Coloring even-faced graphs in the torus and the Klein bottle

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Abstract

We prove that a triangle-free graph drawn in the torus with all faces bounded by even walks is 3-colorable if and only if it has no subgraph isomorphic to the Cayley graph $C(Z_{13}; 1, 5)$. We also prove that a non-bipartite quadrangulation of the Klein bottle is 3-colorable if and only if it has no non-contractible separating cycle of length at most four and no odd walk homotopic to a non-contractible two-sided simple closed curve. These results settle a conjecture of Thomassen and two conjectures of Archdeacon, Hutchinson, Nakamoto, Negami and Ota.

1 Introduction

Our main motivation comes from the following classical theorem of Grötzsch [5].

Theorem 1. *Every triangle-free planar graph is* 3*-colorable.*

Until the seminal work of Thomassen this has been regarded as a very difficult theorem. However, Thomassen [12, 13] found two reasonably simple

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proofs, and extended Theorem 1 to other surfaces. It is easy to see that a minimal counterexample to Theorem 1 cannot have a face bounded a cycle of length four. Furthermore, Theorem 1 can be strengthened to allow the precoloring of a face of length at most five; then a minimal counterexample to the stronger version has no 4-cycles at all. Thus the hard part of Theorem 1 is to show that every planar graph of girth at least five is 3-colorable. This result has been extended in two different directions. Thomassen proved that the same conclusion holds for projective planar and toroidal graphs, and Thomas and Walls [9] proved the same for graphs embeddable in the Klein bottle. For a fixed general surface Σ Thomassen [14] showed that Σ includes only finitely many 4-critical graphs of girth at least five. In the other direction Thomassen [13] proved that every planar graph of girth at least five is 3-choosable.

In this paper we are concerned with a somewhat different generalization of Theorem 1. The statement of Theorem 1 cannot be extended to any surface other than the sphere, because the graphs obtained from an odd cycle by applying Mycielski's construction [3, Section 8.5] embed in every nonplanar surface. Incidentally, it is an interesting open problem [4] whether 3-colorability of triangle-free graphs on a fixed surface Σ can be tested in polynomial time. When Σ is the sphere this is, of course, trivial by Theorem 1, and when Σ is the projective plane a polynomial-time algorithm follows from Theorem 4 below. For all other surfaces the problem is currently open.

Thus we set a more modest goal: We restrict ourselves to triangle-free quadrangulations of the torus or the Klein bottle, and, more generally, to embeddings of graphs in the torus or the Klein bottle with all faces of even size.

Let us be more precise now. *Graphs* may have parallel edges, but no loops. By a *surface* we mean a compact 2-dimensional manifold with no boundary. A *drawing* of a graph G in a surface Σ refers to an embedding of G in Σ with no crossings, and a *subdrawing* is a restriction of the embedding to a subgraph of G. We apply standard graph-theoretic terminology to drawings and speak about cycles in drawings, colorings of drawings, etc. Two drawings G_1 and G_2 are *isomorphic* if there exists an isomorphism between the graphs of G_1 and G_2 that preserves face boundaries. (Please note that we do not assume that G_1 and G_2 are drawings in the same surface.) A drawing G in a surface Σ is a *quadrangulation* if every face is bounded by a walk of length four, and we say that a drawing G is *even-faced* if every face is bounded by a walk of even length.

Thomassen [15] conjectured that every triangle-free quadrangulation of the torus is 3-colorable. We prove that this conjecture holds, with the following exception: the quadrangulation $Q_{13,5,1}$ depicted in Figure 5 is a counterexample, as pointed out by Archdeacon, Hutchinson, Nakamoto, Negami and Ota [2]. However, our first main result states that $Q_{13,5,1}$ is essentially the only counterexample, even for the more general class of even-faced drawings.

Theorem 2. *A triangle-free even-faced drawing in the torus is* 3*-colorable if and only if it has no subdrawing isomorphic to* $Q_{13,5,1}$.

The *edge-width* of a drawing is the length of the shortest non-contractible cycle, or infinity if the drawing has no non-contractible cycle. The *representativity* of a drawing G in a surface Σ is the maximum integer k such that every non-contractible simple closed curve in Σ meets G at least k times. Since $Q_{13,5,1}$ has a non-contractible cycle of length five, the following is an immediate corollary. It settles a conjecture of Archdeacon, Hutchinson, Nakamoto, Negami and Ota [2], who proved the same result for drawings of representativity at least 9. An earlier result of Hutchinson [6] proves this with 6 replaced by 25.

Corollary 3. *Every even-faced drawing in the torus of edge-width at least six is* 3*-colorable.*

Since every triangle-free even-faced drawing G in the torus is 4-colorable by [2, Theorem 6], Theorem 2 gives a polynomial-time algorithm to compute the chromatic number of G.

