Charles University in Prague

Faculty of Mathematics and Physics

Doctoral Thesis

On XY mappings

Tension-continuous and related types of mappings

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Preface

This is a treatise on certain mappings between graphs, defined by means of cycle structure of the respective graphs. We study these mappings from various perspectives, in Chapter 2 we start by comparing them to graph homomorphisms, this may be viewed as a type of reconstruction problems, as we study to what extent is a graph determined by its cycle structure. In Chapter 3 (and part of Chapter 7) we take the point of view of category theory and study the structure that these mappings impose on the class of all graphs. Next, we get to more applicable aspects of the mappings under study. In Chapter 4 we use them to prove certain relaxation of Pentagon Problem due to Nešetřil. In Chapter 5 we introduce a new graph invariant which promises to be useful more generally for study of graph homomorphisms. In Chapter 6 we use our mappings to bring new understanding to various conjectures concerning cycle structure of graphs (particularly to Cycle double cover conjecture).

Chapter 1 is introductory and should be read first (at least Definition 1.2.1), its first part provides more detailed motivation for and overview of this work. The other chapters can be read in any order; there are, however, many dependencies between them.

Core of the thesis is based on the following papers:

[1] Matt DeVos and Robert Šámal, *High-girth cubic graphs map to the Clebsch graph*, (submitted), arXiv:math.CO/0602580.

[2] Jaroslav Nešetřil and Robert Šámal, *Tension-continuous maps—their structure and applications*, (submitted), arXiv:math.CO/0503360.

[3] Jaroslav Nešetřil and Robert Šámal, *On tension-continuous mappings*, (submitted), arXiv:math.CO/0602563.

[4] Robert Šámal, *Fractional covering by cuts*, Proceedings of 7th International Colloquium on Graph Theory (Hyères, 2005), no. 22, 2005, pp. 455–459.

Papers [2,3] form most of Chapters 2 and 3 and also part of Chapter 7. Paper [1] appears here as Chapter 4. Finally, Chapter 5 is expanded version of [4].

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I want to thank to all the people who discussed with me various topics related to this research. They are way too many to name all of them. Still, I want to mention particularly Matt DeVos. Our collaboration (that lead to paper [1] and to Chapter 4 of this thesis) was a joyful experience.

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Most importantly, I wish to thank my family for staying with me in the tough times of finishing the thesis. I cannot forget my son Antonín for the inspiring hours spent trying to make him fall asleep. My deepest thanks belong to my wife Terka, for help with proofreading the thesis and for her endless caring support.

Contents

Chapter 1

Introduction

1.1 Motivation and overview of the results

In this section we will give motivation for the notions we are going to study in this thesis and bring an overview of the main results. We present only simplified version of definitions here, full version will be given in Section 1.2.

We will study certain mappings between edge-sets of graphs. One way that leads naturally to these mappings is an attempt to generalize flows and tensions on graphs. Recall that a mapping φ from the set of edges of a (directed) graph to an additive structure (usually a group) is a *flow* (a *tension*) if the values of φ 'around each vertex' (or 'along each circuit') sum to zero. We will extend this notion by replacing a group with an additive structure based on a graph. That is, an H*-valued flow* (*tension*, respectively) is a mapping from edges of a given graph to edges of H such that the 'image' of a cut (circuit) is an edge-disjoint union of (arbitrarily oriented) circuits of H, arguably the most natural analogue of zero in our setting (we will call an edge-disjoint union of circuits a *cycle*). Here 'image' means the set of such edges of H to which an odd number of edges map. (In another version of the definition we ask for orientation of cuts/cycles to be preserved as well, this will be formalized by considering images and preimages of tensions/flows.)

The above definition will be formulated precisely (and more generally) as Lemma 1.2.9. The cycle–cut duality can be used to obtain an equivalent definition (Definition 1.2.1), which is in fact the one we will start with in Section 1.2. Instead of asking that the image of any cycle is a cycle, we can (equivalently) require that the preimage of any cut is a cut—hence H-valued tensions are usually called cut-continuous mappings (to H), and these are special case of tension-continuous, or TT mappings. By switching from cuts to cycles in one (or both) parts of this definition, we get FT mappings (that is, H-valued flows), FF (cycle-continuous) mappings, and TF mappings.

To motivate a notion by generalization of flows and tensions may perhaps seem inadequate; it turns out, however, that these mappings have important combinatorial meaning.

To start with, cut-continuous mappings (that is TT mappings, generalized tensions) are in many respects similar to graph homomorphisms. In Chapter 2 we study a class of graphs for which existence of homomorphisms and TT mappings coincides (*homotens graphs*). The main result is that, surprisingly, random graphs are homotens with probability tending to 1. In Chapter 3 we pursue the similarity of homomorphisms and TT mappings further and show that both types of mappings share important structural properties (universality, density, antichain extension).

In Chapter 4 we (motivated by previous chapters) prove a weaker version of Nešetřil's Pentagon Problem: We show (Theorem 4.1.3) that a cubic graph of sufficiently high girth admits a cut-continuous mapping to C_5; the original problem asks for a homomorphism to C_5. In Chapter 5 we introduce a new graph invariant, monotone with respect to homomorphisms and cut-continuous mappings. This invariant resembles circular chromatic number and is related to MAXCUT and bipartite subgraph polytope.

In Chapter 6 we finally come to the most appealing reason for study of the quadruple of XY mappings, that is to the use of them as a tool to approach various conjectures about cycle structure of graphs, including Cycle double cover (shortly CDC) conjecture, Tutte's 5-flow conjecture, and Berge-Fulkerson conjecture. (This approach was pioneered by Jaeger [43].) Among others, we use FF and FT mappings to understand proofs of existence of CDC for special types of graphs by Tarsi [82], Häggkvist and McGuinness [33] and to clarify the relation between CDC conjecture and Jaeger's conjecture on Petersen coloring.

We close by Chapter 7 with several smaller results. Of them, particularly worth mentioning is Section 7.1 that brings surprising connection to error correcting codes and Section 7.4 where factorizations of TT and FF mappings are studied and used to prove a variant of Lovász' theorem on a complete system of invariants.

1.2 Definitions & basic properties

1.2.1 Basic notions: flows and tensions

Our terminology is standard, with the exception of terms 'cut', 'cycle', and 'circuit', where we follow the usage common in the study of flows on graphs, rather than the usual one. We refer to [21, 40] for basic notions on graphs and their homomorphisms.

By a graph we mean a finite directed or undirected graph with multiple edges and loops allowed. We use (u, v) for a directed and $\{u, v\}$ for an undirected edge from u to v (one of them, if there are several parallel edges). When there is no danger of confusion we use uv to mean either (u, v) or $\{u, v\}$. A *circuit* in a graph is a connected subgraph in which each vertex is adjacent to two edges. For a circuit C, we let C^+ and C^- be the sets of edges oriented in either direction. We will say that (C^+, C^-) is a *splitting* of C and write $C = (C^+, C^-)$. (Of course we can not tell which direction is which, so we may exchange C^+ and C^-.) A *cycle* is an edge-disjoint union of circuits. *Splitting of a cycle* is determined by splittings of the individual circuits.

Given a graph G and a set U of its vertices, we let $\delta(U)$ denote the set of all edges with one end in U and the other in $V(G) \setminus U$; we call each such edge set a *cut* in G. If G is directed, then $\delta^+(U)$ contains edges leaving U and $\delta^-(U)$ edges entering U; if T is a cut we write $T = (T^+, T^-)$ to indicate the sets of edges in either direction.

Let M be a ring (by this we mean an associative ring with unity), let G be a directed graph for the rest of this section. We say that a function $\varphi : E(G) \to M$ is an *M-flow on G* if for every vertex $v \in V(G)$

$$\sum_{e \text{ enters } v} \varphi(e) = \sum_{e \text{ leaves } v} \varphi(e)\,.$$

A function $\tau : E(G) \to M$ is an *M-tension on G* if for every circuit C in G (with splitting (C^+, C^-)) we have

$$\sum_{e \in C^+} \tau(e) = \sum_{e \in C^-} \tau(e)\,.$$

Note that the definition of flow can be equivalently expressed as that of a tension, we just do the summation over a cut $C = (C^+, C^-)$.

Whenever B is a subset of M, we say a function is an (M, B)-flow (an (M, B)-tension) if it is an M-flow (an M-tension) which attains values in B only.

We remark that the definition of flows and tensions can be stated for any abelian group; in Section 7.2 we justify the restriction to rings. More concretely, in the proof of Lemma 3.1.10, we present a way how results about general abelian groups can be inferred from finitely generated rings.

All M-tensions on a graph G form a module over M (or even a vector space, if M is a field). Its dimension is $|V(G)| - k(G)$, where $k(G)$ denotes the number of components of G. This module will be called the *M-tension module* of G.

For a cut $C = \delta(U)$ we define

$$\tau_C(uv) = \begin{cases} 1, & \text{if } uv \in C^+, \text{ that is } u \in U \text{ and } v \notin U \\ -1, & \text{if } uv \in C^-, \text{ that is } u \notin U \text{ and } v \in U \\ 0, & \text{otherwise}\,. \end{cases}$$

Any such τ_C is called an *M-tension determined by cut C*, or simply a cut-tension. If $U = \{u\}$ than we call $\tau_u = \tau_{\{u\}}$ a vertex-tension. We also name the vertex-tensions as *elementary tensions*: it is easy to prove that elementary M-tensions generate the M-tension module.

Remark that every M-tension is of form δp, where $p : V(G) \to M$ is any mapping and $(\delta p)(uv) = p(v) - p(u)$ (in words, tension is a difference of a potential).

We define here a related construction. Let M be an (abelian) group, $B \subseteq M$. By *Cayley graph* $\mathrm{Cay}(M, B)$ we mean a directed graph with vertex set M and with such edges (x, y) for which $y - x \in B$. If we let $p : V(\mathrm{Cay}(M, B)) \to M$ be the identity, then δp is an (M, B)-tension on $\mathrm{Cay}(M, B)$. (We will see an application of this observation in Proposition 1.2.12.) If $-B = B \not\ni \{0\}$ then $\mathrm{Cay}(M, B)$ is a symmetric orientation of an undirected graph, which will sometimes be caled an undirected Cayley graph.

For M-flows the situation is similar to M-tensions: If $C = (C^+, C^-)$ is a circuit (a cycle) then we define

$$\varphi_C(e) = \begin{cases} 1, & \text{if } e \in C^+ \\ -1, & \text{if } e \in C^- \\ 0, & \text{otherwise} \end{cases}$$

and call it a *flow determined by C*, or simply a circuit-/cycle-flow. Each circuit-flow will be also called an *elementary flow*. All M-flows on G form a module (the *M-flow module* of G) of dimension $|E(G)| - |V(G)| + k(G)$; it is generated by elementary flows and orthogonal to the M-tension module.

The above are the basic notions of algebraic graph theory. For a more thorough introduction to the subject see [21] or [27]; we only mention two more basic observations:

A cycle can be characterized as a support of a $\mathbb{Z}_2$-flow and a cut as a support of a $\mathbb{Z}_2$-tension. If G is a plane graph then each cycle in G corresponds to a cut in its dual G^*; each flow on G corresponds to a tension on G^*.

1.2.2 XY mappings

Next we define the principal notion of this thesis. For conciseness we will use term *F-mapping* instead of *flow* and *T-mapping* instead of *tension*. Also we let F^* mean T and T^* mean F.

Definition 1.2.1 *Let M be a ring, let G, H be directed graphs and let $f : E(G) \to E(H)$ be a mapping between their edge sets. Let $X, Y \in \{F, T\}$. We say f is an* XY_M mapping *if for every Y-mapping $\varphi : E(H) \to M$, the composed mapping φf is an X-mapping on G. Explicitly,*

- *f is TT_M iff $\varphi \circ f$ is an M-tension on G for every M-tension φ on H,*
- *f is FT_M iff $\varphi \circ f$ is an M-flow on G for every M-tension φ on H,*
- *f is TF_M iff $\varphi \circ f$ is an M-tension on G for every M-flow φ on H,*
- *f is FF_M iff $\varphi \circ f$ is an M-flow on G for every M-flow φ on H,*

Clearly, it is enough to verify the condition of Definition 1.2.1 on a basis of the M-flow (M-tension) module of H, e.g., for elementary flows and tensions. The scheme below illustrates this definition.

$$\begin{tikzcd} E(G) \arrow[r, "f"] \arrow[dr, "\varphi \circ f"] & E(H) \arrow[d, "\varphi"] \\ & M \end{tikzcd}$$

We write $f : G \xrightarrow{XY_M} H$ if f is an XY_M mapping from G to H (or, more precisely, from $E(G)$ to $E(H)$). In the important case when $M = \mathbb{Z}_n$ we write XY_n instead of $XY_{\mathbb{Z}_n}$, when M is clear from the context we omit the subscript.

We remark that more generally we could define XY mapping between any two matroids. Then an FF mapping is simply a TT mapping between the dual matroids and similarly in the other cases. We do not follow this approach (although it would save us some technical inconveniences), as we wish to keep our treatment confined to the realm of graphs. The reason for this is that from the point of view of problems we are willing to study (such as Cycle double cover), graphical matroids are rather an exception: indeed, the Cycle double cover conjecture is false e.g. for the uniform matroid $U_{2,4}$.

Of course if $M = \mathbb{Z}_2$ then the orientation of edges does not matter. Hence, if G, H are undirected graphs and $f : E(G) \to E(H)$ any mapping, we say that f is an XY_2 mapping if for some (equivalently, for every) orientation $\overrightarrow{G}$ of G and $\overrightarrow{H}$ of H, f is XY_2 mapping from $\overrightarrow{G}$ to $\overrightarrow{H}$. As cuts correspond to $\mathbb{Z}_2$-tensions and cycles to $\mathbb{Z}_2$-flows, with this provision TT_2 (FT_2) mappings of undirected graphs are mappings for which preimage of any cut is a cut (a cycle). By Lemma 1.2.9, these are exactly the generalized tensions (flows) from introduction. Also, for FF_2 mappings preimage of a cycle is a cycle; hence, we call TT_2 mappings *cut-continuous* and TT_2 mappings *cycle-continuous* (in analogy with continuous mappings between topological spaces). Note that most important choices of the ring are $M = \mathbb{Z}_2$ and $M = \mathbb{Z}$ (as exemplified in Section 6.2). However, developing the theory for general rings presents no difficulty, so we prefer this unified treatment.

For general ring M, the orientation is important. Still, we define that a mapping $f : E(G) \to E(H)$ between undirected graphs G, H is XY_M if for some orientation $\overrightarrow{G}$ of G and $\overrightarrow{H}$ of H, f is XY_M mapping from $\overrightarrow{G}$ to $\overrightarrow{H}$. This definition may seem a bit arbitrary, but in fact it is a natural one: clearly it is equivalent to require that for each $\overrightarrow{H}$ there is an $\overrightarrow{G}$ such that f is an XY_2 mapping from $\overrightarrow{G}$ to $\overrightarrow{H}$ (we just change orientation of edges of $\overrightarrow{G}$ according to the change of orientation of edges of $\overrightarrow{H}$). We will elaborate more on this in Proposition 1.2.2.

Convention. Unless specifically mentioned, our results hold for both the directed and undirected case.

Recall that $h : V(G) \to V(H)$ is called a *homomorphism* if for any $uv \in E(G)$ we have $f(u)f(v) \in E(H)$; we write shortly $h : G \xrightarrow{hom} H$. It is customary to investigate homomorphisms in the context of a quasiorder $\preccurlyeq_h$ defined on the class of all graphs by

$$G \preccurlyeq_h H \iff \text{there is a homomorphism } h : G \xrightarrow{hom} H.$$

Homomorphisms generalize colorings: a k-coloring is exactly a homomorphism $G \xrightarrow{hom} K_k$, hence $\chi(G) \le k$ iff $G \preccurlyeq_h K_k$. For an introduction to the theory of homomorphisms consult [40].

Motivated by the homomorphism order $\preccurlyeq_h$, we define for a ring M orders $\preccurlyeq_M^t$ and $\preccurlyeq_M^f$ by

$$G \preccurlyeq_M^t H \iff \text{there is a mapping } f : G \xrightarrow{TT_M} H;$$
$$G \preccurlyeq_M^f H \iff \text{there is a mapping } f : G \xrightarrow{FF_M} H.$$

These are indeed quasiorders, see Lemma 1.2.3. We write $G \approx_M^t H$ iff $G \preccurlyeq_M^t H$ and $G \succcurlyeq_M^t H$, and similarly we define $G \approx_M^f H$ and $G \approx_h H$; we say G and H are TT_M-equivalent, FF_M-equivalent, or hom-equivalent, respectively. Sometimes, we also write $G \xrightarrow{XY_M} H$ to denote existence of an XY_M mapping from G to H. If we wished to define partial orders instead of quasiorders, we would have two options: either to work with equivalence classes (of $\approx_M^t$ or $\approx_M^f$) of graphs, or to choose one representative from each such class, e.g. so-called *cores*, similarly as in the case of homomorphisms. We touch this topic briefly in Proposition 7.4.6.

In addition to orders $\preccurlyeq_h$ and $\preccurlyeq^t$ we will often study homomorphisms and TT mappings in terms of the corresponding categories. We write $\mathcal{G}ra_{hom}$ for the category with all finite graphs as objects and all homomorphisms among them as morphisms. Category $\mathcal{G}ra_{TT_M}$ has the same class of objects, its morphisms are TT_M mappings.

If G is an undirected graph, its T-*symmetric*[1] *orientation* $\overleftrightarrow{G}_T$ is a directed graph with the same set of vertices and with each edge replaced by an oriented 2-cycle, we will say these two edges are opposite. We obtain the F-*symmetric orientation* $\overleftrightarrow{G}_F$ by subdividing each edge into two and by orienting the resulting two edges in opposite directions (out of the new vertex). The following result clarifies the role of orientations.

Proposition 1.2.2 *Let G, H be undirected graphs, let M be a ring. Then the following are equivalent.*

1. *For some orientation $\overrightarrow{G}$ of G and $\overrightarrow{H}$ of H it holds that $\overrightarrow{G} \xrightarrow{XY_M} \overrightarrow{H}$.*
2. *For each orientation $\overrightarrow{H}$ of H there is an orientation $\overrightarrow{G}$ of G such that $\overrightarrow{G} \xrightarrow{XY_M} \overrightarrow{H}$.*
3. *For symmetric orientations $\overleftrightarrow{G}_X$ of G and $\overleftrightarrow{H}_Y$ of H it holds that $\overleftrightarrow{G}_X \xrightarrow{XY_M} \overleftrightarrow{H}_Y$.*

[1] or just symmetric

Proof: If $M = \mathbb{Z}_2^k$ then all statements are easily equivalent, so suppose this is not the case. Suppose 1 holds and take a mapping $f_1 : \overrightarrow{G} \xrightarrow{XY_M} \overrightarrow{H}$. We may suppose that $\overrightarrow{G} \subseteq \overleftrightarrow{G}_X$ and $\overrightarrow{H} \subseteq \overleftrightarrow{H}_Y$. If e', e'' are opposite edges and $e' \in E(\overrightarrow{G})$, then we let $f_3(e')$ be $f_1(e')$ and $f_3(e'')$ be the edge opposite to $f_1(e')$. Any Y-mapping φ on $\overleftrightarrow{H}_Y$ gives opposite values to opposite edges, thus $\varphi \circ f_3$ is a mapping that results from $\varphi \circ f_1$ by extending to opposite edges by opposite values. Thus $\varphi \circ f_3$ is an X-mapping and f_3 is XY_M. Next, take any $\overrightarrow{H}$, suppose again $\overrightarrow{H} \subseteq \overleftrightarrow{H}_Y$, and let opposite edges e', e'' of $\overleftrightarrow{G}_X$ correspond to $e \in E(G)$. By Lemma 1.2.9, the edges $f_3(e')$, $f_3(e'')$ receive opposite values in each Y-mapping on H; therefore at least one of them agrees with some edge $\bar{e}$ of $\overrightarrow{H}$ in each Y-mapping on H. (That is, if $Y = T$ then these edges connect the same pair of vertices in the same direction, if $Y = F$ then they are in the same circuits of H in the same direction.) We let this one of e', e'' to be an edge of $\overrightarrow{G}$ and let f_2 map it to $\bar{e}$. Clearly, f_2 is an XY_M mapping; therefore 3 implies 2. Finally, 2 implies 1 is trivial. □

1.2.3 Basic properties

In this section we summarize some properties of XY mappings that will be needed in the sequel. Throughout the chapter, X, Y, and Z stand for either F or T.

Lemma 1.2.3 *For any mappings $f : G \xrightarrow{XY_M} H$ and $g : H \xrightarrow{YZ_M} K$ the composition $g \circ f$ is an XZ_M mapping.*

Proof: Let $\varphi : E(K) \to M$ be a Z-mapping. Then $\varphi \circ g$ is a Y-mapping, hence $\varphi \circ g \circ f$ is an X-mapping. For undirected graphs we use part 2 (or 3) of Proposition 1.2.2. □

$$\begin{tikzcd} & E(H) \arrow[dr, "YZ"] & \\ E(G) \arrow[ur, "XY"] \arrow[rr, "XZ"] & & E(K) \end{tikzcd}$$

In the next lemma we will see how the cut–cycle duality of planar graphs translates to XY mappings. Compare also Lemma 6.2.5, which is an analogue for graphs embedded on a non-planar surface.

Lemma 1.2.4 *Let G be a plane graph, G^* its planar dual, and $d_G : E(G) \to E(G^*)$ the mapping that sends e to its corresponding edge e^*. Let M be any ring. Then d_G is an FT_M and TF_M mapping.*

Let H be another plane graph with dual H^*. Suppose $f : E(G) \to E(H)$ is a TT_M mapping. Then

- *$e \mapsto f(e^*)$ is an FT_M mapping,*
- *$e \mapsto f(e)^*$ is a TF_M mapping, and*
- *$e \mapsto f(e^*)^*$ is an FF_M mapping.*

$$\begin{tikzcd}
E(G) \arrow[r, "TT"] \arrow[d, "d_G"'] \arrow[dr, "TF"] & E(H) \arrow[d, "d_H"] \\
E(G^*) \arrow[r, "FF"] \arrow[ur, "FT"'] & E(H^*)
\end{tikzcd}$$

Proof: For the first part we only need to recall that φ is a flow/tension on G if and only if $\varphi \circ d_G$ is a tension/flow on G^*. (This well-known claim is implied by the fact that d_G maps cuts of G to cycles of G^* and vice versa.) The second part is a consequence of the first one and of Lemma 1.2.3. For undirected graphs it is enough to pick an orientation of G and G^* simultaneously, that is the edge e^* of G^* connects the face to the left of e to the one to the right of e. $\square$

In the sequel we need to define a notion dual to that of a subgraph. Let G be a graph and e one of its edges. Then G/e is obtained from G by *contracting* edge e, that is by deleting e and identifying its end-vertices, preserving loops and multiple edges. For each $E \subseteq E(G)$ the graph G/E results from G by contracting every edge of E (in any order). Any graph $H = G/E$ for $E \subseteq E(G)$ is called *a contraction of* G, we write $H \overset{c}{\subseteq} G$. Note that dual notion (in the sense of random graph) is deletion of edges (denoted $G - e$, $G - E$) and the subgraph relation. We also remark that the commonly used notion H is a *minor* of G means that H is a subgraph of a contraction of G.

Lemma 1.2.5 *Let M be a ring, G, H graphs.*

If $H \subseteq G$ that the identity mapping is $H \xrightarrow{TT_M} G$.

If $H \overset{c}{\subseteq} G$ that the identity mapping is $H \xrightarrow{FF_M} G$.

Proof: It is enough to observe that restriction of a tension to a subgraph (or of a flow to a contraction) of G is a tension (a flow). $\square$

(In Lemma 2.1.2 we will generalize the fact that inclusion provides a TT mapping for any homomorphism in place of the inclusion.) The above lemma is complemented by the following result—the epimorphism-monomorphism factorization of XY mappings. In Section 7.4 we will in formula (7.5) derive a quantitative version, which will be a crucial part of proof of Theorem 7.4.9,

Proposition 1.2.6 *Let $f : G \xrightarrow{XY} H$. Then there is a graph H' and mappings $f_1 : G \xrightarrow{XY} H'$, $f_2 : H' \xrightarrow{YY} H$ such that f_1 is surjective and f_2 injective.*

In case $Y = T$ we may choose the graph H' as a subgraph of H, in case $Y = F$ as a contraction of H.

Proof: Let $R = E(H) \setminus f(E(G))$. We put $H' = H \setminus R$ (if $Y = T$) and $H' = H/R$ (if $Y = F$). We define f_1, f_2 in the obvious way, it remains to prove that these are XY and YY mappings, respectively.

By Lemma 1.2.5 mapping f_2 is YY_M. Next, observe that any tension on $H \setminus R$ (any flow on H/R) may be extended to a tension (a flow) on the graph H. Consequently, f_1 is XY_M because f is XY_M. $\square$

If we do not require a factorization, that is if we only want to reduce the target graph, then we can use either a subgraph or a contraction for this reduction, regardless what type of mapping are we considering.

Lemma 1.2.7 *Let $f : G \xrightarrow{XY_M} H$, let H' be a subgraph (or a contraction) of H that contains all edges $f(e)$ for $e \in E(G)$. Then $f : G \to H'$ is XY_M as well.*

Proof: If $Y = T$ and H' is a subgraph, or $Y = F$ and H' is a contraction then by Proposition 1.2.6 and Lemma 1.2.5 we have $G \xrightarrow{XY_M} f(G) \xrightarrow{YY_M} H'$, so it is enough to use composition, Lemma 1.2.3.

In the other two cases take any Y-mapping $\tau' : E(H') \to M$. To extend τ' to a Y-mapping τ on H it is enough to define $\tau(e) = \tau'(e)$ for $e \in E(H')$ and $\tau(e) = 0$ otherwise. As τ agrees with τ' on $E(H')$, we have $\tau' \circ f = \tau \circ f$, and as $\tau \circ f$ is an X-mapping, $\tau' \circ f$ is an X-mapping, as well. $\square$

Another simple (but useful) way to modify an XY_M mapping is by adding parallel edges, respectively subdividing edges. The next result shows, that we may in many respects restrict ourselves to bijective XY_M mappings (this approach was taken by most authors who studied similar notions before, see Section 1.4).

Lemma 1.2.8 *Let $f : G \xrightarrow{XY_M} H$ be an XY_M mapping of (directed or undirected) graphs. Then there is a graph H' and a mapping $f' : E(G) \to E(H')$ such that*

- *f' is XY_M;*
- *f' is a bijection;*
- *we can get H' by adding parallel edges and deleting edges from H in case $Y = T$, by subdividing and contracting edges in H in case $Y = F$;*
- *for each edge $a \in E(G)$ the edge $f'(a)$ is parallel to/subdivision of $f(a)$.*

Proof: Suppose $Y = T$. For an edge $e \in E(H)$ we let $c(e) = |f^{-1}(e)|$ be the number of edges that map to e. We replace each edge of H by $c(e)$ parallel edges (that is we delete e if $c(e) = 0$); in case of directed graphs we add the new edges in the same direction as e. We keep all vertices and let H' denote the resulting graph. We define $f'(a)$ to be any one of the parallel edges that replaced $f(a)$, making sure that f' is injective (therefore bijective). Clearly, for any $p : V(H) = V(H') \to M$, if we consider the M-tensions $\tau = \delta p$ of H and $\tau' = \delta p$ of H', then $\tau \circ f = \tau' \circ f'$. Thus as f is an XT_M mapping, f' is XT_M as well.

If $Y = F$, we proceed in a similar way; in this case we replace each edge $e \in E(H)$ by an oriented path of $c(e)$ edges (that is if no edge of G maps to e, we contract e). The rest of the proof is the same as for $Y = T$. □

An alternative definition of tension-continuous mappings is often useful. It was proved in [19], we present a proof for the reader's convenience. For mappings $f : E(G) \to E(H)$ and $\varphi : E(G) \to M$ we let φ_f denote the *algebraical image of* φ: that is we define a mapping $\varphi_f : E(H) \to M$ by

$$\varphi_f(e') = \sum_{e \in f^{-1}(e')} \varphi(e)\,.$$

Recall that $F^* = T$ and $T^* = F$.

Lemma 1.2.9 *Let $f : E(G) \to E(H)$ be a mapping. Then f is XY_M if and only if for every X^*-mapping $\varphi : E(G) \to M$, its algebraical image φ_f is a Y^*-mapping. Moreover, it is enough to verify this property for a basis of the flow/tension module (for instance, for elementary flows/tensions).*

We formulate this explicitly for $M = \mathbb{Z}_2$ and TT mappings. Mapping f is TT_2 (cut-continuous) if and only if for every cycle C in G, the set of edges of H, to which an odd number of edges of C maps, is a cycle.

Proof: Recall that a mapping $E(G) \to M$ is a flow/a tension iff it is orthogonal to each tension/flow with respect to the scalar product defined by $\langle \varphi, \tau \rangle = \sum_{e \in E(G)} \varphi(e)\tau(e)$ and the same holds for H. Observe that by definition of the algebraical image φ_f it follows that for any $\varphi : E(G) \to M$ and $\tau : E(H) \to M$ we have

$$\langle \varphi, \tau \circ f \rangle = \langle \varphi_f, \tau \rangle . \tag{1.1}$$

Now if f is XY mapping than for any Y-mapping τ and any X^*-mapping φ the left-hand side of (1.1) is zero. Therefore the right-hand side is zero, too, and φ_f is a Y^*-mapping, as claimed. If on the other hand, f satisfies the condition of the lemma, then for any Y-mapping τ the right-hand side of (1.1) is zero for each X^*-mapping φ, hence $\tau \circ f$ is an X-mapping and f is an XY_M mapping. □

If C is a circuit/a cut and $C = (C^+, C^-)$, then we say that C is *M-balanced* if $|C^+| - |C^-|$ is divisible by the characteristic of M, that is if we get 0 by adding $|C^+| - |C^-|$ instances of 1. Otherwise, we say C is *M-unbalanced*. Let $g_M(G)$ ($\lambda_M(G)$) denote the length of the shortest M-unbalanced circuit (cut, respectively) in G, if there is none we put $g_M(G) = \infty$ ($\lambda_M(G) = \infty$). For the particular case $M = \mathbb{Z}_2$, a circuit (cut) is M-balanced if it is of even size, hence $g_{\mathbb{Z}_2}(G)$ is the odd-girth of G and $\lambda_{\mathbb{Z}_2}(G)$ the size of the smallest odd cut of G.

Note that if φ is a flow determined by a circuit (by a cut) C in G, then $\sum_{e \in E(G)} \varphi(e)$ is zero iff C is M-balanced. Thus g_M and λ_M can be defined equivalently as the smallest size of a support of a flow/tension with nonzero sum. To emphasize this (and to enable cleaner formulations) we introduce another notation: $F_M(G) := g_M(G)$ and $T_M(G) := \lambda_M(G)$.

Lemma 1.2.10 *Let M be a ring, G, H directed graphs, suppose $f : G \xrightarrow{XY_M} H$. Let C be an M-unbalanced circuit (if $X = T$) or cut (in case $X = F$) in G. Then $f(C)$ contains an M-unbalanced circuit as a subgraph (if $Y = T$) or cut as a contraction (if $Y = F$).*

Proof: By Lemma 1.2.7 we may suppose $H = f(C)$. Let $\varphi : E(G) \to M$ be the X^*-mapping determined by C. By Lemma 1.2.9, φ_f is a Y^*-mapping. Now

$$\sum_{e \in E(H)} \varphi_f(e) = \sum_{e \in E(G)} \varphi(e) \neq 0$$

as C is M-unbalanced. If follows that if we express φ_f as a sum of elementary Y^*-mappings, at least one of them is determined by an M-unbalanced circuit/cut. □

Lemma 1.2.11 *Suppose G, H are directed graphs such that $G \xrightarrow{XY_M} H$. Then we have $X^*_M(G) \geq Y^*_M(H)$. In particular, if $XY = TT$ then $g_M(G) \geq g_M(H)$.*

Proof: If $X^*_M(G) = \infty$, the conclusion holds. Otherwise, let C be an M-unbalanced circuit/cut of size $X^*(G)$ in G. By Lemma 1.2.10, the subgraph/contraction of H induced by $f(C)$ contains an M-unbalanced circuit as a subgraph/an M-unbalanced cut as a contraction, hence the same holds for H. This circuit/cut is of size at least $Y^*_M(H)$ and at most $X^*_M(G)$. □

Let $H = \mathrm{Cay}(M, B)$ be a Cayley graph, $\varphi : E(G) \to E(H)$ an XT mapping, and let δ be the 'canonical' tension $E(H) \to B \subseteq M$ given by $\delta(uv) = v - u$. Then if $X = F$, $\varphi\delta$ is an (M, B)-flow, if $X = T$ is an (M, B)-tension. Thus, FT mappings into Cayley graphs can be thought of as a generalization of flows and TT mappings into Cayley graphs as a generalization of tensions, justifying the motivation given in Section 1.1. It is a tantalizing question to find out whether, conversely, each generalized tension/flow to a Cayley graph is a tension/flow. For tensions this is fully answered by the next proposition, an extension of it leads to the notion of right homotens graphs (Section 2.3). In the case of flows, however, the situation is unclear; we will propose a conjecture in Section 6.2.

Proposition 1.2.12 *Let $H = \mathrm{Cay}(M, B)$ be a Cayley graph. Then the following are equivalent.*

1. $G \xrightarrow{hom} H$
2. $G \xrightarrow{TT_{\mathbb{Z}}} H$
3. $G \xrightarrow{TT_M} H$
4. *G has a B-tension.*

Moreover, in 1–3 we can take the same mapping.

Proof: By Lemma 2.1.2 we see that 1 implies 2 and by Lemma 7.2.2 follows that 2 implies 3. We proved that 3 implies 4 before stating the proposition. Finally, if τ is an (M, B)-tension on G then we express τ as δp for $p : V(G) \to M$ and observe that p is in fact a homomorphism to H, which proves 1. □

1.3 Basic examples

We start by trivial examples, direct analogues of homomorphism to a graph containing a loop. Here (as in the whole section), G, H are any (directed or undirected) graphs, X, Y stand for F or T.

- If H is formed by loops only, then any mapping to H is XT.
- If H is formed by cut-edges only (i.e., H is a forest), then any mapping to H is XF.
- If G is formed by loops only, then any mapping from G is FY.
- If G is formed by cut-edges only (i.e., G is a forest), then any mapping from G is TY.

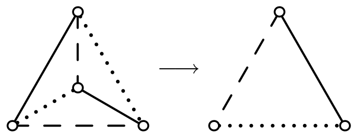


Figure 1.1: A mapping $K_4 \xrightarrow{TT_2} K_3$ that is not induced by mapping of vertices.

More interesting instances of XY mappings are obtained by deleting/contracting edges, see Lemma 1.2.5. As an example of this, let K_2^3 be a graph with two vertices and three edges between them. Clearly $K_2^3 \overset{c}{\subset} K_4$, hence $K_2^3 \xrightarrow{FF_{\mathbb{Z}}} K_4$ and by duality (Lemma 1.2.4) or by another application of Lemma 1.2.5 we have $K_3 \xrightarrow{TT_{\mathbb{Z}}} K_4$. The mapping in the other direction, $K_4 \overset{TT_{\mathbb{Z}}}{\not\longrightarrow} K_3$ does not exist (by Lemma 2.2.5). However, if we consider TT_2 mappings, then (perhaps surprisingly) we find one.

Consider the 1-factorization of K_4 (see Figure 1.1). This constitutes a mapping $K_4 \xrightarrow{TT_2} K_3$, hence $K_4 \approx_2^t K_3$. For complete graphs this example is the only one (Corollary 2.2.9). On the other hand, there are several other examples demonstrating that the existence of TT mapping is not a very restrictive relation. For instance the well-known graphs depicted in Figure 1.2 are all TT_2-equivalent. (In Chapter 2 we will study conditions which guarantee that a TT mapping is induced by a mapping of vertices.)

These mappings (and many others) may be obtained using the following construction: Given an (undirected) graph $G = (V, E)$ write $\Delta(G)$ for the graph

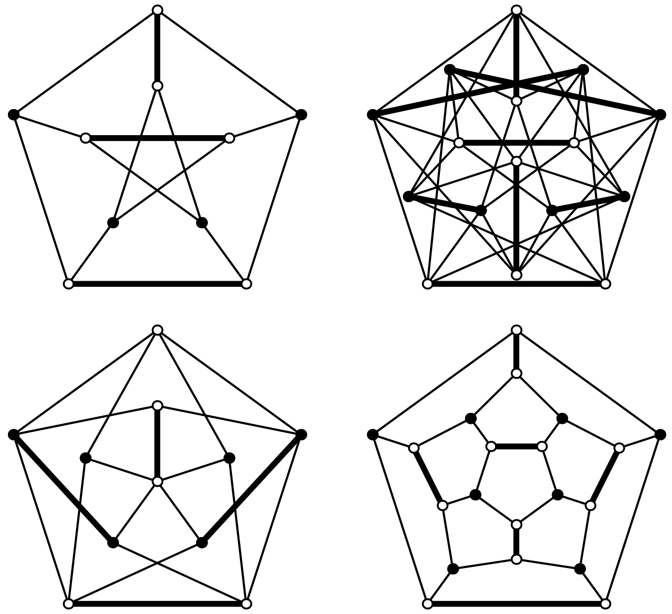


Figure 1.2: Examples of graphs that are TT_2-equivalent to C_5: Petersen graph, Clebsch graph, Grötsch graph, and graph of the dodecahedron. One color class is emphasized, the respective bipartition of the vertex set is depicted, too. The other four color classes are obtained by a rotation.

$(\mathcal{P}(V), E')$, where $AB \in E'$ iff $A \,\Delta\, B \in E$ (here $\mathcal{P}(V)$ denotes the set of all subsets of V and $A \,\Delta\, B$ the symmetric difference of sets We can formulate this construction for rings $M \neq \mathbb{Z}_2$, this is done in Section 2.3.1. Here we only state a special case of Lemma 2.3.3 (which is proved in [19]).

Lemma 1.3.1 *Let G, H be undirected graphs. Then $G \xrightarrow{TT_2} H$ iff $G \xrightarrow{hom} \Delta(H)$.*

Proposition 1.3.2 *Let Pt be the Petersen graph, Cl the Clebsch graph, Gr the Grötsch graph, D the dodecahedron (see Figure 1.2). Then $\mathrm{Pt} \approx_2^t Cl \approx_2^t Gr \approx_2^t D \approx_2^t C_5$. On the other hand, in the homomorphism order no two of these graphs are equivalent. Moreover, Cl is the largest (in $\preccurlyeq_h$) graph that admits a TT_2 mapping to C_5.*

Proof: We have $C_5 \subset \mathrm{Pt} \subset Cl$, $C_5 \subset D$, and $C_5 \subset Gr \subset Cl$. As inclusion is a homomorphism and hence it induces a TT mapping, we only need to provide mappings $Cl \xrightarrow{TT} C_5$ and $D \xrightarrow{TT} C_5$ (though we provide a mapping to C_5 from the other graphs as well). In Figure 1.2, we emphasize some edges in each graph. Let G be the considered graph and $A \subseteq E(G)$ the set of bold edges. Put $A_1 = A$ and let A_2, A_3, A_4, A_5 denote the sets obtained from A by rotation, so that the sets A_i partition $E(G)$. Define a mapping $E(G) \to E(C_5) = \{e_1, \dots, e_5\}$ by sending all edges in A_i to e_i.

Note that 4-edge subgraphs of C_5 generate its $\mathbb{Z}_2$-tension space. Hence it is enough to verify that after deleting any color class of G we are left with a cut. Due to symmetry we only need to check that $E(G) \setminus A$ is a cut in G. This is straightforward to verify, the corresponding bipartition of vertices is depicted in Figure 1.2.

If $G \xrightarrow{TT_2} C_5$ then by Lemma 1.3.1 it holds $G \xrightarrow{hom} \Delta(C_5)$ and it is a routine to verify that $\Delta(C_5)$ consists of two components, both of which are isomorphic to Cl (compare also Section 2.3). Consequently, $G \preccurlyeq_h Cl$ as claimed. $\square$

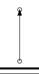
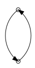
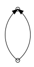
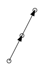
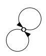
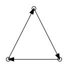
There are quite a few connections of XY mappings to well-established notions of graph theory. To start with, by Lemma 6.2.5 and Figure 6.2 there is a mapping $\mathrm{Pt} \xrightarrow{FT_2} K_6$ and preimages of vertex cuts in K_6 give a list of 6 cycles that cover each edge of Pt exactly twice (Lemma 6.2.2). Cycle double cover conjecture (Conjecture 6.1.3) states that such list of cycles exists for each bridgeless graph.

