Midsummer Combinatorial Workshop 2005 and DIMACS, DIMATIA, Rényi Workshop 2005

Jan Kára, ed.

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Preface

The Twelfth Prague Midsummer Combinatorial Workshop was held from July 24 to July 29, 2005 in our newly reconstructed building Malostranské náměstí 25. This of course contributed to the comfort of the participants as all the activities (including the lunches) could be taken on the same site. Besides, as it was expressed by several participants, the renovated faculty building surely belongs to the most beautiful math and CS departments in the world! The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with the DIMATIA center. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Ralph McKenzie, Zoltán Füredi, Gyula Katona and Robin Thomas among the participants. The list of participants is included in this booklet.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students. For example five undergraduate students from the USA and three undergraduate students from Charles University, together with their mentors Martin Bálek and Lara Pudwell, took part in the workshop, within a joint DIMATIA-DIMACS program International REU (supported jointly by NSF and Czech Ministry of Education).

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest some of the atmosphere at the workshop from these proceedings, and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

This volume mirrors not only Midsummer Combinatorial Workshop but also following DDR 2005 workshop. DIMACS–DIMATIA–Rényi (shortly DDR) cooperation is a joint project of NSF and national grant agencies of Hungary and Czech Republic in the field of combinatorics, graph theory and applications. It has been very active for several years.

This volume was edited by Jan Kára. Most of the contributions were supplied by the authors in an electronic form. In a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

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and the publication of these series is supported by the newly approved ITI 1M0021620808. DIMATIA was the main organizer.

The year 2006 is an exceptional year: there will be no Midsummer Combinatorial Workshop. This is due to the fact that from July 10 to July 15 we have the Sixth Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications. Thus we hope to meet again in 2006 and then in 2007, the same midsummer week.

Jaroslav Nešetřil

Midsummer Combinatorial Workshop
July 25 – 29, 2005

The Lifting Model for Reconfiguration

Sergey Bereg
University of Texas, Dallas, USA

Given a pair of start and target configurations, each consisting of n pairwise disjoint disks in the plane, what is the minimum number of moves that suffice for transforming the start configuration into the target configuration? In one move a disk is lifted from the plane and placed back in the plane at another location, without intersecting any other disk. We discuss efficient algorithms for this task and estimate their number of moves under different assumptions on disk radii. We then extend our results for arbitrary disks to systems of pseudodisks, in particular to sets of homothetic copies of a convex object.

The Complexity of Linear Constraint Languages

Manuel Bodirsky
Humboldt University, Berlin, Germany

(Joint work with Jan Kára)

In a constraint satisfaction problem we are given a set of variables and a set of constraints on that variables, and want to find an assignment of values to the variables such that all the constraints are satisfied. We are interested in the computational complexity of the constraint satisfaction problem depending on the constraint language that we are allowed to use in the instances of the constraint satisfaction problem; see e.g. [1] for an introduction to the state-of-the-art of the techniques used to study the computational complexity of constraint satisfaction problems.

Formally, we can define constraint satisfaction problems (CSPs) as *homomorphism problems* for relational structures. Let Γ be a (not necessarily finite) structure with a relational signature τ. Then the constraint satisfaction problem CSP(Γ) is a computational problem, where we are given a *finite* τ-structure S and want to know whether there is a homomorphism from S to Γ. It is easy to see that the class of constraint satisfaction problems equals the class of problems that is closed under so-called *inverse homomorphisms* (if we add constraints to an unsatisfiable instance it stays unsatisfiable) and *disjoint unions* (two satisfiable constraints on distinct variables have a joint solution). We show several examples.

Example 1. Let Γ be the relational structure $(\mathbb{Q}, <)$, where $<$ is a binary relation for the dense linear order of the rational numbers $\mathbb{Q}$. Then CSP(Γ) is the computational problem of digraph acyclicity, which is tractable.

Example 2. Let Γ be the relational structure $(\mathbb{Q}, R)$, where R is the ternary relation $\{(x, y, z) \in \mathbb{Q}^3 \mid x < y < z \vee z < y < z\}$. Here, the problem CSP($\Gamma$) is the NP-complete problem *Betweennness* [3].

Example 3. Let Γ be the relational structure $(\mathbb{Q}; =, \neq)$. Then CSP(Γ) is the computational problem to decide for a given set of equalities and inequalities on a finite set of variables whether the variables can be mapped to the natural numbers such that variables with a constraint $x = y$ are mapped to the same, and variables with a constraint $x \neq y$ are mapped to distinct values.

Clearly, this problem can be solved by an algorithm in polynomial time. The algorithm iteratively identifies variables with an equality constraint. If it has to identify two variables with an inequality constraint, it outputs that the constraint has no solution. Otherwise, we know that we can finally map all the remaining variables to distinct values and satisfy all the constraints.

Example 4. Let Γ be the relational structure $(\mathbb{Q}; \neq, Q)$, where Q is the relation $Q := \{(x, y, z) \in \mathbb{Q}^3 \mid x = y \vee y = z\}$. Here the problem CSP($\Gamma$) turns out to be NP-complete.

In general, we consider in the following the subclass of constraint satisfaction problems for templates of the form $\Gamma = (\mathbb{Q}; R_1, \ldots, R_k)$ where $\mathbb{Q}$ denotes the rational numbers and each relation R_i, $1 \leq i \leq k$, is a Boolean combination of atoms of the form $x < y$. (A Boolean combination is an formula built from atomic formulas with the usual logical connectives of conjunction, disjunction, and negation.) We say that such a relational structure defines a *linear constraint language.* If all the relations are Boolean combination of atoms of the form $x = y$, we say that it defines an *equality constraint languages.* Note that all the four examples shown above are linear equality languages. Moreover, Example 3 and 4 are also equality constraint languages. Our main result is the following.

