which of the general structural results translate to this setting.*

Finally, it is apparent that the functions $\lambda(G; x, 1)$ are neither concave-up nor concave-down. Indeed, they seem quite the opposite. In the examples shown in this paper, starting from $x = 0$, the graph sections alternately increase and decrease in slope. We know that this kind of behavior is not common to all functions $\lambda(G; x, 1)$ (an example is the function associated with wheels W_n described in Theorem 15) but we think that there should be a reason for this type of behavior common to most of the functions $\lambda(G; x, 1)$.

Problem 5. *What is the explanation for the "up-down" behavior of the functions $\lambda(G;x,1)$ for most graphs G?*

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