On a similar note, if G is a cubic graph then the number of mappings $G \xrightarrow{FT_2} K_3$ equals the number of 1-factorizations of G (as this is for cubic graph the same as the number of triples of cycles covering each edge twice). By Lemma 1.2.4 this yields again the mapping $K_4 \xrightarrow{TT_2} K_3$ from Figure 1.1; on the other hand it proves that if G is a snark (a cubic graph that is not 3-edge-colorable) than $G \overset{FT_2}{\not\longrightarrow} K_3$, equivalently $G \overset{FF_2}{\not\longrightarrow} K_2^3$. In particular we have that $\mathrm{Pt} \overset{FF_2}{\not\longrightarrow} K_2^3$. An important conjecture due to Jaeger (Conjecture 6.1.7) claims that this is the 'worst case': whenever G is a bridgeless graph, then $G \xrightarrow{FF_2} \mathrm{Pt}$.

For a different example, let T be a graph with two vertices, one edge connecting them and one loop. It is known that the number of homomorphisms $f : G \xrightarrow{hom} T$ equals to the number of independent sets of the graph G, a graph parameter that is important and hard to compute. The corresponding parameter, the number of cut-continuous mappings $g : G \xrightarrow{TT_2} T$ is simple to compute (but still interesting): it is equal to the number of cuts in G, that is to $2^{|V(G)|-k}$, where

k is the number of components of G. The number of mappings $G \xrightarrow{TT_{\mathbb{Z}}} \overrightarrow{K}_2$ equals the number of $(\mathbb{Z}, \{1\})$-tensions, that is of homomorphisms of G to an infinite directed path (with one vertex of each component fixed).

We collect more such results for small target graphs in Table 1.1. We omit the (usually straightforward) proofs.

H	FF	FT	TF	TT
	G is Eulerian (has $(\mathbb{Z},\{1\})$-flow)	any G	$G \xrightarrow{hom} \overrightarrow{P}_\infty$ (exists $(\mathbb{Z},\{1\})$-tension)	any G
	any G	G is Eulerian (has $(\mathbb{Z},\{1\})$-flow)	any G	$G \xrightarrow{hom} \overrightarrow{P}_\infty$ (exists $(\mathbb{Z},\{1\})$-tension)
	G is Eulerian (has $(\mathbb{Z},\{1\})$-flow)	G has even degrees ($(\mathbb{Z},\{\pm1\})$-flow)	$G \xrightarrow{hom} \overrightarrow{P}_\infty$ (exists $(\mathbb{Z},\{1\})$-tension)	$G \xrightarrow{hom} \overleftrightarrow{P}_\infty$ (exists $(\mathbb{Z},\{\pm1\})$-tension)
	G has even degrees ($(\mathbb{Z},\{\pm1\})$-flow)	G is Eulerian (has $(\mathbb{Z},\{1\})$-flow)	$G \xrightarrow{hom} \overleftrightarrow{P}_\infty$ (exists $(\mathbb{Z},\{\pm1\})$-tension)	$G \xrightarrow{hom} \overrightarrow{P}_\infty$ (exists $(\mathbb{Z},\{1\})$-tension)
	the same as			
	the same as 			
	any G, counts the number of Eulerian subgraphs of G	any G, counts the number of Eulerian subgraphs of G	any G, counts the number of homomorphisms $G \xrightarrow{hom} \overrightarrow{P}^l_\infty$	any G, counts the number of homomorphisms $G \xrightarrow{hom} \overrightarrow{P}^l_\infty$
	G is Eulerian	G can be decomposed into 3 postman joins	$G \xrightarrow{hom} \overrightarrow{P}_\infty$	$G \xrightarrow{hom} \overrightarrow{C}_3$

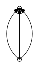
	G can be decomposed into 3 postman joins	G is Eulerian	$G \xrightarrow{hom} \overrightarrow{C}_3$	$G \xrightarrow{hom} \overrightarrow{P}_\infty$

Table 1.1: Characterization of directed graphs G such that $G \xrightarrow{XY_{\mathbb{Z}}} H$, for small instances of H. (In the seventh row $\overrightarrow{P}^{\circlearrowleft}_\infty$ denotes infinite oriented path with a loop on each vertex; the homomorphisms are counted upto a shift.)

1.4 Relevant literature

This thesis continues and generalizes the work started in a paper by DeVos, Raspaud, and Nešetřil [19]. This paper extends Jaeger's [43] approach to classical conjectures on the structure of cycles in a graph (such as Berge-Fulkerson conjecture, Cycle double cover conjecture, Tutte's 5-flow conjecture); we will discuss this approach in more detail in Chapter 6. Perhaps more importantly, Jaeger's approach is in [19] put into more general framework of tension-continuous and flow-continuous (or TT and FF, in our terminology) mappings.

Prior to that, special cases of XY mappings were studied (sometimes implicitly):

- Whitney's [89, 90] classical 2-isomorphism theorem (Theorem 7.4.4) can be restated in our language: For 3-connected graphs G and H, any bijection $f : E(G) \to E(H)$ such that both f and f^{-1} are TT_2[2] is induced by an isomorphism. (A characterization for non-3-connected graphs is given as well.)
- Kelmans [49] generalized Whitney's theorem by introduction of circuit and cocircuit semi-isomorphisms and semi-dualities of graphs. These are equivalent to definition of XY mapping, although the notion is only defined when the mapping is a bijection.
- Linial, Meshulam, and Tarsi [55] study five classes of bijective mappings between edge-sets of graphs. These are chromatic (induced by a homomorphism), cyclic (TT_2), orientable cyclic ($TT_{\mathbb{Z}}$), strong and weak mappings.[3]

[2] or, equivalently, FF_2.

[3] The last two types of mappings are frequently studied in a matroidal setting, in contrary with TT_2 and FF_2 mappings which can be defined for matroids, too.

Among else, they prove that each of these mappings includes the next one and consider versions of chromatic number defined by means of these mapping (compare Section 7.1 and Corollary 2.3.9).

- Shih [80] views a bijection $G \xrightarrow{TT_2} H$ as an identification $E(G) = E(H)$ for which the cycle-space of G is a subspace of cycle-space of H (compare Lemma 1.2.9). He characterizes the pairs of graphs for which dimensions of these two spaces differ by at most 1.
- Tarsi [83] studies bijective FT mappings and discusses their relevance to Cycle double cover conjecture.

In each of the above-mentioned papers, only bijective mappings are studied. As explained in Lemma 1.2.8, this restriction does not loose any generality (although we believe that it is advantageous to allow non-bijective mappings, too).

Several other papers used methods that can be nicely expressed and/or motivated by our context, see for example Section 6.3 and 7.1.

Chapter 2

TT mappings & homomorphisms: homotens graphs

In this chapter we will start to inquire the relationship between homomorphisms and TT mappings. We will see that homomorphisms constitute a typical example of TT mappings and, moreover, that for many graphs these two types of mappings coincide (although much evidence in the other direction will be given, too). We will study properties of such graphs (called left and right homotens graphs) and use these graphs to embed the category $\mathcal{G}ra_{hom}$ of graphs and homomorphisms to the category $\mathcal{G}ra_{TT}$ of graphs and TT mappings.

2.1 Introduction

It is a traditional mathematical theme to study the question when a map between the sets of substructures is induced (as a lifting) by a mapping of underlying structures. In a combinatorial setting (and as one of the simplest instances of this general paradigm) this question takes the following form:

Question 2.1.1 *Given undirected graphs G, H and a mapping $g : E(G) \to E(H)$, does there exist a mapping $f : V(G) \to V(H)$ such that $g(\{x, y\}) = \{f(x), f(y)\}$ for every edge $\{x, y\} \in E(G)$?*

In the positive case we say that g *is induced by* f. It is easy to see that such mapping f is a homomorphism $G \xrightarrow{hom} H$ and that to each homomorphism corresponds exactly one induced mapping g. Thus Question 2.1.1 asks which mappings g between edge sets are induced by a homomorphism. Various instances of this problem were considered for example by Whitney [89, 90], Nešetřil [65], Kelmans [49], and by Linial, Meshulam, and Tarsi [55]. The following necessary condition for a mapping $g : E(G) \to E(H)$ to be induced by a homomorphism was isolated in the above mentioned paper [55] and recently in broader context by DeVos, Nešetřil, and Raspaud [19]:

$$\text{For every cut } C \subseteq E(H) \text{ the set } g^{-1}(C) \text{ is a cut of } G. \tag{2.1}$$

If we use analogy with a continuous mapping between topology spaces (for which preimage of any open set is an opet set), we may say that mapping g is *cut-continuous*. In terminology of Definition 1.2.1 this is equivalent to being TT_2. Still, we will often use term 'cut-continuous' instead of TT_2 for this important special case.

Cut-continuous mappings extend the notion of a homomorphism and the relationship of these two notions is the central topic of Chapters 2 and 3. We provide evidence in both directions. We present various examples of cut-continuous mapping that are not induced, in particular in Proposition 2.1.9 we construct such mappings between highly connected graphs. On the other hand, as described in Section 2.2, for most of the graphs all cut-continuous mappings are induced.

2.1.1 Homotens graphs

For a homomorphism (of directed or undirected graphs) $h : V(G) \to V(H)$ we let $h^\sharp$ denote the *mapping induced by the homomorphism* h on edges, that is $h^\sharp((u,v)) = (h(u), h(v))$, or $h^\sharp(\{u,v\}) = \{h(u), h(v)\}$. If we consider directed graphs and h is an *antihomomorphism*, that is if for every edge $(u,v) \in E(G)$ we have $(h(v), h(u)) \in E(H)$ (h reverses every edge), we define $h^\sharp((u,v)) = (h(v), h(u))$ and call it a mapping induced by antihomomorphism. If H has parallel edges, then $h^\sharp$ is not unique: we just ask that $h^\sharp$ maps each of the edges (u,v) to some of the edges $(h(u), h(v))$; similarly for homomorphisms of undirected graphs and for antihomomorphisms. If H has oriented 2-cycles then it may happen that h is both a homomorphism and an antihomomorphism. In this case we choose on each component of G whether we will use the definition of $h^\sharp$ for homomorphisms or for antihomomorphisms. We do not allow, however, to use the 'homomorphism' definition for some edge and the 'antihomomorphism' definition for another edge of the same component.

To simplify expressing, we will use the term (anti)homomorphism for a mapping that is on each component either a homomorphism or an antihomomorphism.

The following easy lemma is the starting point of our investigation in this chapter.

Lemma 2.1.2 *Let G, H be (directed or undirected) graphs, M a ring. For every (anti)homomorphism f from G to H the induced mapping $f^\sharp$ ($f^\flat$, respectively) from G to H is TT_M. Consequently, from $G \preccurlyeq_h H$ follows $G \preccurlyeq^t_M H$.*

Proof: It is enough to prove Lemma 2.1.2 for homomorphisms of directed graphs. So let $f : G \to H$ be such homomorphism, $\varphi : V(H) \to M$ a tension. We may assume that φ is a cut-tension determined by $\delta(X)$. Then the cut $\delta(f^{-1}(X))$ generates precisely the cut-tension $\varphi \circ f$. $\square$

The main theme of Chapters 2 and 3 is to find similarities and differences between orders $\preccurlyeq_h$ and $\preccurlyeq^t_M$. In particular we are interested in when the converse to Lemma 2.1.2 holds. Now, we present a more precise version of Question 2.1.1 stated in the introduction.

Problem 2.1.3 *Let $f : E(G) \to E(H)$. Find suitable conditions for f, G, H that will guarantee that whenever f is TT_M, then it is induced by an (anti)homomorphism; i.e., there is $g : V(G) \to V(H)$ such that on each component of G, the mapping g is either a homomorphism or an antihomomorphism and $f = g^\sharp$.*

Shortly, we say a mapping is *induced* if it satisfies the condition of Problem 2.1.3. This problem leads us to the following definitions.

Definition 2.1.4 *We say a graph G is* left M-homotens *if for every loopless*[1] *graph H every TT_M mapping from G to H is* induced *(that is induced by a homomorphism or an antihomomorphism on each connected component). For brevity we will often call left M-homotens graphs just M-homotens (following [77]).*

On the other hand, H is a right M-homotens *graph if for every graph G statements $G \xrightarrow{hom} H$ and $G \xrightarrow{TT_M} H$ are equivalent.*

[1] It is easy to observe that if H contains a loop (and it is not a single loop), then for almost every G there are non-induced TT_M mappings from G to H.

We should note here, that the precise analogy of left M-homotens graphs—every TT_M mapping to H is induced—is not interesting, as this is much too strong requirement. For simplicity, suppose $M = \mathbb{Z}_2$. Let H be such graph, let $\Delta(H)$ be as defined before Lemma 1.3.1. The mapping $f : \Delta(H) \xrightarrow{TT_2} H$ given by $f(\{A, B\}) = A \,\Delta\, B$ is induced by an (anti)homomorphism, say g. Now this can happen only if for every $A \in V(\Delta(H))$ vertex $g(A)$ is adjacent to every edge e of H. (To see this, note that $f(\{A, A \,\Delta\, e\}) = e$, therefore $g(A)$ is one of the end vertices of e.) And this in turn can happen only if H is edgeless, or contains at most one edge.

Definition of left M-homotens makes sense for both directed and undirected graphs.

If $M = \mathbb{Z}_2^k$ then there are only trivial directed M-homotens graphs (namely an orientation of a matching). Thus, we restrict to study of undirected homotens graphs in this case. For other rings, Proposition 2.1.5 states that the orientation does not play any role; this will be useful in Section 2.2 in our study of directed M-homotens graphs.

For $M \neq \mathbb{Z}_2^k$ we might study undirected M-homotens graphs, too. The relationship between these two notions (undirected graph is homotens versus some its orientation is homotens) is not clear. For every M, the latter notion implies the former one; however, somewhat surprisingly, both notions are equivalent for many rings M, at least for such, in which the equation $x + x = 0$ has no nonzero solution. (For right homotens graphs, the above discussion applies, too.)

Proposition 2.1.5 *Let G_1, G_2 be two (directed or undirected) graphs, such that we can get G_2 from G_1 by changing directions of edges (in the case of directed graphs), deleting edges and adding multiple ones. Let M be a ring. Then G_1 is left M-homotens if and only if G_2 is left M-homotens.*

Proof: Suppose G_1 is not homotens, that is there exists a graph H_1 and a mapping $f_1 : G_1 \xrightarrow{TT_M} H_1$ that is not induced. By Lemma 1.2.8 we may suppose that f_1 is injective. We modify f_1 and H_1, to get a non-induced mapping $f_2 : G_2 \xrightarrow{TT_M} H_2$. If we change an orientation of an edge, we change an orientation of the corresponding edge in H_1. If we add an edge parallel to some edge e of G_1 then we map it to a new edge of H_1, parallel to $f_1(e)$. It is clear, that we get a TT_M mapping that is not induced. $\square$

We close this section by a lemma that somewhat simplifies the definition of left homotens graphs (and particularly shows that to test if a graph is homotens is a finite problem). (Parallel result for right homotens graphs will be proved as Proposition 2.3.5.)

Proposition 2.1.6 *For any finite graph G there is a finite graph G' such that for any ring M the following are equivalent.*

1. *G is left M-homotens.*
2. *any mapping $G \xrightarrow{TT_M} G'$ is induced.*

Proof: Let $m = |E(G)|$ and let G' be a graph such that every graph with at most m edges appears as a subgraph of G'.

Clearly condition 1 implies condition 2. For the converse implication, take a graph H and a mapping $f : G \xrightarrow{TT_M} H$. Graph induced by $f(E(G))$ contains at most m edges, hence it is a subgraph of G'. By Lemma 1.2.7 and 2.1.2 we may suppose that f is in fact a mapping from G to G', hence f is induced. □

2.1.2 Examples

We illustrate the complex relationship of homomorphisms and TT mappings by several examples presenting the similarities and (mainly) the differences in concrete independent settings. Towards the former, we provide an infinite chain and antichain of $\preccurlyeq_2^t$, thereby exhibiting a similar behavior of homomorphisms and TT mappings. On the other hand, we show that arbitrarily high connectivity of the source and target graphs does not force $TT_{\mathbb{Z}}$ mappings (much the less TT_M mappings) and homomorphisms to coincide. Finally, we show that an equivalence class of $\approx_2^t$ can contain exponentially many equivalence classes of $\approx_h$. In the next section, we will see that these results are in fact an exception, in the sense of random graphs (Theorem 2.2.9).

Proposition 2.1.7 appears already in [19], we include a proof for convenience of the reader. Note that this proposition will be strongly generalized by Theorems 2.2.9, 2.2.15, and 3.1.6.

Proposition 2.1.7 *Graphs K_{2^t} form a strictly increasing chain in $\preccurlyeq_2^t$ order, that is*

$$K_4 \prec_2^t K_8 \prec_2^t K_{16} \prec_2^t \cdots .$$

There are graphs G_1, G_2, … that form an infinite antichain: there is no mapping $G_i \xrightarrow{TT_2} G_j$ for $i \neq j$.

Proof: The following equivalence is a special case of Proposition 2.3.8. (It appears as Proposition 6.6 in [19] with a direct proof.) For any graph G holds

$$G \xrightarrow{hom} K_{2^k} \iff G \xrightarrow{TT_2} K_{2^k} \, . \tag{2.2}$$

This implies the first part. For the second part, let G_t be the Kneser graph $K(n,k)$ with $k = t(2^t - 2)$ and $n = 2k + 2^t - 2$. It is known [59] that $\chi(G_t) = n - 2k + 2 = 2^t$. This by equivalence (2.2) implies that $G_i \stackrel{TT_2}{\not\longrightarrow} G_j$ for $i > j$. The remaining part follows from Lemma 1.2.11: It is known (and easy to verify) that the length of the shortest odd cycle in $K(n,k)$ is the smallest odd number greater than or equal to $n/(n-2k)$, which means that $g_{\mathbb{Z}_2}(G_t) = 2t + 1$. □

The differences of TT_2 mappings and homomorphisms are easy to find. For example (see Section 1.3) we have $K_4 \xrightarrow{TT_2} K_3$ but obviously there is no homomorphism $K_4 \to K_3$. (More such small examples are collected in Proposition 1.3.2.) On the contrary, $TT_{\mathbb{Z}}$ mappings are more restricted and, indeed, there is no $TT_{\mathbb{Z}}$ mapping from K_4 to K_3.

As a further evidence, in the next proposition we give an infinite class of graphs where homomorphisms and TT_2 mappings differ. In particular for every n we present (vertex) n-connected graphs that are not homotens.

Proposition 2.1.8 *Let n be odd. Denote G_n one of the (two isomorphic) components of $\Delta(K_n)$. Graphs K_n and G_n are TT_2-equivalent and both are $(n-1)$-connected. Finally, $G_n \stackrel{hom}{\not\longrightarrow} K_n$ for $n = 2^k - 1$.*

Proof: Using Lemma 1.3.1 for $G = H = K_n$ we get $K_n \xrightarrow{hom} \Delta(K_n)$. From connectivity of K_n and from Lemma 2.1.2 it follows $K_n \xrightarrow{TT_2} G_n$. Using Lemma 1.3.1 for $G = H = \Delta(K_n)$ we get $\Delta(K_n) \xrightarrow{TT_2} K_n$, hence also $G_n \xrightarrow{TT_2} K_n$.

Graph K_n is $(n-1)$-connected. Easily $\Delta(K_n) = Q_n^{(2)}$, where Q_n is the n-dimensional hypercube and $Q_n^{(2)}$ means that we are connecting by an edge the vertices at distance two in the hypercube. It is well-known and straightforward to verify that Q_n is $(n-1)$-connected. The vertices with odd (even) number of 1's among their coordinates form the two components of $Q_n^{(2)}$; for an odd n these two components are isomorphic by a mapping $\vec{x} \mapsto (1,1,\dots,1) - \vec{x}$. Observe that if we take a path in Q_n and leave every second vertex out, we obtain a path in $Q_n^{(2)}$. So $Q_n^{(2)}$ is $(n-1)$-connected since Q_n is.

For the last part of the theorem, it follows from the remarks in the Section 7.1 that $\chi(G_n) = n + 1$ for $n = 2^k - 1$. $\square$

To find differences of $TT_{\mathbb{Z}}$ and homomorphisms is a bit more complicated. Any two oriented trees are $TT_{\mathbb{Z}}$-equivalent, hence we have plenty of 1-connected graphs for which $\preccurlyeq_{\mathbb{Z}}^t$ and $\preccurlyeq_h$ differ. For 2-connected examples, consider any permutation $\pi : E(\overrightarrow{C}_n) \to E(\overrightarrow{C}_n)$. This is $TT_{\mathbb{Z}}$, but (except for n of them) is not induced by a homomorphism. We may now use the replacement operation of [40] (see also proof of Proposition 2.1.10), that is we replace every edge of $\overrightarrow{C}_n$ by a suitable graph (for every edge we use a different graph). In this way we produce from the oriented circuit two graphs G and H, such that there is only one mapping $G \xrightarrow{TT_{\mathbb{Z}}} H$ and it respects one of the permutations $\pi : E(\overrightarrow{C}_n) \to E(\overrightarrow{C}_n)$. So if we choose π that is not a cyclic shift, we obtain graphs such that $G \approx_{\mathbb{Z}}^t H$ and $G \not\approx_h H$. These graphs may have arbitrarily large edge-connectivity, they are not vertex 3-connected, however. Whitney's theorem (Theorem 7.4.4) seems to suggest, that this situation may not repeat for graphs of higher connectivity. Therefore, the following result may be a bit surprising.

Proposition 2.1.9 *For every k there are vertex k-connected graphs G, H such that $G \xrightarrow{TT_{\mathbb{Z}}} H$ but $G \stackrel{hom}{\not\longrightarrow} H$. Therefore, for each k exists a k-connected graph that is not $\mathbb{Z}$-homotens.*

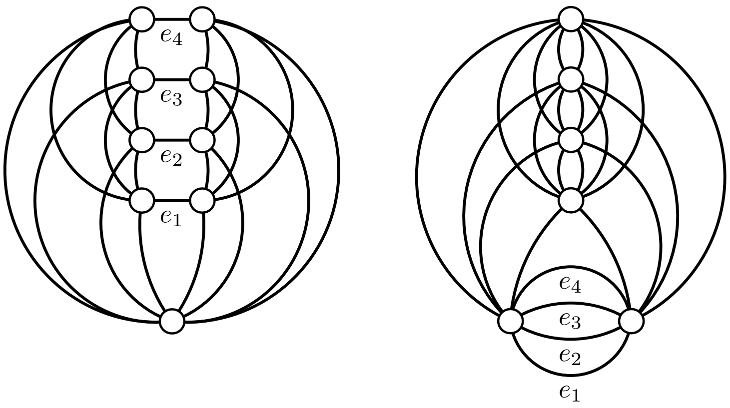


Figure 2.1: The left graph is an example of highly connected graph that is not $\mathbb{Z}$-homotens; the right one is a witness for this fact.

Proof: Fix a k, let G, H be graphs illustrated for $k = 4$ in Figure 2.1.[2] The construction is due to Shih [80].

Clearly both G and H are k-connected and there is no homomorphism between them. The natural bijection between G and H—we identify the left K_{k+1}'s in G and H, the right K_{k+1}'s in G and H, and the edges e_i as depicted in the figure—is easily checked to be $TT_{\mathbb{Z}}$. □

Many more graphs give negative answer to Problem 2.1.3, here we only recall the perhaps most spectacular example: Petersen graph admits a TT_2 mapping to C_5 (Proposition 1.3.2).

We conclude this section by a more quantitative example.

Proposition 2.1.10 *There are 2^{cn} undirected graphs with n vertices that form an antichain in the homomorphism order, yet all of them are TT_2-equivalent.*

Proof: To simplify notation, we will construct $\binom{n}{\lfloor n/2 \rfloor}$ graphs with $sn+1$ vertices, this clearly proves the proposition. We use the *replacement operation* of [40]. Let H be a graph (we explain later how do we choose it), let a, b, x_1, …, x_5 be pairwise distinct vertices of H. Next, we take an oriented path with n edges and replace each of them by a copy of H. That is, we take H_1, …, H_n—isomorphic copies of H—and identify vertex b of H_i with a of H_{i+1} (for every $i = 1, \dots, n-1$). Let G be the resulting graph.

Finally, for each $t \in \{0,1\}^n$ we present a graph G_t. We let F_i be a copy of the Petersen graph Pt if $t_i = 1$, and a copy of the prism of C_5—graph R in Figure 2.1.2—if $t_i = 0$. We construct the graph G_t as a vertex-disjoint union of G, F_1, …, F_t plus some 'connecting edges': for every $i = 1, \dots, n$ and $j = 1, \dots, 5$ we let x_i^j denote the copy of x_j in $H_i \subset G$ and u_i^j the copy of u_j in F_i; we let $x_i^j u_i^j$ be an edge of G_t. Note that each G_t has $(|V(\mathrm{Pt})| + |V(H)| - 1)n + 1$ vertices.

Claim 1. H can be chosen so that the only homomorphism $G \to G$ is the identity. Moreover the vertices x_i can be chosen so that the distance between any two of them is at least 4.

This follows immediately from techniques of [40]: we can take H_9 from the Figure 4.9 of [40] (depicted here in the middle of Figure 2.1.2) as our graph H.

Claim 2. If $G_t \xrightarrow{hom} G_{t'}$ then $t_i \le t'_i$ holds for each i.

[2] If we wish to construct directed graphs, consider any orientation of them, such that corresponding edges of G and of H are oriented in the same way, except of the edges e_i which are oriented differently in G and in H.

Take any homomorphism $f : G_t \xrightarrow{hom} G_{t'}$, fix an i, and let F_i (F'_i) be the copy of Pt or R that constitute the i-th part of graph G_t ($G_{t'}$ respectively). By Claim 1, f maps the vertices of G identically, in particular $f(x_i^j) = x_i'^j$. As the only path of length 3 connecting vertices x_i^j and $x_i^{j \bmod 5+1}$ is the one containing vertices u_i^j and $u_i^{j \bmod 5+1}$, mapping f satisfies $f(u_i^j) = u_i'^j$ as well. Consequently, f maps vertices of F_i to vertices of F'_i. To show $t_i \le t'_i$ it remains to observe that there is no homomorphism $\mathrm{Pt} \xrightarrow{hom} R$.

Claim 3. For every t, t' we have $G_t \xrightarrow{TT_2} G_{t'}$.

We map every edge of G and every edge $x_i^j u_i^j$ and $u_i^j u_i^{j \bmod 5+1}$ identically (we call such edges *easy edges*). We map edges of F_i in G_t to edges of the outer pentagon of F'_i in $G_{t'}$ by sending an edge to the outer edge with the same number in Figure 2.1.2. To check that this is indeed a TT_2 mapping we use Lemma 1.2.9: if C is a cycle contained in some F_i then we easily check that algebraical image of C is a cycle. If C contains only easy edges that it is mapped identically, so its algebraical image is again a cycle. As every cycle can be written as a symmetric difference of these two types, we conclude that we have constructed a TT_2 mapping.

Now we are ready to finish the proof. Consider a set A containing all vertices of $\{0,1\}^n$ with $\lfloor n/2 \rfloor$ coordinates equal to 1. By Claim 2, graphs $G_t, G_{t'}$ are homomorphically incomparable for distinct $t, t' \in A$. On the other hand, by Claim 3, all of the graphs are TT_2-equivalent. □

In this proof we can use other building blocks instead of Petersen graph and the pentagonal prism. To be concrete, we can take graphs G, H from Proposition 2.1.9 and use graphs $G \dot\cup H$ and $H \dot\cup H$. If we slightly modify the construction, we can prove version of Proposition 2.1.10 for $TT_{\mathbb{Z}}$ mappings, and therefore for TT_M mappings for arbitrary M. Moreover, by another small change of the construction, we can guarantee that all of the constructed graphs are k-connected (for any given k).

It would be interesting to know if 2^{cn} from Proposition 2.1.10 can be improved. Note that in the homomorphism order $\preccurlyeq_h$ the maximal antichain has full cardinality [53], that is there are

$$\frac{1}{n!}\binom{\binom{n}{2}}{\lfloor \frac{1}{2}\binom{n}{2} \rfloor}(1 - o(1))$$

homomorphically incomparable graphs with n-vertices (Theorem 2.2.13, compare Corollary 2.2.14). Proposition 2.1.10 claims that at least 2^{cn} of these graphs are contained in one equivalence class of $\approx_M^t$.

2.2 Left homotens graphs

In this section we look where homomorphisms and TT_M mappings meet: we study a class of graphs that force any TT_M mapping from them to be induced, that is class of left homotens graphs. We prove a surprising result that most graphs have this property. In Section 2.2.2 we use these graphs to find an embedding of category of graphs and homomorphism to the category of graphs and TT_M mappings, simplifying and generalizing a result of Chapter 3.

2.2.1 A sufficient condition

Recall (Definition 2.1.4) that a graph G is left M-homotens if every TT_M mapping from G (to any graph) is induced. The characterization of left M-homotens graphs seems to be a difficult problem; in this section we obtain a general sufficient condition in terms of *nice* graphs.

In Proposition 2.1.9 we saw that high connectivity does not imply homotensness. In Corollary 2.2.17 we will see that every vertex of a homotens graph is incident with a triangle. In view of this, a condition sufficient for homotens has to be somewhat restrictive.

Definition 2.2.1 *We say that an undirected graph G is* nice *if the following holds*

1. *every edge of G is contained in some triangle*
2. *every triangle in G is contained in some copy of K_4*
3. *every copy of K_4 in G is contained in some copy of K_5*
4. *for every K, K' that are copies of K_4 in G there is a sequence of vertices v_1, v_2, …, v_t such that*
   - $V(K) = \{v_1, v_2, v_3, v_4\}$,
   - $V(K') = \{v_t, v_{t-1}, v_{t-2}, v_{t-3}\}$,
   - v_iv_j *is an edge of G whenever* $1 \le i < j \le t$ *and* $j \le i + 3$.

We say that a graph is weakly nice *if conditions 1, 2, and 4 in the list above are satisfied. Finally, we say that a* directed *graph is (weakly) nice, if the underlying undirected graph is (weakly) nice.*

Theorem 2.2.2 *Let G, H be undirected graphs, let G be nice, and let $M = \mathbb{Z}_2^r$ for some r. Suppose $f : G \xrightarrow{TT_M} H$. Then f is induced by a homomorphism of the underlying undirected graphs. Shortly, every undirected nice graph is $\mathbb{Z}_2$-homotens.*

Theorem 2.2.3 *Let G, H be (directed or undirected) graphs, let G be weakly nice, and let $M \neq \mathbb{Z}_2^r$ be a ring. Suppose $f : G \xrightarrow{TT_M} H$. Then f is induced by an (anti)homomorphism. Shortly, every weakly nice graph is M-homotens.*

We take time out for two lemmata that describe possible TT mappings from a small complete graph. The first one uses technique of fractional covering by cuts, which is further developed in Chapter 5. The second one is more technical, as it does not restrict to the case $M = \mathbb{Z}_2$.

Lemma 2.2.4 *Let $f : K_5 \xrightarrow{TT_2} H$, where H is any undirected loopless graph. Then f is induced by an injective homomorphism. Moreover, this homomorphism is uniquely determined.*

Proof: Suppose $f(K_5)$ is a four-colorable graph. A composition of TT_2 mapping $f : K_5 \xrightarrow{TT_2} f(K_5)$ with a TT_2 mapping induced by a homomorphism $f(K_5) \xrightarrow{hom} K_4$ gives $K_5 \xrightarrow{TT_2} K_4$. Consider three cuts of size 4 in K_4; they cover every edge exactly twice. Hence, their preimages are three cuts in K_5 that cover every edge exactly twice. But K_5 has 10 edges, while the largest cut has only $2 \cdot 3 = 6$ edges.

Hence, the chromatic number of $f(K_5)$ is at least five. As it has at most 10 edges, the chromatic number is exactly five. Let $V_1, \ldots, V_5$ be the color classes of $f(K_5)$. There is exactly one edge between two distinct color classes (otherwise the graph is four-colorable). Hence, f is a bijection. Next, $|V_i| = 1$ for every i (as otherwise we can split one color-class to several pieces and join these to the other classes; again, the graph would be four-colorable). Consequently, $f(K_5)$ is isomorphic to K_5.

We call star a set of edges sharing a vertex. We know that preimage of every star is a star, hence as f is a bijection, also image of every star is a star. Stars sharing an edge map to stars sharing an edge, hence f is induced by a homomorphism. □

Lemma 2.2.5 *Let M be a ring that is not isomorphic to a power of $\mathbb{Z}_2$. Let $f : \overrightarrow{K_4} \xrightarrow{TT_M} H$, where H is any loopless directed graph and $\overrightarrow{K_4}$ any orientation* *of K_4. Then f is induced by an injective (anti)homomorphism. Moreover, this (anti)homomorphism is uniquely determined.*

Proof: Suppose first that $f(\overrightarrow{K_4})$ is a three-colorable graph, i.e., that there is a homomorphism $h : f(\overrightarrow{K_4}) \to \overleftrightarrow{K}_3$, where $\overleftrightarrow{K}_3$ is the symmetric[3] orientation of K_3, that is a directed graph with three vertices and all six oriented edges among them. A composition of TT_M mapping $f : \overrightarrow{K_4} \xrightarrow{TT_M} f(\overrightarrow{K_4})$ with $h^\sharp$ gives $g : \overrightarrow{K_4} \xrightarrow{TT_M} \overleftrightarrow{K}_3$. Consider the three cuts of size 4 in $\overleftrightarrow{K}_3$: X_1, X_2, X_3. As M is not a power of $\mathbb{Z}_2$, $1+1 \neq 0$; let φ_i be M-tension that attains value ± 1 on X_i and 0 elsewhere. We can choose φ_i so, that for every $e \in E(\overleftrightarrow{K}_3)$ we have $\{\varphi_1(e), \varphi_2(e), \varphi_3(e)\} = \{0, \pm 1\}$. As g is TT_M, mappings $\psi_i = \varphi_i \circ g$ are M-tensions and for every $e \in E(\overrightarrow{K_4})$ we have $\{\psi_1(e), \psi_2(e), \psi_3(e)\} = \{0, \pm 1\}$. (*)

Call an M-tension *simple* if it attains only values 0 and ± 1. We will show that three simple M-tensions ψ_1, ψ_2, ψ_3 on $\overrightarrow{K_4}$ with property (*) do not exist.

To this end, we will characterize sets $\operatorname{Ker}\psi = \{e \in E(\overrightarrow{K_4}), \psi(e) = 0\}$ for simple M-tensions ψ. Let ψ be such tension. Pick $v \in V(\overrightarrow{K_4})$ and let e_1, e_2, e_3 be adjacent to v. Note that ψ is determined by its values on e_1, e_2, e_3. We may suppose that each e_i is going out of v; otherwise we change orientation of some edges and the sign of ψ on them. Further, we may suppose that $|\{i, \psi(e_i) = 1\}| \geq |\{i, \psi(e_i) = -1\}|$; otherwise we consider $-\psi$. Thus, we distinguish the following cases (see Figure 2.3).

- $\psi(e_i) \in \{0, 1\}$ for each i.
  Let z be the number of e_i such that $\psi(e_i) = 0$. Then ψ is determined by a cut with $z+1$ vertices on one side of the cut. Therefore, the set $\operatorname{Ker}\psi$ is either the edge set of a $\overrightarrow{K_4}$, of a triangle, or it is a pair of disjoint edges.

- $\psi(e_1) = 1, \psi(e_2) = 0, \psi(e_3) = -1$.
  In this case $\operatorname{Ker}\psi$ is a single edge. Note, that this case (and the next one) may occur only if $1+1+1 = 0$.

- $\psi(e_1) = \psi(e_2) = 1, \psi(e_3) = -1$.
  In this case too, $\operatorname{Ker}\psi$ is a single edge.

Hence, $E(\overrightarrow{K_4})$ is partitioned into three sets, whose sizes are in $\{1, 2, 3, 6\}$. Therefore, there are two possibilities:

[3] T-symmetric according to the definition of Section 1.2.2

- $6 = 3 + 2 + 1$: The complement of a triangle is a star of three edges, there are no two disjoint edges in it.

- $6 = 2{+}2{+}2$: In this case, all three ψ_i's are determined by a cut. Suppose $\overrightarrow{K_4}$ is oriented as in Figure 2.3, the values of ψ_1 are indicated. It is not possible to fulfill the condition (*) on both edges from $\operatorname{Ker}\psi_1$, as one of φ_2, φ_3 will assign these edges the same (nonzero) value and the other one opposite (nonzero) values.

So far we have proved, that the chromatic number of $f(\overrightarrow{K_4})$ is at least four. As $f(\overrightarrow{K_4})$ has at most 6 edges, its chromatic number is exactly four. Let $V_1, \dots, V_4$ be the color classes. There is exactly one edge between two distinct color classes (otherwise the graph is three-colorable). Thus, f is a bijection. Next, $|V_i| = 1$ for every i (as otherwise, we can split one color-class to several pieces and join these to the other classes; again, the graph would be three-colorable). Consequently, $f(\overrightarrow{K_4})$ is some orientation of K_4.

We call *star* a set of edges sharing a vertex. If we let φ be a simple M-tension on $f(\overrightarrow{K_4})$ corresponding to a cut which is a star, then $\varphi \circ f$ is a simple tension that is nonzero exactly on three edges (f is a bijection). By the characterization of zero sets of simple tensions we see that preimage of each star is a star. As f is a bijection and preimage of every star is a star, also image of every star is a star. This allows us to define a vertex bijection $g : V(\overrightarrow{K_4}) \to V(f(\overrightarrow{K_4}))$ by letting $g(u) = u'$ iff the f-image of the star with u as the central vertex is the star centered at u'. Stars sharing an edge map to stars sharing an edge, hence f is induced by g, which is either a homomorphism or an antihomomorphism. $\square$

Proof of Theorem 2.2.2 and 2.2.3: It is convenient to suppose that G contains no parallel edges (Proposition 2.1.5). Let K be a copy of K_4 in G (if G is a directed graph, then we mean by this that K is some orientation of K_4). For Theorem 2.2.3 (that is when M is not a power of $\mathbb{Z}_2$) we use Lemma 2.2.5 to see that the restriction of f to K is induced by an (anti)homomorphism. When proving Theorem 2.2.2 we know that G is nice, therefore K is part of a copy of K_5 in G. By Lemma 2.2.4 the restriction of f to this copy of K_5 is induced by an homomorphism, therefore the same applies for K. Let this homomorphism from K to H be denoted by h_K, that is we assume $f|_{E(K)} = h_K^\sharp$ (or $f|_{E(K)} = h_K^\flat$).

As every edge of G is contained in some copy of K_4, it is enough to prove that there is a common extension of all mappings $\{h_K \mid K \subseteq G,\ K \cong K_4\}$ (we may define it arbitrarily on isolated vertices of G).

We say that h_K and $h_{K'}$ *agree* if for any $v \in V(K) \cap V(K')$ we have $h_K(v) = h_{K'}(v)$ and either both h_K, $h_{K'}$ are homomorphisms or both are antihomomorphisms. Thus, we need to show that any two mappings h_K, $h_{K'}$ (for K, K' from the same component of G) agree.

First, let K, K' be copies of K_4 that intersect in a triangle. Then h_K and $h_{K'}$ agree (note that this does not necessarily hold if the intersection is just an edge, see Figure 2.3), moreover either both h_K, $h_{K'}$ are homomorphisms or both are antihomomorphisms.

Now suppose K, K' are copies of K_4 that have a common vertex v. Since G is a weakly nice graph, we find vertices $v_1, v_2, \dots, v_t$ as in Definition 2.2.1. Let $G_i = G[\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}]$: every G_i is a copy of K_4, $G_1 = K$ and $G_{t-3} = K'$. Suppose $v = v_l = v_r$, where $l \in \{1, 2, 3, 4\}$, $r \in \{t-3, t-2, t-1, t\}$. Consider a closed walk $W = v_l, v_{l+1}, \dots, v_{r-1}, v_r$. Let $v'_i = h_{G_i}(v_i)$ for $l \le i \le t-3$ and $v'_i = h_{G_{t-3}}(v_i)$ for $t-3 \le i \le r$. Mappings h_{G_i} and $h_{G_{i+1}}$ agree, hence $v'_i v'_{i+1} = f(v_i v_{i+1})$ is an edge of H. So $W' = v'_l, v'_{l+1}, \dots, v'_{r-1}, v'_r$ is a walk in H.

Let φ be 'a ± 1-flow around W', formally

$$\varphi(e) = \sum_{\substack{l \le i \le r-1 \\ e = (v_i, v_{i+1})}} 1 - \sum_{\substack{l \le i \le r-1 \\ e = (v_{i+1}, v_i)}} 1 \,.$$

Clearly φ is an M-flow. Similarly, define $\varphi'(e)$ from W'. We have $\varphi' = \varphi_f$, hence φ' is a flow (Lemma 1.2.9). This can happen only if W' is a closed walk, that is $v'_l = v'_r$.