Theorem 1 *An equality constraint language with template* Γ *is tractable if* Γ *has a constant endomorphism or an injective homomorphism from* Γ^2 *to* Γ*. Otherwise CSP*(Γ) *is NP-complete.*

In Theorem 1, the containment in NP is easy to see: a nondeterministic algorithm can guess which variables in an instance S denote the same element in Γ, and can verify whether this gives rise to a solution for S. Both the hardness result and the algorithmic tractability result in Theorem 1 are nontrivial. The hardness proof relies on the algebraic approach to constraint satisfaction, which was previously mainly applied to constraint satisfaction with finite templates (see e.g. [1]). However, a general result from [2] states that the algebraic approach can also be applied to linear constraint languages. In particular, the computational complexity of linear constraint languages with a template Γ is determined by the homomorphisms from powers of Γ to Γ. The main open problem is the following:

Conjecture 1 *Every linear constraint language is either tractable or NP-complete.*

The algorithm for the problems with an injective binary polymorphism is of a new type, as compared to the known algorithms that are used to solve tractable constraint satisfaction problems with finite templates. Note that there are examples that show that the 'simple' algorithm which was described in the beginning and solves the problem in Example 1 does not work in general for problems that are closed under an injective binary polymorphism.

References

[1] A. Bulatov and A. Krokhin and P. G. Jeavons: Classifying the complexity of constraints using finite algebras. In: SIAM Journal on Computing (34), 720-742, 2005.

[2] Manuel Bodirsky and Jaroslav Nešetřil: Constraint Satisfaction with Countable Homogeneous Templates. In: Proceedings of CSL'03, Vienna, 44-57, 2003.

[3] Garey and Johnson: *A Guide to NP-completeness*. CSLI Press, Stanford, 1978.

The Infinite Locally Random Graph[1]

Anthony Bonato
Wilfrid Laurier University, Waterloo, Canada

1 Introduction

A novel feature of countably infinite graphs is that they are limits of finite graphs. Hence, properties of the finite graphs in the limit may influence the resulting infinite graph. A well known instance of this is the *infinite random graph*, written R. The graph R may be defined as a limit as follows. Let R_0 be a fixed finite graph. For some $t \geq 0$, assume that R_t is a finite graph containing R_0. For each subset S of $V(R_t)$, add a new node x_S joined only to the nodes of S. The graph R_t along with the new nodes x_S defines the graph R_{t+1}. Let R be the graph with vertices $\bigcup_{t\in\mathbb{N}} V(R_t)$ and edges $\bigcup_{t\in\mathbb{N}} E(R_t)$. We will write $R = \lim_{t\to\infty} R_t$. See the surveys [4, 5] for other presentations of R.

The graph R satisfies the *existentially closed* or *e.c.* adjacency property. A graph is e.c. if for all disjoint finite sets of nodes A, B, there is a node $z \notin A \cup B$ joined to each node of A and to no node of B. By a straightforward back-and-forth argument, a countably infinite graph is e.c. if and only if it is isomorphic to R. The e.c. property therefore supplies a powerful tool for studying R. For example, using the e.c. property one may easily derive that R is *inexhaustible*: for each node x of R, $R - x \cong R$.

The graph R arises naturally via the following infinite random process which inspires its name. We add new nodes over countably many discrete time-steps. Fix $p \in (0, 1)$. At time $t = 0$ start with any fixed finite graph. At time step $t+1$, add in a new node x_{t+1}. For each of the existing nodes y, add the edge yx_{t+1} independently with probability p. Erdős and Rényi proved in [9] that with probability 1, a limit generated by this random process is isomorphic to R. This instance of a random process with a seemingly deterministic conclusion has made R the centre of much research activity.

[1]The author gratefully acknowledges the support from an NSERC Discovery grant and from a MITACS grant.

In real-world, self-organizing networks like the web graph (where nodes correspond to web pages, and edges represent links between pages), each node acts as an independent agent, which will base its decision on how to link to the existing network on local knowledge. As a result, the neighbourhood of a new node will often be an imperfect copy of the neighbourhood of an existing node. Both the *copying models* [1, 10] of the web graph, and the *duplication model* [7] of biological networks incorporate this notion of copying in their design.

Through the study of self-organizing networks, new interesting limit graphs have been recently discovered; see [2]. For a node y of a graph G, define $N(y) = \{x \in V(G) : xy \in E(G)\}$ and $N[y] = N(y) \cup \{y\}$. Fix a finite graph H and let X_0 be isomorphic to H. For some $t \geq 0$, assume that X_t is a finite graph containing X_0. For each node y of X_t, and each subset S of $N(y)$ in $V(R_t)$, add a new node $x_{y,S}$ joined only to the nodes of S. The graph X_t along with all the nodes $x_{y,S}$ defines X_{t+1}. Let $R_H = \lim_{t\to\infty} X_t$. We define $\uparrow H$ similarly, but replacing $N(y)$ by $N[y]$. Hence, $\uparrow H$ is formed by extending all the subsets of closed neighbour sets.

The graphs R_H and $\uparrow H$ were first studied in [2], which includes the following result. A graph G is *inexhaustible* if for all nodes x of G, $G-x \cong G$.

Theorem 1 1. *For all finite graphs H and J, $\uparrow H \cong \uparrow J$.*

2. *For all finite graphs H, R_H and $\uparrow H$ are inexhaustible.*

3. *For all finite graphs H, R_H and $\uparrow H$ have one- and two-way Hamilton paths.*

By Theorem 1 (1), the initial graph H has no impact on the limit graph $\uparrow H$. Hence, we name this unique isomorphism type the *infinite locally random graph*, written R_N. In contrast, there are exactly $\aleph_0$ many isomorphism types of graphs R_H.

The graph R_N has other striking properties. For example, for each node x of R_N, the subgraph induced by $N(x)$ is isomorphic to R. This justifies the name of the graph. As R is an induced subgraph of R_N, the graph R_N is $\aleph_0$-*universal*; that is, it embeds all countably infinite graphs as induced subgraphs. Further, R_N consists of infinitely many disjoint, connected, pairwise isomorphic graphs whose isomorphism type we name $c(R_N)$. The graph $c(R_N)$ arises naturally by extending only nonempty sets S in the construction of R_N given above.

The graphs R_H were studied in [2, 3]. The goal of the present article is to summarize some new results on the graph R_N and pose some problems surrounding it. Proofs and additional details may be found in the forthcoming journal version of this extended abstract.