A characterization of 3-colorable quadrangulations of the torus and the Klein bottle (including those that have triangles) in terms of "essential diagonal curves" was given by Archdeacon, Hutchinson, Nakamoto, Negami and Ota [2]. It is not clear whether their theorem can used to design an efficient algorithm, though.

What can we say about other orientable surfaces? Hutchinson [6] proved that for every orientable surface Σ there exists an integer k such that every even-faced drawing in Σ of edge-width at least k is 3-colorable. By analogy with Theorem 2 and the results of [11, 14] one could speculate whether there exist only finitely many 4-critical even-faced drawings on any fixed orientable surface Σ. Unfortunately that is not true, unless Σ is the sphere or the torus. The graphs obtained from an odd cycle by means of the Mycielski's construction serve as counterexamples.

Let us turn to nonorientable surfaces now. From the vertex-coloring point of view triangle-free drawings in the projective plane are completely understood. First of all, Euler's formula implies that they have a vertex of degree at most three, and hence they are always 4-colorable. Youngs [16] discovered the remarkable fact that no quadrangulation of the projective plane has chromatic number exactly three, and Gimbel and Thomassen [4] extended that result to a characterization of 3-colorable triangle-free drawings in the projective plane:

Theorem 4. *A drawing in the projective plane with no contractible cycles of length three is* 3*-colorable if and only if it has no subdrawing isomorphic to a non-bipartite quadrangulation of the projective plane.*

Thus there are infinitely many triangle-free quadrangulations of the Klein bottle: take two quadrangulations of the projective plane such that at least one of them is not bipartite, in each of them select a facial cycle, and identify those cycles. The resulting quadrangulation of the Klein bottle is not 3-colorable, because it has a subdrawing isomorphic to a nonbipartite quadrangulation of the projective plane.

There is another fundamental reason why a quadrangulation of the Klein bottle (and more generally, any non-orientable surface) may fail to be 3-colorable. A closed walk or cycle in a drawing G in the Klein bottle is called *meridian* if it is homotopic to a 2-sided simple closed curve that does not separate the surface. The following is proved in [2] for odd meridian cycles, but the proof extends to walks.

Theorem 5. *Let G be a quadrangulation in the Klein bottle, and assume that G contains an odd meridian walk. Then G is not* 3*-colorable.*

We show in Section 3 that the above two constructions are the only obstructions to 3-colorability of quadrangulations of the Klein bottle. By an *equator* in a drawing G in the Klein bottle we mean a non-contractible cycle in G that separates the surface.

Theorem 6. *A non-bipartite quadrangulation of the Klein bottle is* 3*-colorable if and only if*
(1) *it has no equator of length at most four, and*
(2) *it has no odd meridian walk.*

Since every quadrangulation of the Klein bottle is 4-colorable by [2, Theorem 5], the above theorem gives a polynomial-time algorithm to compute the chromatic number of a given quadrangulation of the Klein bottle. For even-faced drawings we have the following corollary. The proof is analogous to the argument used at the end of Section 2, and is omitted.

Theorem 7. *Let G be an even-faced drawing in the Klein bottle with no equator of length at most four and no odd meridian walk. Then G is 3-colorable.*

The next immediate corollary settles another conjecture of Archdeacon, Hutchinson, Nakamoto, Negami and Ota [2], who proved the same result for drawings of representativity at least 7.

Corollary 8. *Let G be an even-faced drawing in the Klein bottle with edge-width at least five and no odd meridian walk. Then G is 3-colorable.*

By Theorem 6 the bound of five is best possible. An earlier result of Mohar and Seymour [7] implies that the above corollary holds for some bound on edge-width. In fact, they prove an analogous statement for an arbitrary non-orientable surface.

The paper is organized as follows. In the next section we prove Theorem 2, in Section 3 we prove Theorem 6, and in Section 4 we comment on some of the questions this paper leaves unresolved. We will need the following lemma, a special case of [4, Theorem 5.3]. Our special case can be easily deduced from first principles.

Lemma 9. *Let G be an even-faced drawing in the sphere, let C be an induced facial cycle of G of length at most six, and let $c : V(C) \to \{1, 2, 3\}$ be a 3-coloring of C such that if C has length six, then $c(u) \neq c(v)$ for some pair of diagonally opposite vertices $u, v \in V(C)$. Then c can be extended to a 3-coloring of G.*

2 Quadrangulations of the Torus

In this section we prove Theorem 2. The hardest part is to prove it for quadrangulations; the extension to even-faced drawings is straightforward. We begin by eliminating non-contractible cycles of length two or four.

Lemma 10. *Let G be a triangle-free even-faced drawing on the torus. If G contains a non-contractible cycle of length two or four, then G is 3-colorable.*

Proof. Let C be a non-contractible cycle of G of length two or four. We cut the torus open along C and obtain a planar graph G' with facial cycles $v_1v_2\dots v_k$ and $v'_1v'_2\dots v'_k$ such that G is obtained from G' by identifying v_i and v'_i for $i = 1, 2, \dots, k$. Since the sizes of all the faces of G' are even, G' is bipartite. We aim to find a 3-coloring of the vertices of G' such that the corresponding vertices v_i and v'_i are assigned the same color.