By definition, $v'_r = h_{K'}(v)$. Mappings h_{G_i} and $h_{G_{i+1}}$ agree, so $h_{G_i}(v_{i+j}) = h_{G_{i+j}}(v_{i+j})$ for $j \le 3$. Consequently, $v'_l = h_K(v)$, which finishes the proof. □

We summarize Theorems 2.2.2 and 2.2.3.

Corollary 2.2.6 *An undirected nice graph is left M-homotens for every ring M. A (directed or undirected) weakly nice graph is left M-homotens for every ring M that is not a power of $\mathbb{Z}_2$.*

In Proposition 2.1.8 we saw highly-connected graphs G_n that are not $\mathbb{Z}_2$-homotens. As a consequence of Corollary 2.2.6 these graphs are not nice. To see this directly, recall that G_n was one component of $\Delta(K_n)$ for a suitable n, and although $\Delta(K_n)$ contains K_n, not every copy of K_4 is contained in a copy of K_5.

Extending our conditions that guarantee that a graph is M-homotens, we give the following lemma, which will be used in Section 2.2.2. (Note that the word 'spanning' is needed.)

Lemma 2.2.7 *Suppose H contains a connected spanning M-homotens graph. Then H is M-homotens.*

Proof: Let $f : H \xrightarrow{TT_M} K$, let G be the connected spanning M-homotens subgraph of H. Restriction of f to $E(G)$ is TT_M, hence $f(e) = g^\sharp(e)$ for each $e \in E(G)$ and some (anti)homomorphism g. Let $e = uv \in E(H) \setminus E(G)$. We have to prove $f(e) = (g(u), g(v))$. Let P be a path from u to v in G. By treating the closed walk $P \cup \{uv\}$ as W in the end of the proof of Theorem 2.2.3, we conclude the proof. $\square$

2.2.2 Applications

In this section we provide several applications of nice graphs (that is, of Corollary 2.2.6). Particularly, we prove that 'almost all' graphs are left M-homotens for every ring M and construct an embedding of category $\mathcal{G}_{hom}$ into $\mathcal{G}_{TT_M}$. Somewhat stronger embedding result for $M = \mathbb{Z}_2$ is proved by an ad-hoc construction in Theorem 3.2.2. Here we follow a more systematic approach—we employ a modification of an edge-based replacement operation (see [40]). As a warm-up we prove an easy, but perhaps surprising result.

Corollary 2.2.8 *For every graph G there is a graph G' containing G as an induced subgraph such that for every ring M every TT_M mapping from G' to arbitrary graph is induced by an (anti)homomorphism (i.e., G' is M-homotens).*

Proof: We take as G' the (complete) join of G and K_5; that is, we let $V(G') = V(G) \cup \{v_1, v_2, \dots, v_5\}$, and $E(G') = E(G) \cup \{\text{all edges containing some } v_i\}$. By Theorem 2.2.3 it is enough to show that G' is nice. Every copy of K_t ($t < 5$) in G' can be extended to K_5 by adding some vertices v_i. One can also show routinely that any two copies of K_4 in G' are 'K_4-connected'—condition 4 in Definition 2.2.1. $\square$

We consider the random graph model $G(n, 1/2)$, that is every (simple undirected) graph with vertices $\{1, 2, \dots, n\}$ has the same probability (in Proposition 2.2.10 we deal with graphs $G(n,p)$ for a general p). As it is usual in the random graph setting, we study if a graph property P holds asymptotically almost surely (a.a.s.), that is whether

$$\lim_{n\to\infty} Pr_{G \in G(n,1/2)}[G \text{ has } P] = 1\,.$$

We also consider the countable random graph $G(\omega, 1/2)$. Surprisingly, it is almost surely isomorphic to a particular graph, the so-called Rado graph. This is a remarkable graph (it is homogeneous and it contains every countable graph as an induced subgraph), see [14] for more detailed discussion.

The following theorem was our main motivation for introducing (weakly) nice graphs.

Theorem 2.2.9 *Let M be a ring.*

1. *Complete graph K_k is M-homotens for $k \ge 5$ (and for $k \ge 4$ if $M \ne \mathbb{Z}_2^r$).*
2. *The random graph $G(n, 1/2)$ is M-homotens a.a.s. The Rado graph is M-homotens.*
3. *The random k-partite graph is M-homotens a.a.s. for $k \ge 5$ (and for $k \ge 4$ if $M \ne \mathbb{Z}_2^r$). Explicitly,*

$$\lim_{n\to\infty} \Pr_{G=G(n,1/2)}[G \textit{ is } M\textit{-homotens} \mid G \textit{ is } k\textit{-partite}] = 1\,.$$

4. *The random K_k-free graph is M-homotens a.a.s. for $k \ge 6$ (and for $k \ge 5$ if $M \ne \mathbb{Z}_2^r$).*

If $M \ne \mathbb{Z}_2^r$ then in each of the statements, any orientation of the considered graph is M-homotens, too.

Proof: It is a routine to verify that K_t is nice (weakly nice for $t = 4$), hence 1 follows by Corollary 2.2.6.

Next, we prove that the random graph $G(n, 1/2)$ is a.a.s. nice, it is possible to prove in the same way that the Rado graph is nice. Therefore we get 2 by another application of Corollary 2.2.6.

For $S \subseteq V(G)$ (where $G = G(n, 1/2)$) write C_S for the event 'there is a common neighbor for all vertices in S'. If $|S| = s$, the probability of C_S clearly is $(1 - \frac{1}{2^s})^{n-s}$. As $\binom{n}{s} \cdot (1 - \frac{1}{2^s})^{n-s}$ tends to zero for any fixed s, C_S holds a.a.s. for all S with size at most 4. This implies the first three conditions on G.

To prove the last condition, let K, K' be two copies of K_4. Denote vertices of K by v_1, v_2, v_3, v_4, and vertices of K' by v_8, v_9, v_{10}, v_{11} (in any order). If we find a triangle that is connected to every vertex in $K \cup K'$, we may denote its vertices by v_5, v_6, v_7 and we are done. For a given three-element set $S \subseteq V(G) \setminus \big(V(K) \cup V(K')\big)$ the probability that S induces a triangle and is connected to all vertices in $V(K) \cup V(K')$ is at least 2^{-21}, hence the probability that there is no such S is at most $(1-2^{-21})^{(n-8)/3}$. As the number of possible pairs (K, K') is at most n^8, this concludes the proof.

By [51], a random K_k-free graph is a.a.s. $(k-1)$-partite, hence 3 implies 4. The proof of 3 is similar to the proof that the random graph is a.a.s. nice, we sketch it for convenience.

Let $A_1, \ldots, A_k$ be the parts of the random k-partite graph. By standard arguments, all A_i's are a.a.s. approximately of the same size, in particular all are nonempty. It is a routine to verify parts 1, 2, and (in case $k \geq 5$) 3 of Definition 2.2.1. For part 4, let $V(K) = \{v_1, \ldots, v_4\}$, $V(K') = \{v_9, \ldots, v_{12}\}$. We pick $i_1, \ldots, i_4$ so that $v_t \notin A_{i_k}$ for each t, except possibly if $t = k$ or $t = k + 8$. We attempt to pick $v_5 \in A_{i_1}, \ldots, v_8 \in A_{i_4}$ to satisfy the condition 4. The probability that a particular 4-tuple fails is at most $\left(1-2^{-18}\right)^{|A_{i_1}|+\cdots+|A_{i_4}|-8} \leq \left(1-2^{-18}\right)^{n/2k}$. Hence, the probability that some copies K, K' of K_4 are 'bad' is at most $n^8 c^n$ (for some $c < 1$). □

From the point of view of random graphs it is natural to study whether random graphs[4] $G(n,p)$ are homotens for a general $p = p(n)$. We can use the approach of Theorem 2.2.9 for $p \ll 1/2$; it is not clear, however, if we can get tight result by using nice graphs. Still, we can use general results on monotone graph properties to prove that there is some tight result. Recall that a function $p_0(n)$ is called a *threshold* for graph property P if

- $\lim_{n\to\infty} Pr_{G\in G(n,p)}[G \text{ has } P] = 1$ whenever $p \gg p_0$, and
- $\lim_{n\to\infty} Pr_{G\in G(n,p)}[G \text{ has } P] = 0$ whenever $p \ll p_0$.

As property 'being homotens' is not monotone (a graph with one edge is homotens, a graph with two connected edges is not), it may not (and does not) have threshold in the above-defined sense. Indeed, in the next result we show that it has two 'local thresholds'.

Proposition 2.2.10 *There is $p_0 = p_0(n)$ satisfying*

$$\frac{c_1}{n^{2/3}} \leq p_0 \leq \frac{c_2}{n^{1/21}}$$

such that for every M

$$\lim_{n\to\infty} Pr_{G\in G(n,p)}[G \textit{ is left } M\textit{-homotens graph}] = \begin{cases} 0 & \textit{if } \frac{1}{n^{3/2}} \ll p \ll p_0, \\ 1 & \textit{if } p \ll \frac{1}{n^{3/2}} \textit{ or } p \gg p_0. \end{cases}$$

[4] with n vertices and each edge present with probability p

Proof: We start by collecting several results about random graphs. The first four of them are well-known (consult [47]).

Claim 1. If $p \gg \log n/n$ then $G(n,p)$ is connected a.a.s.

Claim 2. If $p \ll 1/n^{3/2}$ then a.a.s. $G(n,p)$ contains no path with two edges.

Claim 3. If $1/n^{3/2} \ll p \ll 1/n$ then $G(n,p)$ is a forest that contains two adjacent edges.

Claim 4. If $p \gg 1/n^{4/3}$ then a.a.s. $G(n,p)$ contains a tree with 4 vertices.

Claim 5. If $p \gg 1/n^{1/21}$ then $G(n,p)$ is nice a.a.s.

This follows by the same proof as part 1 of Theorem 2.2.9.

Claim 6. If $p \ll 1/n^{2/3}$ then $G(n,p)$ a.a.s. contains no odd wheel.

Let X be the random variable counting number of odd wheels. Clearly

$$\mathbb{E}X \leq \sum_{k \geq 3} n^{k+1} p^{2k} = \frac{n \cdot (p^2 n)^3}{1 - p^2 n}$$

which tends to zero. By Markov inequality, there is a.a.s. no odd wheel.

Let H be the graph property 'being left homotens', C 'to be connected' and put $P = H \cap C$. By Lemma 2.2.7 property P is monotone (i.e., it is preserved by adding edges), hence it has a threshold by a result of [11] (consult also [47]). Let p_0 be this threshold.

No forest (except of a matching) is homotens. This, together with Corollary 2.2.17, Claim 3, 4, and 6 implies that if $n^{-3/2} \ll p \ll n^{-2/3}$ then $G(n,p)$ is not homotens a.a.s. For larger p, Claim 1 states $G(n,p)$ has C a.a.s., so $H = H \cap C$ a.a.s. This implies that p_0 is a threshold for H as well and that $p_0 \geq c_1 n^{-2/3}$. By Claim 5 we see $p_0 \leq c_2 n^{-1/21}$. Finally, by Claim 2 the graph $G(n,p)$ is homotens for $p \ll n^{-3/2}$. □

Problem 2.2.11 *Determine the threshold for property 'being connected left M-homotens graph', that is the threshold p_0 from Proposition 2.2.10.*

Next, we mention an easy corollary of Theorem 2.2.9. However simply and naturally does this result look, it is not easy to prove directly. An extension of argument used in Lemma 2.2.4 yields only $K_4 \prec^t K_6 \prec^t K_8 \prec^t \cdots$ (Theorem 5.2.4), use of Lemma 1.3.1 yields $K_4 \prec^t K_8 \prec^t K_{16} \prec^t \cdots$ (for $M = \mathbb{Z}_2$, similarly for other M). In [55] this result is proved (for $M = \mathbb{Z}_2$) by use of strong maps between matroids.

Corollary 2.2.12 *For every M the complete graphs K_n form an ascending chain in $\prec_M^t$, with the exception of $K_3 \approx_2^t K_4$. That is*

$$K_2 \prec_M^t K_3 \preccurlyeq_M^t K_4 \prec_M^t K_5 \prec_M^t K_6 \prec_M^t \cdots ,$$

where the inequality $K_3 \preccurlyeq_M^t K_4$ is strict whenever M is not a power of $\mathbb{Z}_2$.

Theorem 2.2.9 enables us also to prove a TT version of the following result about homomorphisms of random graphs. (The original theorem appears in [53], see also Section 3.6 of [40].) Note that Theorem 2.2.13 (and thus also Corollary 2.2.14) is asymptotically tight, as if we have more graphs then by Sperner's theorem there is even an inclusion between two of them.

Theorem 2.2.13 ([53]) *Random (undirected) graph is almost surely rigid (with respect to homomorphism). There are*

$$\frac{1}{n!}\binom{\binom{n}{2}}{\lfloor \frac{1}{2}\binom{n}{2} \rfloor}(1 - o(1))$$

graphs on n vertices with no homomorphism between any two of them and with only identical homomorphism on each of them.

Corollary 2.2.14 *Let M be a ring. Random (undirected) graph is almost surely TT_M-rigid. There are*

$$\frac{1}{n!}\binom{\binom{n}{2}}{\lfloor \frac{1}{2}\binom{n}{2} \rfloor}(1 - o(1))$$

pairwise TT_M-incomparable TT_M-rigid graphs on n vertices.

We finish Section 2.2.2 by another application of Corollary 2.2.6 —we show that the structure of TT_M mappings is at least as rich as that of homomorphisms.

Theorem 2.2.15 *There exists a mapping F that assigns (directed or undirected) graphs to graphs (of the same type), such that for any ring M and for any graphs G, H (we stress that we consider loopless graphs only) holds*

$$G \preccurlyeq_h H \iff F(G) \preccurlyeq_M^t F(H) .$$

Moreover F can be extended to a 1-1 correspondence for mappings between graphs: if $f : G \to H$ is a homomorphism, then $F(f) : F(G) \to F(H)$ is a TT_M mapping and any TT_M mapping between $F(G)$ and $F(H)$ is equal to $F(f)$ for some homomorphism $f : G \xrightarrow{hom} H$.

In terms of category theory, F is an embedding of the category $\mathcal{G}ra_{hom}$ of all graphs and their homomorphisms into the category $\mathcal{G}ra_{TT_M}$ of all graphs and all TT_M-mappings between them.

Proof: We will use a modification of edge-based replacement (see [40]). Let I be the graph in Figure 2.4 with arbitrary (but fixed) orientation. To construct $F(G)$, we will replace each of the vertices of G by a triangle and each of the edges of G by a copy of I, gluing different copies on the triangles corresponding to the end-vertices of the edge. More precisely, let $U = V(G) \times \{0, 1, 2\}$, for every edge $e \in E(G)$ let I_e be a separate copy of I. If $e = (u, v)$ then we identify vertex u_i ($i \in \{0, 1, 2\}$) with (u, i) in U, and vertex v_i with (v, i) in U. Let $F(G)$ be the resulting graph; we write shortly $F(G) = G * I$. If $f : V(G) \to V(H)$ is a homomorphism then we define $F(f) : E(F(G)) \to E(F(H))$ as follows: let $e = (u, v)$ be an edge of G and a an edge of $E(I_e)$. Let e' be the image of e under f. In the isomorphism between I_e and I'_e the edge a gets mapped to some a'. We put $F(f)(a) = a'$. It is easily seen that $F(f)$ is a $TT_{\mathbb{Z}}$ (thus TT_M) mapping that is induced by a homomorphism, we let $\varphi(f)$ denote this homomorphism. Now, we turn to the more difficult step of proving that every TT_M mapping from G to H is $F(f)$ for some $f : G \xrightarrow{hom} H$. We will need several auxiliary claims.

Claim 1. I is critically 6-chromatic.

Take any K_5 in I, color in by 5 colors. There is a unique way how to extend it, which fails, so $\chi(I) \geq 6$. Clearly 6 colors suffice. Moreover, if we delete any vertex of I then it is possible to color the remaining vertices consecutively $1, 2, 3, 4, 5, 1, 2, \ldots, 5$.

Claim 2. I is rigid.

That is, the only homomorphism $f : I \to I$ is the identity. By Claim 1, f cannot map I to its subgraph, hence f is an automorphism. There is a unique vertex x of degree 9, so f fixes it. There is a unique Hamilton cycle $x = x_1, \ldots, x_{16}$ such that $x_i x_j$ is an edge whenever $|i - j| \leq 4$, therefore this cycle has to be fixed by f too. This leaves two possibilities, but only one of them maps the 'diagonal' edge properly.

Claim 3. I is K_5-connected.

That is, for every two vertices a, b of I there is a path $a = a_1, a_2, \ldots, a_k = b$ such that $a_i a_j$ is an edge whenever $|i - j| \leq 4$.

Claim 4. Whenever H is a graph and $g : I \xrightarrow{hom} H * I$ a homomorphism, there is an edge $e \in E(H)$ such that g is an isomorphism between I and I_e.

If g maps all vertices of I to one of the I_e's, then we are done by Claim 2. If not, let a, b be vertices of I such that $g(a)$ is a vertex of I_e (for some edge

$e = uv \in E(H)$) and $g(b)$ is not. Choose a path $a = a_1, a_2, \ldots, a_k = b$ as in Claim 3, and let a_i be the last vertex on this path that is a vertex of I_e. Not all three vertices $a_{i-1}, a_{i-2}, a_{i-3}$ can be in the 'connecting triangle' $\{v\} \times \{0, 1, 2\}$, on the other hand each of them is connected to a_{i+1}, a contradiction.

Claim 5. For every graph H the graph $H * I$ is M-homotens.

This is an easy consequence of Lemma 2.2.7: if we delete the 'diagonal' edge from each copy of I, the resulting graph is nice.

To finish the proof, let $h : F(G) \xrightarrow{TT} F(H)$ be a TT_M mapping. As graph $G * I$ is M-homotens, h is induced by a homomorphism, say $g : F(G) \xrightarrow{hom} F(H)$. By Claim 4, g maps an I_e to an $I_{e'}$, therefore there is a homomorphism $f : V(G) \to V(H)$ such that $g = \varphi(f)$ and $h = F(f)$, as claimed. □

2.2.3 A necessary condition

In this section we present a necessary condition for a graph to be $\mathbb{Z}$-homotens.[5] As mentioned earlier, circuits are the simplest examples of graphs that are not $\mathbb{Z}$-homotens. Similarly, no graph with a vertex of degree 2 is $\mathbb{Z}$-homotens, except of a triangle. This way of thinking can be further strengthened and generalized, yielding Theorem 2.2.16. To state our result in a compact way, we introduce a definition from [28]. We say that a graph G is *chromatically k-connected* if for every $U \subseteq V(G)$ such that $G - U$ is disconnected the induced graph $G[U]$ has chromatic number at least k. It [28] another (equivalent) formulation is given: G is chromatically k-connected if and only if every homomorphic image of G is k-connected.

Theorem 2.2.16 *Let M be a ring. If a (directed or undirected) graph is connected and M-homotens then it is chromatically 3-connected.*

Proof: Suppose G is a counterexample to the theorem. Hence, vertices of G can be partitioned into sets A, B, U, L, such that $A \cup B$ separates U from L (that is there is no edge from U to L), moreover A, B are independent sets. We may suppose $A \cup B$ is a minimal set that separates U from L. We are going to prove that G is not $\mathbb{Z}$-homotens, therefore by Theorem 7.2.12 not M-homotens as well.

We identify all vertices of A to a single vertex a, and all vertices of B to a vertex b. Let F be the resulting graph, and $f : G \to F$ be the identifying

[5]By Theorem 7.2.12 the presented condition is necessary for a graph to be M-homotens for each M, too.

homomorphism. We define a $TT_{\mathbb{Z}}$ mapping[6] g from F as follows. For $u \in U$ we map edge ua (if it exists) to bu, au to ub, ub to au, and bu to ua. For $u, v \in U$ we map edge uv (if it exists) to vu. Every other edge is mapped to itself. We let F' denote the resulting graph (it has the same set of vertices as F). It is straightforward to use Lemma 1.2.9 to verify that g is indeed $TT_{\mathbb{Z}}$.

Consequently, $g \circ f^\sharp$ is a $TT_{\mathbb{Z}}$ mapping; we need to show that it is not induced. At least one of A, B is non-empty. Moreover, as $A \cup B$ is a minimal separating set, there are vertices $x \in A \cup B$, $u \in U$, $l \in L$ such that, without loss of generality, xu, xl are edges of G. By definition of g we have $g \circ f^\sharp(xl) = xl$ and $g \circ f^\sharp(xu) = uy$. Therefore $g \circ f^\sharp$ maps two adjacent edges to two nonadjacent edges, hence it is not induced. □

The following corollary gives a simpler necessary condition, though a weaker one: We can prove that the graph of icosahedron is not $\mathbb{Z}$-homotens by using Theorem 2.2.16 (the neighborhood of an edge is a C_6), but not by using Corollary 2.2.17.

Corollary 2.2.17 *Let G be a connected graph with at least four vertices. Suppose the neighborhood of some $v \in V(G)$ induces a bipartite graph. Then G is not M-homotens for any ring M.*

Consequently, every vertex of a homotens graph is incident with an odd wheel[7] (in particular with a triangle), except if it is contained in a component of size at most three.

Proof: Let A, B be the color-classes of neighborhood of v. If there is a vertex nonadjacent to v, then we can use Theorem 2.2.16. So suppose v is connected to every vertex of G. Then every other vertex has a bipartite neighborhood. The only case that stops us from using Theorem 2.2.16 is when $|A|, |B| \le 1$, that is when G has at most three vertices. □

A somewhat surprising consequence of Corollary 2.2.17 is that no triangle-free graph (except of a matching) is homotens. This implies the following corollary.

Corollary 2.2.18 *A cubic graph is M-homotens if and only if each of its components is isomophic to K_4 and M is not a power of $\mathbb{Z}_2$.*

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[6] This operation is sometimes called an (oriented) Whitney twist, compare Corollary 7.4.5.

[7] A wheel with k spokes, W_k, is a graph that consists of a circuit C_k and a central vertex that is connected to each vertex of the circuit.

Among regular graphs of higher degree it is possible to find homotens graphs (e.g., the complete graphs). Still, these are not typical (see also Theorem 2.2.9 and Proposition 2.2.10).

Corollary 2.2.19 *Let $r \geq 3$ be integer, M ring. The probability that a random r-regular graph is M-homotens tends to zero if the size of the graph grows to infinity.*

Proof: It is known (see Lemma 2.7 of [91]) that for any fixed graph F with more edges than vertices the probability that random r-regular graph on n vertices contains F tends to zero. If we apply this for all odd wheels with at most r spokes in place of F, we see that by Corollary 2.2.17 the result follows. □

Corollary 2.2.17 also indicates that complete graphs involved in the definition of nice graphs are necessary, at least to some extent. However, the condition of Corollary 2.2.17 (or Theorem 2.2.16) is far from being sufficient: for example the graph from Proposition 2.1.9 is chromatically k-connected and not $\mathbb{Z}$-homotens. In particular, we do not know whether there are K_4-free homotens graphs. By [51], a random K_4-free graph is a.a.s. 3-partite, hence not chromatically 3-connected, hence by Theorem 2.2.16 not $\mathbb{Z}$-homotens. Still, it is possible that K_4-free $\mathbb{Z}$-homotens graphs exist, promising candidates are Kneser graphs $K(4n-1, n)$, which are for large n chromatically 3-connected [28].

Question 2.2.20 *Is the Kneser graph $K(4n-1, n)$ left $\mathbb{Z}$-homotens, if n is large enough?*

2.3 Right homotens graphs

In this section we complement Section 2.2 by study of graphs which, when used as target graphs, make existence of TT mappings and of homomorphisms coincide. Recall (Definition 2.1.4) that a graph H is called right M-homotens if the existence of a TT_M mapping from an arbitrary graph to H implies the existence of a homomorphism. Right homotens graphs (in comparison with left homotens ones) provide more structure; in this section we characterize them by means of special Cayley graphs and state a question aiming to find a better characterization.

2.3.1 Free Cayley graphs

Free Cayley graphs were introduced by Naserasr and Tardif [64] (see also thesis of Lei Chu [16]) in order to study chromatic number of Cayley graphs. They will serve us as a tool to study TT mappings, in particular we will use them to study right homotens graphs and to prove density in Section 3.1.3.

Let M be a ring, let H be a graph. For a vertex $v \in V(H)$ we let $e_v : V(H) \to M$ be the indicator function, that is $e_v(u) = 1$ if $v = u$ and $e_v(u) = 0$ otherwise. We define graph[8] $\Delta_M(H)$ with vertices $M^{V(H)}$, where (f, g) is an edge iff $g - f = e_v - e_u$ for some edge $(u, v) \in E(H)$. We can see that $\Delta_M(H)$ is a Cayley graph, it is called the *free Cayley graph* of H. We begin our study of free Cayley graphs with a simple observation and with a useful lemma, which is due to Naserasr and Tardif (for a proof, see [16]).

Proposition 2.3.1 *Graph $\Delta_M(H)$ contains H as an induced subgraph.*

Proof: Take functions $\{e_v \mid v \in V(H)\} \subseteq V(\Delta_M(H))$. □

Lemma 2.3.2 *Let M be a ring, H a Cayley graph on M^k (for some integer k) and G an arbitrary graph. Then any homomorphism $G \xrightarrow{hom} H$ can be (uniquely) extended to a mapping $\Delta_M(G) \to H$ that is both graph and ring homomorphism.*

The next lemma appears in [19] (although without mentioning graphs Δ_M).

Lemma 2.3.3 *$G \xrightarrow{TT_M} H$ is equivalent with $G \xrightarrow{hom} \Delta_M(H)$.*

Note that Lemma 1.3.1 is a special case of Lemma 2.3.3, as graphs $\Delta(G)$ defined in Section 2.1.2 are isomorphic to $\Delta_{\mathbb{Z}_2}(G)$. Lemmata 2.3.2 and 2.3.3 have as immediate corollary an embedding result that nicely complements Theorem 2.2.15. In contrary with Theorem 2.2.15 though, our embedding is not functorial, it is just embedding of quasiorder $(\mathcal{G}, \preccurlyeq_M^t)$ in $(\mathcal{G}, \preccurlyeq_h)$.

Corollary 2.3.4 *$G \xrightarrow{TT_M} H$ is equivalent with $\Delta_M(G) \xrightarrow{hom} \Delta_M(H)$.*

[8] More precisely, we define $\Delta_M(H)$ to be a directed graph. However, if $\overleftrightarrow{H}$ is a symmetric orientation of an undirected graph H, then $\Delta_M(\overleftrightarrow{H})$ is a symmetric orientation of some undirected graph H', we may let $\Delta_M(H) = H'$. The whole Section 2.3.1 may be modified for undirected graphs by similar changes.

Proof: If $G \xrightarrow{TT_M} H$ then by Lemma 2.3.3 we have $G \xrightarrow{hom} \Delta_M(H)$ and by Lemma 2.3.2 the result follows. For the other implication, by Proposition 2.3.1 graph G maps homomorphically to $\Delta_M(H)$, and application of Lemma 2.3.3 yields $G \xrightarrow{TT_M} H$. □

We remark that Corollary 2.3.4 provides an embedding of category of TT_M mappings to category of Cayley graphs with mappings that are both ring and graphs homomorphisms.

2.3.2 Right homotens graphs

We start our description of right homotens graphs by two simple observations concerning right homotens graphs. The first one is a characterization of right homotens graphs by means of Δ_M. It does not, however, give an efficient method (polynomial algorithm) to verify whether a given graph is right homotens, neither a good understanding of right homotens graphs. Hence, we will seek better characterizations (compare with Corollary 2.3.7 and Question 2.3.12).

Proposition 2.3.5 *A graph H is right M-homotens if and only if $\Delta_M(H) \xrightarrow{hom} H$.*

Proof: For the 'only if' part it is enough to observe that $\Delta_M(H) \xrightarrow{TT_M} H$ for every graph H: clearly $\Delta_M(H) \xrightarrow{hom} \Delta_M(H)$ and we use Lemma 2.3.3. For the other direction, if $G \xrightarrow{TT_M} H$ then by Lemma 2.3.3 we have $G \xrightarrow{hom} \Delta_M(H)$ and by composition we have $G \xrightarrow{hom} H$. □

Lemma 2.3.6 *Assume $H \xrightarrow{hom} H'$ and $H' \xrightarrow{TT_M} H$. If H is right M-homotens then H' is right M-homotens as well.*

Proof: If H is right M-homotens, then $\Delta_M(H) \xrightarrow{hom} H$. By Corollary 2.3.4 from $H' \xrightarrow{TT_M} H$ we deduce that $\Delta_M(H') \xrightarrow{hom} \Delta_M(H)$. By composition,

$$\Delta_M(H') \xrightarrow{hom} \Delta_M(H) \xrightarrow{hom} H \xrightarrow{hom} H',$$

hence H' is right M-homotens. □

Corollary 2.3.7 *Let H, H' be homomorphically equivalent graphs (i.e., $H \xrightarrow{hom} H'$ and $H' \xrightarrow{hom} H$). Then H is right M-homotens if and only if H' is right M-homotens.*

Note that TT_M-equivalence is not sufficient in Corollary 2.3.7: each graph H is TT_M-equivalent with $\Delta_M(H)$ and the latter is always a right M-homotens graph (for each M), as we will see from the next proposition. Also note that the analogy of Corollary 2.3.7 does not hold for left homotens graphs.

Next, we consider a class of right M-homotens graphs that is central to this topic. We will say that H is an *M-graph* if it is a Cayley graph on some power of M ($\mathbb{Z}_2$-graphs are also called cube-like graphs; they have been introduced by Lovász (cf. [35], see also Lemma 5.4.5) as an example of graphs, for which every eigenvalue of the adjacency matrix is an integer).

Proposition 2.3.8 *Any M-graph is right M-homotens.*

Proof: Let H be an M-graph. As $H \xrightarrow{hom} H$, by Lemma 2.3.2 we conclude that $\Delta_M(H) \xrightarrow{hom} H$. □

In analogy with chromatic number we define the TT_M number $\chi_{TT_M}(G)$ to be the minimum n for which there is a graph H with n vertices such that $G \xrightarrow{TT_M} H$. As any homomorphism induces a TT_M mapping, we see that $\chi_{TT_M}(G) \leq \chi(G)$ for every graph G. Continuing our project of finding similarities between TT_M mappings and homomorphisms, we prove that the TT_M number cannot be much smaller than the chromatic number. Note that the second claim of the next result is proved (essentially by the same method) in [55].

Corollary 2.3.9 *Let G be a graph, M a ring of characteristic p, and q the smallest prime dividing p.*

1. *If $p > 0$ then $1 \leq \chi(G)/\chi_{TT_M}(G) < q$.*
2. *If $p = 0$ then $\chi(G)/\chi_{TT_M}(G) = 1$.*

Proof: First we prove that $\chi(G) < q \cdot \chi_{TT_M}(G)$ for any finite ring M of size q. To this end, consider a Cayley graph on M^k with the generating set $M^k \setminus \{\vec{0}\}$—that is a complete graph K_{q^k} with every edge in both orientations. This is an M-graph, hence by Proposition 2.3.8 it is right M-homotens.

Now, choose k so that $q^{k-1} < \chi_{TT_M}(G) \leq q^k$. It follows that $G \xrightarrow{TT_M} K_{q^k}$, and as K_{q^k} is right M-homotens, $G \xrightarrow{hom} K_{q^k}$. Therefore, $\chi(G) \leq q^k < q \cdot \chi_{TT_M}(G)$.

Next, if $p > 0$ is the characteristic of M and q divides p, then M contains $\mathbb{Z}_q$ as a subring. This by Lemma 7.2.2 implies that any TT_M mapping is $TT_{\mathbb{Z}_q}$, thus $\chi_{TT_M}(G) \geq \chi_{TT_{\mathbb{Z}_q}}(G)$, and the result follows.

For the second part suppose $\chi_{TT_M}(G) = n$. Note that a ring of characteristic 0 contains $\mathbb{Z}$. We use Lemma 7.2.2 again to infer that any TT_M mapping is $TT_{\mathbb{Z}_n}$. As above, complete graph K_n is right $\mathbb{Z}_n$-homotens, hence the mapping $G \xrightarrow{TT_n} K_n$ is induced, therefore $\chi(G) \leq n$ as required. □

How good is the bound given by Corollary 2.3.9 is an interesting and difficult question. Even in the simplest case $M = \mathbb{Z}_2$ this is widely open; perhaps surprisingly it is related to the quest for optimal error correcting codes (see Section 7.1 for details). Another corollary of Proposition 2.3.8 is a characterization of right homotens graphs.

Corollary 2.3.10 *A graph is right M-homotens if and only if it is homomorphically equivalent to an M-graph.*

Proof: The 'if' part follows from Corollary 2.3.7 and Proposition 2.3.8. For the 'only if' part, notice that $\Delta_M(H)$ is a M-graph, $H \subseteq \Delta_M(H)$, and if H is right M-homotens then $\Delta_M(H) \xrightarrow{hom} H$. □

Corollary 2.3.10 is not very satisfactory, as it does not provide any useful algorithm to verify if a given graph is right homotens. Indeed, it is more a characterization of graphs that are hom-equivalent to some M-graph, than the other way around: Suppose we are to test whether a given graph is hom-equivalent to some (arbitrarily large) M-graph. It is not obvious if there is a finite process to decide this; however Corollary 2.3.10 reduces this task to decide whether $\Delta_M(H) \xrightarrow{hom} H$. The latter condition can be checked by an obvious brute-force algorithm.

We hope that a more helpful characterization of right homotens graphs will result from considering the core of a given graph. As a core of a graph H is hom-equivalent with H, it is right homotens if and only if H is. Therefore, we attempt to characterize right homotens cores, leading to an easy proposition and an adventurous question. We note that one part of the proof of Proposition 2.3.11

is basically the well-known fact that the core of a vertex-transitive graph is vertex-transitive, while the other part is a generalization of an argument used by [34] to prove that K_n is right $\mathbb{Z}_2$-homotens (or, in the language of [34], that the injective chromatic number of the cube Q_n is n) if and only if n is a power of 2. However, we include the proof for the sake of completeness.

Proposition 2.3.11 *Let H be a right M-homotens graph that is a core. Then*

- *$|V(H)|$ is a power of $|M|$, and*
- *H is vertex transitive. If $M = \mathbb{Z}_2$, then for every two vertices of H, there is an automorphism exchanging them.*

Proof: For a function $g \in M^{V(H)}$ we let H_g denote the subgraph of $\Delta_M(H)$ induced by the vertex set $\{g + e_v; v \in V(H)\}$. Observe that each H_g is isomorphic with H. Let $f : \Delta_M(H) \to H$ be a homomorphism and for each $u \in V(H)$, define $V_u = \{v \in V(\Delta_M(H)); f(v) = u\}$. Now f restricted to H_g is a homomorphism from H_g to H. As H is a core, every homomorphism from H to H is a bijection. Consequently, for every g the graph H_g contains precisely one vertex from each V_u. By considering all graphs H_g we see that all sets V_u are of the same size $|M|^{|V(H)|}/|V(H)|$. Therefore, $|V(H)|$ is a power of $|M|$, as claimed.

For the second part let u, v be distinct vertices of H. We know $\Delta_M(H) \xrightarrow{hom} H$. As $H \cong H_{\vec{0}}$ ($\vec{0}$ being the identical zero), we have a homomorphism $f : \Delta_M(H) \xrightarrow{hom} H_{\vec{0}}$. As H is a core, we know that f restricted to $H_{\vec{0}}$ is an automorphism of $H_{\vec{0}}$. By composition with the inverse automorphism, we may suppose that f restricted to $H_{\vec{0}}$ is the identity. Next, consider the isomorphism $\varphi : \Delta_M(H) \xrightarrow{hom} \Delta_M(H)$ given by $g \mapsto g + e_v - e_u$. A composed mapping $f \circ \varphi$ is a homomorphism $H_{\vec{0}} \xrightarrow{hom} H_{\vec{0}}$ (therefore an automorphism) that maps u to v. Moreover, if $M = \mathbb{Z}_2$ then $f \circ \varphi$ maps v to u as well. □

The previous proposition suggests that a stronger result might be true, and that this may be a way to a characterization of right homotens graphs. In particular, we ask the following question, which is of independent interest.

Question 2.3.12

1. *Suppose H is a right M-homotens graph and a core. Is H an M-graph?*
2. *Is the core of each M-graph an M-graph?*

We note that even the (perhaps easier to understand) case $M = \mathbb{Z}_2$ is open. But one can see easily that 1 and 2 in Question 2.3.12 are equivalent: If H is a right M-homotens core, then H is the core of the M-graph $\Delta(H)$; hence 2 implies 1. Conversely, let K be an M-graph and H its core. Graph K is right M-homotens by Proposition 2.3.8, therefore by Corollary 2.3.7 its core H is right M-homotens. If 1 is true, then H is an M-graph, as claimed in 2.

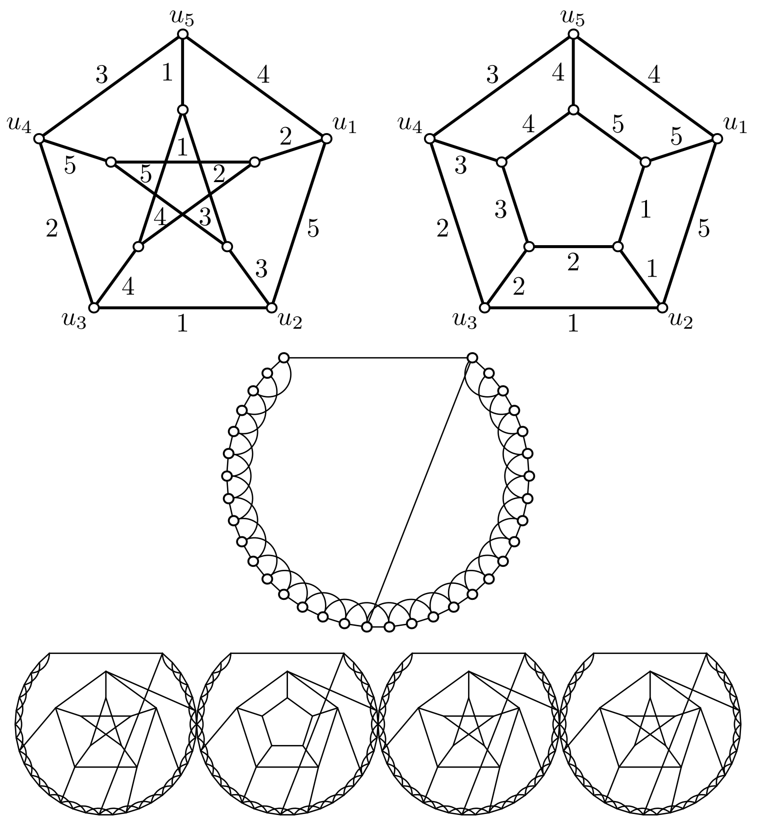

Figure 2.2: Petersen graph, Pt, and the prism R of C_5—two TT_2-equivalent graphs used in the proof of Proposition 2.1.10. Below is a graph that we use as a 'frame' to hold one of the two graphs above, and an example of the construction for $n = 4, t = (1, 0, 1, 1)$.

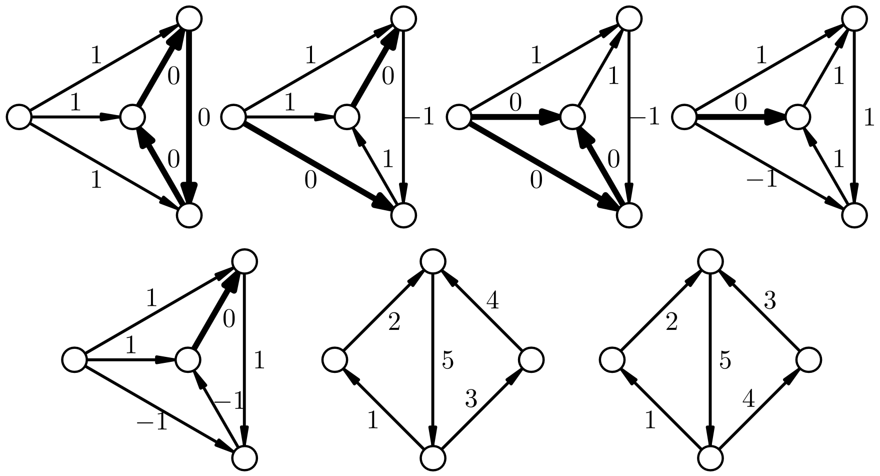

Figure 2.3: Illustration of proof of Lemma 2.2.5, Theorem 2.2.2 and 2.2.3.