2 Isomorphic representations

Suppose that H, J are finite graphs such that H is an induced subgraph of J, and let $v \in V(J)$. Define $H \preccurlyeq_v J$ if there is a node u in $J - v$ such that $N(v) \subseteq N[u]$, and $H = J - v$. We write $H \preccurlyeq J$ if there is a nonnegative integer m, graphs $H_0 = H$, $H_1, \ldots, H_m = J$, and nodes $v_0, \ldots, v_{m-1} \in V(J)$ so that $H_i \preccurlyeq_{v_i} H_{i+1}$ for all $0 \leq i \leq m-1$. We say that the graph J *strongly folds onto* H. For example, $K_1 \preccurlyeq K_n$ for all $n \in \mathbb{N}$, but $K_1 \not\preccurlyeq C_5$. This ordering has been studied in the context of domination elimination orderings of graphs; see [6, 8].

We extend folding to countable graphs as follows. Let H and J be countable graphs. The relation $H \preccurlyeq_v J$ is defined exactly as in the finite case. Fix I as either $\mathbb{N}$ or one of the sets $\{0, 1, \ldots n\}$, where $n \in \mathbb{N}$. We write $H \preccurlyeq J$ if there exists a sequence of countable graphs $(H_t : t \in I)$ so that $H_0 = H$, $H_t \preccurlyeq_v H_{t+1}$ for all $t \in I$, and $J = \lim_{t \to \infty} H_t$ if $I = \mathbb{N}$, or $J = H_n$ if I is of the form $\{0, 1, \ldots n\}$. For example, $K_1 \preccurlyeq K_{\aleph_0}, \overline{K_{\aleph_0}}$. Note that for all $t > 0$, $H \preccurlyeq X_t$, and $X_t \preccurlyeq X_{t+1}$. Hence, $H \preccurlyeq R_N$, and so R_N folds to every graph by Theorem 1 (1).

A graph G is *locally closed e.c.* if for each node y of G, for each finite $X \subseteq N[y]$, and each finite $Y \subseteq V(G) \setminus X$, there exists a node z not in $\{y\} \cup X \cup Y$ that is joined to X and not to Y. The locally e.c. property is therefore a variant of the e.c. property that applies only to sets contained in the closed neighbour set of a node. The graphs R_N and R satisfy the locally closed e.c. property. In contrast to the e.c. property, it was proved in [2] that there are $2^{\aleph_0}$ many non-isomorphic locally closed e.c. countable graphs (note that in [2], the locally closed e.c. property is referred to as property (A)).

The following theorem ties together the relation $\preccurlyeq$, the graph $c(R_N)$, and the locally closed e.c. property.

Theorem 2 *The graph $c(R_N)$ is the unique isomorphism type of connected countable graph with the property that it is locally closed e.c. and strongly folds to a finite graph.*

For the *closed duplication model* (based on the duplication model of [7]), we add new nodes over a countable number of discrete time-steps. Fix $p \in (0,1)$. At time $t = 0$ start with any fixed finite graph. At time step $t+1$, choose a node u uniformly at random from the nodes of time t. Add a new node x_{t+1}, and for each of the nodes y in $N[u]$, add the edge yx_{t+1} independently with probability p.

Theorem 3 *With probability* 1, *a nontrivial connected component of a limit generated by the closed duplication model is isomorphic to* $c(R_N)$.

If we take H in $\uparrow H$ to be countable, then the graph $\uparrow H$ remains countable and locally closed e.c. A graph H is *finitely approximated* if H folds to some finite graph. For example, the graphs $\overline{K_{\aleph_0}}$ and $K_{\aleph_0}$ are finitely approximated. By definition, R_N and each R_H are finitely approximated. Finitely approximated graphs always generate R_N.

Theorem 4 *If H is a finitely approximated countable graph, then* $\uparrow H \cong R_N$.

Not all countable graphs, however, are finitely approximated. For example, consider the graph H formed from the infinite one-way path, where each node of the path is attached to a distinct C_5. The graph H is not finitely approximated, but we do not know if $\uparrow H \cong R_N$.

3 Isometric subgraphs and indestructibility

The graph R_N displays rich metric properties, unlike R which is of diameter 2. A graph G *isometrically embeds* in H if there is an embedding of G into H that preserves distances. A graph is *isometric* $\aleph_0$-*universal* if it isometrically embeds all countable graphs. A graph G is *isometrically constructible* if G is a limit of sequence $(G_t : t \in \mathbb{N})$ with the property that G_t isometrically embeds in G_{t+1}, for all $t \in \mathbb{N}$.

Theorem 5 *The graph R_N is isometrically constructible and isometrically* $\aleph_0$-*universal.*

Theorem 5 gives an alternate proof of an early result of Pach [11].

Corollary 1 *Every countable graph isometrically embeds in a countable isometrically constructible graph.*

We know from results of [2] that R_N is inexhaustible. Inexhaustibility is only one type of fractal property a graph may satisfy. For example, the infinite random graph R satisfies the *pigeonhole property*: if S is a set of nodes so that the subgraph induced by S is not isomorphic to R, then $R - S \cong R$. Hence, inexhaustibility is the case where S is finite. The pigeonhole property is sometimes called a *fractal* or *vertex partition property* of a graph.

For R_N, the picture is more complex. For example, for any node x the subgraph induced by $N(x)$ is isomorphic to R, but $R_N - N(x)$ contains an isolated node and so is not isomorphic to R_N. Despite this example, certain subsets S may be deleted leaving an isomorphic copy of R_N.

Theorem 6 *If S is a clique in R_N, then $R_N - S \cong R_N$.*

It is not known exactly which sets of nodes S satisfy $R_N - S \cong R_N$. We leave this as an open problem. It would be interesting to know the answer in the case where S is an infinite independent set.

Some things are known about algebraic properties of R_N. For example, R_N is vertex- and edge-transitive, and the automorphism group and endomorphism monoid of R_N are an $\aleph_0$-universal group and monoid, respectively. Unlike the situation for R, the automorphism group of R_N is not oligomorphic. Further details will be included in the journal version of this extended abstract.

References

[1] A. Adler, M. Mitzenmacher, Towards compressing web graphs, In: *Proceedings of the Data Compression Conference*, 2001.