Let us color the vertices of G' black and white. We may assume without loss of generality that the vertices v_i with odd i are colored white and those with even i are colored black. If the vertices v'_i with odd i are colored white, then the graph G is bipartite and 2-colorable. We may therefore assume that the vertices v'_i with odd i are colored black and those with even i are white. Let us recolor the vertices v_i and v'_i with $i = 2, 4$ by the grey color, the vertices v'_1 and v'_3 white and all the neighbors of v'_1 or v'_3 in G' by the grey color. Note that all the pairs of the vertices v_i and v'_i with the same index i now have the same color.

We verify that the coloring thus obtained is a proper 3-coloring of G'. If it is not a proper coloring, then there is a neighbor w of v'_j, $j \in \{1, 3\}$, in G' colored grey that is also a neighbor of v_i, $i \in \{2, 4\}$. However, the vertices v_iwv_j form a 3-cycle in the original graph G. This contradicts our assumption that G is triangle-free. Hence, the coloring of G' we constructed gives rise to a proper 3-coloring of G, as desired. □

We will need the following operation, which we call *face contraction*: if $abcd$ is a face of a quadrangulation G, $a \neq c$, and a and c are not adjacent, then $G.ac$ is the quadrangulation obtained from G by adding an edge ac, contracting it and deleting one edge from each facial cycle of length two that results. We say that $G.ac$ is obtained from G by a *face contraction* of the vertices a and c. A triangle-free quadrangulation G of the torus is *triangle-prone* if for every face $abcd$ with $a \neq c$ there is a path in G of length three with ends a and c (in which case the face contraction $G.ac$ produces a triangle).

Lemma 11. *Let G be a triangle-prone quadrangulation of the torus with every face bounded by a cycle. Then G has no vertices of degree three.*

Proof. Suppose for a contradiction that G is a triangle-prone quadrangulation of the torus with every face bounded by a cycle, and let v be a vertex of G of degree three.

Let a, c and e be the three neighbors of v and let $abcv$, $cdev$ and $efav$ be the three faces incident with v (see Figure 1). Since G is triangle-prone, the

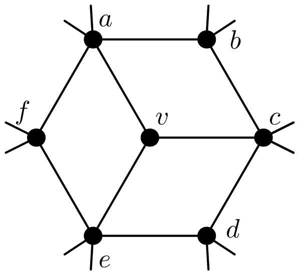

Figure 1: The notation used in the proof of Lemma 11.

vertices v and b are joined by a path of length three, say $vxyb$. If the vertex x were a or c, then the original graph would have contained a triangle xyb. Hence, $x = e$. The vertex y is neither d nor f (otherwise, G would contain a triangle abf or acd). We conclude that G contains a vertex $y = w_{be}$ that is a common neighbor of b and e and is different from all the vertices v, a, b, c, d, e and f. Similarly, there is such a common neighbor w_{ad} of the vertices a and d and a common neighbor w_{cf} of the vertices c and f. In addition, the three vertices w_{ad}, w_{be} and w_{cf} are pairwise distinct: if, e.g., $w_{ad} = w_{be}$, then G contains a triangle comprised of the vertices a, b and $w_{ad} = w_{be}$. It also follows that the vertices a, b, c, d, e, f are pairwise distinct.

However, the subdrawing of G formed by the vertices v, a, b, c, d, e, f, w_{ad}, w_{be} and w_{cf} is unique up to isomorphism of drawings (because none of the 5-cycles $vaw_{ad}dc$, $vbw_{be}ed$ and $vaw_{cf}fe$ is contractible). This drawing is depicted in Figure 2. Note that the 9-cycle $aw_{ad}dcw_{cf}few_{be}b$ is contractible. This contradicts the fact that G is a quadrangulation of the torus and each of its contractible cycles is even. $\square$

Dealing with vertices of degree two or less in triangle-prone quadrangulations is easy, and so we now turn our attention to 4-regular quadrangulation of the torus. Those were completely described by Altshuler [1]. Let us state his theorem. By a *lattice* we mean the set of all integral linear combinations of two linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$. We say that the lattice is *generated* by $\mathbf{u}$ and $\mathbf{v}$. We wish to consider the infinite planar square grid: its vertex-set is the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ and two vertices (i, j) and (i', j') are adjacent if and only if $|i - i'| + |j - j'| = 1$. If L is a lattice, then $\mathbb{Z}^2/L$ stands for the quotient graph obtained by identifying vertices $\mathbf{u}$ and

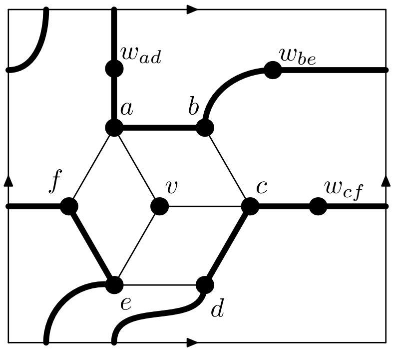

Figure 2: The unique embedding of the subgraph of G formed by the vertices v, a, b, c, d, e, f, w_{ad}, w_{be} and w_{cf} on the torus and the contractible 9-cycle $aw_{ad}dcw_{cf}few_{be}b$ (drawn bold).