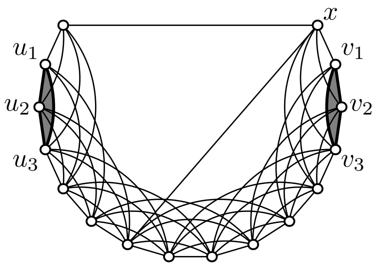

Figure 2.4: The graph I used in triangle-based replacement (proof of Theorem 2.2.15).

Chapter 3

TT mappings & homomorphisms: structural properties

In this chapter we compare homomorphisms and TT mappings from a different perspective: we prove that partial orders defined by existence of a homomorphism (a TT_M mapping respectively) share several important properties. We start with the density and give two proofs, one using a new structural Ramsey theorem, and another by a $\Delta(G)$ construction. Then we provide an embedding of homomorphisms to TT_2 mappings. Unlike the proof in previous chapter (where we used nice graphs), we restrict to the case $M = \mathbb{Z}_2$; on the other hand, we use triangle-free graphs in our construction, an impossibility with the approach via nice graphs. We finish by several smaller results on the structure of TT orders.

3.1 Density

To recall, we say that a partial order $<$ is dense, if for every A, B satisfying $A < B$ there is an element C for which $A < C < B$.

It is known [40, 88, 72] that the homomorphism order (with all hom-equivalence classes of finite graphs as elements and with the relation $\prec_h$) is dense, if we do not consider graphs without edges. The parallel result for the order defined by TT_M mappings is given by Corollary 3.1.7. In fact we prove a stronger property (proved in [68]) that every finite antichain in a given interval can be extended (Theorem 3.1.6), density is the special case $t = 0$.

The usual proofs of density of the homomorphism order rely on the fact, that the category of graphs and homomorphisms has products. We prove in Proposition 3.1.12, that this is not true for TT_M mappings; therefore another approach is needed. In Section 3.1.1 we develop a new structural Ramsey-type theorem to overcome the non-existence of products and in Section 3.1.2 we apply it. Finally, in Section 3.1.3 we use the construction Δ_M for a different (and shorter) proof of density.

3.1.1 A Ramsey-type theorem for locally balanced graphs

In Section 3.1.1 we deal with undirected graphs only. We prove a Ramsey-type theorem that will be used in Section 3.1.2 as a tool to study $\prec_M^t$ (on directed graphs).

An *ordered graph* is an undirected graph with a fixed linear ordering of its vertices. The ordering will be denoted by $<$, an ordered graph by $(G, <)$, or shortly by G. We say that two ordered graphs are *isomorphic*, if the (unique) order-preserving bijection is a graph isomorphism. An ordered graph $(G, <)$ is said to be a *subgraph* of $(H, <')$, if G is a subgraph of H, and the two orderings coincide on $V(G)$.

A circuit $C = v_1, \dots, v_l$ in an ordered graph is *balanced* iff

$$\left|\{i; v_i < v_{(i \bmod l)+1}\}\right| = \left|\{i; v_i > v_{(i \bmod l)+1}\}\right| .$$

This can be reformulated using the notion preceding Lemma 1.2.10. Let $\overrightarrow{G}$ be a directed graph with $V(\overrightarrow{G}) = V(G)$ and $E(\overrightarrow{G}) = \{(u, v); uv \in E(G) \text{ and } u < v\}$. (We can say that all edges are oriented 'up'.) Then a circuit in G is balanced iff the corresponding circuit in $\overrightarrow{G}$ is $\mathbb{Z}$-balanced. Note that a circuit in $\overrightarrow{G}$ is $\mathbb{Z}_2$-balanced iff its length is even.

Denote by Cyc_p the set of all ordered graphs that contain no odd circuit of length at most p. Denote by Bal_p the set of all ordered graphs that contain no unbalanced circuit of length at most p.

Nešetřil and Rödl [69] proved the following Ramsey-type theorem.

Theorem 3.1.1 *Let k, p be positive integers. For any ordered graph $(G, <) \in \mathrm{Cyc}_p$ there is an ordered graph $(H, <) \in \mathrm{Cyc}_p$ with the following property: for every coloring of $E(H)$ by k colors there is a monochromatic subgraph $(G', <)$, isomorphic to $(G, <)$.*

We will need a version of this theorem for Bal_p. By the discussion above, this means that we consider $\mathbb{Z}$-balanced (instead of $\mathbb{Z}_2$-balanced) circuits.

Theorem 3.1.2 *Let r, p be positive integers. For any ordered graph $(G,<) \in \mathrm{Bal}_p$ there is an ordered graph $(H,<) \in \mathrm{Bal}_p$ with the the following property: for every edge coloring of H by r colors there is a monochromatic subgraph $(G',<)$, isomorphic to G. This conclusion will be shortly written as $(H,<) \to (G,<)^2_r$.*

Proof: The proof of Theorem 3.1.2 uses a variant of the amalgamation method (partite construction) due to the first author and Rödl (see, e.g., [70, 66]), which has many applications in structural Ramsey theory.

For the purpose of this proof we slightly generalize the notion of ordered graph. We work with graphs with a quasiordering $\leq$ of its vertices; such graphs are called *quasigraphs*, $\leq$ is called the standard ordering of G. Alternatively, a quasigraph $(G,\leq)$ is a graph $G=(V,E)$ with a partition $V_1 \cup V_2 \cup \cdots \cup V_a$ of V: each V_i is a set of mutually equivalent vertices of V and $V_1 < V_2 < \cdots < V_a$. The number a of equivalence classes of $\leq$ will be fixed throughout the whole proof. In this case we speak about *a-quasigraphs*. It will be always the case that every V_i is an independent set of G.

An embedding $f : (G,\leq) \to (G',\leq')$ is an embedding (i.e. an isomorphism onto an induced subgraph) $G \to G'$ which is moreover monotone with respect to the standard orderings $\leq$ and $\leq'$. Explicitly, such an embedding f is an embedding of G to G' for which there exists an increasing mapping $\iota : \{1,2,\ldots,a\} \to \{1,2,\ldots,a'\}$ such that $f(V_i) \subseteq V'_{\iota(i)}$ for $i=1,\ldots,a$. (Here $V'_1 < V'_2 < \cdots < V'_{a'}$ are equivalence classes of the quasiorder $\leq'$.) By identifying the equivalent vertices of a quasigraph G we get a graph $\bar{G}$ and a homomorphism $\pi : G \to \bar{G}$; graph $\bar{G}$ is called the *shadow* of G, mapping π is called *shadow projection*.

We prove Theorem 3.1.2 by induction on p. The case $p=1$ is the Ramsey theorem for ordered graphs and so we can use Theorem 3.1.1 for $p=1$. In the induction step ($p \to p+1$) consider arbitrary ordered graph $(G,\leq)$, let $G=(V,E)$, $|V|=n$, and $G \in \mathrm{Bal}_{p+1}$. By the induction assumption there exists an ordered graph $(K,\leq) \in \mathrm{Bal}_p$ such that

$$K \to (G)^2_r\,.$$

Let $V(K)=\{x_1 < \cdots < x_a\}$ and $E(K)=\{e_1,\ldots,e_b\}$. In this situation we shall construct (by induction) a-quasigraphs P^0, P^1, …, P^b (called usually 'pictures'). Then the quasigraph P^b will be transformed to the desired ordered graph $(H,\leq) \in \mathrm{Bal}_p$ satisfying

$$(H,\leq) \to (G,\leq)^2_r\,.$$

We proceed as follows. Let $(P^0, \leq^0) \in \mathrm{Bal}_{p+1}$ be a-quasiordered graph for which for every induced subgraph G' of K, such that $(G', \leq)$ is isomorphic to $(G, \leq)$, there exists a subgraph G_0 of P^0 with the shadow G'. Clearly $(P^0, \leq^0)$ exists, as it can be formed by a disjoint union of $\binom{a}{n}$ copies of G with an appropriate quasiordering.

In the induction step $k \to k+1$ ($k \geq 0$) let the picture $(P^k, \leq^k)$ be given. Write $P^k = (V^k, E^k)$ and let $V_1^k < V_2^k < \cdots < V_a^k$ be all equivalence classes of $\leq^k$. Consider the edge $e_{k+1} = \{x_{i_{k+1}}, x_{j_{k+1}}\}$ of K ($x_{i_{k+1}} < x_{j_{k+1}}$). To simplify the notation, we will write $i = i_{k+1}$, $j = j_{k+1}$. Let $B^k = (V_i^k \cup V_j^k, F^k)$ be the bipartite subgraph of P^k induced by the set $V_i^k \cup V_j^k$. We shall make use of the following lemma.

Lemma 3.1.3 *For every bipartite graph B there exists a bipartite graph B' such that*

$$B' \to (B)^2_r\,.$$

(The embeddings of bipartite graphs map the upper part to the upper part and the lower part to the lower part.)

This lemma is easy to prove and it is well-known, see for example [66].

Continuing our proof, let

$$B'^k \to (B^k)^2_r \tag{3.1}$$

be as in Lemma 3.1.3 and put explicitly $B'^k = (V_i^{k+1} \cup V_j^{k+1}, F^{k+1})$. Let also $\mathcal{B}_k$ be the set of all induced subgraphs of B'^k, which are isomorphic to B^k. Now we are in the position to construct the picture $(P^{k+1}, \leq^{k+1})$.

We enlarge every copy of B^k to a copy of $(P^k, \leq^k)$ while keeping the copies of P^k disjoint outside the set $V_i^{k+1} \cup V_j^{k+1}$. The quasiorder $\leq^{k+1}$ is defined from copies of quasiorder $\leq^k$ by unifying the corresponding classes. While this description perhaps suffices to many here is an explicit definition of P^{k+1}:

Put $P^{k+1} = (V^{k+1}, E^{k+1})$, where $V^{k+1} = V^k \times \mathcal{B}/\sim$. The equivalence $\sim$ is defined by

$$(v, B) \sim (v', B') \quad \Longleftrightarrow \quad v = v' \in V_i^{k+1} \cup V_j^{k+1} \quad \text{or} \quad v = v' \text{ and } B = B'.$$

Denote by $[v, B]$ the equivalence class of $\sim$ containing (v, B). We define the edge set by putting $\{[v, B], [v', B']\} \in E^{k+1}$ if $\{v, v'\} \in E^k$ and $B = B'$. Define quasiorder $\leq^{k+1}$ by putting

$$[v, B] \leq^{k+1} [v', B'] \Longleftrightarrow v \leq^k v'\,.$$

It follows that $\leq^{k+1}$ has a equivalence classes $V_1^{k+1} < \cdots < V_a^{k+1}$. (Note that this is consistent with the notation of classes V_i^{k+1}, V_j^{k+1} of B'^k.)

Continuing this way, we finally define the picture $(P^b, \leq^b)$. Put $H = P^b$ and let $\leq$ be an arbitrary linear ordering that extends the non-symmetric part of the quasiorder $\leq^b$. We claim that the graph H has the desired properties. To verify this it suffices to prove:

(i) $(H, \leq) \in \mathrm{Bal}_{p+1}$ and

(ii) $(H, \leq) \rightarrow (G, \leq)^2_r$.

The statement (i) will be implied by the following claim.

Claim 3.1.4 *1.* $(P^0, \leq^0) \in \mathrm{Bal}_{p+1}$.

2. If $(P^k, \leq^k) \in \mathrm{Bal}_{p+1}$, *then* $(P^{k+1}, \leq^{k+1}) \in \mathrm{Bal}_{p+1}$.

Proof of Claim: The first part follows from the construction. In the second part, suppose that P^{k+1} contains an unbalanced circuit $C = u_1, u_2, \ldots, u_l$ of length $l \leq p+1$. Let $\pi : V(P^{k+1}) \rightarrow V(K)$ be the projection, that is for $u \in V_s^{k+1}$ we have $\pi(u) = x_s$. From the construction it follows that π is a homomorphism $P^{k+1} \xrightarrow{hom} K$, in other words that K is the shadow of P^{k+1}.

Consider the closed walk $C_\pi = \pi(u_1), \pi(u_2), \ldots, \pi(u_l)$ in K. As C_π is unbalanced closed walk, it contains an unbalanced circuit of length $l' \leq l$. Since $K \in \mathrm{Bal}_p$, we have $l' = l = p+1$, that is $\pi(u_1), \ldots, \pi(u_{p+1})$ are all distinct. Let $u_s = [v_s, B_s]$. If $B_1 = B_2 = \cdots = B_l$, that is the whole C is contained in one copy of P^k, we have a contradiction as $P^k \in \mathrm{Bal}_{p+1}$.

Now we use the construction of P^{k+1} as an amalgamation of copies of P^k: If $B_t \neq B_{t+1}$ (indices modulo l), then $\pi(u_t) \in \{x_{i_{k+1}}, x_{j_{k+1}}\}$. As the vertices $\pi(u_1), \ldots, \pi(u_l)$ are pairwise distinct, this happens just for two values of t. Consequently, the whole C' is contained in two copies of P^k and there are indices α, β such that $\pi(u_\alpha) = x_{i_{k+1}}$ and $\pi(u_\beta) = x_{j_{k+1}}$.

The circuit C is a concatenation of P' and P''—two paths between u_α and u_β, each of them properly contained in one copy of P^k. No copy of P^k contains whole C, therefore both P' and P'' have at least two edges, hence at most $p-1$ edges. Let $\bar{P}'$, $\bar{P}''$ denote the shadows of P' and P''. Both $\bar{P}' \cup \{e_{k+1}\}$ and $\bar{P}'' \cup \{e_{k+1}\}$ are closed walks in K containing at most p edges. As $K \in \mathrm{Bal}_p$, both of them are balanced, so C is balanced as well, a contradiction. $\square$

We turn to the proof of statement (ii). We use a standard argument that is the core of the amalgamation method. Let $E(H) = E(P^b) = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_r$ be a fixed coloring. We proceed by backwards induction $b \to b-1 \to \cdots$ and we prove that there exists a quasisubgraph P_0^k of P^b isomorphic to P^k such that for any $l > k$, any two edges of P_0^k with shadow e_l get the same color. This is easy to achieve using the Ramsey properties (3.1) of graphs B'^k. Finally, we obtain a copy P_0^0 of P^0 in P^b such that the color of any of its edges depends only on its shadow (in K). However $K \to (G)^2_r$ and as for any copy G' of G in K there exists a subgraph G_0 of $\bar{P}^0$ such that its shadow is G' we get that there exists a monochromatic copy of G in P^b. This concludes the proof. □

3.1.2 Density: first proof

In this section we prove the density of TT_M order. For this we first prove the 'Sparse Incomparability Lemma', Lemma 3.1.5 (analogous statement for homomorphisms was proved in [71, 73], it is stated here as Lemma 3.1.11). The proof follows similar path as in the homomorphism case, the main step is considerably harder, though. To overcome the nonexistence of products in the category of TT_M mappings, we use the Ramsey-type theorem from the previous section.

Lemma 3.1.5 *Let M be a ring, let l, $t \geq 1$ be integers. Let $G_1, G_2, \ldots, G_t$, H be graphs such that $H \overset{TT_M}{\not\longrightarrow} G_i$ for every i and at least one of $E(G_i)$ is nonempty. Then there is a graph G such that*

1. *all circuits in G shorter than l are M-balanced, and*
2. *$G \preccurlyeq^t_M H$, moreover $G \preccurlyeq_h H$,*
3. *$G \overset{TT_M}{\not\longrightarrow} G_i$ for every $i = 1, \ldots, t$.*

Proof: Choose an odd integer p larger than $\max\{|E(H)|, l\}$. Pick any linear ordering of $V(H)$ to make H into an ordered graph $(H, <)$ and subdivide each edge to increase the girth. More precisely, we replace every edge e of H by an oriented path $P(e) = e_1, e_2, \ldots, e_p$; the ordering of $V(H)$ is extended to the new vertices so, that e_j goes up iff j is odd, see Figure 3.1. When we do this for every edge of H, we forget the orientation of the edges and let $(H', <)$ denote the resulting ordered graph. It is $(H', <) \in \mathrm{Bal}_p$.

Put $r = \max_i |E(G_i)|^{|E(H)|}$. Using Theorem 3.1.2 we find a graph $(R, <) \in \mathrm{Bal}_p$ satisfying $(R, <) \to (H', <)^2_r$. As every circuit of $(R, <)$ is balanced, it is also M-balanced.

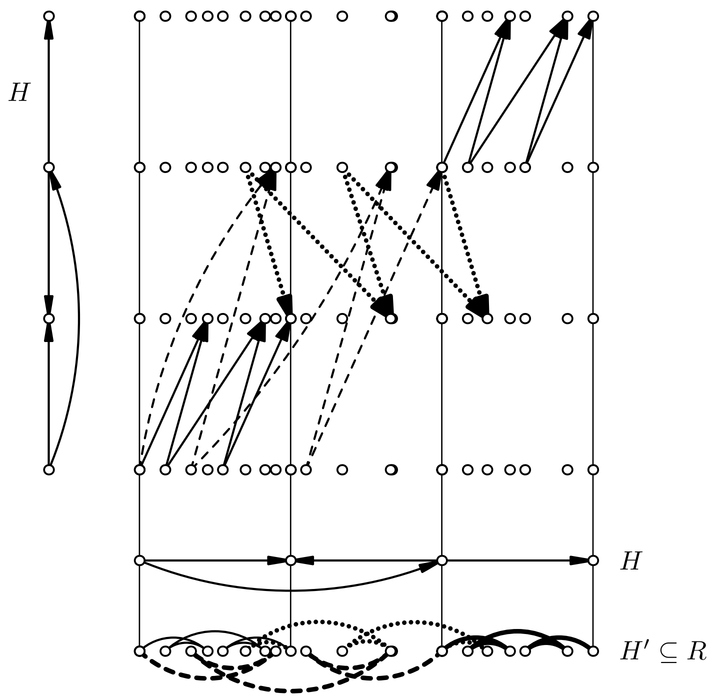

Figure 3.1: An illustration of the proof of Lemma 3.1.5 (here $s = 5$). Only a part of the graph $G = H \times R$ is shown.

We orient all edges of R up (that is towards the vertex larger in $<$), and set $G = H \times R$ (see Figure 3.1). Formally, $V(G) = V(H) \times V(R)$, and for edges $e = uv$ of H and $e' = u'v'$ of R we have an edge from (u, u') to (v, v') (this edge will be denoted by (e, e')).

Now $G \xrightarrow{TT_M} R$ (there is even a homomorphism—the projection), so by Lemma 1.2.10 there is no short M-unbalanced circuit in G. This gives part 1 of the statement. The other projection of G gives $G \xrightarrow{TT_M} H$, and indeed even $G \xrightarrow{hom} H$. This proves part 2, it remains to prove part 3.

For the contrary, suppose there is an index i and a TT_M mapping $f : G \xrightarrow{TT_M} G_i$. As $G = H \times R$, this induces a coloring c of edges of R by elements of $E(G_i)^{E(H)}$ (where $c(e')$ sends e to $f((e, e')))$. As we have chosen R to be a Ramsey graph for H', there is a monochromatic copy of H' in R. To ease the notation we will suppose this copy is just H', let g be the color of edges of H'. We will show that g is a TT_M mapping $H \to G_i$, and this will be our desired contradiction.

We will use Lemma 1.2.9, hence for any flow $\varphi : E(H) \to M$ we need to show that φ_g is a flow. Clearly it is enough to verify this for φ being an elementary flow, as elementary flows generate the M-flow space on H. So let C be a circuit in H that is the support of φ. The corresponding circuit $\bar{C}$ in $H \times R$ has edge set

$$E(\bar{C}) = \bigcup \{\{e\} \times P(e), e \in E(C)\} .$$

Let $\bar{\varphi}$ be the elementary flow on $H \times R$ corresponding to φ. Explicitly,

$$\bar{\varphi} : (e, e_i) \mapsto \begin{cases} \varphi(e), & \text{if } i \text{ is odd,} \\ -\varphi(e), & \text{if } i \text{ is even.} \end{cases}$$

As H' is g-monochromatic, $f((e, e_i)) = g(e)$ for every i. Consequently $\varphi_g = \bar{\varphi}_f$, so φ_g is a flow. □

Theorem 3.1.6 *Let M be a ring, let $t \geq 0$ be an integer. Let G, H be graphs such that $G \prec_M^t H$ and $E(G) \neq \emptyset$. Let $G_1, G_2, \ldots, G_t$ be pairwise incomparable (in $\prec_M^t$) graphs satisfying $G \prec_M^t G_i \prec_M^t H$ for every i. Then there is a graph K such that*

1. *$G \prec_M^t K \prec_M^t H$,*
2. *K and G_i are TT_M-incomparable for every $i = 1, \ldots, t$.*

If in addition $G \preccurlyeq_h H$ then we have even $G \prec_h K \prec_h H$. If we consider undirected graphs, then we get undirected graph K.

Proof: Choose $l > \max\{|E(H)|, |E(G_i)|, i = 1, \ldots, t\}$. We use Lemma 3.1.5 to get a graph G' without short unbalanced circuits such that $G' \xrightarrow{hom} H$, $G' \stackrel{TT_M}{\not\longrightarrow} G_i$ and $G' \stackrel{TT_M}{\not\longrightarrow} G$, then we put $K = G + G'$. Easily $G \preccurlyeq_M^t K \preccurlyeq_M^t H$ and $K \stackrel{TT}{\not\longrightarrow} G_i$, $K \stackrel{TT}{\not\longrightarrow} G$ (as G' has this property). It remains to show $F \stackrel{TT}{\not\longrightarrow} K$ for $F \in \{H, G_1, \ldots, G_t\}$. Note that it is not enough to show $F \stackrel{TT}{\not\longrightarrow} G$ and $F \stackrel{TT}{\not\longrightarrow} G_i$, we have to proceed more carefully.

So suppose we have a TT_M mapping $f : F \xrightarrow{TT} G + G'$. Pick an edge $e_0 \in E(G)$, and define $g : E(F) \to E(G)$ as follows:

$$g(e) = \begin{cases} f(e), & \text{if } f(e) \in E(G) \\ e_0, & \text{otherwise.} \end{cases}$$

We prove that g is TT_M which will be a contradiction. So let τ be an M-tension on G, we are to prove that τg is an M-tension on F. By the choice of l, graph $f(F) \cap G'$ doesn't contain an M-unbalanced circuit (there is no that short unbalanced circuit in G'), hence any constant mapping is an M-tension. So we may choose a tension τ' on $G + G'$ that equals a constant $\tau(e_0)$ on $f(F) \cap G'$ and extends τ. Clearly τg is the same function as $\tau' f$, hence it is a tension.

For the last part of statement of the theorem, $G \xrightarrow{hom} G + G' \xrightarrow{hom} H$ follows immediately (using Lemma 3.1.5, part 2). If we had $H \xrightarrow{hom} K$ or $K \xrightarrow{hom} G$, then by Lemma 2.1.2 the homomorphism induces a TT_M mapping $H \xrightarrow{TT_M} K$ (or $K \xrightarrow{TT_M} G$, respectively), which we already excluded.

To prove the result for undirected graphs, we have two options. One of them is to use Lemma 3.1.10 from the next section that improves Lemma 3.1.5. Another is to modify the above proof as follows. We apply Lemma 3.1.5 for symmetric orientations $\overleftrightarrow{H}$, $\overleftrightarrow{G}_i$, $\overleftrightarrow{G}$, that is we replace each edge of the undirected graph by a directed 2-cycle. Let $\overrightarrow{G}'$ be the graph we get and G' be its underlying undirected graph. Again, we put $K = G + G'$. Now $G \xrightarrow{TT_M} K$ is immediate, $K \xrightarrow{TT_M} H$ follows by Proposition 1.2.2 (as $\overrightarrow{G}' \xrightarrow{TT_M} \overleftrightarrow{G}'$). If $G' \xrightarrow{TT_M} X$ (for $X \in \{G, G_i\}$) then $\overrightarrow{G}' \xrightarrow{TT_M} \overleftrightarrow{G}' \xrightarrow{TT_M} \overleftrightarrow{X}$ (again Proposition 1.2.2), a contradiction with the choice of $\overrightarrow{G}'$. It remains to show that for $F \in \{H, G_i\}$ we have $F \not\xrightarrow{TT_M} K$. Suppose the contrary, it follows that for some orientation $\overrightarrow{G}$ and the (given) orientation $\overrightarrow{G}'$ there is an orientation $\overrightarrow{F}$ such that $\overrightarrow{F} \xrightarrow{TT_M} \overrightarrow{G} + \overrightarrow{G}'$. Now $\overrightarrow{F}$ contains an M-unbalanced circuit (otherwise $F \xrightarrow{TT_M} G$, a contradiction) while $\overrightarrow{G}'$ does not contain such short M-unbalanced circuits (by Lemma 3.1.5). Therefore we may proceed as above for the case of directed graphs and conclude $\overrightarrow{F} \xrightarrow{TT_M} \overrightarrow{G}$, a contradiction. $\square$

To state Theorem 3.1.6 in a concise form we define open and closed intervals in order $\prec^t_M$. Let $(G, H)_M = \{G' \mid G \prec^t_M G' \prec^t_M H\}$ and $[G, H]_M = \{G' \mid G \preccurlyeq^t_M G' \preccurlyeq^t_M H\}$. Similarly, define $(G, H)_h$ and $[G, H]_h$—intervals in order $\prec_h$. Lemma 2.1.2 implies that $[G, H]_h \subseteq [G, H]_M$ for any ring M. On the contrary, none of the two possible inclusions between $(G,H)_h$ and $(G,H)_M$ is valid for every G, H. Therefore the additions in the following corollaries do indeed provide a strengthening, we will use this strengthening in Section 3.3.3.

Corollary 3.1.7 *Suppose $G \prec^t_M H$ and $E(G) \neq \emptyset$. Then $(G,H)_M$ is nonempty. If in addition $G \prec_h H$ then $(G,H)_M \cap (G,H)_h$ is nonempty.*

Corollary 3.1.8 *Suppose $G \prec^t_M H$ and $E(G) \neq \emptyset$. Then in partial order $\prec^t_M$ restricted to (G,H), any finite antichain can be extended. If in addition $G \prec_h H$ then any finite antichain of $\prec^t_M$ restricted to $(G,H)_M \cap (G,H)_h$ can be extended.*

Remark 3.1.9 *Throughout this section we need to assume $E(G) \neq \emptyset$: for example in Corollary 3.1.7 there is no graph K satisfying $K_1 \prec^t_M K \prec^t_M \overrightarrow{K_2}$ (if K has no edge then it maps to K_1, otherwise $\overrightarrow{K_2}$ maps to it). We may say that $(K_1, \overrightarrow{K_2})$ is a* gap *of the partial order $\prec^t_M$.*

3.1.3 Density: second proof

In this section, we give a different proof of Theorem 3.1.6; or, in fact, only of Lemma 3.1.5, that was the key step of the proof. We prove here even a slightly stronger version:

Lemma 3.1.10 (Sparse incomparability lemma for TT_M) *Let M be an abelian group (not necessarily a finitely generated one), let l, $t \geq 1$ be integers. Let G_1, G_2, …, G_t, H be (finite directed non-empty[1]) graphs such that $H \overset{TT_M}{\not\longrightarrow} G_i$ for every i. Then there is a graph G such that*

1. *$g(G) \geq l$ (that is G contains no circuit shorter than l),*
2. *$G \prec_h H$,*
3. *$G \overset{TT_M}{\not\longrightarrow} G_i$ for every $i = 1, \dots, t$.*

(For undirected graphs we get undirected graph G.)

In the proof we will use variant of Sparse incomparability lemma for homomorphisms in the following form (it has been proved for undirected graphs in [71, 73], the version we present here follows by the same proof).

[1] that is with non-empty edge set

Lemma 3.1.11 (Sparse incomparability lemma for homomorphisms) *Let l, $t \geq 1$ be integers, let H, G_1, ..., G_t be (finite directed non-empty) graphs such that $H \overset{hom}{\not\longrightarrow} G_i$ for every i. Let c be an integer. Then there is a (directed) graph G such that*

- *$g(G) \geq l$ (that is G contains no circuit shorter than l),*
- *$G \prec_h H$, and*
- *$G \overset{hom}{\not\longrightarrow} G_i$ for every i.*

(For undirected graphs we get undirected graph G.)

Proof of Lemma 3.1.10: First, suppose that M is a finite ring; by Lemma 2.3.3 we know that $H \overset{hom}{\not\longrightarrow} \Delta_M(G_i)$ for every i. Therefore, we may use Lemma 3.1.11 to obtain G' of girth greater than l such that $G' \preccurlyeq_h H$ and $G' \not\preccurlyeq_h \Delta_M(G_i)$. Consequently $G' \overset{TT}{\not\longrightarrow}_M G_i$ for every i.

Next, let M be an infinite, finitely generated group, that is a ring. Then $M \simeq \mathbb{Z}^\alpha \times \prod_{i=1}^k \mathbb{Z}_{n_i}^{\beta_i}$ for some integers k, n_i, β_i, α (this classical result is stated as Theorem 7.2.1). As M is infinite, we have $\alpha > 0$, therefore $M \geq \mathbb{Z}$. By Lemma 7.2.2 we conclude that for any mapping it is equivalent to be TT_M and to be $TT_\mathbb{Z}$, hence we may suppose $M = \mathbb{Z}$. By Lemma 7.2.8, there is only finitely many integers n for which holds $H \xrightarrow{TT_n} G_i$ for some i or $H \xrightarrow{TT_n} \overrightarrow{K}_2$. Pick some n for which neither of this holds. By the previous paragraph for ring $\mathbb{Z}_n$ we find a graph G' such that $G' \overset{TT_n}{\not\longrightarrow} G_i$ for every $i = 1, \ldots, t$. It follows from Lemma 7.2.2 that also $G' \overset{TT_M}{\not\longrightarrow} G_i$.

Finally, let M be a general abelian group. For each mapping $f : E(H) \to X$ (where $X \in \{G_1, \ldots, G_t\}$) there is an M-tension φ_X on X which certifies that f is not a TT_M mapping. Let $A = \big\{\varphi_X(e) \mid e \in E(X), X \in \{G_1, \ldots, G_t\}\big\}$ be the set of all elements of M that are used for these certificates. Let M' be the subgroup of M generated by A; by the choice of A we have $H \overset{TT_{M'}}{\not\longrightarrow} G_i$. By the previous paragraph there is a graph G' that meets conditions 1, 2, and $G' \overset{TT_{M'}}{\not\longrightarrow} G_i$ for every i. Consequently, $G' \overset{TT_M}{\not\longrightarrow} G_i$ for every i, which concludes the proof. □

We finish this section by a proposition explaining why proof of density for TT_M mappings has to follow different path than in the case of homomorphisms.

Proposition 3.1.12 *Category $\mathcal{G}_{TT_M}$ of (directed or undirected) graphs and TT_M mappings does not have products for any ring M.*

Proof: We will formulate the proof for the undirected version, although for the directed version the same proof goes through. We show that there is no product $C_3 \times C_3$. Suppose, to the contrary, that P is the product $C_3 \times C_3$. Let $\pi_1, \pi_2 : P \xrightarrow{TT} C_3$ be the projections, let $E(C_3) = \{e_1, e_2, e_3\}$.

We look first at mappings $f_i : \overrightarrow{K_2} \to C_3$ sending the only edge of $\overrightarrow{K_2}$ to e_i. If we consider mapping f_i to the first copy of C_3 and f_j to the second one, by definition of the product there is exactly one edge $e \in E(P)$ such that $\pi_1(e) = e_i$ and $\pi_2(e) = e_j$. We let $e_{i,j}$ denote this e. So, $E(P)$ consists of nine edges $e_{i,j}$, for $1 \le i, j \le 3$.

As π_1, π_2 are TT mappings, by Lemma 1.2.9 there are no loops in P. There are no parallel edges either: suppose e, f are parallel edges in P. Then without loss of generality $\pi_1(e) \neq \pi_1(f)$, hence we get a contradiction by Lemma 1.2.9.

Finally, for a $\rho \in S_3$ let $f_\rho : C_3 \to C_3$ send e_i to $e_{\rho(i)}$. Using the definition of product for mapping f_{id} and f_ρ, Lemma 1.2.9, and the fact that there are no parallel edges in P we find that $E_\rho = \{e_{1,\rho(1)}, e_{2,\rho(2)}, e_{3,\rho(3)}\}$ are edges of a cycle. Considering $\rho = \mathrm{id}$ and $\rho = (1,3,2)$ we find that part of P looks as in the Figure 3.2 (in the directed case, the orientation may be arbitrary, if $M = \mathbb{Z}_2^k$).

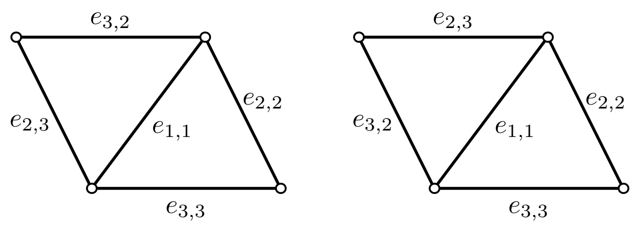

Figure 3.2: Proof of Proposition 3.1.12.

Consider the first case. As E_ρ is a cycle for $\rho = (2,3,1)$, the edges $e_{1,2}$ and $e_{2,3}$ are adjacent. By taking $\rho = (2,1,3)$, we find that $e_{1,2}$ and $e_{2,3}$ are adjacent. As there are no parallel edges in P, we have $e_{1,2} = xy$ or $e_{1,2} = yx$. Hence, $e_{1,2}$, $e_{2,3}$, $e_{2,2}$ forms a cycle. As π_1 is TT mapping, we obtain a contradiction by Lemma 1.2.9. In the second case we proceed in the same way with edge $e_{2,1}$, we prove that it is adjacent with $e_{3,2}$ and $e_{3,3}$ and yield a contradiction with π_2 being a TT mapping. $\square$

3.2 Universality of TT_2 order

In this section we restrict our attention to TT_2 mappings and consequently to undirected graphs. We first construct a particular TT_2-rigid graph. (By Corollary 2.2.14 such graph exists, but we need some additional properties.) Then we use this graph to provide an embedding of $\mathcal{G}ra_{hom}$ to $\mathcal{G}ra_{TT_2}$. Another proof of this result (for general TT_M mappings) was given in Section 2.2.2 as an application of study of homotens graphs.

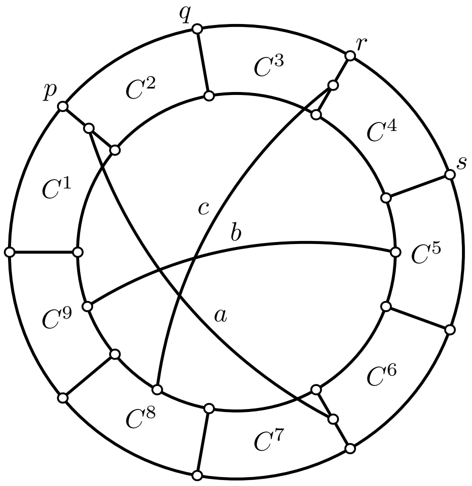

Figure 3.3: A TT_2-rigid graph

Lemma 3.2.1 *Let S be the graph in Figure 3.3.*

1. *S is TT_2-rigid, i.e. the only TT_2 mapping $S \to S$ is the identity.*
2. *Suppose G is a graph that contains edge-disjoint copies of S: $S_1, \ldots, S_t$. Suppose G does not contain triangles nor pentagons, except those pentagons that are contained in some S_i. Then the only TT_2 mapping $S \to G$ is the identity mapping to some S_i.*

Proof: We will prove the second part, which implies the first (by taking $G = S$). Consider a TT_2 mapping $f : S \to G$. Let pentagons in S be denoted $C^1, \ldots, C^9$ as in the figure, note that there are no other pentagons in S. As there are no triangles in G and the only pentagons are contained in some S_k, we can deduce by Lemma 1.2.9 that each C^i maps to a pentagon in some S_k (possibly different k for different i).

Pentagon C^i shares an edge with C^j iff i and j differ by 1 (modulo 9). As sharing an edge is preserved by any mapping and since different copies of S in G are edge-disjoint, we conclude that there is a copy of S in G (to simplify the notation, we will identify this copy with S) and a bijection $p : [9] \to [9]$ such that $f(C^i) = C^{p(i)}$ for each i; moreover p preserves the cyclic order. Next we note that the size of the intersection of neighboring pentagons is preserved too. There are exactly three pairs of pentagons that share two edges: $\{C^1, C^2\}$, $\{C^3, C^4\}$, $\{C^6, C^7\}$. As the pairs $\{C^1, C^2\}$ and $\{C^3, C^4\}$ are adjacent, the pairs $\{C^5, C^6\}$ and $\{C^3, C^4\}$ have a common neighboring pentagon, while the pairs $\{C^5, C^6\}$ and $\{C^1, C^2\}$ do not, we see that p is the identity; that is $f(C^i) = C^i$ for each i.

We still have to prove that f does not permute edges in the respective pentagons. Let C^o be the outer cycle and note it is the only 9-cycle in S that shares exactly one edge with each C^i. Hence, f is an identity on $E(C^o)$. This means that f can only permute two edges that share an endpoint of some of the edges a, b, and c.

Edge a is a part of a 7-cycle C^a that has four edges in common with C^o. Now, C^o is preserved by f, and there is no other 7-cycle in S with the same intersection with C^o. Thus, C^a is preserved as well, in particular a and the edges incident with it are preserved. Edge b is a part of a 7-cycle C^b that intersects C^5, C^6, C^7, C^8 and C^9. Since the edges it has in common with C^6, C^7, and C^8 are preserved by f (at least set-wise), and there is no other 7-cycle including these edges, C^b is preserved too, in particular b and the edges incident with it are preserved. Similarly, c is contained in an 8-cycle that has five of its edges fixed, hence it is fixed by f. $\square$

Theorem 3.2.2 *There is a mapping F that assigns (undirected) graphs to (undirected) graphs, such that for any graphs G, H (we stress that we consider loopless graphs only) holds*

$$G \xrightarrow{hom} H \iff F(G) \xrightarrow{TT_2} F(H)\,.$$

Moreover F can be extended on mappings between graphs: if $f : G \to H$ is a homomorphism, then $F(f) : F(G) \to F(H)$ is a TT mapping and any TT mapping between $F(G)$ and $F(H)$ is equal to $F(f)$ for some homomorphism $f : G \xrightarrow{hom} H$. (In category-theory terms F is an embedding of the cate-

gory of all graphs and their homomorphisms into the category of all graphs and all TT_2-mappings between them.)

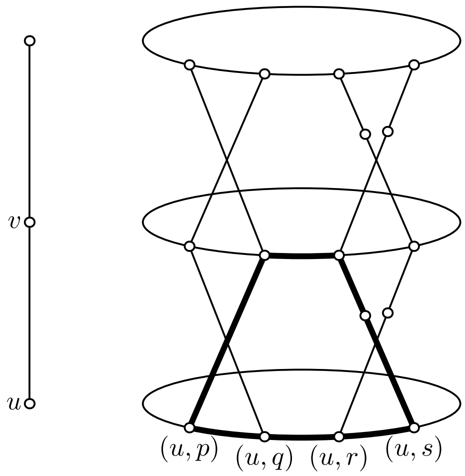

Figure 3.4: Example of construction of $F(G)$ for $G = P_2$. The 7-cycle used in the proof of Theorem 3.2.2 is emphasized.

Proof: Let S be the graph from Lemma 3.2.1, let p, q, r, s be its vertices as denoted in Figure 3.3. For a graph G, let the vertices of $F(G)$ be $(V(G) \times V(S)) \dot\cup (E(G) \times \{1,2\})$. On each set $\{v\} \times V(S)$ we place a copy of S, it will be denoted by S_v. For an edge uv of G we introduce edges $(u,p)(v,q)$, $(u,q)(v,p)$ (we refer to them as to *add-on edges*) and paths of length two from (u,r) to (v,s) and from (u,s) to (v,r) (we refer to these as to add-on paths, the middle vertices of these paths are $(uv,1)$ and $(uv,2)$). There are no other edges in $F(G)$. See Figure 3.4 for an example of the construction. As we wish to apply Lemma 3.2.1, we first show that $F(G)$ contains no triangles and only those pentagons that are contained in some S_v. Suppose C is a cycle violating this. If C contains some add-on path, it is easy to check that the length of C is at least six. If it is not then C has to contain some add-on edges (as S is triangle-free). If it contains only add-on edges and copies of the edge pq then it has even length; otherwise it has length at least seven.

It is clear how to define $F(f)$ for a homomorphism $f : G \to H$: mapping $F(f)$ sends each S_v in G to $S_{f(v)}$ in H in the only way, the edges between different copies of S are mapped in the 'canonical' way. Clearly $F(f)$ is a TT mapping induced by a homomorphism.