[2] A. Bonato, J. Janssen, Infinite limits of copying models of the web graph, *Internet Mathematics* **1** (2004) 193-213.

[3] A. Bonato, J. Janssen, Infinite limits of self-organizing networks, submitted to *Random Structures and Algorithms.*

[4] P.J. Cameron, The random graph, In: *Algorithms and Combinatorics* **14** (R.L. Graham and J. Nešetřil, eds.), Springer Verlag, New York (1997) 333-351.

[5] P.J. Cameron, The random graph revisited, In: *European Congress of Mathematics* Vol. I (C. Casacuberta, R. M. Miró-Roig, J. Verdera and S. Xambó-Descamps, eds.), Birkhauser, Basel (2001) 267-274.

[6] V. Chepoi, On distance-preserving and domination elimination orderings, *SIAM Journal on Discrete Mathematics* **11** (1998) 414–436.

[7] F.R.K. Chung, G. Dewey, D.J. Galas, Duplication models for biological networks, *Journal of Computational Biology*, **10** (2003) 677-688.

[8] E. Dahlhaus, P. Hammer, F. Maffray, S. Olariu, On domination elimination orderings and domination graphs, In: *Proceedings of the 20th International Workshop on Graph-Theoretic Concepts in Computer Science*, 1994.

[9] P. Erdős, A. Rényi, Asymmetric graphs, *Acta Mathematica Academiae Scientiarum Hungaricae* **14** (1963) 295-315.

[10] R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins, E. Upfal, Stochastic models for the web graph, In: *Proceedings of the 41th IEEE Symposium on Foundations of Computer Science*, 2000.

[11] J. Pach, On metrical properties of countable graphs, *Matematikai Lapok* **26** (1975) 305–310.

Covering point sets with two disjoint disks

Sergio Cabello
Institute for Mathematics, Physics and Mechanics, Ljubljana, Slovenia

(Joint work with J. Miguel Díaz-Báñez, Carlos Seara, J. Antoni Sellarès, Jorge Urrutia and Inma Ventura)

Consider a point set R with at most n *red* points, a point set B with at most n *blue* points, and let C_R and C_B denote a red and a blue unit disk, respectively. We consider the following geometric optimization problem: place C_R and C_B on the plane such that the number of red points covered by C_R plus the number of blue points covered by C_B is maximized. We allow C_B and C_R to cover some red (resp. blue) points, but require them to have disjoint interiors. The requirement for disjoint interiors is relevant, for example, in facility location problems where the facilities may interfere negatively or when their influence area is not allowed to overlap.

We provide a solution to this problem that needs $O(n^{8/3} \log^2 n)$ time. The key ingredient is the following result, which can be seen as a generalization of the Szemerédi-Trotter Theorem for incidences between points and lines in the place.

Theorem 1 (Katz and Sharir, 1997) *Let M be a set of k congruent annuli and let P be a set of k points, both in the plane. One can compute the incident annuli-point pairs $\{(A,p) \mid A \in M, p \in P, p \in A\}$ as a collection of $\{M_s \times P_s\}_{s \in S}$ of complete edge-disjoint bipartite graphs in $O(k^{4/3} \log k)$ time and space. Moreover, it holds that $\sum_s |M_s|$, $\sum_s |P_s| = O(k^{4/3} \log k)$.*

Finally, observe that if we do not require disjoint interiors, then the problem can be solved for each one of the colors independently. We may consider also consider the monochromatic variant of the problem, where we only have one point set and we seek placing two unit disks with disjoint interiors that maximize the number of covered points. Observe that our problem is more general than the monochromatic version: given a set of points for the monochromatic problem, we replace each point by a red and a blue point, and find the solution to the bichromatic problem. However, in the monochromatic version, if we do not require disjoint interiors, then we have a substantially different problem, whose main difficulty is to avoid the double counting in the intersection between the disks.

A Simple Proof for Open Cups and Caps

Jakub Černý
Charles University, Prague, Czech Republic

All sets of points through out this abstract will be in general position in the plane. By *general position* we mean that no three points lie on a line and no two points have the same x-coordinate. Let X be a set of n points and denote its points by p_1, p_2, $\dots p_n$ according to the increasing x-coordinate. Let $Y \subseteq X$ be a set of points q_1, q_2, $\dots q_k$ again ordered by the x-coordinate. For $i = 1{,}2, \dots k-1$, let s_i be the slope of the line $q_i q_{i+1}$. The set $Y = \{q_1, \dots, q_k\}$ is a *k-cup* or a *k-cap* if the sequence s_1, s_2, $\dots$, s_k is increasing or decreasing, respectively (see Figure 1). In other word if the points lie on the graph of a convex, resp. concave function. The set Y is *open* in X if there is no point $p \in X$ with $x(q_1) < x(p) < x(q_k)$ lying above the polygonal line $p_1 p_2 \dots p_k$.

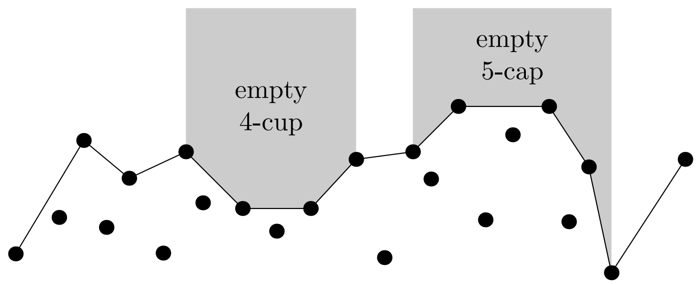

Figure 1: The set of points on the polygonal line is open. There is also the empty 4-cup and the empty 5-cap in the figure.

Erdős-Szekeres theorem [1] says that for every positive integer k there exists positive integer N such that any N-point set contains k points that are vertices of a convex polygon. There are several proofs of the theorem using Ramsey theory and a proof using cups and caps. The latter proof gives much better upper bound on N.

Define $f(k, l)$ to be the smallest positive integer for which X contains a k-cup or an l-cap whenever X has at least $f(k, l)$ poinst. Erdős and Szekeres

[1] proved that $f(k,l) = \binom{k+l-4}{k-2} + 1$.