$\mathbf{v}$ whenever $\mathbf{u}-\mathbf{v} \in L$ and identifying the corresponding edges. The natural planar embedding of the grid defines a drawing of $\mathbb{Z}^2/L$ in the torus; this drawing is a quadrangulation of the torus (see Figure 3). Conversely, the following theorem of Altshuler [1] states that every 4-regular quadrangulation of the torus arises this way.

Theorem 12. *Let G be a 4-regular quadrangulation of the torus. Then G is isomorphic to $\mathbb{Z}^2/L$ for some lattice L.*

Let $x, y \geq 1$ and $r \geq 0$ be integers. We define $Q_{x,y,r}$ to be the toroidal quadrangulation $\mathbb{Z}^2/L$, where L is generated by the vectors $(x, 0)$ and (r, y). Theorem 12 may be used to describe all triangle-prone quadrangulations of the torus. However, for our purposes it suffices to find those that do not have a non-contractible cycle of length at most four.

Lemma 13. *Every 4-regular triangle-prone quadrangulation of the torus with no non-contractible cycles of length at most four is isomorphic to $Q_{13,5,1}$, $Q_{14,4,1}$, or $Q_{17,4,1}$.*

Proof. Let G be a 4-regular triangle-prone quadrangulation of the torus. By Theorem 12, G is isomorphic to $\mathbb{Z}^2/L$ for some lattice L. Let v_{ij} denote the vertex of G corresponding to the vertex (i, j) of the infinite grid. Consider the face $v_{11}v_{21}v_{22}v_{12}$ of G. See Figure 4.

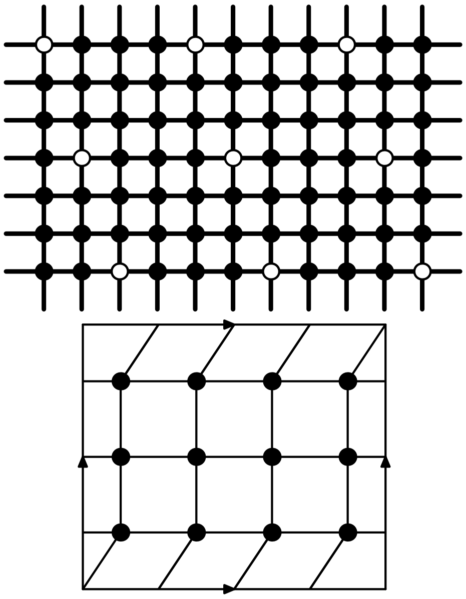

Figure 3: $\mathbb{Z}^2/L$ for the lattice L generated by $(4,0)$ and $(-1,3)$ with one of the sets of identified vertices marked. The corresponding graph $Q_{4,3,1}$ and its embedding on the torus is depicted in the right part of the figure.

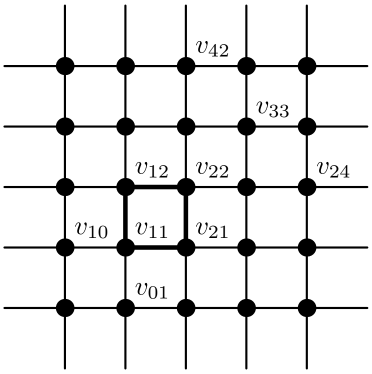

Figure 4: Face contractions in Lemma 13.

Since G is triangle-prone, the face contraction of v_{11} and v_{22} yields a triangle in G. Hence, one of the vertices v_{01} and v_{10} is identified with one of the vertices v_{24}, v_{33} and v_{42}. Hence, the lattice L contains at least one of the following four vectors: $(4, 1)$, $(3, 2)$, $(2, 3)$, and $(1, 4)$. Since interchanging the coordinates does not change the (isomorphism type of the) quadrangulation $\mathbb{Z}^2/L$, we may assume that L contains the vector $(4, 1)$ or $(3, 2)$. If we consider the face contraction of the vertices v_{12} and v_{21}, we obtain that L includes one of the vectors $(4, -1)$, $(3, -2)$, $(2, -3)$, or $(1, -4)$.