The only difficult part is to show, that for every $g : F(G) \xrightarrow{TT} F(H)$ there is an $f : G \xrightarrow{hom} H$ such that $g = F(f)$. So let g be such a mapping. By Lemma 3.2.1 each copy of S is mapped to a copy of S, to be precise, there is a mapping $f : V(G) \to V(H)$ such that g maps S_v to $S_{f(v)}$. Let uv be an edge of G. First, we show that $f(u) \neq f(v)$. Suppose the contrary and consider the 7-cycle (u, p), (u, q), (u, r), (u, s), x, (v, r), (v, q) (x is the middle vertex of an add-on path). Since S is rigid, edges $(u, q)(u, r)$ and $(v, q)(v, r)$ map to the same edge, hence the algebraical image of the other five edges is a cycle. However, there is no cycle of length at most five containing edges pq and rs, a contradiction.

Considering again the image of the same cycle shows that $f(u)$ and $f(v)$ are connected by an edge of H, which finishes the proof. □

Remark 3.2.3 *It is worth noting that graphs $F(G)$ are all triangle-free. We believe that the construction from Theorem 3.2.2 can be modified to work for other rings than $\mathbb{Z}_2$, some modification can possibly produce even graphs of girth at least g, for any given g. If we consider graphs containing complete graphs then the situation becomes easier, which allowed us to prove embedding result for a general ring in Chapter 2 (Theorem 2.2.15).*

3.3 Miscellanea

3.3.1 Complexity

Let $\mathrm{TT}_M(H)$ denote the problem of decision, whether for a given graph G there is a TT_M mapping $G \xrightarrow{TT_M} H$. The complexity of the related problem $\mathrm{HOM}(H)$ (that is the testing of the existence of a homomorphism to H) is now well understood, at least for undirected graphs: $\mathrm{HOM}(H)$ is NP-complete if and only if H contains an odd circuit, otherwise it is in P (as it is equivalent to decide whether G is bipartite), see [39]. In the same spirit, we wish to determine the complexity of the problem $\mathrm{TT}_M(H)$.

Theorem 3.3.1 *Let H be an undirected graph. Then $\mathrm{TT}_{\mathbb{Z}_2}(H)$ is NP-complete if H contains an odd circuit; otherwise it is polynomial.*

Proof: By Theorem 1.3.1, problems $\mathrm{TT}_2(H)$ and $\mathrm{HOM}(\Delta(H))$ have the same answer for any graph G, hence they have the same complexity. Observe that $\Delta(H)$

is bipartite iff H is bipartite: H and $\Delta(H)$ are TT_2 equivalent and any graph is bipartite iff it admits a TT_2 mapping to $\overrightarrow{K}_2$. Consequently, $\mathrm{TT}_2(H)$ is NP-complete iff H contains an odd circuit. □

For $M \neq \mathbb{Z}_2$ (or $\mathbb{Z}_2^k$), we may still reduce $\mathrm{TT}_M(H)$ to $\mathrm{HOM}(H')$ for a suitable graph H'. However, now we deal with directed graphs, where the complexity of HOM is not characterized. Another obstacle is that for $M = \mathbb{Z}$ the graph H' is infinite. (For infinite graph H, the complexity of $\mathrm{HOM}(H)$ was investigated in [9].)

We can also study the complexity of the decision problem for other types of XY mappings. For TF_2 mappings we can again reduce the $\mathrm{TF}_2(H)$ problem to HOM, yielding that it is polynomial precisely if H is Eulerian. And, again, in this case the problem is trivial—equivalent to decision if the given graph is Eulerian.

On the other hand, the situation for FT and FF mappings is unclear. Their existence is equivalent to the existence of certain flows on the given graph. The complexity of determining, whether a given graph admits an (M, B)-flow is, however, not well-understood, compare [50] for a partial answer.

Question 3.3.2 *Let H be a graph. What is the complexity of decision, for a given graph G, whether $G \xrightarrow{FT} H$ ($G \xrightarrow{FF} H$, respectively)?*

3.3.2 Dualities in the TT order

Dualities were introduced as an example of good characterization which can help to solve $\mathrm{HOM}(H)$ for some graphs H. We say that a tuple $(F_1, \dots, F_t; H)$ forms a duality if for every G

$$G \xrightarrow{hom} H \iff (\forall i \in \{1, \dots, t\}) F_i \overset{hom}{\not\longrightarrow} G\,.$$

It is well-known that G has a homomorphism to $\overrightarrow{T}_n$ (transitive tournament with n vertices) iff it does not contain $\overrightarrow{P}_{n+1}$ (path with $n+1$ vertices). Hence, the pair $(\overrightarrow{P}_{n+1}; \overrightarrow{T}_n)$ is a duality. If $(F_1, \dots, F_t; H)$ is a duality, we can solve $\mathrm{HOM}(H)$ in polynomial time; moreover, it means that the class of graphs admitting a homomorphism to H is first-order definable. Dualities are studied in a sequence of papers, see [72], [40] and references there. We present a sample of results:

- for undirected graphs there are only trivial dualities $(K_2; K_1)$ and $(K_1; K_0)$.

- for directed graphs, for any t and any trees $F_1, \ldots, F_t$, there is an H such that $(F_1, \ldots, F_t; H)$ is a duality; there are no other dualities.
- similarly as for directed graphs, it is possible to characterize all dualities for arbitrary relational systems.

Here we adopt proof of the homomorphic case (for undirected graphs) to characterize dualities for TT_M, that is we characterize all tuples $(F_1, \ldots, F_t; H)$ for which

$$G \xrightarrow{TT} H \iff (\forall i \in \{1, \ldots, t\}) F_i \not\xrightarrow{TT} G\,. \tag{3.2}$$

We suppose $M \neq \mathbb{Z}_1$ to avoid trivialities.

Theorem 3.3.3 *For every ring M, there are no dualities in the TT_M order, up to the trivial ones, that is $H \approx_M^t K_1$ and for some i we have $F_i \approx_M^t \overrightarrow{K}_2$.*

Proof: Let $(F_1, \ldots, F_t; H)$ be a duality. Put $g = \max\{g_M(F_1), \ldots, g_M(F_t)\}$. If $g = \infty$, then there is an i such that $F_i \xrightarrow{TT_M} \overrightarrow{K}_2$. In this case, the right-hand side of (3.2) holds iff G is edgeless. This is equivalent to $G \xrightarrow{TT_M} H$ exactly when H is edgeless, that is $H \approx_M^t K_1$.

If g is finite, we consider a graph G such that $\chi(G) > c$ (c will be specified later) and all circuits in G are longer than g. (Such graphs exist by the celebrated theorem of Erdős.) We orient the edges of G arbitrarily. Now $F_i \not\xrightarrow{TT_M} G$ by Lemma 1.2.11, it remains to prove $G \not\xrightarrow{TT_M} H$. So suppose the contrary; by Lemma 7.2.4 and 7.2.2 we may suppose M is finite. By Theorem 1.3.1 (and the remarks following it), there is a finite directed graph H' such that $G \xrightarrow{TT_M} H$ iff $G \xrightarrow{hom} H'$. Hence it is enough to choose $c = \chi(H')$. $\square$

We remark that there may be more 'restricted dualities'. For example Conjecture 4.1.4 may be expressed as the duality

$$G \xrightarrow{TT_2} C_5 \iff C_3 \not\xrightarrow{TT_2} G\,.$$

for G that has maximum degree at most 3.

3.3.3 Bounded antichains in the TT order

In [19], the following question is posed (for $M = \mathbb{Z}_2$) as Problem 6.9.

Problem 3.3.4 ([19]) *Is there an infinite antichain of order $\preccurlyeq^t_M$, that consists of graphs with bounded chromatic number?*

One motivation for this question stems from the fact that it is easy to produce infinite antichain of unbounded chromatic number (similarly to Proposition 2.1.7). More importantly, the unsolved analogy for $\preccurlyeq^f_M$ (Problem 6.1.8) is relevant for Jaeger's conjecture 6.1.7. Our approach provides a straightforward answer in a very strong form.

Corollary 3.3.5 *For every M, there is an infinite antichain in the order $\preccurlyeq^t_M$, that consists of graphs with chromatic number at most 3.*

Proof:
Let $G = \overrightarrow{K}_2$ and choose a 3-colorable H such that $H \succ^t_M \overrightarrow{K}_2$: we can take $H = \overrightarrow{C}_3$ whenever M is not a power of $\mathbb{Z}_3$. In that case we choose $H = \overrightarrow{C}_5$.

Denote $I = (G,H)_h \cap (G,H)_M$. By Corollary 3.1.7, I is nonempty, hence choose $G_0 \in I$. Now we inductively find (using Corollary 3.1.8) graphs G_1, G_2, $\dots$ from I such that for every k, G_0, $\dots$, G_k is an antichain in the order $\prec^t_M$. Hence $\{G_n, n \geq 0\}$ is an infinite antichain, and as for every i, $G_i \xrightarrow{hom} H$, every G_i is 3-colorable. $\square$

Remark 3.3.6 *An alternative proof is provided by Theorem 2.2.15: the homomorphism order is known to have infinite antichain of bounded chromatic number, this is mapped to an infinite antichain in $\preccurlyeq^t_2$ of bounded χ_{TT_2}, hence of bounded chromatic number ($\chi \leq c_m \chi_{TT_M}$, see Corollary 2.3.9).*

In contary with this, in the presented proof we obtain 3-colorable graphs for every M. Moreover, we can choose H more carefully, namely we can let $H = \overrightarrow{C}_p$ where p is a large enough prime (so that $\overrightarrow{C}_p \overset{TT_M}{\not\longrightarrow} \overrightarrow{K}_2$). In this way, we obtain for any given $\varepsilon > 0$ an infinite antichain of $\prec^t_M$ that consists of graphs with circular chromatic number bounded by $2 + \varepsilon$.

Chapter 4

Cut-continuous mappings of high-girth cubic graphs

In this chapter we give a (computer assisted) proof that the edges of every graph with maximum degree 3 and girth at least 17 may be 5-colored (possibly improperly) so that the complement of each color class is bipartite. Equivalently, every such graph admits a homomorphism to the Clebsch graph (Figure 1.2), and a cut-continuous mapping to C_5.

Hopkins and Staton [42] and Bondy and Locke [13] proved that every (sub)cubic graph of girth at least 4 has an edge-cut containing at least $\frac{4}{5}$ of the edges. The existence of such an edge-cut follows immediately from the existence of a 5-edge-coloring as described above, so our theorem may be viewed as a kind of coloring extension of their result (under a stronger girth assumption).

Every graph which has a homomorphism to a cycle of length five has an above-described 5-edge-coloring; hence our theorem may also be viewed as a weak version of Nešetřil's Pentagon Problem: Every cubic graph of sufficiently high girth maps to C_5.

4.1 Introduction

Throughout this chapter all graphs are assumed to be finite, undirected and simple. Recall that if G is a graph and $U \subseteq V(G)$, we put $\delta(U) = \{uv \in E(G) : u \in U \text{ and } v \notin U\}$, and we call any subset of edges of this form a *cut*. The maximum size of a cut of G, denoted $\mathrm{MAXCUT}(G) = \max_{U \subseteq V} |\delta(U)|$ is a parameter which has received great attention. Next, we normalize and define

$$b(G) = \frac{\mathrm{MAXCUT}(G)}{|E(G)|}\,.$$

Determining $b(G)$ (or equivalently $\mathrm{MAXCUT}(G)$) for a given graph G is known to be NP-complete, so it is natural to seek lower bounds. It is an easy exercise to show that $b(G) \geq 1/2$ for any graph G and $b(G) \geq 2/3$ whenever G is cubic (that is 3-regular). The former inequality is almost attained by a large complete graph, the latter is attained for $G = K_4$: any triangle contains at most two edges from any bipartite subgraph, and each edge of K_4 is in the same number of triangles (namely in two). This suggests that triangles play a special role, and raises the question of improving this bound for cubic graphs with higher girth. In the 1980's, several authors independently considered this problem [13, 42, 93], the strongest results being

- $b(G) \geq 4/5$ for G with maximum degree 3 and no triangle [13]
- $b(G) \geq 6/7 - o(1)$ for cubic G with girth tending to infinity [93]

On the other hand, cubic graphs exist with arbitrarily high girth and satisfying $b(G) < 0.94$ (see Section 4.2).

Define a set of edges C from a graph G to be a *cut complement* if $C = E(G) \setminus \delta(U)$ for some $U \subseteq V(G)$. Then the problem of finding a cut of maximum size is exactly equivalent to that of finding a cut complement of minimum size. A natural relative of this is the problem of finding many disjoint cut complements. Indeed, packing cut complements may be viewed as a kind of coloring version of the maximum cut problem.

There are a variety of interesting properties which are equivalent to the existence of $2k + 1$ disjoint cut complements, so after a handful of definitions we will state a proposition which reveals some of these equivalences. This proposition is well known, but we have provided a proof of it in Section 4.4 for the sake of completeness. For every positive integer n, we let Q_n denote the n-dimensional cube, so the vertex set of Q_n is the set of all binary vectors of length n, and two such vertices are adjacent if they differ in a single coordinate. The n-dimensional projective cube,[1] denoted PQ_n, is the simple graph obtained from the $(n + 1)$-dimensional cube Q_{n+1} by identifying pairs of antipodal vertices (vertices that differ in all coordinates). Equivalently, the projective cube PQ_n can be described as a Cayley graph, see Section 4.4.

[1] sometimes called folded cube

Recall that a mapping $g : E(G) \to E(H)$ is *cut-continuous* if the preimage of every cut is a cut. An alternative term for a cut-continuous mapping is TT_2 mapping, but we will not use this shorter name in this chapter. Now we are ready to state the relevant equivalences.

Proposition 4.1.1 *For every graph G and nonnegative integer k, the following properties are equivalent.*

1. *There exist $2k$ pairwise disjoint cut complements.*
2. *There exist $2k + 1$ pairwise disjoint cut complements with union $E(G)$.*
3. *$G \xrightarrow{hom} PQ_{2k}$ (G has a homomorphism to PQ_{2k}).*
4. *$G \xrightarrow{TT_2} C_{2k+1}$ (G has a cut-continuous mapping to C_{2k+1}).*

Perhaps the most interesting conjecture concerning the packing of cut complements—or equivalently homomorphisms to projective cubes, cut-continuous mappings to odd circuits—is the following conjectured generalization of the Four Color Theorem. Although not immediately obvious, the formulation we give here is equivalent to Seymour's conjecture on edge-coloring of planar r-graphs for odd values of r.

Conjecture 4.1.2 (Seymour) *Each planar graph in which all odd cycles have length at least $2k + 1$ has a cut-continuous mapping to C_{2k+1} (a homomorphism to PQ_{2k}).*

Since the graph PQ_2 is isomorphic to K_4, the $k = 1$ case of this conjecture is equivalent to the Four Color Theorem. The $k = 2$ case of this conjecture concerns homomorphisms to the graph PQ_4 which is also known as the Clebsch graph (see Figure 1.2). This case was resolved in the affirmative by Naserasr [63] who deduced it from a theorem of Guenin [31]. The following theorem is the main result of this chapter; it shows that graphs of maximum degree three without short cycles also have homomorphisms to PQ_4. The *girth* of a graph is the length of its shortest cycle, or ∞ if none exists.

Theorem 4.1.3 *Every graph of maximum degree 3 and girth at least 17 admits a cut-continuous mapping to C_5. Equivalently, it has a homomorphism to PQ_4 (also known as the Clebsch graph), and 5 disjoint cut complements. Furthermore, there is a linear time algorithm which computes the cut-continuous mapping, the homomorphism, and the cut complements.*

Clearly no graph with a triangle can map homomorphically to the triangle-free Clebsch graph (equivalently, have 5 disjoint cut complements), but we believe this to be (for cubic graphs) the only obstruction. We highlight this and one other question we have been unable to resolve below.

Conjecture 4.1.4 *Every triangle-free cubic graph has a homomorphism to PQ_4.*

Problem 4.1.5 *What is the largest integer k with the property that all cubic graphs of sufficiently high girth have a homomorphism to PQ_{2k}?*

As we mentioned before, there are high-girth cubic graphs with $b(G) < 0.94$. Such graphs do not admit a homomorphism to PQ_{2k} for any $k \geq 8$, so there is indeed some largest integer k in the above problem. At present, we know only that $2 \leq k \leq 7$.

Another topic of interest for cubic graphs of high girth is circular chromatic number, a parameter we now pause to define. For any graph G, we let $G^{\geq k}$ denote the simple graph with vertex set $V(G)$ and two nodes adjacent if they have distance at least k in G. The *circular chromatic number* of G, is $\chi_c(G) = \inf\{\frac{n}{k} : G \text{ has a homomorphism to } C_n^{\geq k}\}$. Every graph satisfies $\lceil \chi_c(G) \rceil = \chi(G)$ so the circular chromatic number is a refinement of the usual notion of chromatic number. The following curious conjecture asserts that cubic graphs of sufficiently high girth have circular chromatic number $\leq \frac{5}{2}$ (since $C_{2k+1} \cong C_{2k+1}^{\geq k}$).

Conjecture 4.1.6 (Nešetřil's Pentagon Conjecture [67]) *If G is a cubic graph of sufficiently high girth then there is a homomorphism from G to C_5.*

It is an easy consequence of Brook's Theorem that the above conjecture holds with C_3 in place of C_5 (every cubic graph of girth at least 4 is 3-colorable). On the other hand, it is known that the conjecture is false if we replace C_5 by C_{11} [52], consequently it is false if we replace C_5 by any C_n for odd $n \geq 11$. Later, it was shown that the conjecture is false also for C_9 [87] and C_7 [36] in place of C_5.

An important extension of Conjecture 4.1.6 is the problem to determine the infimum of real numbers r with the property that every cubic graph of sufficiently high girth has circular chromatic number $\leq r$. The above results show that this infimum must lie in the interval $[\frac{7}{3}, 3]$, but this is the extent of our knowledge. It is tempting to try to use the fact that girth ≥ 17 cubic graphs map to the Clebsch graph and girth ≥ 4 cubic graphs map to C_3 to improve the upper bound, but the circular chromatic numbers of C_3, the Clebsch graph, and their direct product are all at least three,[2] so no such improvement can be made. Neither were we able to use our result to improve upper bounds on fractional chromatic numbers of cubic graphs. This is conjectured to be at most $14/5$ for triangle-free cubic graphs (Heckmann and Thomas [38]), and proved to be at most $3 - 3/64$ (Hatami and Zhu [37]).

It is easy to prove directly that Conjecture 4.1.6, if true, implies Theorem 4.1.3 (perhaps with a stronger assumption on the girth). This follows from part 4 of Proposition 4.1.1 and from Lemma 2.1.2.

In Chapter 2 we did study when existence of a cut-continuous mapping from G to H implies the existence of a homomorphism from G to H. It was proved there that this happens for most graphs G (Theorem 2.2.9) but for no triangle-free ones (Corollary 2.2.17). Therefore these techniques can not be used to extend Theorem 4.1.3 to attain Conjecture 4.1.6.

We finish the introduction with another conjecture due to Nešetřil (personal communication) concerning the existence of homomorphisms for cubic graphs of high girth.

Conjecture 4.1.7 *For every integer k there is a graph H of girth at least k and an integer N, such that for every cubic graph G with girth at least N we have*

$$G \xrightarrow{hom} H\,.$$

Our theorem shows this conjecture to be true for $k \le 5$, but the other cases remain open.

4.2 Bipartite density of random cubic graphs

The aim of this short section is to show that there exist cubic graphs of arbitrary high girth with bipartite density bounded from 1. This result was announced by McKay (and is referred to as [61] by [93]); however, it never appeared in print. Althought nowadays proof of this is more an exercise in the use of random regular graphs, we include it for completeness as this proposition yields a limit to extension of Theorem 4.1.3. We present only a straightforward calculation, more careful approach yields better bounds, see thesis of Hladký [41].

Proposition 4.2.1 *There is $\varepsilon > 0$ such that for every l cubic graph G exists that contains no circuit shorter than l and satisfies $b(G) < 1 - \varepsilon$.*

[2] The only nontrivial case is the product $PQ_4 \times K_3$. By a theorem of [29] this graph is uniquely 3-colorable; consequently $\chi_c(PQ_4 \times K_3) = 3$.

Proof: We will use the *pairing model* (introduced by Bollobás [10] and also called configuration model) of random cubic graphs. That is, we take a random perfect matching on $[n] \times [3]$ (with n even) and project it to $[n]$. The result is a cubic multigraph on n vertices—a sample from $\mathcal{G}^*_{n,3}$. Conditioned that the result is a simple graph (which has probability tending to e^{-2}), we obtain an element of $\mathcal{G}_{n,3}$, that is a uniformly random cubic graph. For more detailed introduction to the use of this model we recommend [91].

Let $\varepsilon > 0$ be sufficiently small (to be specified later). We will prove that $b(G_{n,3}) < 1 - \varepsilon$ for $G_{n,3} \in \mathcal{G}_{n,3}$ a.a.s. As random regular graphs have girth greater than g with a positive probability (for every g), this will prove our claim. Equivalently, we want to prove $b(G^*_{n,3}) < 1 - \varepsilon$ for $G^*_{n,3} \in \mathcal{G}^*_{n,3}$ a.a.s. a.a.s. To this end, we put $l = (1-\varepsilon)\frac{3}{2}n$ and use $f(t)$ for the number of perfect matchings on t vertices. We will work with $G^*_{n,3}$ instead of $G_{n,3}$, as this allows us to calculate in fact with random matching M on $[n] \times 3$.

$$
\begin{aligned}
&Pr[b(G^*_{n,3}) \geq 1 - \varepsilon] = \\
&= Pr_{G^*}[(\exists A \subseteq V)\ |\delta(A)| \geq l] \\
&\leq \sum_{A \subseteq V, |A| \leq n/2} Pr_{G^*}[|\delta(A)| \geq l] \\
&= \sum_{A \subseteq V} Pr_M[\text{number of edges of } M \text{ leaving } A \times [3] \text{ is at least } l] \\
&= \sum_{a \leq n/2} \sum_{k \geq l} \binom{n}{a} \binom{3a}{k} \binom{3(n-a)}{k} \frac{k! f(3a-k) f(3(n-a)-k)}{f(3n)}
\end{aligned}
$$

If we show that the limit of the last sum is 0 (as n grows to ∞), we are done. To do that, we need two estimates. The first one is the folklore estimate of binomial coefficient (note that the $o(1)$ term can be chosen independently on p and that p can depend on n):

$$\binom{n}{pn} = 2^{n(H(p)+o(1))} .$$

Next, we observe that (for even t) $f(t) = \frac{t!}{(t/2)! 2^{t/2}}$. Using Stirling formula, we find that $f(t) = \sqrt{t!} \cdot 2^{o(t)}$ (the error term is in fact roughly $t^{-1/4}$).

Now, we can manipulate part of the estimated sum:

$$\binom{3a}{k} f(3a-k) = \frac{(3a)!}{k! \sqrt{(3a-k)!}} \cdot 2^{o(3a-k)} = \sqrt{\binom{3a}{k} \frac{(3a)!}{k!}} \cdot 2^{o(3a-k)} .$$

After handling similarly the other binomial coefficient, we get

$$Pr[b(G^*_{n,3}) \geq 1-\varepsilon] \leq \sum_{a \leq n/2} \sum_{k \geq l} \binom{n}{a} \sqrt{\frac{\binom{3a}{k}\binom{3(n-a)}{k}}{\binom{3n}{3a}}} \cdot 2^{o(n)} .$$

Note that only terms with $k \leq 3a$ are nonzero. Let $\alpha = \frac{a}{n/2}$, $\delta = \frac{k}{3n/2}$. We have $1-\varepsilon \leq \delta \leq \alpha \leq 1$. Define $\varphi(a,d) = 3\frac{\alpha}{2}H(\frac{\delta}{\alpha}) + 3(1-\frac{\alpha}{2})H(\frac{\delta}{2-\alpha}) - H(\frac{\alpha}{2})$. Also suppose that $\varphi(\alpha,\delta) \leq -\sigma < 0$ for all $1-\varepsilon \leq \delta \leq \alpha \leq 1$. If we replace each of the terms in the estimated sum by the maximal one, we can see that

$$Pr[b(G^*_{n,3}) \geq 1-\varepsilon] \leq n^2 \cdot 2^{-n(\sigma/2+o(1))} .$$

Clearly, the last term converges to 0.

As φ is a continuous function and as $\varphi(1,1) = -1$, we see that for small enough ε, we get the desired result. Numerical computation gives $\varepsilon \geq 0.0614$, or $1-\varepsilon \leq 0.9386$. (McKay and Hladký report $1-\varepsilon \leq 0.9351$.) □

4.3 The proof

The goal of this section is to prove the main result of this chapter, Theorem 4.1.3. We begin with a lemma which reduces our task to cubic graphs.

Lemma 4.3.1 *If Theorem 4.1.3 holds for every cubic G then it holds for every subcubic G, too.*

Proof: Let G be a subcubic graph of girth at least 17. We will find a cubic graph G' such that girth of G' is at least 17 and $G' \supseteq G$. The lemma then follows, as restriction of any homomorphism $G' \xrightarrow{hom} PQ_4$ to $V(G)$ is the desired homomorphism $G \xrightarrow{hom} PQ_4$.

To construct G', put $r = \sum_{v \in V(G)}(3-\deg(v))$. Let H be an r-regular graph of girth at least 17 (it is well-known that such graphs exists, see, e.g., [8] for a nice survey). We take $|V(H)|$ copies of G. For every edge uv of H we choose two vertices of degree less than 3, one from a copy of G corresponding to each of u and v; then we connect these by an edge. Clearly, this process will lead to a cubic graph containing G and with girth at most equal to the minimum of girths of G and H. □

To show that cubic graphs of girth ≥ 17 have homomorphism to the Clebsch graph, we shall use property 1 from Proposition 4.1.1. Accordingly, we define a *labeling* of a graph G to be a four-tuple $X = (X_1, X_2, X_3, X_4)$ so that each X_i is a subset of $E(G)$. We call a labeling X a *cut labeling* if every X_i is a cut, and a *cut complement labeling* if every X_i is a cut complement. If $X_i \cap X_j = \emptyset$ whenever $1 \leq i < j \leq 4$ we say that the labeling is *wonderful*.

Define function $a : \{0, 1, \ldots, 4\} \to \mathbb{Z}$ by $a(0) = 0$, $a(1) = 1$, $a(2) = 10$, $a(3) = 40$, and $a(4) = 1000$. Now, for any labeling X, we define the *mark* of an edge e (with respect to X) to be $m_X(e) = \{i \in \{1, 2, 3, 4\} : e \in X_i\}$, the *weight* of e to be $w_X(e) = |m_X(e)|$, and the *cost* of e to be $\mathrm{cost}_X(e) = a(w_X(e))$. Finally, we define the *cost* of X to be $\mathrm{cost}(X) = \sum_{e \in E(G)} \mathrm{cost}_X(e)$.

The structure of our proof is quite simple: we prove that any cut complement labeling of minimum cost in a cubic graph of girth ≥ 17 is wonderful. To show that such a labeling is wonderful, we shall assume it is not, and then make a small local change to improve the cost—thus obtaining a contradiction. The observation below will be used to make our local changes. For any sets A, B we let $A \,\triangle\, B = (A \setminus B) \cup (B \setminus A)$ be the symmetric difference. If $X = (X_1, \ldots, X_4)$ and $Y = (Y_1, \ldots, Y_4)$ are labelings, then we let $X \,\triangle\, Y = (X_1 \,\triangle\, Y_1, \ldots, X_4 \,\triangle\, Y_4)$. We say we obtain $X \,\triangle\, Y$ from X by *switching* Y.

Observation 4.3.2 *If C is a cut and D is a cut complement, then $C \,\triangle\, D$ is a cut complement. Similarly, if X is a cut complement labeling and Y is a cut labeling, then $X \,\triangle\, Y$ is a cut complement labeling.*

Proof: Let $C = \delta(U)$ and $D = E(G) \setminus \delta(V)$. Then $C \,\triangle\, D = E(G) \setminus (\delta(U) \,\triangle\, \delta(V)) = E(G) \setminus \delta(U \,\triangle\, V)$ so it is a cut complement. For labelings we consider each coordinate separately. $\square$

The graphs we consider will have high girth, so they will look like trees locally. Our proof will exploit this by using the above observation to make changes on a tree. To state our method precisely, we now introduce a family of rooted trees. Let T_i denote a rooted tree of 'depth i' in which all vertices have degrees 1 and 3, and the root vertex, denoted r, has degree 1. Explicitly, we let T_1 be an edge (with one end being the root). Having defined T_i, we form T_{i+1} by joining two copies of T_i by identifying their root vertices and then connecting this common vertex to a new vertex, which will be the new root. The unique edge incident with the root we shall call the *root edge*. We let $2T_i$ denote the tree obtained from two copies of T_i by identifying their root edges in the opposite direction (the resulting edge will be called the *central* edge of $2T_i$). A vertex of T_i or $2T_i$ is *interior* if either it has degree 3, or it is the root of T_i. A cut C (cut labeling X) of T_i or $2T_i$ is called *internal* if $C = \delta(Z)$ ($X = (\delta(Z_1), \dots, \delta(Z_4))$) for some set Z (sets $Z_1, \dots, Z_4$) of interior vertices. Now we are ready to state and prove a lemma that forms the first step of the proof: it will be used to show that any cut complement labeling of minimum cost has no edges of weight > 2.

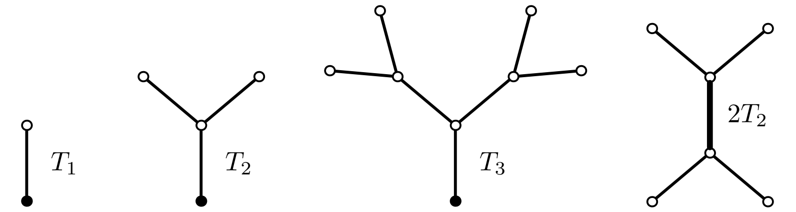

Figure 4.1: Illustration of definitions, root vertex/central edge are emphasized.

Lemma 4.3.3 *Let X be a labeling of the tree $2T_2$ and assume that the weight of the central edge is > 2. Then there exists an internal cut labeling Y of $2T_2$ so that* $\mathrm{cost}(X \,\Delta\, Y) < \mathrm{cost}(X)$.

Proof: Let e be the central edge, let x be a vertex incident with e, let f, g be the other edges incident with x, and let $A = m_X(e)$, $B = m_X(f)$, and $C = m_X(g)$. We will construct a cut labeling $Y = (\delta(Z_1), \dots, \delta(Z_4))$ (where each Z_i is either $\emptyset$ or $\{x\}$) so that $\mathrm{cost}(X \,\Delta\, Y) < \mathrm{cost}(X)$. For convenience, we shall say that we *switch* a set $I \subseteq \{1, 2, 3, 4\}$ if we set $Z_i = \{x\}$ if $i \in I$ and $Z_i = \emptyset$ otherwise.

If $S = A \cap B \cap C$ is nonempty then we may switch S, thereby reducing the cost of each of e, f, g. Hence we may suppose S is empty.

Case 1. $|A| = 4$**:** If $B = C = \emptyset$ then we switch $\{1\}$ decreasing the cost from $a(4)$ to $a(3) + 2a(1)$. Otherwise we switch $B \cup C$; this leads to a mark $\{1, 2, 3, 4\} \setminus (B \cup C)$ on e, C on f and B on g, reducing the cost again.

Case 2. $|A| = 3$**:** We may suppose $A = \{1, 2, 3\}$ and $|A| \geq |B| \geq |C|$ Moreover, $|C| < 3$ for otherwise $A \cap B \cap C$ is nonempty. If B and A have a common element, then we switch it. This changes the weights of edges in T from 3, $|B|$, $|C|$ to 2, $|B| - 1$, $|C| + 1$ and as $|C| < 3$, this is an improvement in the total cost. It remains to consider the cases when both B and C are subsets of $\{4\}$. In each of these cases we switch $\{1\}$, this reduces the cost from at least $a(3)$ to at most $3a(2)$. $\square$

The next lemma, which provides the second step of the proof, is analogous to the previous one, but is considerably more complicated to prove.

Lemma 4.3.4 *Let X be a labeling of the tree $2T_9$ and assume that every edge has weight ≤ 2 and that the central edge has weight exactly 2. Then there exists an internal cut labeling Y of $2T_9$ so that $\operatorname{cost}(X \,\Delta\, Y) < \operatorname{cost}(X)$.*

Before discussing the proof of this lemma we shall use it to prove the main theorem.

Proof: It follows from Lemma 4.3.1 and Proposition 4.1.1 that it suffices to prove that all cubic graphs with girth at least 17 have wonderful cut complement labelings. Let G be such graph and let X be a cut complement labeling of G of minimum cost. It follows immediately from Lemma 4.3.3 that every edge of G has weight ≤ 2. Suppose there is an edge e of weight 2. Then it follows from our assumption on the girth that G contains a subgraph isomorphic to $2T_9$ (possibly with some of the leaf vertices identified) where e is the central edge. Now Lemma 4.3.4 gives us an *internal* cut labeling Y of $2T_9$ (hence a cut labeling of G) such that $\operatorname{cost}(X \,\Delta\, Y) < \operatorname{cost}(X)$. This contradiction shows that X is wonderful, and completes the proof.

Next we give a short description of a linear-time algorithm that finds the partition. We start with a cut complement labeling $(E(G), E(G), E(G), E(G))$. Then we repeatedly pick a bad edge e—that is an edge for which $w(e) > 1$. By Lemma 4.3.3 and 4.3.4 we can decrease the total cost by switching a cut labeling that contains only edges at distance at most 8 from e. We can therefore find the cut labeling in constant time (we can even use brute force, if we do not try to minimize the constant)—we only have to use efficient representation of the graph, namely a list of edges, list of vertices, and pointers between the adjacent objects. As the cost of the starting coloring is $a(4) \cdot |E(G)|$ and at each step the decrease is at least by 1, it remains to handle the operation 'pick a bad edge' in constant time. For this, we maintain a linked list of bad edges, for each element of the list there is a pointer from and to the corresponding edge in the main list of edges. This allows us to change the list of bad edges after each switch in constant time (although, we repeat, the constant is impractically large). $\square$

It remains to prove Lemma 4.3.4, and our proof of this requires a computer. Unfortunately, both the number of labelings and the number of possible cuts is far too large for a brute-force approach: There are $2(2^9 - 1) - 1$ edges of $2T_9$, which means more than 11^{1000} labelings, even if we use Lemma 4.3.3 to eliminate labeling with edges of weight 3 or 4. Moreover, there are roughly $(2^{2 \cdot 2^8})^4$ internal cut labelings in $2T_9$, hence we cannot use brute-force even for one labeling. To overcome the second problem we shall recursively compute all of the necessary information, called a 'menu' on the subtrees, leading to an efficient algorithm for a given labeling. To solve the first problem, instead of enumerating all labelings of $2T_9$ and computing the menu for them, we will iteratively find all menus corresponding to all labelings of T_1, T_2, ..., T_8. This way we avoid considering the same 'partial labeling' several times. To further reduce the computational load, we will consider only 'worst possible menus' in each T_i. Now, to the details.

If $S \subseteq \{1, 2, 3, 4\}$, we define an internal cut labeling Y of T_i to be an *internal S-swap* if $Y = (\delta(Z_1), \dots, \delta(Z_4))$ where every Z_i is a set of interior nodes (note that the root is interior) and $S = \{i \in \{1, 2, 3, 4\} : r \in Z_i\}$. Informally, an internal S-swap 'switches S between the root and the leaves'. A *menu* is a mapping $M : \mathcal{P}([4]) \to \mathbb{Z}$. If T_i is a copy of a rooted tree with root r and X is a labeling of T_i then the *menu corresponding to X* is defined as follows

$$M_X(S) = \min\{\mathrm{cost}(X \,\Delta\, Y) - \mathrm{cost}(X) : Y \text{ is an internal } S\text{-swap}\} . \tag{4.1}$$

Thus, the menu M_X associated with X is a function which tells us for each subset $S \subseteq \{1, 2, 3, 4\}$ the minimum cost of making an internal S-swap. This is enough information to check whether we can decrease the cost of a given labeling: if T_1, T_2, T_3 are trees meeting at a vertex and X_i is the restriction of a labeling X to T_i, then we can decrease the cost by a local swap if we have $M_{X_1}(S) + M_{X_2}(S) + M_{X_3}(S) < 0$ for some $S \in \mathcal{P}([4])$.

For menus M, N and a set $R \subseteq \{1, 2, 3, 4\}$ we let $\mathrm{Parent\_menu}(M, N, R) : \mathcal{P}([4]) \to \mathbb{Z}$ be the following mapping:

$$\begin{aligned} &\mathrm{Parent\_menu}(M, N, R)(S) = \\ &\qquad \min_{Q \in \mathcal{P}([4])} \Big(M(Q) + N(Q) + a(|R \,\Delta\, S \,\Delta\, Q|) - a(|R|) \Big) . \end{aligned} \tag{4.2}$$

The motivation for this definition is the following observation, which is the key to our recursive computation.

Observation 4.3.5 *Let X be a labeling of the tree T_i where $i \geq 2$. Assume that T_i is composed of the root edge e and two copies of T_{i-1} denoted T' and T'', and let X' and X'' be the restrictions of the labeling X to the trees T' and T''. Then*

$$M_X = \mathrm{Parent\_menu}(M_{X'}, M_{X''}, m_X(e)) .$$

Proof: Let v be the end of the edge e which is distinct from r and pick $S, Q \in \mathcal{P}([4])$. Now choose a cut labeling $Y = (\delta(Z_1), \dots, \delta(Z_4))$ so that $\mathrm{cost}(X \,\Delta\, Y) - \mathrm{cost}(X)$ is minimal subject to the following constraints

(i) Z_i is internal for $1 \leq i \leq 4$,

(ii) $S = \{i \in \{1,2,3,4\} : r \in Z_i\}$, and

(iii) $Q = \{i \in \{1,2,3,4\} : v \in Z_i\}$.

Then $m_{X \,\Delta\, Y}(e) = m_X(e) \,\Delta\, S \,\Delta\, Q$ and we find that

$$\begin{aligned}\operatorname{cost}(X \,\Delta\, Y) - \operatorname{cost}(X) = {} & M_{X'}(Q) + M_{X''}(Q) \\ & + a(|m_X(e) \,\Delta\, S \,\Delta\, Q|) - a(|m_X(e)|)\,.\end{aligned}$$

It follows from this that $M_X = \mathrm{Parent\_menu}(M_{X'}, M_{X''}, m_X(e))$ as desired. □

Using the above observation, it is relatively fast to compute the menu associated with a fixed labeling of a tree T_i. However, for our problem, we need to consider all possible labelings of T_i. Accordingly, we now define a few collections of menus which contain all of the information we need to compute to resolve Lemma 4.3.4. Prior to defining these collections, we need to introduce the following partial order on menus: if M_1 and M_2 are menus, we write $M_1 \preccurlyeq M_2$ if $M_1(s) \le M_2(s)$ for every $s \in \mathcal{P}([4])$.

We let $\mathcal{M}_i$ be the set of all M_X, where X is a labeling of T_i, and every $e \in E(T_i)$ satisfies $w_X(e) \le 2$. We let $\mathcal{W}_i$ denote the set of maximal ('worst') elements (with respect to $\preccurlyeq$) of $\mathcal{M}_i$. Further, we define two subsets of these sets: $\mathcal{M}'_i$ denotes the set of menus corresponding to those labelings X of T_i where each edge is of weight at most 2 and where the root edge is marked by $\{1,2\}$. Finally, $\mathcal{W}'_i$ is the set of maximal elements of $\mathcal{M}'_i$. The following observation collects the important properties of these sets.

Observation 4.3.6 *For every $i \ge 2$ we have*

1. $\mathcal{M}_i = \big\{\mathrm{Parent\_menu}(M,N,s) \mid M,N \in \mathcal{M}_{i-1}, s \in \mathcal{P}([4]), |s| \le 2\big\}$
2. $\mathcal{W}_i = \max\limits_{\textit{in} \preccurlyeq}\big\{\mathrm{Parent\_menu}(M,N,s) \mid M,N \in \mathcal{W}_{i-1}, s \in \mathcal{P}([4]), |s| \le 2\big\}$
3. $\mathcal{W}'_i = \max\limits_{\textit{in} \preccurlyeq}\big\{\mathrm{Parent\_menu}(M,N,\{1,2\}) \mid M,N \in \mathcal{W}_{i-1}\big\}$

Proof: Part 1 follows immediately from Observation 4.3.5. The second part follows from this and from the fact that the mapping Parent_menu is monotone with respect to the order $\preccurlyeq$ on menus. Part 3 follows by a similar argument. □

Next we state the key claim proved by our computer check.