Erdős also asked if for every k there exists N such that any N-point set X contain k vertices of an empty convex polygon. Empty polygon is a polygon with no point of X in its interior. We say that $Y \subseteq X$ is a k-hole if Y lies in the vertices of an empty convex k-gon. His conjecture holds up to $k = 5$. In 1983 Horton [3] showed that it is not true for all $k \geq 7$. The question for $k = 6$ was open for a long time. Using a computer Overmars [7] found a configuration of 29 points without empty hexagon and very recently Gerken [2] showed that the conjecture holds also for $k = 6$.

What is the sufficient condition for existence of k-hole? The set X is l-convex if and only if every triangle determined by points of X contains at most l points of X in its interior. The l-convex sets were introduced by Valtr [8] and he also showed the following theorem:

Theorem 1 (Valtr) *For every positive integers k and l there exists positive integer N such that any l-convex N-point set X contains a k-hole.*

Denote by $n(k,l)$ the smallest positive integer N such that any l-convex N-point set contains a k-hole. Károlyi, Pach and Toth [4](2001) proved this theorem for $l = 1$. Later Karolyi, Valtr [5] determined the exact value of $n(k,1)$. The first proof for general l was given by Valtr [8]. He was followed by Kun and Lippner [6](2002) who improved the bound to $n(k,l) \leq (l+2)^{(l+2)^k-1}$. Finally Valtr [9](2004) again improved the bound to $n(k,l) \leq 2^{\binom{k+l}{k+2}-1}$. The last Valtr's proof generalizes Erdős-Szekeres results on cups and caps to open cups and open caps.

Theorem 2 (Valtr) *For every positives integers k and l there exits positive integer N such that any N-point set in the plane contains an open k-cup or an open l-cap.*

We show a simple proof of theorem 1. The theorem 1 for l-convex sets is a corrolary of theorem 2. If we have an $(l-3)$-convex N-point set X and we want to find a $(k+1)$-hole, we use the projective transformation, which sends one point on the convex hull of X to the infinity. We obtain an $(N-1)$-point set $\bar{X}$. We apply the theorem 2 on set $\bar{X}$ and receive either open k-cup or open l-cap. In the backward projective transformation the open k-cup corresponds to a $(k+1)$-hole and the open l-cap corresponds to a triangle cointaing at least $(l-2)$-points, but that contradicts the $(l-3)$-convexity of the set X. See Valtr [9] for the details.

We define $g(k,l)$ as the smallest number N such that any N-point set in general position contains an open k-cup or an open l-cap. Valtr [9] showed the following bounds:

$$2^{\binom{\lfloor k/2\rfloor+\lfloor l/2\rfloor-2}{\lfloor k/2\rfloor-1}} \le g(k,l) \le 2^{\binom{k+l-4}{k-2}}.$$

References

[1] P. Erdős and Gy. Szekeres: *A combinatorial problem in geometry*, Compositio Math. 2 (1935), 464–470.

[2] T. Gerken: *Empty convex hexagon always exist!*, submitted.

[3] J. D. Horton: *Sets with no empty* 7*-gons*, Canadian Math. Bull. 26 (1983), 482–484.

[4] Gy. Károlyi, J. Pach and G. Tóth: *A modular version of the Erdős-Szekeres theorem*, Studia Sc. Math Hungar. 38 (2001), 245-259.

[5] Gy. Károlyi and P. Valtr: *Empty convex polygons in* 1*-convex sets*, submitted.

[6] G. Kun and G. Lippner: *Large convex empty polygons in k-convex sets*, Period. Math. Hungar. 46 (2003), 81–88.

[7] M. Overmars, B. Scholten, I. Vincent: *Sets without empty convex 6-gons*, Bull. European Assoc. Theor. Comput. Sci. No 37 (1989), 160–168.

[8] P. Valtr: *A sufficient condition for the existence of large empty convex polygons*, Discrete Comput. Geom. 28 (2002), 671–682.

[9] P. Valtr: *Open caps and cups in planar point sets*, submitted.

Planar Graphs without 7-cycles are 4-choosable

Babak Farzad[1]
University of Toronto, Toronto, Canada

Choosability of planar graphs has been extensively studied. Thomassen [7] proved that every planar graph is 5-choosable. Voigt [8], Gutner [3] and Mirzakhani [6] presented examples of non-4-choosable planar graphs. So, characterizing 4-choosable planar graphs turned out to be one of the most interesting problems in the choosability of planar graphs.

A graph G is *d-degenerate* if every subgraph H of G has a vertex of degree at most d in H. It is easy to see that every d-degenerate graph is $(d+1)$-choosable. An easy argument using Euler's formula shows that every planar graph without 3-cycles is 3-degenerate. Hence, every planar graph without 3-cycles is 4-choosable. Wei-fan and Lih [10] proved that every planar graph without 5-cycles is 3-degenerate. Recently, Fijavz, Juvan, Mohar and Skrekovski [2] proved that every planar graph without 6-cycles is 3-degenerate. In summary:

Theorem 1 *Let k be an integer, $k = 3, 5,$ or 6. If G is a planar graph with no cycle of length k, then G is 3-degenerate.*

The lack of 4-cycles does not imply the 3-degeneracy of a planar graph, e.g. the line graph of a dodecahedron. However, Lam, Xu and Liu [5] proved that every planar graph without 4-cycles is 4-choosable. Thus, we know:

Theorem 2 *Let k be an integer, $3 \leq k \leq 6$. If G is a planar graph with no cycle of length k, then G is 4-choosable.*

Fijavz, Juvan, Mohar, Skrekovski [2] and Wei-Fan, Lih [9] independently conjectured that the above theorem can be extended to $k = 7$, i.e. a planar graph with no 7-cycle is 4-choosable. In proving the above conjecture, one may not hope to prove the 3-degeneracy of planar graphs without 7-cycles. In fact, Choudum [1] constructed 4-regular 3-connected planar graphs without k-cycles for each $k \geq 7$. We use the discharging method to prove the above conjecture, that is:

Theorem 3 *If G is a planar graph without 7-cycles, then G is 4-choosable.*

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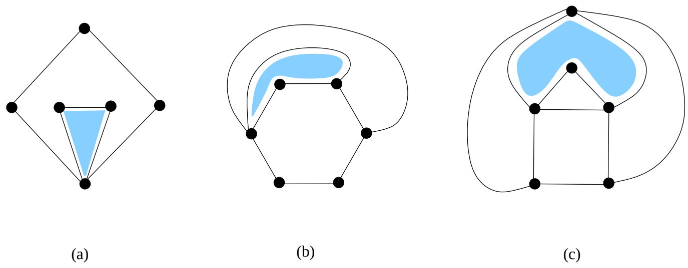

Figure 1: Examples of unexpected configurations in planar graphs without 7-cycles. The shaded areas may contain more vertices and edges.