If $(4, 1) \in L$ and $(3, -2) \in L$, then $(1, 3) = (4, 1) - (3, -2) \in L$. Hence, G contains a non-contractible 4-cycle which is impossible. Similarly, L cannot include any of the following vectors: $(3, 2) - (3, -2) = (0, 4)$, $(4, 1) - (4, -1) = (0, 2)$ and $(3, 2) - (4, -1) = (-1, 3)$. Hence, L includes neither the vector $(4, -1)$ nor the vector $(3, -2)$, and therefore it includes $(2, -3)$ or $(1, -4)$.

Assume now that $(4, 1) \in L$ and $(2, -3) \in L$. Let L' be the lattice generated by $(4, 1)$ and $(2, -3)$. Note that $(14, 0) = 3(4, 1) + (2, -3) \in L'$. If L is a proper subset of L', then, since $(4, 1) \in L$, $(2, 0) \in L$ or $(7, 0) \in L$. In the former case, G has a non-contractible cycle of length two. In the latter case $(3, -1) = (7, 0) - (4, 1) \in L$ which implies that G has a non-contractible cycle of length four. Both cases are impossible. Hence, $L = L'$ and G is isomorphic to the graph $Q_{14,4,1}$.

If $(4, 1) \in L$ and $(1, -4) \in L$, then $(17, 0) = 4(4, 1) + (1, -4) \in L$. This implies that G is isomorphic to $Q_{17,4,1}$. If $(3, 2) \in L$ and $(2, -3) \in L$, then $(13, 0) = 3(3, 2) + 2(2, -3) \in L$. Since $(5, -1) = (3, 2) + (2, -3)$ is also contained in L, the graph G is isomorphic to $Q_{13,5,1}$. Finally, if $(3, 2) \in L$ and $(1, -4) \in L$, then $(-1, 4) \in L$, and using the transformation $(x, y) \mapsto (y, -x)$ we see that this leads to the same outcome as the case considered at the beginning of the previous paragraph. This completes the proof of the lemma. □

We are now ready to prove Theorem 2. For convenience we first prove the nontrivial implication for quadrangulations.

Theorem 14. *Every triangle-free quadrangulation of the torus with no subdrawing isomorphic to $Q_{13,5,1}$ is* 3*-colorable.*

Proof. We proceed by induction on $|V(G)|$. Since there is no quadrangulation of the torus with at most one vertex we may assume that G is a non-3-colorable triangle-free quadrangulation of the torus, and that the theorem

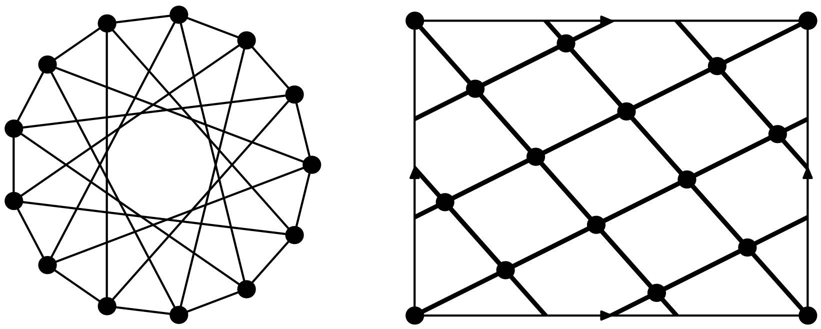

Figure 5: The drawing $Q_{13,5,1}$ in the torus.

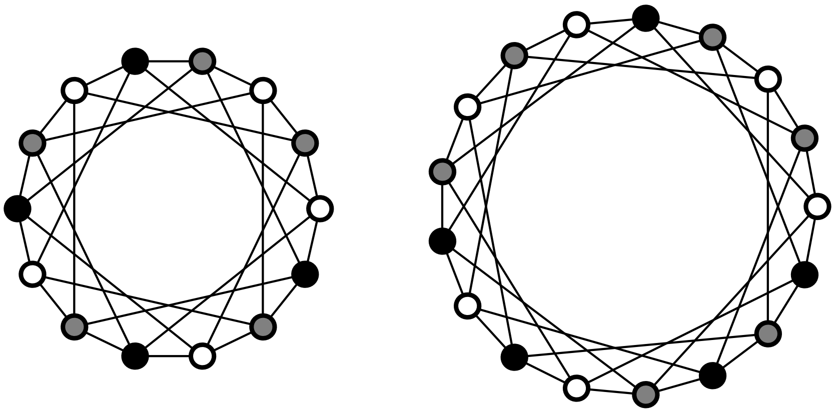

Figure 6: The graphs of the drawings $Q_{14,4,1}$ and $Q_{17,4,1}$ and their 3-colorings.

holds for all quadrangulations on strictly fewer than $|V(G)|$ vertices.

The minimality of G implies that G is 2-connected. This and Lemma 10 imply that G has no cycles of length two, and hence every face of G is bounded by a cycle. The minimality of G and Lemma 9 further imply that every contractible 4-cycle in G bounds a face.