Claim 4.3.7 (verified by computer) *For every $W_1 \in \mathcal{W}_9'$, and $W_2, W_3 \in \mathcal{W}_8$ there exists $S \in \mathcal{P}([4])$ such that $W_1(S) + W_2(S) + W_3(S) < 0$.*

We use the Observation 4.3.6 to give a practical scheme for computing the collections $\mathcal{W}_8$ and $\mathcal{W}_9'$ followed by a simple test for each possible triple. Further details are described in Section 4.5. With this, we are finally ready to give a proof of Lemma 4.3.4.

Proof of Lemma 4.3.4: Let X be an edge labeling of $2T_9$ as in the lemma; we may suppose the central edge uv is labeled by $\{1, 2\}$. Let T^1, T^2, T^3 be the three distinct maximal subtrees of $2T_9$ which have v as a leaf, and assume that T^1 contains the central edge. Let X_j denote the restriction of X to T^j, and write $M_j = M_{X_j}$. Choose $W_1 \in \mathcal{W}_9'$, $W_2, W_3 \in \mathcal{W}_8$ so that $M_j \preccurlyeq W_j$ holds for each j. By Claim 4.3.7, we may choose $S \in \mathcal{P}([4])$ for which $W_1(S) + W_2(S) + W_3(S) < 0$ and by definition of $\preccurlyeq$ we have $M_1(S) + M_2(S) + M_3(S) < 0$, too. Let X_j be the internal S-swap for which the minimum in the definition of M_j (equation (4.1)) is attained. Then $Y = X_1 \,\Delta\, X_2 \,\Delta\, X_3$ is an internal cut labeling of $2T_9$ and $\mathrm{cost}(X \,\Delta\, Y) - \mathrm{cost}(X) = M_1(S) + M_2(S) + M_3(S) < 0$. This completes the proof. $\square$

Remark 4.3.8 *In the definition of cost of a coloring, the values of parameters $a(i)$ can be chosen in a variety of ways—provided we do penalize edges of weight 1. Perhaps it seems more natural to have $a(1) = 0$ but this straightforward approach does not work. Consider the edge labeling of $2T_4$ the upper part of which is depicted in Figure 4.2. It is rather easy to verify, that switching any local cut labeling does not get rid of edge of weight 2. Moreover, this labeling can be extended to arbitrary $2T_n$ by the 'growing rules' depicted in the figure (a, b, c, d stand for $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ in any order). On the other hand, by switching $\{2\}$ and $\{3\}$ on the cuts depicted in the figure, we decrease the cost of the coloring by $a(1)$.*

Remark 4.3.9 *Note that it is possible to prove Lemma 4.3.3 by the same method as Lemma 4.3.4; in fact a simple modification of the code verifies both of these lemmata at the same time. The reason we put Lemma 4.3.3 separately is that it allows for an easy proof by hand, and this hopefully makes the proof easier to understand.*

Another remark is that an easy modification of our method to verify Claim 4.3.7 could decrease the running time by 30%. We did not want to obscure the main proof for this relatively small saving, but we wish to mention the trick here. In the

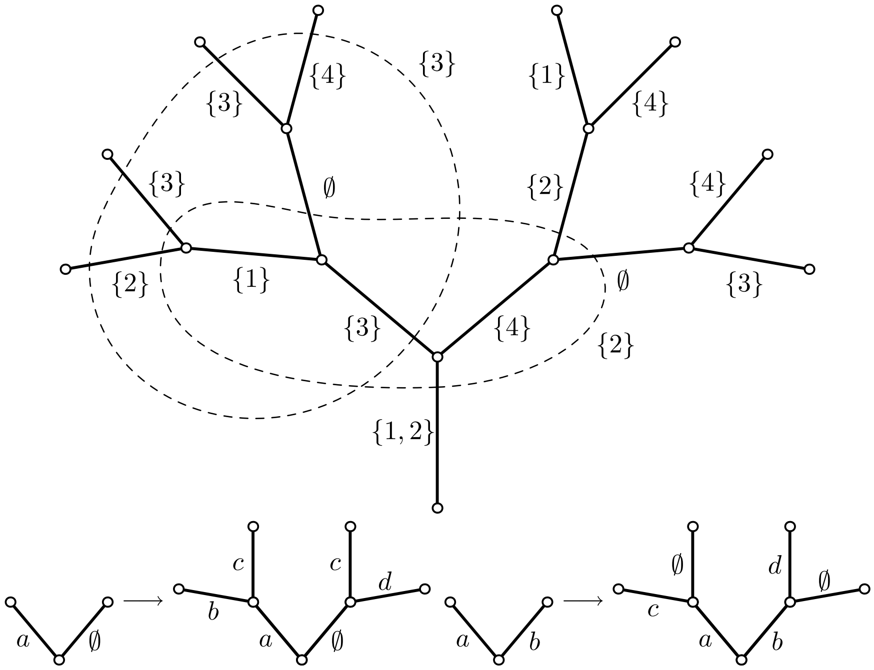

Figure 4.2: A difficult labeling of T_4.

process of enumerating the sets $\mathcal{W}_i$, we can throw away all menus M that satisfy $M(\emptyset) < 0$. It is not hard to show that we still consider all 'hard cases'.

Remark 4.3.10 *The necessity to use computer for huge amount of checking is not entirely satisfying (although this point of view may be rather historically conditioned aesthetic criterion). It would be interesting to find a proof of Lemma 4.3.4 without extensive case-checking, perhaps by a careful inspection of the sets $\mathcal{W}_i$.*

4.4 Some equivalences

The goal of this section is to prove Proposition 4.1.1 from the Introduction (restated here for convenience as Proposition 4.4.2), which gives several graph properties equivalent to the existence of a homomorphism to a projective cube PQ_{2k}. To prove this, it is convenient to introduce another family of graphs first. For every positive integer n, let H_n denote the graph with all binary vectors of length n forming the vertex set and with two vertices being adjacent if they agree in exactly one coordinate (note that H_n is a Cayley graph on $\mathbb{Z}_2^n$).

For odd n, the graph H_n has exactly two components, one containing all vertices with an even number of 1's, and the other all vertices with an odd number of 1's; we call the components H_n^e and H_n^o, respectively.

Observation 4.4.1 *For every $k \geq 1$ we have $H_{2k+1}^e \cong H_{2k+1}^o \cong PQ_{2k}$.*

Proof: The mapping that sends each binary vector to its complementary vector gives an isomorphism between H_{2k+1}^o and H_{2k+1}^e. Thus, the simple graph obtained from H_{2k+1} by identifying complementary vectors is isomorphic to H_{2k+1}^e (and to H_{2k+1}^o). However, this graph is also isomorphic to PQ_{2k}, since viewing the vertices of each as a pair of complementary vectors, we see that u and v will be adjacent if and only if one vector associated with u and one vector associated with v differ in exactly 1 coordinate. $\square$

Now we are ready to prove the proposition.

Proposition 4.4.2 *For every graph G and nonnegative integer k, the following properties are equivalent.*

1. *There exist $2k$ pairwise disjoint cut complements.*
2. *There exist $2k+1$ pairwise disjoint cut complements with union $E(G)$.*
3. *G has a homomorphism to PQ_{2k}.*
4. *G has a cut-continuous mapping to C_{2k+1}.*

Proof: We shall show $1 \implies 2 \implies 3 \implies 4 \implies 1$.

To see that $1 \implies 2$, let $S_1, S_2, \ldots, S_{2k}$ be pairwise disjoint cut complements, and for every $1 \leq i \leq 2k$ let $W_i = E(G) \setminus S_i$. Now setting $S_{2k+1} = E(G) \setminus \cup_{1 \leq i \leq 2k} S_i = E(G) \setminus \Delta_{1 \leq i \leq 2k} W_i$ we have 2.

Next we shall show that $2 \implies 3$. Let S_1,S_2,$\ldots$,S_{2k+1} be $2k+1$ disjoint cut complements with union $E(G)$ and for every $1 \leq i \leq 2k+1$ choose $U_i \subseteq V(G)$ so that $S_i = E(G) \setminus \delta(U_i)$. Now assign to each vertex v a binary vector x^v of length $2k+1$ by the rule $x_i^v = 1$ if $x \in U_i$ and $x_i^v = 0$ otherwise. This mapping gives a homomorphism from G to H_{2k+1}, so by Observation 4.4.1 we conclude that G has a homomorphism to PQ_{2k}.

Next we prove that $3 \implies 4$. Since the composition of two cut-continuous mappings is cut-continuous, it follows from Lemma 2.1.2 and Observation 4.4.1 that it suffices to find a cut-continuous mapping from H_{2k+1} to C_{2k+1}. To construct this, we let $E(C_{2k+1}) = \{e_1, e_2, \dots, e_{2k+1}\}$ and define a mapping $g : E(H_{2k+1}) \to E(C_{2k+1})$ by the rule that $g(uv) = e_i$ if u and v agree exactly in coordinate i. We claim that g is a cut-continuous mapping. To see this, let R be a cut of C_{2k+1}, let $J = \{i \in \{1, 2, \dots, 2k+1\} : e_i \in R\}$, and note that $|J|$ is even. Now let X be the set of all binary vectors with the property that there are an even number of 1's in the coordinates specified by J. Then $g^{-1}(R) = \delta(X)$ so our mapping is cut-continuous as required.

To see that $4 \implies 1$, simply note that the preimage of any edge of C_{2k+1} is a cut complement, so the preimages of the $2k+1$ edges are $2k+1$ disjoint cut complements. □

We can extract the key idea of the above proof as follows. Let $E_i \subseteq E(H_{2k+1})$ be the set of edges uv such that u and v agree in exactly the i-th coordinate.[3] The sets $E_1, \dots, E_{2k+1}$ form a partition of $E(H_{2k+1})$ into disjoint cut complements.

4.5 Code listing

In this section we present the code used to verify Claim 4.3.7. It is written in C and can be downloaded at `http://kam.mff.cuni.cz/~samal/papers/clebsch/`. It runs for about 30 minutes on a 2 GHz processor. We have tested it with compilers gcc (version 3.0, 3.3), Intel C, and Borland C++ on several computers to minimize the possibility of error in the proof due to wrong computer hardware/software.

We use Observation 4.3.6 to iteratively compute $\mathcal{W}_{i+1}$ from $\mathcal{W}_i$, this is accomplished by function W_update. By the same function we compute $\mathcal{W}'_9$ from $\mathcal{W}_8$. Finally, we use final_test to check whether all triples of menus satisfy the inequality of Claim 4.3.7. To simplify and speed up the code, we use static data structures for $\mathcal{W}_i$'s—that is, the elements of the set $\mathcal{W}_i$ are stored as `W[i][j]`— with a limit MAX=20000 on the number of elements, if this number turned out to be too small, the program would output an error message (this does not happen).

Marks of edges, that is elements of $\mathcal{P}([4])$ are represented as integers from 0 up to 15. For convenience variables that hold marks have type **mark** (which is a new name for **short**). Symmetric difference of marks corresponds to bitwise xor—'^'.

[3] If you think of H_n as of a Cayley graph, then E_i consists of edges corresponding to the i-th element of the generating set. We thank to Reza Naserasr for this comment.

Cost of edges are stored in variables of type **cost** (a new name for **int**). From equation (4.2) it is easy to deduce that $\mathrm{Parent\_menu}(M, N, R)(S) \leq M(S) + N(S)$. Consequently, the largest coordinate of an element of $\mathcal{W}_i$ is at most $2^{i-1}a(4)$, and as we only use sets $\mathcal{W}_i$ for $i \leq 9$, we will not have to store larger numbers than an **int** can hold. Other new data types are **menu** (array of 16 **cost**'s used to represent a menu), and **comparison**—variables of that type are assigned values -1, 0, 1, or INCOMP=2 if the result of a corresponding comparison (of two menus) is $\prec$, $=$, $\succ$ or incomparable.

When we need to compute $M = \mathrm{Parent\_menu}(M_1, M_2, c)$, this is implemented as add_menus(M_1,M_2,child); p_menu(child, parent, M). Here child corresponds to the sum $M_1 + M_2$, parent is a menu corresponding to coloring of the single edge of T_1 by color c. Then we insert the menu in the set $\mathcal{W}_i$ (array W[i]) by calling insert_menu. Note that if we implemented the deletion of 'small' menus in this function in a more straightforward manner ('move everything left'), the running time would approximately double.

```
#include <stdio.h>
#define MAX 20000     // limit on size of the sets W_i

typedef short mark;
typedef int cost;
typedef cost menu[16];

cost a[5]={0,1,10,40,1000};
cost markcost[16];
// cost of edge marked by each possible mark
menu one_mark[1];
// W'_1, i.e. one_mark[0] corresponds to T1 marked by {1,2}
menu W[9][MAX];
menu Wprime[MAX];     // W'_9
int Wsize[10];
// Wsize[i] is the number of elements of W[i]
int Wprimesize;       // the number of elements of Wprime

typedef short comparison;
comparison INCOMP = 2;

void menu_from_mark(mark Q, menu M) {
// M will be the menu corresponding to T1 marked by Q
  mark s;
  for(s=0; s<16; s++)
```

```
    M[s] = markcost[Q ^ s] - markcost[Q];
}

void init_variables() {
  mark s;
  for (s=0; s<16; s++)
    markcost[s] = a[(s&1) + ((s>>1) & 1) +
                  ((s>>2) & 1) + ((s>>3)&1)];
// the right hand side is a[n], where n is the number of ones
// in binary representation of s

  menu_from_mark(3,one_mark[0]);

  Wsize[1]=0;
  for(s=0; s<16; s++)
    if (markcost[s] < a[3])
      menu_from_mark(s,W[1][++Wsize[1]]);
}

void add_menus(menu M1, menu M2, menu sum) {
  mark s;
  for (s=0; s<16; s++)
    sum[s] = M1[s]+M2[s];
}

comparison sign(int n) {
  if (n > 0) return 1;
  if (n < 0) return -1;
  return 0;
}

comparison compare_menus(menu M1, menu M2) {
// returns -1, 0, 1, INCOMP, depending on
// whether M1<M2, M1=M2, M1 > M2, or they are incomparable
  mark s;
  comparison t, current=0;

  for (s=0; s < 16; s++) {
    t = sign (M1[s] - M2[s]);
    if ((t != 0) && (t == -current)) return INCOMP;
    if (current == 0) current = t;
  }
```

```
  return current;
}

void p_menu(menu child, menu parent, menu output) {
// child is the sum of two childs
// parent corresponds to the new edge
  mark s, q;
  cost new, current_best;

  for (s=0; s<16; s++) {
    current_best = a[4];
    for (q = 0; q < 16; q++) {
      new = child[q] + parent[s ^ q];
      if (new < current_best) current_best = new;
    }
    output[s] = current_best;
  }
}

void insert_menu(menu *book, int *booksize, menu M) {
// book is an array of menus
// booksize is the number of elements of book
// we are inserting M
  int i;
  mark s;
  comparison t=0;

  for (i=0; i < *booksize; i++) {
    t = compare_menus(M, book[i]);
    if (t <= 0) return;  // we will not insert small menu
    if (t == 1) break;   // we will delete book[i]
  }

// we will delete all elements of book that are <= M
  if (t==1)   // i.e. M > book[i]
    for ( ; i < *booksize; i++) {
      while (i < *booksize &&
             compare_menus(M, book[i])==INCOMP)
        i++;
      while (*booksize > i &&
             compare_menus(book[*booksize-1],M) <= 0 )
        (*booksize)--; // we abandon small menus at the end
```

```
      if (*booksize <= i) break;
    // and move big menu from end to the place of book[i]:
      (*booksize)--;
      for (s = 0; s<16; s++)
        book[i][s] = book[*booksize][s];
    }

// we insert M as the last element of book
  if (*booksize == MAX) printf("too short array!\n");
  else {
    for (s = 0; s<16; s++)
      book[*booksize][s] = M[s];
    (*booksize)++;
  }
}

void W_update(menu *oldW, int oldsize, menu *root_edge,
              int rootsize, menu *newW, int *newsize) {
  menu N, child;
  int i, j, k;

  *newsize = 0;
  for (i=0; i < oldsize; i++)
    for (j=i; j < oldsize; j++) {
      add_menus(oldW[i],oldW[j],child);
      for (k=0; k < rootsize; k++) {
        p_menu(child,root_edge[k],N);
        insert_menu(newW, newsize, N);
      }
    }
}


int final_test(menu *C, int Csize, menu *P, int Psize) {
  int i, j, k;
  mark s;
  int counter=0;
  menu child;

  for (i=0; i < Csize; i++)
    for (j=i; j < Csize; j++) {
      add_menus(C[i],C[j],child);
```

```
      for (k=0; k < Psize; k++) {
        counter ++;
        for (s=0; s<16; s++)
          if (child[s]+P[k][s] < 0) { counter--; break;}
        // Claim 2.7 holds for C[i],C[j],P[k]
        // we proceed by testing another triple
      }
    }
  return counter;
}

int main() {
  int i;
  init_variables();

  for (i=1; i<8; i++) {
    W_update(W[i],Wsize[i],W[1],Wsize[1],W[i+1],&Wsize[i+1]);
    printf("The size of W%d is: %d\n",i+1,Wsize[i+1]);
  }
  W_update(W[8],Wsize[8],one_mark,1,Wprime,&Wprimesize);

  if (final_test(W[8],Wsize[8],Wprime,Wprimesize) == 0)
    printf("\nProof is finished.\n\n");

  return 0;
}
```

Chapter 5

Cubical coloring (Fractional covering by cuts)

In this chapter we introduce a new graph invariant that measures fractional covering of a graph by cuts. Besides being interesting in its own, it is useful for study of TT_2 mappings and homomorphisms. We pursue connections with fractional chromatic number and with bipartite subgraph polytope. For the sake of simplicity we restrict to the case of $M = \mathbb{Z}_2$. For other rings, we may proceed similarly; compare Lemma 2.2.4 and 2.2.5.

5.1 Introduction

All graphs we consider are undirected loopless; to avoid trivialities we do not consider edgeless graphs in this chapter. Recall that for a set $W \subseteq V(G)$ we let $\delta(W)$ denote the set of edges leaving W and that we call any set of form $\delta(W)$ a *cut*.

Definition 5.1.1 *Let us call (cut)* n/k-cover *of* G *an* n*-tuple* $(X_1, \dots, X_n)$ *of cuts in G such that every edge of G is covered by at least* k *of them. We define*

$$\chi_q(G) = \inf\left\{\frac{n}{k} \mid \textit{exists } n/k\textit{-cover of } G\right\}$$

and call $\chi_q(G)$ *the* cubical chromatic number *of G. (Motivation for this terminology will be given in the discussion preceding Lemma 5.1.2.)*

First recall that if $k = 1$, i.e. if we want to cover every edge by some cut then we need at least $\lceil \log_2 \chi(G) \rceil$ of them (see, e.g., [19]). Here we consider a fractional version. In this context we may find it surprising that $\chi_q(G) < 2$ for every G (Corollary 5.2.3).

From another perspective, $\chi_q(G)$ is the fractional chromatic number of a certain hypergraph: it has $E(G)$ as points and odd cycles of G as hyperedges. This suggests that $\chi_q(G)$ is a solution of a linear program, see Lemma 5.1.3.

As the last of the introductory remarks, we note that $\chi_q(G)$ is a certain type of chromatic number, but instead of complete graphs (or Kneser graphs or circulants) which are used to define chromatic (fractional resp. circular chromatic) number it uses another graph scale. Let $Q_{n/k}$ denote a graph with $\{0,1\}^n$ as the set of vertices, where xy forms an edge iff $d(x,y) \ge k$ (here $d(x,y)$ is the Hamming distance of x and y). It is easy to see that G has n/k-cover if and only if it is homomorphic to $Q_{n/k}$. That means that an alternative definition is

$$\chi_q(G) = \inf\left\{ \frac{n}{k} \mid G \xrightarrow{hom} Q_{n/k} \right\}. \tag{5.1}$$

Let $H^{\ge k}$ denote the graph with vertices $V(H)$ and edges uv for any $u, v \in V(H)$ with distance in H at least k. Further let Q_n denote the n-dimensional cube. Then $Q_{n/k} = Q_n^{\ge k}$. This corresponds to the definition of circular chromatic number, where the target graph is $C_n^{\ge k}$. To stress this similarity we use the term cubical chromatic number and notation $\chi_q(G)$.

The original motivation for study of $\chi_q(G)$ was the following lemma, we use it in the next section to prove non-existence of certain TT_2 mappings.

Lemma 5.1.2 *Let G, H be graphs. Then* $G \preccurlyeq_2^t H \implies \chi_q(G) \le \chi_q(H)$.

Proof: It suffices to show that whenever H has an n/k-cover, G has it as well. So let f be some TT_2 mapping from G to H and let $X_1, \dots, X_n$ be an n/k-cover and consider X_i'—a preimage of a cut X_i under f. By definition, X_i' is also a cut. If e is an edge of G, $f(e)$ is an edge of H hence it is covered by at least k of the cuts X_i. Thus e is covered by at least k of the cuts X_i'. $\square$

Note that graph $Q_{n/k}$ are Cayley graph on some power of $\mathbb{Z}_2$, hence $\mathbb{Z}_2$-graphs in terminology of Section 2.3.2. Proposition 2.3.8 implies that these graph are right $\mathbb{Z}_2$-homotens, that is $G \xrightarrow{hom} Q_{n/k}$ and $G \xrightarrow{TT_2} Q_{n/k}$ are equivalent properties for every graph G. Consequently, we may as well use TT_2 mapping to $Q_{n/k}$ in equation (5.1). This provides an indirect proof of Lemma 5.1.2.

The next lemma provides another characterization of cubical chromatic number, analogical to fractional chromatic number.

Lemma 5.1.3 *The parameter $\chi_q(G)$ is the solution of the following linear program ($\mathcal{C}$ denotes the family of all cuts in G).*

$$\textit{minimize} \sum_{X \in \mathcal{C}} w(X) \textit{ subject to: for every edge } e, \sum_{X, e \in X \in \mathcal{C}} w(X) \geq 1$$

Proof: Let $x(G)$ be the optimal solution to the linear program. For every n/k-cover there is a feasible solution w of the linear program which assigns value $1/k$ to each of the given cuts and 0 to all other cuts. This shows $x(G) \leq \chi_q(G)$. To prove the converse inequality, recall that any linear program has optimal solution with rational coordinates, let w be this solution and write $w(X) = c(X)/N$, with $c(X)$ being a (non-negative) integer. If we take $c(X)$ copies of every cut X, we obtain a $\bigl(\sum_X c(X)\bigr)/N$-cover. As $\sum_X c(X)/N = \sum_X w(X) = x(G)$, we get $\chi_q(G) \leq x(G)$. $\square$

We conclude that we can replace $\inf$ by $\min$ in the definition of $\chi_q(G)$—the infimum is always attained. We can also consider the dual program

$$\text{maximize} \sum_{e \in E(G)} y(e) \text{ subject to: for every cut } X, \sum_{e, e \in X} y(e) \leq 1 \ . \tag{5.2}$$

This program is useful for computation of $\chi_q(G)$ for some G. Moreover, in Section 5.5 we use this dual program to discuss yet another definition of $\chi_q(G)$ in terms of bipartite subgraph polytope.

There is another possibility to dualize the notion of fractional cut covering, namely *fractional cycle covering*. Bermond, Jackson and Jaeger [5] proved that every bridgeless graph has a cycle $7/4$-cover, and Fan [24] proved that it has a $10/6$-cover. An equivalent formulation of Berge-Fulkerson conjecture (Conjecture 6.1.5) claims that every cubic bridgeless graph has a $6/4$-cover. On the other hand, Edmonds characterization of the matching polytope implies that every cubic bridgeless graph has a $3k/2k$-cover (for some k).

5.2 Basic properties of $\chi_q(G)$

As in Chapter 4 we let $\mathrm{MAXCUT}(G)$ be the number of edges in the largest cut in G and write $b(G) = \mathrm{MAXCUT}(G)/|E(G)|$.

Lemma 5.2.1 *For any graph G, $\chi_q(G) \geq 1/b(G)$. If G is edge-transitive, then equality holds.*

Proof: Suppose $\chi_q(G) = n/k$ and let $X_1, \ldots, X_n$ be an n/k-cover. Then $\sum_{i=1}^{n} |X_i| \leq n \cdot b(G)|E(G)|$, on the other hand this sum is at least $k \cdot |E(G)|$, as every edge is counted at least k times. This proves the first part of the lemma. To prove the second part, let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be all cuts of the maximal size (i.e. $|X_i| = b(G)|E(G)|$). From the edge-transitivity follows that every edge is covered by the same number (say k) of elements of $\mathcal{X}$. Now $k \cdot |E(G)| = \sum_{i=1}^{n} |X_i| = n \cdot b(G)|E(G)|$, which finishes the proof. □

Corollary 5.2.2 *1.* $\chi_q(K_{2n}) = \chi_q(K_{2n-1}) = 2 - 1/n$

2. $\chi_q(C_{2k+1}) = 1 + 1/(2k)$

3. $\chi_q(\mathrm{Pt}) = 5/4$

Corollary 5.2.3 *For any graph G,*

$$1 + \frac{1}{g_o(G) - 1} \leq \chi_q(G) \leq 2 - \frac{1}{\lceil \chi(G)/2 \rceil} .$$

In particular, $\chi_q(G) \in [1, 2)$.

Proof: Let $l = g_o(G)$, i.e., C_l is the shortest cycle that is a subgraph of G. Then by Lemma 5.1.2 and Corollary 5.2.2 we have $1+1/(l-1) \leq \chi_q(G)$. For the upper bound, note that G maps to $K_{\chi(G)}$ homomorphically, thus also by a TT_2 mapping. By Lemma 5.1.2 and 5.2.2 we have $\chi_q(G) \leq \chi_q(K_{\chi(G)}) = 2 - 1/\lceil \frac{\chi(G)}{2} \rceil$. □

The above results imply that the function $\frac{2}{2-\chi_q(G)}$ more resembles a version of chromatic number— for K_n it equals n or $n + 1$; this partly explains Theorem 5.2.7. However, we prefer to work with a function that has nicer properties (among else is a solution of a linear program).

By combining Lemma 5.1.2 and Corollary 5.2.2 we get the following theorem. It is surpassed by results of Chapter 2, still it presents a simple and direct proof of a reasonable result (previously, it was only known that graphs K_{2^n} form a strictly increasing chain). In fact, the proof of Lemma 2.2.4 (a stepping stone towards Corollary 2.2.12) implicitly uses cubical chromatic number and Lemma 5.1.2.

Theorem 5.2.4 *Graphs K_{2n} form a strictly ascending chain in the order $\prec_{cc}$. In other words, $K_2 \prec_2^t K_4 \prec_2^t K_6 \prec_2^t \cdots$.*

The next lemma shows that χ_q enjoys some of the properties of other chromatic numbers. ($G_1 \Box G_2$ denotes the cartesian product of graphs, $G_1 \times G_2$ the categorial one.)

Lemma 5.2.5

1. *$\chi_q(G) = \max\{\chi_q(G') \mid G'$ is a component of $G\}$*
2. *$\chi_q(G) = \max\{\chi_q(G') \mid G'$ is a 2-connected block of $G\}$ for a connected graph G.*
3. *$\chi_q(G_1 \Box G_2) = \max\{\chi_q(G_1), \chi_q(G_2)\}$*
4. *$\chi_q(G_1 \times G_2) \le \min\{\chi_q(G_1), \chi_q(G_2)\}$*

Proof: We will prove that if G', G'' are graphs that share at most one vertex, then $\chi_q(G' \cup G'') = \max\{\chi_q(G'), \chi_q(G'')\}$. Clearly, this proves 1 and 2. Let X_1', …, X_n' be an optimal cover of G', X_1'', …, X_m'' an optimal cover of G'', thus $\chi_q(G') = n/k$, and $\chi_q(G'') = m/l$. Consider the collection of mn cuts $\{X_i' \cup X_j''\}$ (these are cuts, indeed, as G' and G'' share at most one vertex). An edge of G' is covered at least mk times, an edge of G'' at least nl times. Hence $\chi_q(G) \le \frac{mn}{\min\{mk,nl\}} = \max\{\frac{n}{k}, \frac{m}{l}\} = \max\{\chi_q(G'), \chi_q(G'')\}$. On the other hand, both G' and G'' are subgraphs of G, hence by Lemma 5.1.2 the other inequality follows.

Part 3 follows from Lemma 5.1.2, as $G_1 \Box G_2$ is TT-equivalent to the disjoint union of G_1 and G_2.

Part 4 follows from Lemma 5.1.2 as there are homomorphisms (and therefore TT mappings) $G_1, G_2 \to G_1 \times G_2$ □

We close this section by a study of cubical chromatic number of random graphs.

Lemma 5.2.6 *Let $c > 0$, let p, δ be functions of n such that $p, \delta \in [0,1]$ and $\delta^2 p = \Omega(n^{c-1})$. Then $b(G(n,p)) \le \frac{1}{2}(1 + O(1/n) + O(\delta))$ a.a.s. In particular*

$$b(G(n,1/2)) \le \frac{1 + O(\frac{1}{n^{1/2-c}})}{2} \qquad \textit{a.a.s.}\,.$$

Proof:

Claim 1. If $\delta^2 p \gg 1/n^2$ then $|E(G(n,p))| > (1-\delta)p\binom{n}{2}$ a.a.s.

To prove this we use Chernoff inequality (as stated in Corollary 2.3 of [47]) for random variable $X = |E(G(n,p))|$. It claims

$$\Pr[X \leq \mathbb{E}X - \delta\mathbb{E}X] \leq 2e^{-\frac{\delta^2}{3}\mathbb{E}X}$$

And as $\mathbb{E}X = p\binom{n}{2}$, Claim 1 follows.

Claim 2. If $\delta^2 p = \Omega(n^{c-1})$ then $\mathrm{MAXCUT}(G(n,p)) < (1+\delta)p\frac{n^2}{4}$ a.a.s.

For a set $A \subseteq V(G(n,p))$ we let X_A be the random variable that is equal to the number of edges leaving A, let $a = |A| \leq n/2$. By another variant of Chernoff inequality

$$\Pr[X_A \geq \mathbb{E}X_A + \delta\mathbb{E}X_A] \leq 2e^{-\frac{\delta^2}{3}\mathbb{E}X_A}$$

and substituting $\mathbb{E}X = pa(n-a)$ we get

$$\Pr[X_A \geq (1+\delta)pn^2/4] \leq 2e^{-\frac{\delta^2}{3}pa(n-a)} \leq 2e^{-\frac{\delta^2 pan}{6}}.$$

It remains to estimate the total probability of a large cut:

$$\begin{aligned}\Pr[(\exists A)X_A \geq (1+\delta)pn^2/4] &\leq \sum_{a=1}^{n/2}\binom{n}{a}2e^{-\frac{\delta^2 pan}{6}}\\ &\leq 2[(1+e^{-\frac{\delta^2 pn}{6}})^n - 1].\end{aligned}$$

For $\delta^2 p = \Omega(n^{c-1})$ the last expression tends to zero, which finishes the proof. $\square$

Theorem 5.2.7 *For any $c > 0$,*

$$2 - O\left(\frac{1}{n^{1/2-c}}\right) \leq \chi_q(G(n,1/2)) \leq 2 - \Omega\left(\frac{\log n}{n}\right) \qquad \textit{a.a.s.}$$

Proof: The lower bound follows by Lemma 5.2.6, the upper one by an application of Corollary 5.2.3 and the well-known fact that $\chi(G(n,1/2)) = O(n/\log n)$. $\square$

Note that we could use the known result on clique number of a random graph for a direct proof of the lower bound in Theorem 5.2.7, but this way we would obtain only $\chi_q(G(n,1/2)) \geq 2 - O(1/\log n)$. On the other hand, by more careful computation we could obtain slightly sharper lower bound by the same method, but still far from the optimal one. It would be interesting to determine precisely the asymptotic behaviour of $\chi_q(G(n,1/2))$.

5.3 Cubical coloring and other graph parameters

In this section we relate $\chi_q(G)$ to various other graph parameters, we start by $\chi_f(G)$—the fractional chromatic number of G. This may be defined by $\chi_f(G) = \inf\{n/k \mid G \xrightarrow{hom} K(n,k)\}$, where $K(n,k)$ is the *Kneser graph*. Its vertex set consists of all k-element subsets of $[n] = \{1, 2, \dots, n\}$, two vertices are connected iff they are disjoint subsets of $[n]$.

Lemma 5.3.1 *Let k, n be integers such that $2k \le n$. Then there is a cut in $K(n,k)$ with $\binom{n-1}{k-1}\binom{n-k}{k}$ edges. Consequently, $b(K(n,k)) \ge 2k/n$.*

Proof: We let $U = \{S \subseteq [n] \mid 1 \in S\}$. Clearly, $\delta(U)$ contains $\binom{n-1}{k-1}\binom{n-k}{k}$ edges. □

Corollary 5.3.2 *For $2k \le n$ we have $\chi_q(K(n,k)) \le n/2k$. Consequently, for any graph G we have $\chi_q(G) \le \frac{1}{2}\chi_f(G)$.*

Proof: As Kneser graphs are edge-transitive, Lemma 5.2.1 and 5.3.1 imply

$$\chi_q(K(n,k)) = \frac{1}{b(K(n,k))} \le \frac{\frac{1}{2}\binom{n}{k}\binom{n-k}{k}}{\binom{n-1}{k-1}\binom{n-k}{k}} = \frac{n}{2k}\,.$$

The rest follows by Lemma 5.1.2 and the definition of fractional chromatic number. □

Corollary 5.3.3 *For every $\varepsilon > 0$ and every integer b there is a graph G such that*

$$\chi_q(G) < 1 + \varepsilon \quad \textit{and} \quad \chi(G) > b\,.$$

Proof: Let $G = K(n,k)$, for $n = 2k + t$, $k = 2^t$ and t large enough. Then by Corollary 5.3.2 $\chi_q(G) \le n/2k = 1 + t/2^{t+1}$ and by [59] and $\chi(G) = n - 2k + 2 = t + 2$. □

By Corollary 5.2.3, we can view Corollary 5.3.3 as a strengthening of the well-known fact that there are graphs with no short odd cycle and with a large chromatic number. It also shows that the converse of Lemma 5.1.2 is far from being true: just take G from the corollary and let $H = K_{b/2}$. Then $\chi_q(G)$ is close to 1 and $\chi_q(H)$

close to 2 (that is as far apart as these values can be), still by an application of Corollary 2.3.9 we don't have $G \preccurlyeq_2^t H$.

It is interesting to find how various graph properties affect $\chi_q(G)$. We saw already, that small $\chi(G)$ makes $\chi_q(G)$ small, while large $\chi(G)$ does not force it to be large. Also small $g_o(G)$ makes $\chi_q(G)$ large. In this context we ask:

Question 5.3.4 *Let G be a cubic graph with no cycle of length $\leq c$. How large can $\chi_q(G)$ be?*

For $c = 3$, it follows from Brook's theorem that $\chi_q(G) \leq \chi_q(K_3) = 3/2$. For $c = 17$ we saw in Chapter 4 that $G \xrightarrow{TT_2} C_5$, hence $\chi_q(G) \leq \chi_q(C_5) = 5/4$. On the other hand, there is $\varepsilon > 0$ such that cubic graphs of arbitrary high girth exist such that $b(G) < 1 - \varepsilon$ (by a result of McKay, see Section 4.2 for further details), hence with $\chi_q(G) > 1 + \varepsilon$.

We finish by a result explaining why we in Question 5.3.4 restrict to cubic graphs. Note that much sharper results on MAXCUT of graphs without short cycles were conjectured in [23] and (some of them) proved in [1, 2].

Theorem 5.3.5 *For any $\varepsilon > 0$ and integer l there is a graph G such that $\chi_q(G) > 2 - \varepsilon$ and G contains no circuit of length at most l.*

Proof: We mimic the famous Erdős' proof of existence of high-girth graphs of high chromatic number. Let $p = n^{\alpha-1}$ (where $\alpha < 1/l$) and consider random graph $G(n,p)$.

The expected number of circuits of length at most l is $O((pn)^l) = o(n)$, therefore by Markov inequality with probability at least 1/2 the graph $G(n,p)$ contains at most n circuits of length at most l. We delete one edge from each of them and let G' be the resulting graph. We use Lemma 5.2.6 for $\delta = n^{-\alpha/3}$ and $c < \alpha/3$, note that by Claim 1 from proof of Lemma 5.2.6, the number of edges of $G(n,p)$ is a.a.s. $\Omega(n^{1+\alpha})$, hence the deletion of n edges creates only another $(1 + o(1))$ factor in the estimate for $b(G(n,p))$. An application of Lemma 5.2.1 and a choice of sufficiently large n finishes the proof. $\square$

5.4 Measuring the scale ($\chi_q(Q_{n/k})$)

In this section we will discuss the 'invariance property' of cubical chromatic number. In analogy with $\chi(K_n) = n$, $\chi_c(C_n^{\geq k}) = n/k$ and 'dimension of product of n complete graphs is n' we would like to prove that $\chi_q(Q_{n/k}) = n/k$. The following lemma shows, that the situation is not that simple for χ_q.

Lemma 5.4.1 *Let $k \leq n$ be integers. Then we have $\chi_q(Q_{n/k}) \leq \frac{n}{k}$. If k is odd, then $\chi_q(Q_{n/k}) \leq \frac{n+1}{k+1}$.*

Proof: For the first part, it suffices to consider the identical homomorphism $Q_{n/k} \xrightarrow{hom} Q_{n/k}$. For the second part, mapping $V(Q_{n/k}) \to V(Q_{(n+1)/(k+1)})$ given by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, x_1 + \dots + x_n \bmod 2)$ is a homomorphism whenever k is odd. □

Another complication is that by Corollary 5.2.3 we have $\chi_q(G) < 2$ for any graph G. But we conjecture that with this exception, Lemma 5.4.1 gives the correct answer.

Conjecture 5.4.2 *Let k, n be integers such that $k \leq n < 2k$. Then $\chi_q(Q_{n/k}) = \frac{n}{k}$ if k is even and $\chi_q(Q_{n/k}) = \frac{n+1}{k+1}$ if k is odd.*

We present two arguments in support of Conjecture 5.4.2. First, observe that $K(n,r)$ is a subgraph of $Q_{n/2r}$. By Lemma 5.1.2 and 5.2.1 we have

$$\chi_q(Q_{n/2r}) \geq \chi_q(K(n,r)) \geq \frac{1}{b(K(n,r))} .$$

In [74] it is claimed that if $2r \leq n \leq 3r$ then Lemma 5.3.1 gives the correct size of $\mathrm{MAXCUT}(K(n,r))$, i.e. $b(K(n,r)) = 2r/n$. This would imply the conjecture for even k less than $3/2 \cdot n$; unfortunately the proof in [74] seems incomplete.

Another promising approach to Conjecture 5.4.2 is to use algebraical graph theory. The following results appear as Lemma 13.7.4 and 13.1.2 of [27].

Lemma 5.4.3 *Let G be a graph with n vertices and m edges, let λ_n be the largest eigenvalue of the Laplacian of G. Then $b(G) \leq \frac{n}{m}\frac{\lambda_n}{4}$.*

Lemma 5.4.4 *Let G be an r-regular graph with n vertices, let eigenvalues of G be $\Theta_1 \geq \Theta_2 \geq \dots \geq \Theta_n$. Then the eigenvalues of the Laplacian of G are given by $\lambda_i = r - \Theta_i$.*

We will also use an expression for spectra of graphs with transitive automorphism group ([58], see also Problem 11.8 in [60]).

Lemma 5.4.5 *Let G be a graph whose automorphism group contains a commutative subgroup Γ. Suppose Γ is regular, that is for each pair $x, y \in V(G)$ there is* *exactly one element $\gamma_{x,y} \in \Gamma$ that moves x to y. Let χ be a character of Γ and u any vertex of V. Then*

$$\sum_{v;uv\in E(G)} \chi(\gamma_{u,v})$$

is an eigenvalue of G; moreover all eigenvalues are of this form.

By Lemma 5.4.5 we find that eigenvalues of Q_n^k are $\sum_{t=0}^{k}(-1)^t\binom{x}{t}\binom{n-x}{k-t}$ for any $x \in \{0,1,\dots,n\}$. This is in fact the definition of Krawtchouk polynomial $K_k^n(x)$, which enables us to use various known results on Krawtchouk polynomials (recurrence relations etc.). None of these methods, however, was sufficient to prove the following desirable inequality. From numerical experiments, however, this inequality is well-justified (in particular it is true whenever $n \le 1000$).