Planar graphs without cycles of specific lengths

The structure of a planar graph without cycles of specific lengths forbids several configurations. However, these can be fairly subtle. For example, in a planar graph G without 7-cycles, it is tempting to conclude that there are no 7-faces. But this is not necessarily true: G may have a 7-face if the 7-face is non-simple (Figure 1(a)). Similarly, G may have a 6-face adjacent to a 3-face, e.g. when the 6-face is non-simple.

Unfortunately, the existence of non-simple faces is not the only source of unexpected configurations. Similar unexpected configurations may occur when two adjacent faces intersect in more than one edge, e.g. graph G without 7-cycles might have a simple 6-face adjacent to a 3-face as shown in Figure 1(b). Unexpected configurations may also occur when two different neighbours of a face f intersect outside of f, e.g. G may have a 4-face adjacent to three 3-faces as shown in Figure 1(c).

We emphasize that we are not just being overly pedantic here. In fact a number of published papers contain fatal errors of this sort. Here is an example:

One of the main results of Lam, Shiu and Xu in [4] is that every planar graph without 6-cycles is $(4m, m)$-choosable. Their proof is based on the following observations on planar graphs without 6-cycles (page 288 of [4]):

(i) a k-vertex, where $k \geq 5$, is incident to at most $\lfloor \frac{3k}{4} \rfloor$ in total, of 3- or 4-faces;

(ii) a 5-face is not adjacent to any 3-face;

(iii) a 4-face is adjacent to at most one 3-face;

(iv) a 3-face is adjacent to at most two 3-faces;

(v) if f_1 and f_2 are adjacent 3-faces, then none of them is adjacent to a 4-face.

All of these observations are incorrect as shown in Figure 2.

References

[1] S. Choudum, Some 4-valent, 3-connected, planar, almost pancyclic graphs, *Discrete Math.*, **18** (1977), 125-129.

[2] G. Fijavz, M. Juvan, B. Mohar, R. Skrekovski, Planar graphs without cycles of specific lengths, *European J. Combin.*, **23** (2002), 377-388.

[3] S. Gutner, The complexity of planar graph choosability, *Discrete Math.*, **159** (1996), 119-130.

[4] P. Lam, W. Shiu, and B. Xu, On structure of some plane graphs with application to choosability, *J. Combin. Theory Ser. B*, **82**, (2001), 285-296.

[5] P. Lam, B. Xu, J. Liu, The 4-choosability of plane graphs without 4-cycles, *J. Combin. Theory Ser. B*, **76** (1999), 117-126.

[6] M. Mirzakhani, A small non-4-choosable planar graph, *Bull. Inst. Combin. Appl.*, **17** (1996), 15-18.

[7] C. Thomassen, Every planar graph is 5-choosable, *J. Combin. Theory Ser. B*, **62** (1994), 180-181.

[8] M. Voigt, List colourings of planar graphs, *Discrete Math.*, **120** (1993), 215-219.

[9] W. Weifan and K. Lih, The 4-choosability of planar graphs without 6-cycles, *Australas. J. Combin.* , **24** (2001), 157-164.

[10] W. Weifan and K. Lih, Choosability and edge choosability of planar graphs without five cycles, *Appl. Math. Lett.*, **15** (2002), 561-565.

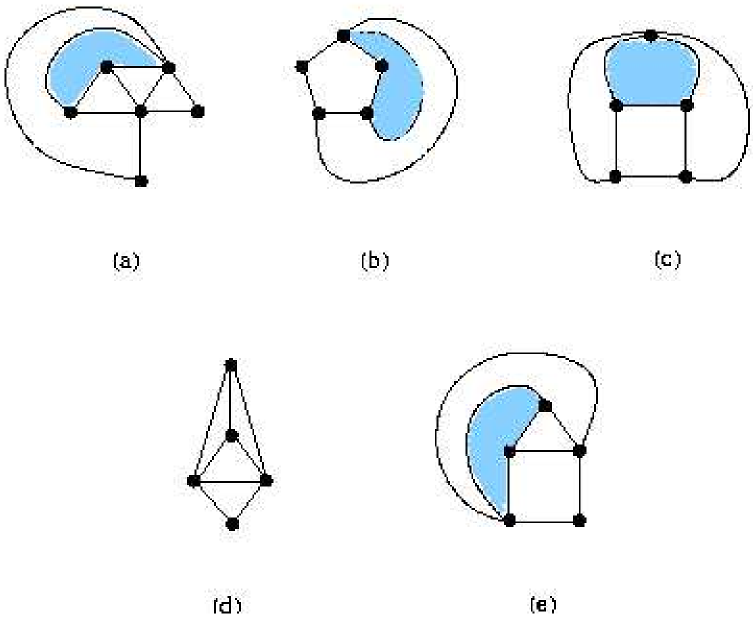

Figure 2: The shaded areas may contain more vertices and edges. (a) a 5-vertex incident to four 3- or 4-faces; (b) a 5-face adjacent to a 3-face; (c) a 4-face adjacent to two non-adjacent 3-faces; (d) a 4-face adjacent to three 3-faces; (e) two adjacent 3-faces where one of them is adjacent to a 4-face.