We claim that G is triangle-prone. To prove this claim we may assume for a contradiction that G has a face $abcd$ such that the drawing $H := G.ac$ is triangle-free. Since a 3-coloring of H can be converted to a 3-coloring of G, we deduce that H is not 3-colorable, and hence, by the induction hypothesis, H has a subdrawing isomorphic to $Q_{13,5,1}$. Since $Q_{13,5,1}$ is not a subdrawing of G, we deduce that G has a subdrawing G' isomorphic to one of the drawings depicted in Figure 7. The figure also shows 3-colorings of these drawings. Please note that each color class of G' forms an independent set in G, because every contractible cycle in G is even and every contractible 4-cycle of G bounds a face of G. By Lemma 9 the 3-colorings pictured in Figure 7 extend to 3-colorings of G, a contradiction. This proves our claim that G is triangle-prone.

If G has a vertex v of degree at most two, then by the induction hypothesis applied to the drawing $G\backslash v$ we deduce that G is 3-colorable, a contradiction. Thus G has minimum degree at least three, and by Lemma 11 it has minimum degree at least four. It follows easily from Euler's formula that G is 4-regular. Using Lemmas 10 and 13 we conclude that G is isomorphic to $Q_{13,5,1}$, because $Q_{14,4,1}$ and $Q_{17,4,1}$ are 3-colorable, as shown in Figure 6. Thus G satisfies the theorem. □

We are now ready to prove Theorem 2 in full.

Proof of Theorem 2. The "only if" part is clear, because $Q_{13,5,1}$ is not 3-colorable, as is easily verified. To prove the "if" part let G be a triangle-free even-faced drawing in the torus with no subdrawing isomorphic to $Q_{13,5,1}$. By deleting selected parallel edges we may assume that G has no faces of size two. We proceed by induction on the number of faces of size at least six. If there is no such face, then G is a quadrangulation, and the theorem follows from Theorem 14. Otherwise we select a face bounded by a walk $v_1v_2\dots v_k$, where $k \geq 6$. Note that k is even. We add to this face two concentric cycles $v'_1v'_2\dots v'_k$ and $v''_1v''_2\dots v''_k$ together with the paths $v_iv'_iv''_i$ for $i = 1, 2, \dots, k$. In addition, we add the edges $v''_{i+1}v''_{k-i}$ for $i = 1, 2, \dots, k/2 - 2$. Let H be the resulting drawing. The construction is illustrated in Figure 8.

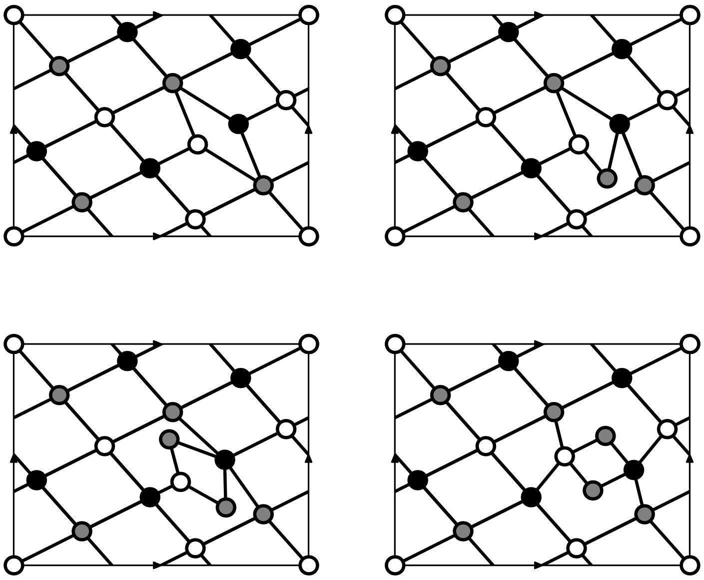

Figure 7: Four drawings used in the proof of Theorem 14.

The drawing H is triangle-free, and has no subdrawing isomorphic to $Q_{13,5,1}$, because every edge of $Q_{13,5,1}$ belongs to a cycle of length five. Thus H is 3-colorable by the induction hypothesis, and hence so is G, as desired. This completes the proof of Theorem 2. □

3 Quadrangulations of the Klein bottle

In this section we prove Theorem 6. We begin by disposing of quadrangulations that have an equator of length six.

Lemma 15. *Let G be a quadrangulation of the Klein bottle with no odd meridian walk. If the length of the shortest equator in G is six, then G is 3-colorable.*

Proof. Let C be an equator in G of length six. Cut the surface open along the cycle C; let G_1 and G_2 be the two resulting drawings, each a drawing in the projective plane with a facial cycle C. Since G has no odd meridian walk it follows that G_1, G_2 are either both bipartite, or both non-bipartite. If they are bipartite, then so is G, and the lemma holds. Thus we may assume that both G_1 and G_2 are non-bipartite.