Conjecture 5.4.6 *Let k, n be integers such that $k \le n < 2k$ and k is even, let x be an integer such that $1 \le x \le n$. Then*

$$\sum_{t=0}^{k}(-1)^t\binom{x}{t}\binom{n-x}{k-t} \ge \binom{n}{k}\left(1-\frac{2k}{n}\right).$$

By Vandermonde's identity an equivalent formulation is

$$\sum_{\text{odd } t}\binom{x}{t}\binom{n-x}{k-t} \le \binom{n-1}{k-1}.$$

Theorem 5.4.7 *Conjecture 5.4.6 implies Conjecture 5.4.2.*

Proof: Suppose first that k is even. By Lemma 5.1.2 and 5.2.1 we have that $\chi_q(Q_{n/k}) \ge \chi_q(Q_n^k) = 1/b(Q_n^k)$. By Lemma 5.4.3 and Lemma 5.4.4 it is enough to determine the smallest eigenvalue Θ of Q_n^k. As Q_n^k is $\binom{n}{k}$-regular, we have

$$\frac{1}{b(Q_n^k)} \ge \frac{|E(Q_n^k)|}{|V(Q_n^k)|}\frac{4}{\binom{n}{k}-\Theta} = \frac{2\binom{n}{k}}{\binom{n}{k}-\Theta}.$$

Now we use Lemma 5.4.5. We suppose $V(Q_n^k) = \mathbb{Z}_2^n$ and take $\Gamma \simeq \mathbb{Z}_2^n$; therefore $\gamma_{u,v}$ corresponds to $u+v$ (operation modulo 2 in each coordinate) and the characters are $\chi_y : v \mapsto (-1)^{\sum_{i=1}^{n} v_i y_i}$ for each $y \in \mathbb{Z}_2^n$. Now put $u = \vec{0}$ and suppose that weight of y is x (that is $y_i = 1$ for exactly x values of i). The sum from Lemma 5.4.5 becomes

$$\sum_{v \text{ of weight } k} \chi_y(v) = \sum_{t=0}^{k}(-1)^t\binom{x}{t}\binom{n-x}{k-t}$$

(here t is the number of bits that y and v have in common). By using the conjectured inequality, we obtain $\chi_q(Q_{n/k}) \geq n/k$ as desired.

For odd values of k we cannot use the same method, as then Q_n^k is bipartite, hence $b(Q_n^k) = 1$. However, observe that $Q_{(n+1)/(k+1)} \xrightarrow{hom} Q_{n/k}$, hence by Lemma 5.1.2 and the result for (even) $k+1$ we have

$$\chi_q(Q_{n/k}) \geq \chi_q(Q_{(n+1)/(k+1)}) \geq (n+1)/(k+1)\,.$$

□

5.5 Bipartite subgraph polytope

For a bipartite subgraph $B \subseteq G$, let c_B be the characteristic vector of $E(B)$. Bipartite subgraph polytope $P_B(G)$ is the convex hull of points c_B, for all bipartite graphs $B \subseteq G$. The study of this polytope was motivated by the max-cut problem: to look for a weighted maximum cut of G simply means to solve a linear program over $P_B(G)$. Thus, for graphs where $P_B(G)$ has simple description, we can have polynomial-time algorithm for max-cut; this in particular happens for weakly bipartite graphs (which include planar graphs), see [30]. We apply P_B to yield yet another definition of χ_q.

Theorem 5.5.1 $\chi_q(G) = \max\{\sum_{e\in E(G)} y_e \mid y \cdot c \leq 1$ *defines a facet of* $P_B(G)\}$

Proof: By LP duality $\chi_q(G)$ is a solution to the program (5.2). This means, that we are maximizing over such y, that for each cut X satisfy $y \cdot c_X \leq 1$. As the convex hull of vectors c_X is P_B, we are maximizing the sum of coordinates of an element of P_B^*. This maximum is attained for some vertex of P_B^*, that is for y such that $y \cdot c \leq 1$ defines a facet of P_B. □

'Natural' facets of $P_B(G)$ are defined by $\sum_{e\in E(H)} y_e \leq \mathrm{MAXCUT}(H)$ for some $H \subseteq G$. (For other subgraphs H, this inequality is satisfied too, but defines a face of smaller dimension.) This proves the following observation (we add a direct proof, too).

Lemma 5.5.2 $\chi_q(G) \geq 1/(\min_{H\subseteq G} b(H))$

Proof: Let $H \subseteq G$. Then $H \xrightarrow{TT_2} G$, which by Lemma 5.1.2 and 5.2.1 implies $1/b(H) \leq \chi_q(G)$. □

Let us return to Lemma 5.2.1 for a while. In general $\chi_q(G)$ and $1/b(G)$ can be as distant as possible: Let G be a disjoint union of a K_n and $K_{N,N}$. Now $\chi_q(G)$ is close to 2 (because G is homomorphically equivalent to K_n, hence $\chi_q(G) = \chi_q(K_n)$) and $b(G)$ is close to 1 (provided N is sufficiently large). This motivates Lemma 5.5.2, which improves the original bound. A natural question is, whether this improvement gives the correct size of χ_q.

For some graphs there are other facets of P_B than the natural ones. In [4] graphs with several other types of facets are constructed (for some of them, the ratio between distinct nonzero coefficients of the facets is of order $|V(G)|^2$). However, none of these constructions yields a facet with the sum of coefficients larger than one of the natural facets. Hence, a conjecture emerges.

Conjecture 5.5.3 $\chi_q(G) = 1/(\min_{H \subseteq G} b(H))$

Perhaps more importantly we ask, whether $\chi_q(G)$ can be computed or approximated efficiently (at least for some graph class). Lemma 5.1.2 can be used as a 'no-homomorphism lemma', therefore this may be of interest for the study of homomorphisms, too.

Chapter 6

FF, FT & CDC

In this chapter we are going to discuss possible use of FF and FT mappings to handle various problems dealing with cycle structure of a graph, particularly Cycle double cover (CDC) conjecture (Conjecture 6.1.3). In the first section we start by summarizing Jaeger's [43] approach to these conjectures and its development by DeVos, Nešetřil, and Raspaud [19]. In the second section we look more closely at the relation between CDC and FT mappings. It the last section we exhibit a way how to construct FT mappings from elementary ones, thereby allowing for illustrative proofs of various results about CDC. We explain how this approach was (implicitly) taken by, e.g., [83, 33].

6.1 FF and Petersen coloring

We start by a list of several important conjectures that are the topic of this section and in fact of this entire area of discrete mathematics. For a more detailed presentation of these conjectures and proofs of the results we will mention we refer the reader to [12, 79, 92, 46, 45].

By a *nowhere-zero k-flow* in a directed (or undirected) graph we mean a $\mathbb{Z}$-flow in the graph (or any of its orientations) that attains only values $\pm 1, \pm 2, \dots, \pm(k-1)$. Surprisingly, by a result of Tutte the existence of such flow is equivalent to existence of a *nowhere-zero M-flow* (that is of an $(M, M \setminus \{0\})$-flow) whenever M is a ring with k elements.

The following conjectures of Tutte [85, 86] are a core part of the study of nowhere-zero flows.

Conjecture 6.1.1 (Tutte's 5-flow conjecture) *Every bridgeless graph has a nowhere-zero 5-flow.*

Conjecture 6.1.2 (Tutte's 3-flow conjecture) *Every 4-connected bridgeless cubic graph has a nowhere-zero 3-flow.*

Motivation for the next two conjectures stems from the observation that in a bridgeless planar graph the face-boundaries are circuits that cover every edge precisely twice. Such system of circuits obviously does not exist if the graph contains a bridge. On the other hand, Seymour [78] and Szekeres [81] independently conjectured that planarity is not needed. (According to [12], this conjecture was known to Tutte already in 1950's.) Celmins [15] and Preissmann [75] conjecture that five cycles suffice for each graph.

Conjecture 6.1.3 (Cycle double cover conjecture) *For every bridgeless graph there is a list of cycles such that each edge is contained in precisely two of them (so-called cycle double cover, or CDC of the graph). More specifically, at most 5 cycles suffice.*

The following yet stronger conjecture is due to [3, 46].

Conjecture 6.1.4 (Orientable cycle double cover conjecture) *For every bridgeless directed graph there is a list of cycles $C_1, \dots, C_t$ with splittings (C_i^+, C_i^-) such that each edge of G is in exactly one of the sets C_i^+ and one of the sets C_i^-. (Such list of cycles is called orientable cycle double cover, or OCDC of the graph). More specifically, at most 5 cycles suffice.*

The following conjecture was made independently by Berge and by Fulkerson [25]. It may be viewed as a fractional relaxation of 3-edge coloring.

Conjecture 6.1.5 (Berge-Fulkerson conjecture) *For every bridgeless cubic graph there is a list of six perfect matchings so that every edge is contained in exactly two of them.*

Jaeger [43] observed that each of these conjectures may be equivalently stated as a conjecture about existence of a certain (M, B)-flow and defined a mapping between graphs that 'preserves flows'. The mapping he defined was exactly a surjective FF_2 mapping. We state here a variant of Jaeger's result as formulated in [19] and prove part of it for the reader's convenience.

Proposition 6.1.6 *Let G, H be graphs.*

- *Suppose $G \xrightarrow{FF_2} H$. If Conjecture 6.1.3 (or 6.1.5) is valid for H then it is valid for G as well.*
- *Suppose $G \xrightarrow{FF_{\mathbb{Z}}} H$. If Conjecture6.1.4 (or 6.1.1, 6.1.2) is valid for H then it is valid for G as well. (For the latter two conjectures it is sufficient if $G \xrightarrow{FF_5} H$, or $G \xrightarrow{FF_3} H$ respectively.)*

Proof: Let $f : G \xrightarrow{FF_2} H$ and let $C_1, \ldots, C_t$ be a CDC of H. By definition of FF mapping, $f^{-1}(C_i)$ are cycles in G. If e is an edge of G then $f(e)$ (as an edge of H) is covered by exactly two of C_i, hence e is covered by two of the preimages. For Conjecture 6.1.5 we observe that equivalent formulation asks for a list of six cycles such that every edge appears in exactly four of them and proceed as above. We omit the other two proofs. $\square$

Proposition 6.1.6 explains the motivation to study FF mappings. Indeed, suppose we find classes $\mathcal{A}$ and $\mathcal{B}$ of graphs such that

- for every graph $A \in \mathcal{A}$ there is a graph $B \in \mathcal{B}$ for which $A \xrightarrow{FF} B$, and
- some of the above conjectures is true for any graph from $\mathcal{B}$.

Then the same conjecture is true for any graph from $\mathcal{A}$ as well. In particular, all of these conjectures would be resolved if the following conjecture holds true. (The first part is due to Jaeger, the second one to DeVos, Raspaud, and Nešetřil).

Conjecture 6.1.7 ([43], [19]) *Any bridgeless graph admits*

1. *an FF_2 mapping to the Petersen graph.*
2. *an $FF_{\mathbb{Z}}$ mapping to the Petersen graph or to K_4.*

It is proved by Jaeger [43] that the first part of the conjecture can be reduced to the case of cubic graphs and that for such graphs existence of an FF_2 mapping to the Petersen graph is equivalent to existence of so-called Petersen edge-coloring: coloring of edges of a cubic graph by edges of Petersen graph so that any three neighboring edges are mapped to three neighboring edges of the Petersen graph. In the next section we will show an equivalent formulation of part 1 of Conjecture 6.1.7 in terms of a certain cycle cover.

In the rest of this section we mention more conjectures describing the structure of FF mappings. Some of them have direct implication for the above 'big' conjectures, and a solution to any of them would bring light to this area of graph theory.

Problem 6.1.8 ([19]) *Is there an infinite antichain for orders $\preccurlyeq_2^f$, $\preccurlyeq_{\mathbb{Z}}^f$? Explicitly, is there an infinite set of graphs with no FF_2 ($FF_{\mathbb{Z}}$) mapping between any two of them?*

An affirmative answer to this problem would apparently make it harder to find a small class of graphs to which every other graph maps. Note that arbitrarily large finite antichains are known to exist for FF_2 mappings [19]. On the positive side, Rizzi [76] made a conjecture about packing T-joins, an equivalent formulation (according to [19]) is as follows. Note that K_2^n denotes an undirected graph with two vertices and n parallel edges between them. Also recall Lemma 1.2.11 which particularly implies that for $(2k+2)$-edge-connected graph G we have $K_2^{2k+1} \stackrel{FF_2}{\not\longrightarrow} G$.

Conjecture 6.1.9 (Rizzi [76]) *Suppose G is a graph and $k \geq 0$ an integer. If $K_2^{2k+1} \stackrel{FF_2}{\not\longrightarrow} G$ then $G \xrightarrow{FF_2} K_2^{2k+1}$. In other words, no graph is incomparable with K_2^{2k+1} in $\preccurlyeq_2^f$.*

This conjecture is immediate for $k = 0$; for $k = 1$ if follows from Jaeger's construction of a nowhere zero 4-flow in each 4-edge-connected graph. For general k, an approach to it is a result of Jaeger [44] and its strengthening by DeVos and Seymour [20].

Theorem 6.1.10 *Let G be a graph and $k \geq 0$ an integer. If G is $4k$-edge-connected (or just $K_2^{4k-1} \stackrel{FF_2}{\not\longrightarrow} G$) then $G \xrightarrow{FF_2} K_2^{2k+1}$.*

Note that $G \xrightarrow{FF_2} K_2^{2k+1}$ is equivalent with $G \xrightarrow{FT_2} C_{2k+1}$ (Lemma 1.2.4); in Theorem 4.1.3 we proved that under certain conditions $G \xrightarrow{TT_2} C_5$, which may be viewed as a dual form of Theorem 6.1.10 for $k = 2$.

Related notion (defined by Jaeger [44]) is that of *modular $(2k+1)$-orientation*, that is such orientation for which a constant 1 is a $\mathbb{Z}_{2k+1}$-flow. It is known that a graph G admits a modular $(2k+1)$-orientation iff $G \xrightarrow{FF_{\mathbb{Z}}} K_2^{2k+1}$, so the following conjecture is a natural strengthening of Theorem 6.1.10. It was proposed by Jaeger, the extension is suggested in [19].

Conjecture 6.1.11 *Let G be a graph and $k \geq 0$ an integer. If G is $4k$-edge-connected (or just $K_2^{4k-1} \stackrel{FF_{\mathbb{Z}}}{\not\longrightarrow} G$) then $G \xrightarrow{FF_{\mathbb{Z}}} K_2^{2k+1}$.*

The general setting of FF mappings enables us to state a whole scale of claims between the known and the open ones (we follow [19] in this presentation). For

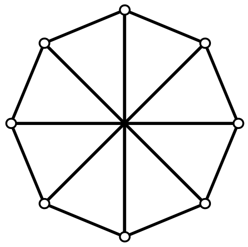

Figure 6.1: The graph V_8.

example, let V_8 be the graph in Figure 6.1. It is possible to show $K_2^3 \prec_{\mathbb{Z}}^f V_8 \prec_{\mathbb{Z}}^f K_4$. By a result of Jaeger, every 4-edge-connected graph has a nowhere-zero $\mathbb{Z}_4$-flow, therefore admits an $FF_{\mathbb{Z}}$ mapping to K_4. On the other hand, if every highly-connected graph maps in $FF_{\mathbb{Z}}$ to K_2^3, then a (weak but still open) version of case $k = 1$ of Conjecture 6.1.11 is true; this would also be a weaker version of Tutte's 3-flow conjecture (Conjecture 6.1.2) with stronger connectivity requirement.

Thus, the following is a reasonable approach to a longstanding open problem.

Conjecture 6.1.12 ([19]) *Is there an integer k such that any k-edge-connected bridgeless graph admits an $FF_{\mathbb{Z}}$ mapping to V_8?*

6.2 FT and CDC

In this section we are going to inquire the intimate relation between cycle covering problems and FT mappings. Although we do not solve any of these problems, hopefully the presented way to view CDC problems sheds some light on some of the folklore observations. Moreover, in Theorem 6.2.7 we present an equivalent formulation of Jaeger's conjecture (Conjecture 6.1.7), which seems not to be known before. We start by a definition of (orientable) cycle covers.

Definition 6.2.1 *Let G be a graph, let $\mathcal{C} = (C_1, \dots, C_t)$ be a collection of cycles in G. We say $\mathcal{C}$ forms a* cycle double cover *(shortly CDC) of G, if each edge of G is contained in exactly two of the cycles C_i. We say that $\mathcal{C}$ forms an* orientable cycle double cover *(shortly OCDC) of G, if there is (for some orientation of G if G is undirected) a splitting (C_i^+, C_i^-) of each cycle such that each edge of G is in exactly one of the sets C_i^+ and one of the sets C_i^-.*

We define a certain dual graph $G^{\mathcal{C}}$ of G based on this cycle cover. The vertices of $G^{\mathcal{C}}$ are $\{1, \dots, t\}$. In case of CDC we let $\{i, j\}$ be an edge iff $E(C_i) \cap E(C_j) \neq \emptyset$. In the OCDC case, (i, j) is an edge iff $E(C_i^-) \cap E(C_j^+) \neq \emptyset$.

We say a CDC (OCDC) $\mathcal{C}$ is a circular CDC (OCDC) *if $G^{\mathcal{C}}$ is a circuit (an orientation of circuit).*

The following lemmata exhibit the relation of FT mappings to cycle double covers (the first one appears—for H being a complete graph—already in [82]).

Lemma 6.2.2 *Let G, H be graphs, let H be loopless. Then the following are equivalent:*

1. $G \xrightarrow{FT_2} H$
2. *There is a CDC $\mathcal{C}$ of G, such that $G^{\mathcal{C}} \subset H$.*

In particular:

- *There is a t-CDC, iff $G \xrightarrow{FT_2} K_t$.*
- *There is a circular t-CDC, iff $G \xrightarrow{FT_2} C_t$.*

Lemma 6.2.3 *Let G, H be graphs, let H be loopless. Then the following are equivalent:*

1. $G \xrightarrow{FT_{\mathbb{Z}}} H$
2. *There is an OCDC $\mathcal{C}$ of G, such that $G^{\mathcal{C}} \subset H$.*

In particular:

- *There is a t-OCDC of G, iff $G \xrightarrow{FT_{\mathbb{Z}}} K_t$.*
- *There is a circular t-OCDC of G, iff $G \xrightarrow{FT_2} C_t$.*

Proof of Lemma 6.2.2 and 6.2.3: Let $\mathcal{C} = (C_1, \dots, C_t)$ be an (O)CDC of an orientation $\overrightarrow{G}$ of G. Let φ_i be the flow determined by C_i, that is $\varphi_i(e) = 1$ if $e \in C_i^+$ and $\varphi_i(e) = -1$ if $e \in C_i^-$. If $\varphi_i(e) = 1$ and $\varphi_j(e) = -1$ then we define $f(e) = ij$. (In case of CDC we compute in $\mathbb{Z}_2$, so $-1 = 1$ and we may not—and don't need to—specify orientation of $f(e)$.) We defined $f : E(G) \to E(G^{\mathcal{C}})$. If τ_i is a vertex-tension on H determined by vertex i then $\tau_i \circ f = \varphi_i$, thus f is indeed an FT mapping, proving $2 \implies 1$. The reverse implication is easy: take vertex-tensions τ_i, and let C_i be the support of the flow $\tau_i \circ f$. $\square$

A corollary of these results (and of Lemma 1.2.4) is that Jaeger's modular $(2k+1)$-orientation (discussed before Conjecture 6.1.11) corresponds to a circular $(2k+1)$-OCDC.

As an application of our formulation of CDC problems we reprove the following well-known property of cycle double covers.

Proposition 6.2.4 *Any graph with a 4-CDC admits a 3-CDC as well.*

Proof: If G admits a 4-CDC, then $G \xrightarrow{FT_2} K_4$. As $K_4 \xrightarrow{TT_2} K_3$ (see Section 1.3), we yield by composition $G \xrightarrow{FT_2} K_3$ which we wanted to prove. $\square$

The fact that $K_4 \overset{TT_{\mathbb{Z}}}{\not\longrightarrow} K_3$ (Lemma 2.2.5) clearly explains, why there is no analogue of Proposition 6.2.4 for oriented covers.

Next, we turn our attention to graphs embedded on surfaces (the reader may consult [62] for an introduction to graph embeddings). We saw in Lemma 1.2.4 that duality forms an FT and TF mapping between a plane graph and its dual. The following result generalizes this for a general surface.

Lemma 6.2.5 *Let G be an undirected graph embedded on a surface S and let G^* be its dual. Then*

1. *$G \xrightarrow{FT_2} G^*$, and*
2. *$G \xrightarrow{FT_{\mathbb{Z}}} G^*$ if S is orientable.*

Proof: Let d be the mapping that assigns to an edge e of G an edge e^* of G^* which connects the faces that e separates; if S is an orientable surface, then we choose such orientation of G and G^* that each edge of G^* connects the face to the left of e to the one to the right of e (not excluding the possibility that G^* contains loops, in which case the two above-mentioned faces are in fact equal).

Suppose that this is the case; we will prove d is $FT_{\mathbb{Z}}$ by using Lemma 1.2.9. Let τ be an elementary $\mathbb{Z}$-tension determined by a vertex v of G. Suppose for the ease of notation that all edges adjacent to v are oriented out of it, so that $\tau(e) = 1$ if e is adjacent to v and $\tau(e) = 0$ otherwise. By definition of the dual graph, τ_d is a $\mathbb{Z}$-flow, and d is $FT_{\mathbb{Z}}$.

If S is a non-orientable surface, then we cannot choose orientation of G and G^* as above. Still, an image of a $\mathbb{Z}_2$-tension is a $\mathbb{Z}_2$-flow, as in this case change of sign does not matter. $\square$

Lemma 6.2.5 explains, why the next conjecture is a generalization of Conjecture 6.1.3 and 6.1.4. Indeed, G^* is loopless if the embedding of G is circular.

Conjecture 6.2.6 ([32, 56, 45]) *Any 2-connected graph admits a circular 2-cell embedding on an (orientable) surface; that is such embedding in which each face is homeomorphic to a disc and its boundary is a circuit.*

Another easy consequence of Lemma 6.2.5 (remarked already by [82]) is the fact that planar graphs admit a 4-OCDC: by the 4-color theorem the planar dual of a graph G admits a homomorphism (and therefore a $TT_{\mathbb{Z}}$ mapping) to K_4, so we have

$$G \xrightarrow{FT_{\mathbb{Z}}} G^* \xrightarrow{TT_{\mathbb{Z}}} K_4 \,.$$

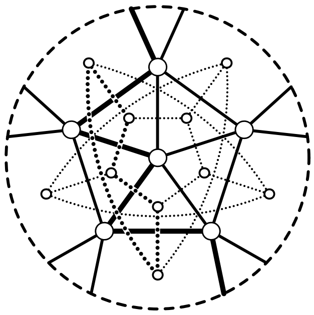

Figure 6.2: Duality of K_6 and the Petersen graph on projective plane.

There are more interesting applications, though. From Figure 6.2 it follows that K_6 and the Petersen graph are dual on projective plane.[1] This implies that $\mathrm{Pt} \xrightarrow{FT_2} K_6$, consequently Pt (and by composition any graph that admits an FF_2 mapping to Pt) admits a 6-CDC (confirm Proposition 6.1.6). This is in itself not exceedingly interesting, as Pt admits even a 5-CDC and therefore $\mathrm{Pt} \xrightarrow{FT_2} K_5$. By a closer look, we can however find the following reformulation of the first part of Conjecture 6.1.7. This in particular means that if the CDC conjecture

[1]The author is thankful to Matt DeVos for making him aware of this embedding.

was resolved in the stronger formulation given by the next theorem, then Berge-Fulkerson conjecture would follow from it.

Theorem 6.2.7 *Let $V(K_6) = \{1, 2, \dots, 6\}$. The following are equivalent.*

1. $G \xrightarrow{FF_2} \mathrm{Pt}$
2. *there is a mapping $f : G \xrightarrow{FT_2} K_6$ such that $f^{-1}(\{12, 23, 34, 45, 51\})$ is a cycle.*
3. *there is a 6-CDC by cycles $C_1, \dots, C_6$ such that*

$$(C_1 \cap C_2) \cup (C_2 \cap C_3) \cup (C_3 \cap C_4) \cup (C_4 \cap C_5) \cup (C_5 \cap C_1)$$

is a cycle, too.

Proof: Let $d : \mathrm{Pt} \xrightarrow{FT_2} K_6$ be the duality mapping, let $\mathcal{F}$ be the cycle space of Pt and $\mathcal{T}$ the cutspace of K_6. As $\dim \mathcal{F} = 6$ while $\dim \mathcal{T}$ is only 5, d is not TF_2. Indeed, the non-contractible circuit of Pt (emphasized in Figure 6.2) is mapped by d to the emphasized 5-circuit of K_6. With proper notation, the edge set of this circuit of K_6 is $A = \{12, 23, 34, 45, 51\}$, we let $\mathcal{T}'$ be the space generated by $\mathcal{T} \cup \{\chi_A\}$. By considering the dimension we see that the bijection d sends $\mathcal{F}$ to $\mathcal{T}'$. As it is enough to verify the condition of Definition 1.2.1 only for generators of the whole space, equivalence of 1 and 2 follows.

If f is a mapping from part 2, then preimages of the elementary vertex-tensions satisfy conditions of 3. In the other direction, the CDC determines a mapping $G \xrightarrow{FT_2} K_6$ (as in Lemma 6.2.2) and the extra condition in 3 exactly proves the extra condition in 2. □

We finish this section by a conjecture on FT mappings. We saw in Proposition 1.2.12 that if $H = \mathrm{Cay}(M, B)$ is a Cayley graph, then TT_M mappings to H precisely correspond to (M, B)-tensions, justifying the term H-valued tensions from Section 1.1. The dual version is true if H is an odd cycle, K_3 or K_4 and if Conjecture 6.2.8 holds generally (or at least for $H = K_6$) then Cycle double cover conjecture follows from it.

Conjecture 6.2.8 *Let $H = \mathrm{Cay}(M, B)$ be a Cayley graph. Then the following are equivalent.*

1. $G \xrightarrow{FT_M} H$
2. *G has an (M, B)-flow*

6.3 Building FT mappings from elementary ones

In this section we develop a method to construct an FT mapping as a sum of 'elementary FT mappings' (Theorem 6.3.2), in a similar way as a flow is a sum of elementary flows (see Section 1.2.1) To be able to do this, we must further extend the notion of FT mappings.

We say a mapping f is an $F\mathfrak{T}_M$ *mapping from* G *to* H if f maps $E(G)$ to $M^{E(H)}$ and for every cut $C = (C^+, C^-)$ the mapping

$$\sum_{e \in C^+} f(e) - \sum_{e \in C^-} f(e) \tag{6.1}$$

is an M-flow on H. (For $M = \mathbb{Z}_2$ this may be expressed shortly: $\sum_{e \in C} f(e)$ is a cycle in H.)

If h is an edge of H then we let $\chi_h \in M^{E(H)}$ be the characteristic function corresponding to h, that is $\chi_h(e') = 1$ if $e' = h$ and $\chi_h(e') = 0$ otherwise. Suppose $g : G \xrightarrow{FT} H$ and put $f(e) = \chi_{g(e)}$. If τ is cut-tension corresponding to a cut C then the expression (6.1) is exactly the image τ_g, hence a flow by Lemma 1.2.9. Consequently, FT mappings are a special case of $F\mathfrak{T}$ mappings; or, in the other way around, $F\mathfrak{T}$ mappings form an 'algebraical extension' of FT mappings, i.e., they form a structure that allows adding and multiplying by a constant. In Theorem 6.3.3 we will see the converse reduction: how to obtain an FT mapping from $F\mathfrak{T}$ mapping.

We let $\vec{0}$ denote the zero of $M^{E(H)}$ and we define operations for functions from $E(G)$ to $M^{E(H)}$ coordinate-wise. The following lemma shows that $F\mathfrak{T}$ mappings form the same structure as flows, namely, an M-module. We let $F\mathfrak{T}_M(G, H)$ denote the set of all $F\mathfrak{T}_M$ mappings from G to H.

Lemma 6.3.1 *Let G, H be directed graphs, M a ring. The set $F\mathfrak{T}_M(G, H)$ is an M-module.*

Proof: Clearly $\big((M^{E(H)})^{E(G)}, +, \vec{0}^{E(G)}\big)$ is an M-module, so we only have to show that $F\mathfrak{T}_M(G, H)$ is closed on addition and multiplication by elements of M. This follows directly from the fact that whenever f, g are M-flows on H and $m \in M$ then $f + g$ and $m \cdot f$ are M-flows as well. □

Pick a circuit $C = (C^+, C^-)$ of G and an edge $h \in E(H)$. We define a mapping $\mathrm{f}_{C,h}$ by

$$\mathrm{f}_{C,h}(e) = \begin{cases} \chi_h, & \text{if } e \in C^+ \\ -\chi_h, & \text{if } e \in C^- \\ \vec{0}, & \text{otherwise.} \end{cases}$$

Further, pick $g \in E(G)$ and a circuit $D = (D^+, D^-)$ in H. Let φ_D be an elementary flow around D, that is $\varphi_D(e) = 1$ if $e \in D^+$, $\varphi_D(e) = -1$ if $e \in D^-$ and $\varphi_D(e) = 0$ otherwise. We define a mapping $\mathrm{f}_{g,D}$ by

$$\mathrm{f}_{g,D}(e) = \begin{cases} \varphi_D, & \text{if } e = g \\ \vec{0}, & \text{otherwise.} \end{cases}$$

It is easy to verify that mappings $\mathrm{f}_{C,h}$ and $\mathrm{f}_{g,D}$ are $F\mathfrak{T}_M$ mappings from G to H. We call each such mapping an *elementary $F\mathfrak{T}_M$ mapping*. The following theorem finishes the parallel between flows and $F\mathfrak{T}$ mappings.

Theorem 6.3.2 *Suppose G, H are directed graphs, and M a ring. Then the module $F\mathfrak{T}_M(G,H)$ is generated by elementary $F\mathfrak{T}_M$ mappings. Explicitly, for every $f \in F\mathfrak{T}_M(G,H)$ there are $k, l \geq 0$, edges $g_i \in E(G)$, $h_j \in E(H)$, circuits C_j of G and D_i of H and finally $a_i, b_j \in M$ (where $1 \leq i \leq k$ and $1 \leq j \leq l$) such that*

$$f = \sum_{i=1}^{k} a_i \mathrm{f}_{g_i,D_i} + \sum_{j=1}^{l} b_j \mathrm{f}_{C_j,h_j} \,. \tag{6.2}$$

Proof: We use induction by the number of edges of G. Let f be an $F\mathfrak{T}_M$ mapping. First suppose $f(e) = \vec{0}$ for some $e \in E(G)$. Then $f(e)$ contributes $\vec{0}$ to each term of expression (6.1), thus the restriction of f to $G \setminus e$ is an $F\mathfrak{T}_M$ mapping, too. So we may assume that $f(e) \neq \vec{0}$ for every e. Pick one $e \in E(G)$. We distinguish two cases.

Case 1. e is a cut of G:
In this case, $f(e)$ is a flow of H, therefore we can write $f(e)$ as $\sum_t m_t \varphi_{D_t}$ for some circuits D_t of H and corresponding flows φ_{D_t}. Now $f - \sum_t m_t \mathrm{f}_{e,D_t}$ is an $F\mathfrak{T}_M$ mapping that is zero on e, therefore it is an $F\mathfrak{T}_M$ mapping on $G \setminus e$ and we may use induction.

Case 2. e is part of a circuit C of G:
$f - \sum_{h \in E(H}(f(e))(h) \cdot \mathrm{f}_{C,h}$ is an $F\mathfrak{T}_M$ mapping that is zero on e, therefore we may use induction as in Case 1. □

So far, we have shown how to generate an $F\mathfrak{T}$ mapping from elementary ones. Now we are going to see how can we obtain 'normal' FT mappings (and thereby CDC's) from $F\mathfrak{T}$ mappings. This is of course not possible for each $F\mathfrak{T}$ mapping (as, e.g., the contant mapping to $\vec{0}$ is an $F\mathfrak{T}$ mapping between any pair of graphs); we have to restrict the considered $F\mathfrak{T}$ mappings somehow. We will say that a mapping $f : G \xrightarrow{F\mathfrak{T}_M} H$ is *simple* if for every $e \in E(G)$ the mapping $f(e)$ is a characteristic mapping of some edge $h_e \in E(H)$, that is $f(e) = \chi_{h_e}$. Furthermore, we say a mapping $f : G \xrightarrow{F\mathfrak{T}_M} H$ is *almost simple* if for every e mapping $f(e)$ differs from a characteristic function of an edge by some flow, that is for some edge $h_e \in E(H)$ and M-flow φ_e on H we have $f(e) = \chi_{h_e} + \varphi_e$. (Remark that in the case $M = \mathbb{Z}_2$ this simply means that $f(e)$ is a characteristic function of a postman join.) We define $\bar{f}(e) = h_e$.

Theorem 6.3.3 *Let G, H be directed graphs, M a ring. Then the following are equivalent.*

1. *There is a mapping $G \xrightarrow{FT_M} H$.*
2. *There is a simple mapping $G \xrightarrow{F\mathfrak{T}_M} H$.*
3. *There is an almost simple mapping $G \xrightarrow{F\mathfrak{T}_M} H$.*
4. *There are circuits C_i of G, edges h_i of H and $b_j \in M$ such that the mapping $\sum_j b_j \cdot \mathrm{f}_{C_j,h_j}$ is almost simple.*

Proof: Suppose 1 and take some $g : G \xrightarrow{FT_M} H$. The mapping $f : e \mapsto \chi_{g(e)}$ is $F\mathfrak{T}_M$ (by Lemma 1.2.9) and simple, proving 2. By the same argument, mapping $g = \bar{f}$ is FT_M whenever f is a simple $F\mathfrak{T}_M$ mapping, so 1 and 2 are equivalent. Implication 2 $\implies$ 3 is obvious, for the reverse one we only observe that if $f : G \xrightarrow{F\mathfrak{T}_M} H$ is almost simple and $f(e) = \sum_t m_t(e) \cdot \varphi_{D_t(e)}$ then $f - \sum_e \sum_t m_t(e) \cdot \mathrm{f}_{D_t(e)}$ is simple and $F\mathfrak{T}$. Finally, 4 $\implies$ 3 follows by Theorem 6.3.2. For the converse it is enough to consider an almost simple mapping expressed in form (6.2): as the first sum assigns a flow to any edge of G, the second sum is almost simple, finishing the proof. $\square$

We illustrate use of Theorem 6.3.3 by two applications. The first one is the following well-known result.

Proposition 6.3.4 *Any hamiltonian graph has a 3-CDC.*

Proof: Let G be a hamiltonian graph and C a Hamilton circuit in it. According to Lemma 6.2.2 we need to show $G \xrightarrow{FT_2} K_3$. Let $E(K_3) = \{a, b, c\}$. For every edge $e \in E(G) \setminus C$ we let C_e be a circuit containing e and no other edge of $E(G) \setminus C$. We define

$$f = \mathrm{f}_{C,a} + \sum_{e \in E(G) \setminus C} \mathrm{f}_{C_e,b} \,.$$

For any $e \in E(G)$ we have $f(e) = \chi_a$ or $\chi_a + \chi_b$, therefore f is an almost simple $F\mathcal{T}_2$ mapping. An application of Theorem 6.3.3 finishes the proof. □

This easy proposition was extended in various ways: Tarsi [82] and Goddyn [26] proved that graphs with a Hamilton path admit a 6-CDC, Goddyn [26] and Häggkvist, McGuinness [33] construct a CDC, among else, in graphs which have a spannig subgraph that is a Kotzig graph. We want to stress that the original proofs of these theorems give the same list of cycles as our method. Still, we believe that our approach explains better the structure of these proofs; we attempt to exhibit this on a more complicated case below.

We say that a graph is a *Kotzig graph* if it is possible to properly color its edges so that each pair of colors induces a Hamilton circuit. For example, it is easy to provide such coloring of K_2^3 and K_4. More generally, if an r-regular grapn H (for odd r) is Kotzig, then $H \xrightarrow{FF_2} K_2^r$ (but the converse implication is false).

Proposition 6.3.5 *Let G be a cubic graph and H its spanning subgraph, so that H is a subdivision of a cubic Kotzig graph. Then G admits a 6-CDC.*

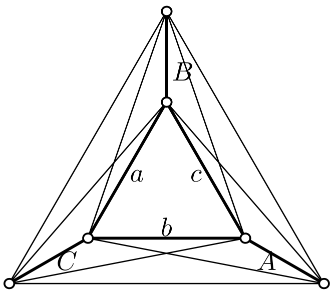

Figure 6.3: Illustration of proof of Proposition 6.3.5.

Proof: We are going to construct an almost simple $F\mathfrak{T}_2$ mapping to K_6 (with edges denoted as in Figure 6.3). Consider the Kotzigian 3-edge-coloring of the cubic Kotzig graph and extend it to H. So obtained coloring is not proper, but still each pair of colors induces a Hamilton circuit. Suppose these three colors are a, b, and c and define $f(e)$ for $e \in E(H)$ accordingly to obtain a simple $F\mathfrak{T}_M$ mapping from H to K_3. We extend it to a mapping $f : G \xrightarrow{F\mathfrak{T}_2} K_3$ by putting $f(e) = \vec{0}$ for $e \in E(G) \setminus E(H)$ (hence f is not simple anymore). Next, we aim to modify f to get an (almost) simple mapping $G \xrightarrow{F\mathfrak{T}} K_6$.

To this end, for each edge $e \in E(G) \setminus E(H)$ we let C_e be some circuit containing e, such that all edges in $C_e \setminus e$ are elements of $E(H)$ which use only two of the colors a, b, c, say, they do not use a for this choice of e. Put $g_e = \mathrm{f}_{C_e,A}$. Now,

$$f + \sum_{e \in E(G)\setminus E(H)} g_e$$

is almost simple: edges outside of H are mapped to characteristic function of $\{A\}$, $\{B\}$, or $\{C\}$, and edge of H which was colored, say, a by the Kotzigian coloring is mapped to characteristic function of one of $\{a\}$, $\{a, B\}$, $\{a, C\}$, $\{a, B, C\}$. It remains to look at Figure 6.3 to see that each of these sets is a postman join. □

Chapter 7

Miscellanea

7.1 Codes and χ/χ_{TT}

In this section we study the relationship between TT_2 mappings and homomorphisms by comparing the 'chromatic numbers' that these mappings define. Surprisingly, error correcting codes come into play. Recall the definition of a notion parallel to $\chi(G)$ that we started to study in Section 2.3.2,

$$\chi_{TT}(G) = \min\{n; G \xrightarrow{TT_2} K_n\}$$

(we concentrate on the case $M = \mathbb{Z}_2$ in this section). For random graphs, Corollary 2.2.9 implies that $\chi_{TT}(G) = \chi(G)$ a.a.s.

For general graph G, Lemma 2.1.2 implies $\chi_{TT}(G) \leq \chi(G)$, on the other hand by Corollary 2.3.9 we have $\chi_{TT}(G) > \chi(G)/2$. More precise information about behavior of $\chi(G)/\chi_{TT}(G)$ is desirable.

Consequently, let $\mathcal{G}_n = \{G \mid G \xrightarrow{TT_2} K_n\}$ and study $\chi(G)$ for $G \in \mathcal{G}_n$. By Lemma 1.3.1, $G \in \mathcal{G}_n$ is equivalent to $G \xrightarrow{hom} \Delta(K_n)$. In other words,

- $\Delta(K_n) \in \mathcal{G}_n$; and
- for every $G \in \mathcal{G}_n$ we have $G \xrightarrow{hom} \Delta(K_n)$.

This reduces the problem of behavior of $\chi_{TT}(G)/\chi(G)$ to special values of G.

Problem 7.1.1 *Study the sequence $r_n = \chi(\Delta(K_n))/n$, in particular determine its limes superior.*

The chromatic number of $\Delta(K_n)$ was studied (with the same motivation) in [55]. In [34], the connection with injective chromatic number of hypercubes is presented. In [22] graphs $\Delta(K_n)$ are studied (as a special type of graphs arising from hypercubes) in the context of embedding of trees. It is claimed there that $\chi(\Delta(K_9)) \geq 13$ (a result that follows by Delsartes' LP bound for size of error correcting codes, [18]). There is also a section on the topic in [48] ('chromatic number of cube-like graphs').