Construction of Transitive Graphs According to Given Degree Structure

Jiří Fiala
Charles University, Prague, Czech Republic

For a graph we encode the degree structure as a partition of its vertices into the smallest number of blocks $B_1, \dots, B_k$ such that whenever two vertices belong to the same block, they have the same number of neighbors inside any block: $u, v \in B_i \Rightarrow \forall j : |N(u) \cap B_j| = |N(v) \cap B_j = m_{ij}$ We arrange constants m_{ij} into the degree matrix or order k.

We ask whether for a given degree matrix M, exists a graph G such that M is the degree matrix of G, and in addition for any two edges $e, f \in E_G$ connecting the same pair of blocks there exists an automorphism of G that swaps edges e and f.

Symmetries in Ramsey Theory: Recent Results and Applications

Willem Fouché
University of South Africa, Pretoria, South Africa

1 Introduction

In this survey we discussed the role of symmetry in structural Ramsey theory. For a survey of this subject, the reader is referred to Nešetřil [14].

In the seventies, largely because of the work by Nešetřil, Rödl [15, 16, 18] and, independently, by Abramson, Harrington [1], remarkable progress was made with the problem of determining the Ramsey objects in various classes of combinatorial configurations. Recently, it was shown by Kechris, Pestov and Todorcevic [12] that these results have, among other things, major applications to the study of the extremely amenable subgroups of the symmetry group S_ω of a countably infinite set, where S_ω has the usual pointwise convergence topology. In particular, they showed how the notion of a Ramsey degree, to be discussed below, can be generalised to infinite Ramsey theory.

2 Ramsey degrees

If r is a natural number, we write $[r]$ for the set $\{1, \cdots, r\}$. The set of non-negative integers is denoted by ω. If $\mathcal{C}$ is a class of finite structures for which we have a notion of a copy (image under an embedding) of an object A in an object B of $\mathcal{C}$, then A is said to be a *Ramsey object* in $\mathcal{C}$ if for each object B and $r < \omega$, there is an object C such that for each partition $\chi : [C, A] \longrightarrow [r]$, where $[C, A]$ is the set of copies of A in C, there is a copy B' of B in C such that all the elements of $[B', A]$ are in one block of the partition.

It is well-known that in the class of finite graphs, the complete graphs and their complements are the only Ramsey objects. In the class of finite posets the only Ramsey objects are the ordinal sums of antichains [17].

In the papers [3, 6, 7] the author studied the following problem: For a given object in a class $\mathcal{C}$, how can one measure the extent to which this is a Ramsey object? We answered this question by finding for various classes $\mathcal{C}$, for each A in $\mathcal{C}$, the *smallest* natural number, $t(A)$, with the following property: For each $r < \omega$, for each B in $\mathcal{C}$, there is some C in $\mathcal{C}$, such that, for each partition $\chi : [C, A] \to [r]$, there is some copy B' of B in C such that χ assumes at most $t(A)$ values on $[B', A]$. The number, $t(A)$, if it exists, is called the *Ramsey degree* of A. Thus A is a Ramsey object iff $t(A) = 1$. In each of the classes, we found that the Ramsey degree of an object A can be expressed in terms of its symmetries. This has the implication that only the most symmetric configurations are Ramsey objects.

If $\mathbf{P} = (X, P)$ is a poset, we frequently write $x < y(P)$ or $x < y(\mathbf{P})$ instead of $(x, y) \in P$. We recall that a total order L on the underlying set X of $\mathbf{P}$ is a *linear extension* of $\mathbf{P}$ if, for all x and y in X we have: $x < y(P) \Rightarrow x < y(L)$, i.e if $P \subseteq L$.

For a finite poset $\mathbf{P}$, let $e(\mathbf{P})$ denote the number of linear extensions of $\mathbf{P}$ and write $A(\mathbf{P})$ for the automorphism group of $\mathbf{P}$. We set

$$t(\mathbf{P}) = e(\mathbf{P})/|A(\mathbf{P})|.$$

It is easily seen that $t(\mathbf{P})$ is always a natural number. An *embedding* of a poset $\mathbf{P}$ into a poset $\mathbf{Q}$ is a one-to-one map $\lambda : \mathbf{P} \to \mathbf{Q}$ such that, for all $x, y \in \mathbf{P}$, we have that $x < y$ iff $\lambda(x) < \lambda(y)$. We refer to an image under an embedding of $\mathbf{P}$ in $\mathbf{Q}$ as a copy of $\mathbf{P}$ in $\mathbf{Q}$. The set of copies of $\mathbf{P}$ in $\mathbf{Q}$ is denoted by $[\mathbf{Q}, \mathbf{P}]$. In [5] we proved the following theorem.

Theorem 1 *For finite posets $\mathbf{P}$ and $\mathbf{Q}$ and a natural number r, there exists a finite poset $\mathbf{R}$ such that, for any r-colouring χ of the copies of $\mathbf{P}$ in $\mathbf{R}$, there is a copy, $\mathbf{Q}'$, of $\mathbf{Q}$ in $\mathbf{R}$ such that χ assumes at most $t(\mathbf{P})$ values on $[\mathbf{Q}', \mathbf{P}]$. Conversely, for any $\mathbf{P}$ and $r \geq t(\mathbf{P})$, there is a poset $\mathbf{Q}$ such that for any $\mathbf{R}$ containing a copy of $\mathbf{Q}$, an r-colouring of $[\mathbf{R}, \mathbf{P}]$ can be found which assumes, on any set of the form $[\mathbf{Q}', \mathbf{P}]$, with $\mathbf{Q}'$ a copy of $\mathbf{Q}$ in $\mathbf{R}$, at least $t(\mathbf{P})$ values.*

It follows that $t(\mathbf{P})$ is the Ramsey degree of the poset $\mathbf{P}$. In order to unfold the meaning of this theorem, we introduce the following notation: If $\mathbf{P}$ and $\mathbf{Q}$ are posets, we write $\mathbf{P} \oplus \mathbf{Q}$ for their ordinal sum. For a natural number n, we write $\mathbf{n}$ for the antichain of size n. The symmetric group on n elements is denoted by S_n. We shall use the one-line notation for the elements of S_n; that is, if π is a permutation of $[n]$ which maps i to π_i, say, for $i = 1, \cdots, n$, then we denote π by the word $\pi_1 \ldots \pi_n$.