If G_1 has a subdrawing H that is a quadrangulation of the projective plane, then the boundary of the face of H that includes the face of G_1 bounded by C is an equator in G of length four, contrary to hypothesis. Thus G_1 has no such subdrawing, and hence is 3-colorable by Theorem 4. Similarly, G_2 is 3-colorable. Fix a 3-coloring c_1 of G_1 and a 3-coloring c_2 of G_2.

Let u, v be two diagonally opposite vertices of C. If $c(u) \neq c(v)$, then c_1 is a 3-coloring of the non-bipartite quadrangulation of the projective plane obtained from G_1 by adding the edge uv inside the face bounded by C, contrary to Theorem 4. Thus $c_1(u) = c_1(v)$ for every pair of diagonally opposite vertices of C, and by symmetry the same holds for c_2. It follows that the colorings c_1 and c_2 may be combined to produce a 3-coloring of G, as desired. □

Next we eliminate triangles.

Lemma 16. *Let G be a quadrangulation of the Klein bottle with no equator of length at most four and no odd meridian walk. If G has a triangle, then it is 3-colorable.*

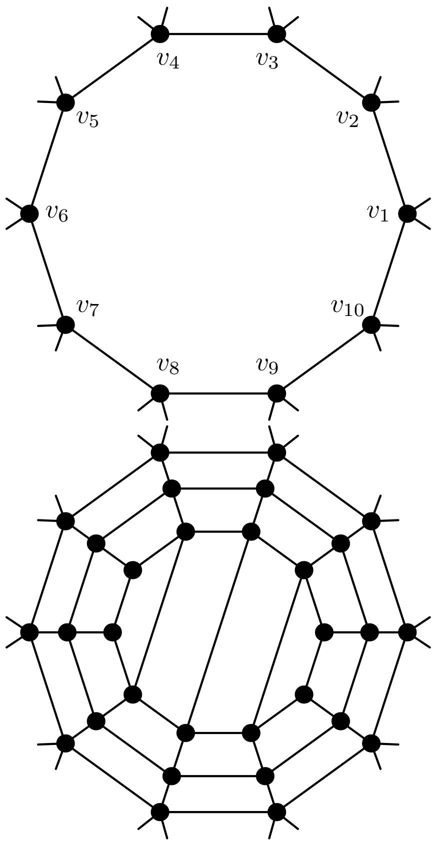

Figure 8: Replacement of the interior a face of length 10 in the proof of Theorem 2.

Proof. Let C be a triangle in G. Since G is even-faced and has no odd meridian walk it follows that C is one-sided. By cutting open along C we obtain a drawing G' in the projective plane with a face bounded by a cycle C' of length six in such a way that G is obtained from G' by identifying diagonally opposite vertices of C'. We claim that G' is not bipartite. For let u, v be two diagonally opposite vertices on C', and let W be a closed non-contractible walk in G' passing through u. (Such a walk exists, because G has no equator of length four.) Then W can be extended along C' to an odd walk in G' from u to v; the latter walk becomes a meridian walk in G, and hence W is odd, proving that G' is not bipartite. If G' has a subdrawing J that is a quadrangulation of the projective plane, then the boundary D of the face of J that includes the face bounded by C' is an equator in G of length four, a contradiction. (Notice that the identifications that produce G from G' do not identify distinct vertices of D, because G is a quadrangulation and has no loops.) By Theorem 4 the drawing G' has a 3-coloring c. Let u, v be two diagonally opposite vertices of C'. If $c(u) \neq c(v)$, then c is a 3-coloring of the non-bipartite quadrangulation of the projective plane obtained from G' by adding the edge uv inside the face bounded by C', contrary to Theorem 4. Thus $c(u) = c(v)$ for every pair of diagonally opposite vertices of C', and hence c gives rise to a 3-coloring of G, as desired. □

Our third lemma is an analogue of Lemma 10.

Lemma 17. *Let G be a triangle-free even-faced drawing in the Klein bottle with no equator of length two and no odd meridian walk. If G has a non-contractible cycle of length two, then G is 3-colorable.*

Proof. Let C be a non-contractible cycle in G of length two. We may assume that G is not bipartite, for otherwise the lemma holds. We claim that C is a meridian cycle. To prove this claim suppose for a contradiction that it is not. Since C is not an equator by hypothesis, it is one-sided. By cutting open along C we obtain a drawing G' in the projective plane with a face bounded by a cycle C' of length four in such a way that G is obtained from G' by identifying diagonally opposite vertices of C'. Since G is not bipartite, neither is G', and hence G' has an odd walk W joining a pair of diagonally opposite vertices of C'. But W gives rise to an odd meridian walk in G, a contradiction. This proves our claim that C is a meridian cycle. The rest of the argument is identical to the proof of Lemma 10. □

Proof of Theorem 6. Let G be a non-bipartite quadrangulation of the Klein bottle. If G has an odd meridian walk, then G is not 3-colorable by Theorem 5. If G has an equator C of length two or four, then C divides G into two quadrangulations G_1 and G_2 of the projective plane. (If C has length two, then we need to delete a parallel edge from G_1 and G_2 to turn them into quadrangulations.) Since G is not bipartite, one of G_1, G_2 is not bipartite, and hence is not 3-colorable by Theorem 4. Thus G is not 3-colorable.