If we view the vertices of $\Delta(K_n)$ as $\{0,1\}^n$ then an independent set forms a 'code'—a set where no two elements have Hamming distance 2. With some more work we can use results from theory of error-correcting codes. This approach was taken in [55] and [34]. After using [7] they obtained the following result.

$$\chi(\Delta(K_n)) = 2^k \qquad \text{for } 2^k - 3 \leq n \leq 2^k \text{ and } k \geq 2 \tag{7.1}$$

In [34] an observation on covering by codes is used to show the following bound:

$$\chi(\Delta(K_{2n+1})) \leq 2\chi(\Delta(K_{n+1}))\,. \tag{7.2}$$

According to [48], Gordon F. Royle did show $\chi(\Delta(K_9)) \leq 14$ by a computer search. By relation (7.2), this implies $\chi(\Delta(K_{2^k+1}) \leq 2^k \cdot 7/4$. Consequently $r_{2^k+1} \leq 7/4$.

On the other hand, Best conjecture [6] claims a result extending that of [7], namely that shortened Hamming code of size n is optimal whenever $3/4 \cdot 2^k \leq n \leq 2^k$ ('half of the time'). This would imply an extension equality (7.1), namely that for such n, $\chi(\Delta(K_n)) = 2^k$. Consequently for minimal such n we would have $r_n = 4/3$. On the other hand, $r_{2^k} = 1$ for every k. In this context, we can speculate: is $\limsup r_n = 4/3$?

We add a new piece of information to the picture: if we restrict our attention to sparse graphs we see the same set of values $\chi_{TT}(G)/\chi(G)$.

Lemma 7.1.2 *Let n, l be integers, $n \geq 3$. There is $G \in \mathcal{G}_n$ such that $\chi(G) = \chi(\Delta(K_n))$ and $g(G) \geq l$.*

Proof: Suppose $\chi(\Delta(K_n)) = t$, hence $\Delta(K_n) \overset{hom}{\not\longrightarrow} K_{t-1}$. Lemma 3.1.11 gives us G with $g(G) \geq l$ such that $G \xrightarrow{hom} \Delta(K_n)$ and $G \overset{hom}{\not\longrightarrow} K_{t-1}$. Hence $G \in \mathcal{G}_n$ and $\chi(G) > t - 1$. On the other hand $\chi(G) \leq \chi(\Delta(K_n)) = t$. □

Remark 7.1.3 *In [55] (and Corollary 2.3.9) it is proved that if we define $\chi_{TT_\mathbb{Z}}$ by means of $TT_\mathbb{Z}$ mappings, then $\chi_{TT_\mathbb{Z}}(G) = \chi(G)$ for every graph G. On the other hand, for finite rings M the behaviour of χ_{TT_M}/χ should be similar to the case $M = \mathbb{Z}_2$.*

7.2 Influence of the ring

In this section we study how the notion of XY_M mapping (and also of M-homotens graph) depends on the ring M. Although the existence of XY_M mappings seems to be strongly dependent on the choice of M, we prove here (in Theorem 7.2.5) that this dependence relates only to the cyclical structure of M.

Throughout this section, G, H will be directed graphs, $f : E(G) \to E(H)$ a mapping, and M, N finitely generated rings. We start, however, by explanation why we can restrict to finitely generated rings instead of using (as usual in study of flows and tensions on graphs) abelian groups.

Most of the time we study finite graphs so we can restrict our attention to finitely generated groups—clearly f is XY_M iff it is XY_N for every finitely generated subgroup of M. Consequently, we can use the classical characterization of finitely generated abelian groups (see, e.g., [54]) given by the next theorem and, in particular, we can define a ring structure on each of the considered groups.

Theorem 7.2.1 *For a finitely generated abelian group M there are integers α, k, β_i, n_i ($i = 1, \dots, k$) so that*

$$M \simeq \mathbb{Z}^\alpha \times \prod_{i=1}^{k} \mathbb{Z}_{n_i}^{\beta_i} . \tag{7.3}$$

For a ring M in the form (7.3), denote $n(M) = \infty$ if $\alpha > 0$, otherwise let $n(M)$ be the least common multiple of $\{n_1, \dots, n_k\}$.

As a first step to complete characterization we consider a specialized question: given an XY_M mapping, when can we conclude that it is XY_N as well?

Lemma 7.2.2 *1. If f is $XY_{\mathbb{Z}}$ then it is XY_M for any M.*

2. Let M be a subring of N. If f is XY_N then it is XY_M.

Proof: 1. This appears (for TT and FF mappings) as Theorem 4.4 in [19], the proof there works for TF and FT, too.

2. Let τ be an M-tension/flow on H. — By this we mean that if $Y = T$ then τ is a tension, if $Y = F$ then τ is a flow. We will use this slightly ambiguous expression throughout this chapter instead of using terms F-mappings and T-mappings (to mean flows and tension). The latter approach (used in Section 1.2) is formally more correct, on the other hand nonstandard and so perhaps confusing. — As $M \leq N$, we may regard τ as an N-tension/flow, hence τf is an

N-tension/flow on G. As it attains only values in the range of τ, hence in M, it is an M-tension/flow, too. □

Lemma 7.2.3 *Let M_1, M_2 be two rings. Mapping f is XY_{M_1} and XY_{M_2} if and only if it is $XY_{M_1 \times M_2}$.*

Proof: As M_1, M_2 are subrings of $M_1 \times M_2$, one implication follows from part 2 of Lemma 7.2.2. For the other implication let τ be an $M_1 \times M_2$-tension/flow on H. Write $\tau = (\tau_1, \tau_2)$, where τ_i is an M_i-tension/flow on H. By assumption, $\tau_i f$ is an M_i-tension/flow on G, consequently $\tau f = (\tau_1 f, \tau_2 f)$ is a tension/flow too. □

The following (somewhat surprising) lemma shows that we can restrict our attention to cyclic rings only.

Lemma 7.2.4 *1. If $n(M) = \infty$ then f is XY_M if and only if it is $XY_{\mathbb{Z}}$.*

2. Otherwise f is XY_M if and only if it is $XY_{n(M)}$.

Proof: By previous lemmata. Note that $\mathbb{Z}_{n(M)}$ is a subrings of $M = \prod_{i=1}^{k} \mathbb{Z}_{n_i}^{\beta_i}$. □

By a theorem of Tutte (see [21]), the number of group-valued nowhere-zero flows on a given graph does depend only on the size of the group (that is, surprisingly, it does not depend on the structure). Before proceeding in the main direction of this section, let us note a consequence of Lemma 7.2.4, which is an analogy of the Tutte's theorem.

Theorem 7.2.5 *Given graphs G, H, the number of XY_M mappings from G to H depends only on $n(M)$.*

Lemma 7.2.4 suggests to define for two graphs the set

$$XY(G, H) = \{n \geq 1 \mid \text{there is } f : E(G) \to E(H) \text{ such that } f \text{ is } XY_n\}$$

and for a particular $f : E(G) \to E(H)$

$$XY(f, G, H) = \{n \geq 1 \mid f \text{ is } XY_n\}.$$

Remark that most of these sets contain 1: $\mathbb{Z}_1$ is a trivial ring, so any mapping is XY_1. Therefore $1 \in XY(f, G, H)$ for every $f : E(G) \to E(H)$, while $1 \in XY(G, H)$ iff there exists a mapping $E(G) \to E(H)$. This happens precisely when $E(H)$ is nonempty or $E(G)$ is empty.

Lemma 7.2.6 *Either $XY(f, G, H)$ is finite or $XY(f, G, H) = \mathbb{N}$. In the latter case f is $XY_{\mathbb{Z}}$.*

Proof: It is enough to prove that f is $XY_{\mathbb{Z}}$ if it is XY_n for infinitely many integers n. To this end, take a $\mathbb{Z}$-tension/flow τ on H. As $\tau_n : e \mapsto \tau(e) \bmod n$ is a $\mathbb{Z}_n$-tension/flow, $\tau_n f = \tau f \bmod n$ is a $\mathbb{Z}_n$-tension/flow whenever f is XY_n. To show τ is a $\mathbb{Z}$-tension/flow consider a circuit/cut C and let s be the '$\pm$-sum' (in $\mathbb{Z}$) along/across C. As $s \bmod n = 0$ for infinitely many values of n, we have $s = 0$. $\square$

Any f induced by a homomorphism provides an example where $TT(f, G, H)$ is the whole $\mathbb{N}$. When G, H are planar graphs, using duality (Lemma 1.2.4) provides us with instances of $XY(f, G', H') = \mathbb{N}$ for every type of XY mapping. For finite sets, the situation is more interesting. By the next theorem the sets $XY(f, G, H)$ are precisely ideals in the divisibility lattice.

Theorem 7.2.7 *Let T be a finite subset of $\mathbb{N}$. Then the following are equivalent.*

1. *There are G, H, f such that $T = XY(f, G, H)$.*
2. *There is $n \in \mathbb{N}$ such that T is the set of all divisors of n.*

Proof: First we show that 1 implies 2. The set T has the following properties

(i) If $a \in T$ and $b|a$ then $b \in T$. (We use the second part of Lemma 7.2.2: if b divides a, then $\mathbb{Z}_b \le \mathbb{Z}_a$.)

(ii) If $a, b \in T$ then the least common multiple of a, b is an element of T. (We use Lemma 7.2.2 and Lemma 7.2.3: if $l = \mathrm{lcm}(a, b)$ then $\mathbb{Z}_l \le \mathbb{Z}_a \times \mathbb{Z}_b$.)

Denote n the maximum of T. By (i), all divisors of n are in T. If there is a $k \in T$ that does not divide n then $\mathrm{lcm}(k, n)$ is element of T larger than n, a contradiction.

For the other implication, let f be the only mapping from $\overrightarrow{C}_n$ to $\overrightarrow{K}_2$. Then we have $TT(f, \overrightarrow{C}_n, \overrightarrow{K}_2) = T$: mapping f is TT_k iff for any $a \in \mathbb{Z}_k$ the constant mapping $E(\overrightarrow{C}_n) \mapsto a$ is a $\mathbb{Z}_k$-tension; this occurs precisely when k divides n. For other XY mappings we again consider the corresponding dual graphs. $\square$

Let us turn to description of sets $XY(G, H)$. We stress here that G, H are finite graphs—in contrary with most of other results, this one is not true for infinite graphs.

Lemma 7.2.8 *Let G, H be* finite *graphs. Then either $XY(G,H)$ is finite or $XY(G,H) = \mathbb{N}$. In the latter case $G \xrightarrow{XY_{\mathbb{Z}}} H$.*

Proof: As in the proof of Lemma 7.2.6, the only difficult step is to show that if $G \xrightarrow{XY_n} H$ for infinitely many values of n, then $G \xrightarrow{XY_{\mathbb{Z}}} H$. As G and H are finite, there is only a finite number of possible mappings between their edge sets. Hence, there is one of them, say $f : E(G) \to E(H)$, that is XY_n for infinitely many values of n. By Lemma 7.2.6 we have $f : G \xrightarrow{XY_{\mathbb{Z}}} H$. □

We start the characterization of sets $XY(G,H)$ by observing that the analogue of Lemma 7.2.3 does not hold: there is an TT_M mapping from $\overrightarrow{C}_9$ to $\overrightarrow{C}_7$ for $M = \mathbb{Z}_2$ (mapping induced by a homomorphism of the undirected circuits) and for $M = \mathbb{Z}_3$ (e.g., a constant mapping), but not the same mapping for both, hence there is no $XY_{\mathbb{Z}_2 \times \mathbb{Z}_3}$ mapping. We will see that the sets $XY(G,H)$ are precisely down-sets in the divisibility poset. First, we prove a lemma that will help us to construct pairs of graphs G, H with a given $XY(G,H)$. Integer cone of a set $\{s_1, \ldots, s_t\} \subseteq \mathbb{N}$ is the set $\{\sum_{i=1}^t a_i s_i \mid a_i \in \mathbb{Z}, a_i \geq 0\}$.

Lemma 7.2.9 *Let A, B be non-empty subsets of $\mathbb{N}$, $a \in \mathbb{N}$, define $G = \bigcup_{a \in A} \overrightarrow{C}_a$, and $H = \bigcup_{b \in B} \overrightarrow{C}_b$. Then there is an TT_n mapping from G to H if and only if*

$$A \text{ is a subset of the integer cone of } B \cup \{n\}\,.$$

Proof: We use Lemma 1.2.9. Consider a flow φ_a attaining value 1 on $\overrightarrow{C}_a$ and 0 elsewhere. Algebraical image of this flow is a flow, hence it is (modulo n) a sum of several flows along the cycles $\overrightarrow{C}_b$, implying a is in integer cone of $B \cup \{n\}$. On the other hand if $a = \sum_i b_i + cn$ then we can map any c edges of $\overrightarrow{C}_a$ to one (arbitrary) edge of H, and for each i any ('unused') b_i edges bijectively to $\overrightarrow{C}_{b_i}$. After we have done this for each $a \in A$ we will have constructed an TT_n mapping from G to H. □

Theorem 7.2.10 *Let T be a finite subset of $\mathbb{N}$. Then the following are equivalent.*

1. *There are G, H such that $T = XY(G,H)$.*
2. *There is a finite set $M \subset \mathbb{N}$ such that*

$$T = \{k \in \mathbb{N}; (\exists m \in M) k | m\}\,.$$

Proof: If T is empty, we take M empty. In the other direction, if M is empty we just consider graphs such that $E(H)$ is empty and $E(G)$ is not. Next, we suppose M is nonempty.

By the same reasoning as in the proof of Theorem 7.2.7 we see that if $a \in T$ and $b|a$ then $b \in T$. Hence, 1 implies 2, as we can take $M = T$ (or, to make M smaller, let M consist of the maximal elements of T in the divisibility relation).

For the other implication we again suppose $XY = TT$, for other XY mappings we take duals of the (planar) graphs we will construct. We pick a prime $p > 4 \max M$ and let $p' \in (1.25p, 1.5p)$ be any integer. Let $A = \{p, p'\}$ and

$$B = \{p - m; m \in M\} \cup \{p' - m; m \in M\} ;$$

note that every element of B is larger than $\frac{3}{4}p$. As in Lemma 7.2.9 we define $G = \bigcup_{a \in A} \overrightarrow{C}_a$, $H = \bigcup_{b \in B} \overrightarrow{C}_b$. We claim that $XY(G, H) = T$. By Lemma 7.2.9 it is immediate that $XY(G, H) \supseteq T$. For the other direction take $n \in XY(G, H)$. By Lemma 7.2.9 again, we can express p and p' in form

$$\sum_{i=1}^{t} b_i + cn \tag{7.4}$$

for integers $c, t \geq 0$, and $b_i \in B$.

- If $t \geq 2$ then the sum in (7.4) is at least $1.5p$; hence neither p nor p' can be expressed with $t \geq 2$.

- If $t = 1$ then we distinguish two cases.

  - $p = (p - m) + cn$, hence n divides m and $n \in T$.
  - $p = (p' - m) + cn$, hence $p' - p \leq m$. But $p' - p > 0.25p > m$, a contradiction.

  Considering p' we find that either $n \in T$ or $p' = (p - m) + cn$.

- Finally, consider $t = 0$. If $p = cn$ then either $n = 1 \in t$ or $n = p$. (We don't claim anything about p'.)

To summarize, if $n \in XY(G, H) \setminus T$ then necessarily $n = p$. For p' we have only two possible expressions, $p' = cn$ and $p' = (p-m)+cn$. We easily check that both of them lead to a contradiction. The first one contradicts $1.25p < p' < 1.5p$. In the second expression $c = 0$ implies $p' < p$ while $c \geq 1$ implies $p' \geq 2p-m \geq 1.75p$, again a contradiction. □

In contrast with the results above, if we put extra conditions on the mapping then the influence of the ring is supressed. We will find the next result useful in Section 7.4.

Lemma 7.2.11 *Let G, H be directed graphs and M a ring that is not a power of $\mathbb{Z}_2$. Suppose $f : E(G) \to E(H)$ is a bijective mapping that is TT_M and FF_M. Then f is $TT_{\mathbb{Z}}$ and $FF_{\mathbb{Z}}$.*

Proof: First, we observe that f maps blocks of G to blocks of H. This is immediate if the block is a bridge. It remains to prove that if G' is a 2-connected subgraph of G then $f(E(G'))$ induces a 2-connected subgraph of H, i.e. that any two of its edges are contained in a common circuit. Pick $e_1, e_2 \in E(G')$, consider a circuit C that contains both of them and the M-flow φ determined by C. Mapping f is TT_M and so φ_f is an M-flow. Consequently there is no bridge in $f(C)$. If $f(C)$ is 2-connected then we are done. Otherwise, let C' be a 2-connected component and observe that restriction of φ_f to C' is an M-flow. Now $\varphi_f \circ f$ is the restriction of φ to $f^{-1}(C')$, hence it is not a flow, a contradiction.

We proved that the image of a block of G is a part of a block of H. By using f^{-1} we prove the converse, so we may consider restriction of f on these blocks separately and suppose that G, H are 2-connected for the rest of the proof. (If both G and H are a single edge then the statement is immediate.)

As M is not a power of $\mathbb{Z}_2$, there is a $k \ge 3$ such that $M \ge \mathbb{Z}_k$, hence f is TT_k and FF_k. To prove f is $FF_{\mathbb{Z}}$, we proceed by Lemma 1.2.9. Take $v \in V(G)$ and let τ be elementary $\mathbb{Z}$-tension determined by v. We may suppose that all edges are oriented out of v. Since τ is a $\mathbb{Z}_k$-tension, τ_f is a $\mathbb{Z}_k$-tension, as well. Let $p : V(H) \to \mathbb{Z}_k$ be such that $\tau_f = \delta p$. We let $E_i = \{xy \in E(H) \mid p(x) = p(y) - 1 = i\}$ (for $i \in \mathbb{Z}_k$) and $Z = \{xy \in E(H) \mid p(x) = p(y)\}$. It follows that each edge adjacent to v is mapped to some E_i and the other edges are mapped to Z. Now consider edges e_1, e_2 adjacent to v and a circuit C containing both of them. Let φ be the elementary $\mathbb{Z}_k$-flow determined by C. As f is TT_k, the image φ_f is a $\mathbb{Z}_k$-flow and since $k \ge 3$ this implies that both e_1 and e_2 map to the same set E_i, suppose to E_0. As this applies for all edges adjacent to v, it follows that the potential p is such that $E_0 \cup Z$ covers all edges of H. Consequently, if we let $\tau' = \delta_{\mathbb{Z}} p$ be the difference of p in $\mathbb{Z}$ (we are not counting modulo k), then in fact $\tau' = \tau_f$, hence τ_f is a $\mathbb{Z}$-tension, as required. To prove that f is $TT_{\mathbb{Z}}$ we consider the inverse mapping and show (as above) that it is $FF_{\mathbb{Z}}$. $\square$

We finish this section by a result that relates the property of being M-homotens for different rings M.

Theorem 7.2.12 *Let G be a finite graph.*

1. *G is left $\mathbb{Z}$-homotens if and only if it is left $\mathbb{Z}_n$-homotens for some n.*
2. *If G is right $\mathbb{Z}_n$-homotens for some n then it is right $\mathbb{Z}$-homotens.*

Proof: By Lemma 7.2.2 any $TT_{\mathbb{Z}}$ mapping is TT_n for each n; this proves part 2 and one implication of part 1. To prove the other, let G be left $TT_{\mathbb{Z}}$-homotens graph and suppose, for the sake of contradiction, that G is not TT_n-homotens for any n. By Proposition 2.1.6 this means that for a fixed finite graph H and for every n there are TT_n mappings $g_n : G \xrightarrow{TT_n} H$ that are not induced. As G, H are finite, there is a mapping $g : E(G) \to E(H)$ that is equal to g_n (and so is TT_n) for infinitely many values of n; hence g is $TT_{\mathbb{Z}}$ by Lemma 7.2.6. This implies g is induced, a contradiction. □

7.3 TT-perfect graphs

For every graph G, its chromatic number $\chi(G)$ is at least as big as the size of its largest clique, $\omega(G)$. Recall, that a graph G is called *perfect* if $\chi(G') = \omega(G')$ holds for every induced subgraph G' of G. A graph is called *Berge* if for no odd $l \geq 5$ does G contain C_l or $\overline{C_l}$ as an induced subgraph. It is easy to see that being perfect implies being Berge; the so-called Strong perfect graph conjecture (due to Claude Berge) claims that the opposite is true, too. Perfect graphs have been a topic of intensive research that recently lead to a proof [17] of the Strong perfect graph conjecture.

As a humble parallel to this development we define a graph G to be TT-*perfect*[1] if for every induced subgraph G' of G we have $\chi_{TT_2}(G') \leq \omega(G')$ (definition of $\chi_{TT_2}(G')$ appears before Corollary 2.3.9). Equivalently, G is TT-perfect if each of its induced subgraphs G' admits a TT_2 mapping to its maximal clique.

Note that we cannot require $\chi_{TT}(G') = \omega(G')$ since $K_4 \xrightarrow{TT} K_3$, and therefore $\chi_{TT}(K_4) = 3$, while $\omega_{TT}(K_4) = 4$.

As any homomorphism induces a TT mapping (see Lemma 2.1.2), $\chi_{TT}(G') \leq \chi(G')$ holds for every graph G'. Consequently, every perfect graph is TT perfect. The converse, however, is false. For example, let $G = \overline{C_7}$. Graph G itself is not perfect. On the other hand $\chi_{TT}(G) = 3$ and every induced subgraph of G is Berge, hence perfect, hence TT-perfect. Let us study TT-perfect graphs in a

[1] more precisely, TT_2-perfect, but we consider only TT_2 mappings in this section

similar manner as Strong perfect graph theorem does for perfect graphs. To this end, we define a graph G to be *critical* if G is not TT-perfect, but each induced subgraph of G is. We start our approach by a technical lemma.

Lemma 7.3.1 *Let $l \geq 3$ be odd. Cycle C_l is not TT-perfect. Graph $\overline{C}_l$ is TT-perfect if and only if $l = 7$.*

Proof: Clearly $\chi_{TT}(C_l) = 3 > \omega(C_l)$. Graph $\overline{C}_7$ was discussed above, $\overline{C}_5$ is isomorphic to C_5. As $\chi(\overline{C}_9) = 5$ and as K_4 is right $\mathbb{Z}_2$-homotens, being a $\mathbb{Z}_2$-graph, we have $\chi_{TT}(\overline{C}_9) = 5 > \omega(\overline{C}_9)$. It is easy to verify that graphs $\overline{C}_l$ for $l \geq 13$ are nice. Thus they are homotens and not TT-perfect, since they are not perfect. The only remaining case is the graph $\overline{C}_{11}$. This is not nice, on the other hand, every edge is contained it a K_5 and all K_5's are 'connected'—there is a chain of all 11 copies of K_5 such that neighboring copies intersect in a K_4. It follows that $\overline{C}_{11}$ is homotens, in particular $\overline{C}_{11} \stackrel{TT}{\not\longrightarrow} K_5$. □

Corollary 7.3.2 *For every odd $l > 3$ graph C_l is critical; if $l \neq 7$ then $\overline{C}_l$ is critical, too. Moreover graphs G_1, G_2, and G_3 in Figure 7.1 are critical.*

Proof: We sketch the proof of G_1 being critical. We have $\chi(G_1) = 1 + \chi(\overline{C}_7) = 5$, therefore Corollary 2.3.9 implies $\chi_{TT}(G_1) = 5 > \omega(G_1)$ and G_1 is not TT-perfect. Let G' be an induced subgraph of G_1. If $G' = \overline{C}_7$ then G' is TT-perfect; otherwise, it is a routine to verify that G' is Berge, consequently perfect and TT-perfect. □

We do not know how many other critical graphs there are, not even if there is an infinite number of them.

7.4 TT and FF mappings as invariants

In this section we will study how can the numbers of TT (or FF) mappings serve as a *system of invariants*—that is, to what extent do these numbers determine the graph. (We will consider any finitely generated ring M throughout this section and omit the subscript in XY_M.) Our guidepost will be the following theorem of Lovász [57].

Theorem 7.4.1 *Let G, H be directed graphs such that*

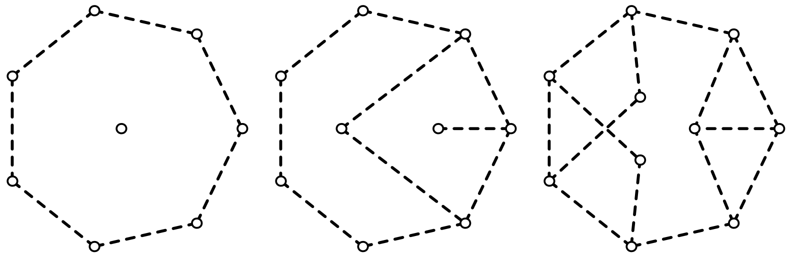

Figure 7.1: Several critical graphs that are not cycles neither complements of cycles. The dashed lines denote precisely the **non-edges** of the graph.

1. *for every graph F the number of homomorphisms from F to G and to H equal;* **or**

2. *for every graph F the number of homomorphisms to F from G and from H equal.*

Then G and H are isomorphic.

Lovász did use his theorem to find 'cancelation properties' of graphs: he proved (among else) that the graph G is determined by $C \times G$ whenever C contains a loop; so we may, in a sense, divide by such graph C. Our results have to wait for such spectacular application (remember that in Proposition 3.1.12 we proved that the category $\mathcal{G}ra_{TT}$ does not have products). Still, it provides an interesting comparison of homomorphisms and TT mappings.

We start by considering what is the proper measure of 'being the same graph' in our situation, that is which relation should replace isomorphism in the conclusion of Theorem 7.4.1. We write $G \equiv H$ if there is a surjective TT mapping from G to H and from H to G. We let $M(G)$ be the cycle matroid of a graph G. The following lemma lists several important equivalent properties.

Lemma 7.4.2 *The following are equivalent for graphs G, H.*

1. *$G \equiv H$, i.e., there is a surjective TT mapping from G to H and from H to G;*

2. *there is an injective TT mapping from G to H and from H to G;*

3. *there is a bijection $f : E(G) \to E(H)$ such that both f and f^{-1} are TT mappings;*

1'. there is a surjective FF mapping from G to H and from H to G;

2'. there is an injective FF mapping from G to H and from H to G;

3'. there is a bijection $f : E(G) \to E(H)$ such that both f and f^{-1} are FF mappings;

4. $M(G)$ and $M(H)$ are isomorphic; if M is not a power of $\mathbb{Z}_2$ then orientation of edges is preserved by this isomorphism.

Proof: Assume 1, and take surjective mappings $f : G \xrightarrow{TT} H$ and $g : H \xrightarrow{TT} G$. It follows that $|E(G)| = |E(H)|$, hence f and g are in fact bijections. Let $X_G \subseteq M^{E(G)}$ be the set of all M-tensions on G, similarly X_H. By definition of TT mappings, f^{-1} maps elements of X_H to X_G and it is an injection (as f is a surjection), thus $|X_H| \leq |X_G|$. Similarly, g^{-1} gives us $|X_H| \geq |X_G|$. Therefore $|X_H| = |X_G|$ and f^{-1} is a bijection of X_H and X_G, so f^{-1} is TT and 3 is proved. The reverse implication is trivial (we can take $g = f^{-1}$). Equivalence of 2 and 3 is proved in exactly the same way.

The equivalence of 1'–3' follows as for 1–3. By Lemma 1.2.9 a bijective mapping is TT iff its inverse is FF, therefore 3 and 3' are equivalent, too.

Finally, 3 and 4 are equivalent. If $M = \mathbb{Z}_2$ (or $\mathbb{Z}_2^k$) then by Lemma 1.2.9 the condition in 3 is equivalent to f being isomorphism between $M(G)$ and $M(H)$. For other M the condition 4 implies 3 for $TT_{\mathbb{Z}}$ mappings, hence for TT_M as well. For the remaining implication we use Lemma 7.2.11.

If G, H are undirected, we proceed similarly, we present the only nontrivial part: $1 \implies 3$. By Proposition 1.2.2 we have orientations G' of G and H', H'' of H such that there are surjective (thus bijective) mappings $f : G' \xrightarrow{TT_M} H'$ and $g : H'' \xrightarrow{TT_M} G'$. As in the directed case we find that $|X_{H'}| \leq |X_{G'}| \leq |X_{H''}|$. The number of M-tensions depends only on the number of vertices and of components. So $|X_{H'}| = |X_{H''}|$ and we as in the directed case conclude that f^{-1} is TT_M. $\square$

Corollary 7.4.3 *For any ring M the equivalence $\equiv_M$ is either*

- *$\equiv_2$ if M is a power of $\mathbb{Z}_2$ or*
- *$\equiv_{\mathbb{Z}}$ otherwise.*

Moreover, $G \equiv_{\mathbb{Z}} H$ implies $G \equiv_2 H$ and, conversely, $G \equiv_2 H$ implies $\overrightarrow{G} \equiv_{\mathbb{Z}} \overrightarrow{H}$ for some orientation $\overrightarrow{G}$ of G and $\overrightarrow{H}$ of H.

The above lemma allows us to use Whitney's 2-isomorphism theorem [89, 90] to understand the equivalence $\equiv$. Prior to stating this result, we must define three graph operations. A *Whitney twist* consists of decomposing a graph along a 2-vertex cut-set $\{u, v\}$ into parts G_1 and G_2 and then identifying vertex u in G_1 with v in G_2 and vice versa. *Vertex identification* consist of identifying two vertices from distinct components of the graph, *vertex cleaving* is the inverse operation. In particular adding/deleting isolated vertices (except of deleting the only vertex of K_1 and adding vertex to an empty graph) are instances of vertex cleaving/identification. We call two graphs *2-isomorphic* if one can be transformed to the other by a sequence of Whitney twists, vertex identification and vertex cleaving. It is easy to verify that 2-isomorphic graphs have isomorphic matroids, Whitney's result claims that the converse is true, too.

Theorem 7.4.4 *Let G and H be undirected graphs such that $M(G)$ and $M(H)$ are isomorphic. Then G and H are 2-isomorphic. Explicitly:*

- *if G is 3-connected then G and H are isomorphic.*
- *if G is 2-connected then it is possible to transform G to H by a sequence of Whitney twists.*
- *it is possible to transform G to H by a sequence of Whitney twists, vertex identification and vertex cleaving.*

For directed graphs we define oriented Whitney twist as Whitney twist above, with the addition that we change orientation of each edge of G_1. The following corollary of Whitney's result is easy to prove. In fact Thomassen [84] gives a much stronger version: he proves the same conclusion for mappings that are only known to preserve directed cycles (for strongly connected graphs).

Corollary 7.4.5 *Let G and H be directed graphs such that $M(G)$ and $M(H)$ are isomorphic and this isomorphism preserves orientation of the edges. Then*

- *if G is 3-connected then G and H are either isomorphic or 'antiisomorphic': that is there is a bijection $f : V(G) \to V(H)$ such that both f and f^{-1} are a homomorphism or an antihomomorphism.*
- *if G is 2-connected then it is possible to transform G to H by a sequence of oriented Whitney twists.*
- *it is possible to transform G to H by a sequence of oriented Whitney twists, vertex identification and vertex cleaving.*

We now digress from the main course of this section to investigate a parallel of the notion of core, which was introduced to the theory of homomorphisms independently by several researchers (see [40]). Let G, H be graphs. We say that H is

- a TT_M*-core* iff $\quad (\forall e \in E(H)) \quad H \not\xrightarrow{TT_M} H \setminus e$, and
- a FF_M*-core* iff $\quad (\forall e \in E(H)) \quad H \not\xrightarrow{FF_M} H/e$.

Further, we say H is *the* XX_M*-core of* G if it is an XX_M-core and $H \subseteq G$ (if $X = T$) or $H \overset{c}{\subseteq} G$ (if $X = F$). In the next proposition we prove that the XX_M-core is uniquely determined (up to $\equiv_M$). Perhaps surprisingly, this equivalence $\equiv_M$ is the same, regardless if we speak of TT_M- or of FF_M-cores.

Proposition 7.4.6 *Let X be T or F, let G_1, G_2 be XX_M-equivalent graphs and let H_i be the XX_M-core of G_i. Then $H_1 \equiv_M H_2$.*

In particular, any two XX_M-cores of a given graph are equivalent with respect to $\equiv_M$.

Proof: By definition, H_i and G_i are XX_M-equivalent, so there are mappings $f_1 : H_1 \xrightarrow{XX_M} H_2$ and $f_2 : H_2 \xrightarrow{XX_M} H_1$. As $f_2 \circ f_1$ is XX_M and H_1 is XX_M-core, mapping f_2 is surjective. Similarly, f_1 is surjective, so it remains to use Lemma 7.4.2. □

Now we are in position to explore the use of number of XX mappings as invariants. We let $\langle G, H\rangle_{TT}$ denote the number of TT mappings from G to H and $\langle G, H\rangle\!\rangle_{TT}$ the number of surjective such mappings. Further, $\langle\!\langle G, H\rangle_{TT}$ denotes the number of injective TT mappings that are an 'embedding': there is a subgraph $H' \subseteq H$ such that the considered mapping is a bijection $G \xrightarrow{TT} H'$ and its inverse is TT as well. (In category theory terms, we count the number of so-called extremal monomorphisms.) In the same way we define $\langle G, H\rangle_{FF}$, $\langle G, H\rangle\!\rangle_{FF}$ and $\langle\!\langle G, H\rangle_{FF}$ (the last one counts equivalences to a contraction of H) for the number of FF mappings. Let $\mathrm{Aut}_{TT}(G)$ be the set of all TT permutations on $E(G)$ (it is easy to verify that $\mathrm{Aut}_{TT}(G)$ is in fact a group and that $\mathrm{Aut}_{TT}(G) = \mathrm{Aut}_{FF}(G)$). By the next lemma, the pairs of graphs characterized in Theorem 7.4.4 and 7.4.5 are the limit of what can TT or FF mappings (used as invariants) distinguish. In the sequel X stands for F or T.

Lemma 7.4.7 *Let $G \equiv H$ be graphs. Then for every graph F*

$$\begin{aligned} \langle F, G\rangle_{XX} &= \langle G, F\rangle_{XX}\,, & \langle F, H\rangle_{XX} &= \langle H, F\rangle_{XX} \\ \langle F, G\rangle\!\rangle_{XX} &= \langle\!\langle G, F\rangle_{XX}\,, & \langle F, H\rangle\!\rangle_{XX} &= \langle\!\langle H, F\rangle_{XX} \\ \langle\!\langle F, G\rangle_{XX} &= \langle\!\langle F, H\rangle_{XX}\,, & \langle G, F\rangle\!\rangle_{XX} &= \langle H, F\rangle\!\rangle_{XX}\,. \end{aligned}$$

Proof: This immediately follows from parts 3 and 3' of Lemma 7.4.2, the only difficult case is $\langle\!\langle F, G\rangle_{XX} = \langle\!\langle F, H\rangle_{XX}$. To prove this, it is sufficient to provide a bijection between the subgraphs/contractions G' of G and H' of H such that $G' \equiv H'$ holds for corresponding graphs. So let $f : G \xrightarrow{XX} H$ be a bijection such that f^{-1} is XX as well. As f is a bijection on edges, to any $G' \subseteq G$ ($G' \overset{\mathrm{c}}{\subseteq} G$) corresponds $H' \subseteq H$ ($H' \overset{\mathrm{c}}{\subseteq} H$). Now $G' \xrightarrow{XX} G \xrightarrow{XX} H$ (by Lemma 1.2.5 and by assumptions of the theorem), so $G' \xrightarrow{XX} H$ and consequently $G' \xrightarrow{XX} H'$ (by Lemma 1.2.7). Clearly, this mapping is a bijection. By changing the roles of G and H in this argument, we obtain a bijective XX mapping $H' \xrightarrow{XX} G'$, finishing the proof. □

The rest of this section is devoted to proving the converse to Lemma 7.4.7. We start with the easy cases.

Proposition 7.4.8 *Let G, H be graphs such that*

1. *for every F we have $\langle G, F\rangle\!\rangle_{XX} = \langle H, F\rangle\!\rangle_{XX}$;* **or**
2. *for every F we have $\langle\!\langle G, F\rangle_{XX} = \langle\!\langle H, F\rangle_{XX}$;* **or**
3. *for every F we have $\langle F, G\rangle\!\rangle_{XX} = \langle F, H\rangle\!\rangle_{XX}$;* **or**
4. *for every F we have $\langle\!\langle F, G\rangle_{XX} = \langle\!\langle F, H\rangle_{XX}$.*

Then $G \equiv H$.

Proof: We prove 1, proof of the other cases is almost identical. Put first $F = G$. As identity is an XX mapping, we have $\langle H, G\rangle\!\rangle_{XX} = \langle G, G\rangle\!\rangle_{XX} \geq 1$, hence there is a surjective XX mapping from H to G. By putting $F = H$ we obtain a surjective mapping in the other direction, hence $G \equiv H$ by Lemma 7.4.2. □

Theorem 7.4.9 *Let G, H be graphs such that*

1. *for every F we have $\langle G, F\rangle_{XX} = \langle H, F\rangle_{XX}$;* **or**

2. *for every F we have $\langle F, G\rangle_{XX} = \langle F, H\rangle_{XX}$.*

Then $G \equiv H$.

Proof: We aim to use Proposition 7.4.8.

For part 1 it is easy to verify that the assumption of part 1 of Proposition 7.4.8 is satisfied: By inclusion-exclusion principle we know that

$$\langle G, F\rangle_{XX} = \langle G, F\rangle\!\rangle_{XX} + \sum_{\emptyset \neq T \subseteq E(F)} (-1)^{|T|} \langle G, F \setminus T\rangle_{XX}\,,$$

consequently $\langle G, F\rangle\!\rangle_{XX} = \langle H, F\rangle\!\rangle_{XX}$.

For the second part, we use induction to verify assumptions of part 4 of Proposition 7.4.8. We let $G \twoheadrightarrow H$ mean that there is a surjective XX-mapping from G to H, and extend the order $\twoheadleftarrow$ to a linear order. As $G \twoheadrightarrow H$ implies that $|E(G)| \geq |E(H)|$, there are only finitely many (up to isomorphism, therefore up to $\equiv$) graphs that precede a given graph in $\twoheadleftarrow$. Consequently, we can choose a linear extension that is a well-ordering (of $\equiv$-equivalence classes). Pick one element from each equivalence class (of $\equiv$), to obtain $F_1, F_2, \dots$ So we have a representative of each $\equiv$-equivalence class in such order, that $F_i \twoheadrightarrow F_j$ implies $i > j$. We will use the following 'factorization formula', which may be thought of as a quantitative version of Proposition 1.2.6.

$$\langle F, G\rangle_{XX} = \sum_{j; F \twoheadrightarrow F_j} \frac{\langle F, F_j\rangle\!\rangle_{XX} \langle\!\langle F_j, G\rangle_{XX}}{|\operatorname{Aut}_{XX}(F_j)|} \tag{7.5}$$

Before we prove this equality, we use it to finish the proof. By Lemma 7.4.7 it is enough to verify condition of part 4 of Proposition 7.4.8 if F is one of the F_j's. That is, we prove (by induction on i) that $\langle\!\langle F_i, G\rangle_{XX} = \langle\!\langle F_i, H\rangle_{XX}$. Suppose this is true whenever we replace i by $j < i$, we prove it for i. Consider formula (7.5) for G and for H in place of G. The left-hand sides are the equal (by assumption of the theorem) and all terms on the right-hand sides for $j < i$ are equal (by the induction hypothesis). Therefore, also the remaining term (for $i = j$) is the same for G and for H, hence $\langle\!\langle F_i, G\rangle_{XX} = \langle\!\langle F_i, H\rangle_{XX}$, as claimed.

It remains to prove formula (7.5). Consider a mapping $f : F \xrightarrow{XX} G$. By Proposition 1.2.6 there is a graph G' such that $f = f_2 \circ f_1$, $f_1 : F \xrightarrow{XX} G'$, $f_2 : G' \xrightarrow{XX} G$, f_1 is surjective, and f_2 injective. In fact, we can take G'

as a subgraph/a contraction of G, hence f_2 is identical mapping. Obviously, there is a unique choice of such G', f_1, and f_2 for every f. We choose j such that $G' \equiv F_j$ and group all XX mappings counted by $\langle F, G\rangle_{XX}$ according to G'. By Lemma 7.4.7 we know that $\langle F, G'\rangle\rangle_{XX} = \langle F, F_j\rangle\rangle_{XX}$. By definition of $\langle\langle F_j, G\rangle_{XX}$ we see that this quantity equals $|\operatorname{Aut}_{XX}(F_j)|$ times the number of G' that are a subgraph/a contraction of G and satisfy $G' \equiv F_j$.

By application of all the facts above, we conclude that the j-th term of the sum in (7.5) counts precisely the number of XX mappings from F to G with image $\equiv$-equivalent to F_j. $\square$

Remark 7.4.10 *Relation 7.5 holds more generally for any factorization system of any category, so what we proved can be reformulated: epimorphisms with extremal monomorphisms form a factorization system in category $\mathcal{G}ra_{TT}$. We avoided this more general way of stating (and proving) our result to keep the presentation on the combinatorial side of the border with category theory.*

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