To prove the converse let G satisfy (1) and (2). We proceed by induction on $|V(G)|$. We may assume that G is 2-connected, for otherwise the lemma follows by induction. By Lemma 16 we may assume that G has no triangles, and by Lemma 17 we may assume that G has no non-contractible cycle of length two. Since G is 2-connected the latter implies that every face of G is bounded by a cycle. Let G' be obtained from G by a face contraction of an arbitrary pair of vertices. The drawing G' well-defined, because G is triangle-free and every face is bounded by a cycle. Since G has no equator of length at most six, the drawing G' has no equator of length at most four (using the fact that G is triangle-free and has no parallel edges), and hence is 3-colorable by the induction hypothesis. But every 3-coloring of G' gives rise to a 3-coloring of G, as desired. □

4 Concluding Remarks

In this section we comment on some of the problems left unresolved. Unlike in Theorem 6, the two hypotheses of Theorem 7 are not necessary. The first two remarks address the possibility of characterizing 3-colorable even-faced drawings in the Klein bottle.

Even-faced drawings in the Klein bottle with an equator of length four. Let C be an equator of length four in an even-faced drawing G in the Klein bottle. Then C divides G into two even-faced drawings in the projective plane, and hence in order to understand the 3-colorability of G we need to study the following two coloring extension problems. Let H be an even-faced drawing in the projective plane, and let x, y be two diagonally opposite vertices on a face of H. When is it the case that no 3-coloring of H gives x and y the same color, and when is it the case that no 3-coloring of H gives x and y different colors? Of course, if x and y have the former property, then the other two vertices on the same face with x and y have the latter property.

Let us describe several obstructions to the coloring extension problems of the previous paragraph. We say that an even-faced drawing H in a surface is a *near-quadrangulation* if H has a face f such that every face other than f is bounded by a walk of length four. If f is bounded by a cycle C of length at least six, then we say that H is a *near-triangulation with hole* C. Now let H be a non-bipartite near-triangulation of the projective plane with hole C. If C has length six, and x and y are at distance two on C, then no 3-coloring of H gives x and y the same color. Similarly, the same holds if C has length eight, and x and y are diagonally opposite on C. On the other hand, if C has length six, and x and y are diagonally opposite on C, then in every 3-coloring of H the vertices x and y receive the same color. Finally, the same holds in the following more complicated scenario. Let C' be a contractible cycle in H of length six such that C belongs to the disk bounded by C', let $z \in V(C) \cap V(C')$, let x be diagonally opposite to z on C, and let y be diagonally opposite to z on C'.

Even-faced drawings in the Klein bottle with an odd meridian walk. Let G be an even-faced drawing in the Klein bottle with no equator of length at most four. By analogy with Theorem 4 one could ask whether G is not 3-colorable if and only if it has a subdrawing that is quadrangulation of the Klein bottle with an odd meridian walk, but that is not true. There is another obstruction to 3-colorability, the following. Let G be a near-quadrangulation of the Klein bottle with a hole C of length six and with an odd meridian walk, and let C' be an equator in G of length six dividing G into two even-faced projective planar drawings G_1 and G_2. Since G has an odd meridian walk we may assume that G_1 is bipartite and G_2 is not. Let the cycle C belong to G_1. Then in every 3-coloring of G the diagonally opposite vertices of both C and C' must receive the same color. However, there could be edges between C and C' joining vertices that must receive the same color. A simple example may be found in Figure 9.

Triangle-free drawings. It would be nice to say something about (not necessarily even-faced) triangle-free drawings in a fixed surface Σ. Can their 3-colorability be tested in polynomial time? The authors of [9] asked whether it is true that for every orientable surface Σ there exists an integer k such that every triangle-free drawing in Σ of edge-width at least k is 3-colorable.

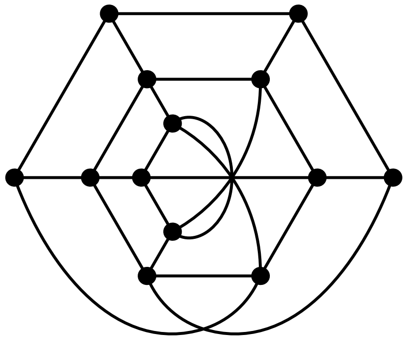

Figure 9: An example of a non-3-colorable near-quadrangulation of the Klein bottle.

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