If $n_1, \cdots, n_k$ are natural numbers then

$$A(\mathbf{n}_1 \oplus \ldots \oplus \mathbf{n}_k) \simeq S_{n_1} \times \ldots \times S_{n_k};$$

moreover, this ordinal sum of antichains clearly has exactly $n_1! \ldots n_k!$ linear extensions. It follows that $\mathbf{n}_1 \oplus \ldots \mathbf{n}_k$ has Ramsey degree one and is, therefore, a Ramsey object in the class of finite posets.

The number $t(\mathbf{P})$ has the following order theoretic interpretation: Let L_1, L_2 be linear extensions of $\mathbf{P}$. We say that L_1 and L_2 are equivalent and write $L_1 \sim L_2$, if the unique order preserving map $(X, L_1) \to (X, L_2)$, where X is the underlying set of $\mathbf{P}$, induces an automorphism of $\mathbf{P}$. Then, $t(\mathbf{P})$ is the number of equivalence classes with respect to $\sim$. To see this, fix some linear extension L of $\mathbf{P}$ and label the elements of $\mathbf{P}$ by natural numbers $1, \cdots, n$ in such a way that L will order the elements of $\mathbf{P}$ as $1 < \ldots < n$. In this context, any linear extension L' of $\mathbf{P}$ corresponds to a permutation π of $[n]$ such that, for all $i, j \in [n]$:

$$i < j(P) \Rightarrow \pi^{-1}(i) < \pi^{-1}(j)(\omega). \tag{1}$$

Indeed, let $\pi = \pi(1) \ldots \pi(n)$ be such that L' orders the elements of $\mathbf{P}$ as $\pi(1) < \ldots \pi(n)$. We then have, for $i, j \in [n]$, that $i < j \leftrightarrow x < y(\omega)$ where x, y are such that $\pi(x) = i$ and $\pi(y) = j$. Conversely, any permutation π of $[n]$ satisfying (1), corresponds to a unique linear extension of $\mathbf{P}$. Let $\mathcal{A}$ be the set of $\pi \in S_n$ satisfying (1). Then $\mathcal{A}$ has exactly $e(\mathbf{P})$ elements. If $\sigma \in A(\mathbf{P}) \leq S_n$ and $\pi \in \mathcal{A}$, then $\sigma\pi \in \mathcal{A}$. Indeed, if $i < j$ in $\mathbf{P}$, then $\sigma^{-1}i < \sigma^{-1}j$ in $\mathbf{P}$ and $\pi^{-1}\sigma^{-1}i < \pi^{-1}\sigma^{-1}j$ in ω, since π satisfies (1). This means that $A(\mathbf{P})$ acts on $\mathcal{A}$; it is obvious that this action is faithful. Now, $\pi_1, \pi_2 \in \mathcal{A}$ correspond to equivalent linear extensions of $\mathbf{P}$ iff $\pi_1 = \sigma\pi_2$ for some $\sigma \in A(\mathbf{P})$. Let $s(\mathbf{P})$ be the number of nonequivalent linear extensions of $\mathbf{P}$. It is now clear that there are exactly $s(\mathbf{P})$ orbits under the group action of $A(\mathbf{P})$ on $\mathcal{A}$. Each orbit with respect to this action has size $|A(\mathbf{P})|$. It follows that $s(\mathbf{P}) = e(\mathbf{P})/|A(\mathbf{P}))| = t(\mathbf{P})$, as required.

It is readily seen that if $\mathbf{P}$ is not an ordinal sum of antichains, then one can find two non-equivalent linear extensions of $\mathbf{P}$. Indeed, for $x \in \mathbf{P}$, write $\rho(x)$ for the maximum size of the chains having x as a maximum element. If $\mathbf{P}$ is not an ordinal sum of antichains, it has elements x, y and z satisfying $\rho(x) = \rho(y)$, $\rho(z) = \rho(y) + 1$; moreover, $y < z(\mathbf{P})$ but x and z are independent. Let L be a linear extension of $\mathbf{P}$ such that $x < y(L)$ and there is no element t such that $x < t < y$ (i.e., y covers x with respect to the order L). Moreover, we require that $x_1 < x_2(L)$ whenever $\rho(x_1) < \rho(x_2)$. (Any total order with this property is a linear extension of $\mathbf{P}$, for if $x < y(\mathbf{P})$, then $\rho(x) < \rho(y)$.) Let L' be the linear order that is the same as L with the only exception that $y < x(L')$. Then L and L' are nonequivalent linear extensions of $\mathbf{P}$.

In this way we recover the result of Nešetřil and Rödl [17] that the ordinal sums of antichains are the only Ramsey objects in the class of finite posets.

The proof of Theorem 1 revolves around the observation that a partial order is the intersection of all its linear extensions. The proof then uses the partition theorems of parameter words of Graham and Rothschild [11].

Kechris et al mentioned in [12] that the result has the following dynamical interpretation: Let $\mathbf{P}_\omega$ be the random poset (i.e., the Fraïssé limit of the class of finite posets) and let L be an appropriate linear extension of $\mathbf{P}_\omega$. Then the automorphism group G of the pair $(\mathbf{P}_\omega, L)$ is extremely amenable, i.e., every compact flow of G has a fixed point. (Here we consider G as a closed subgroup of S_ω where the latter has the pointwise convergence topology.) Equivalently, the universal minimal flow $M(G)$ of G is actually trivial, a singleton. Moreover, the universal minimal flow of the automorphism group of the random poset $\mathbf{P}_\omega$ can be parametrised by the space of all linear extensions of $\mathbf{P}_\omega$. In [12] it is shown that one can also express the extreme amenability of G as follows:

For any open subgroup V of G, every colouring $c : G/V \to [k]$, of the set of left cosets hV of V in G, and every finite $A \subset G/V$, there is some $g \in G$ and $1 \leq i \leq k$, such that $c(g.a) = i$, for all $a \in A$, where G acts on G/V in the usual way: $g.hV = ghV$.

In [12] the reader will also find a dynamical interpretation of a vast array of results by Nešetřil and Nešetřil, Rödl.

In collaboration with Pretorius and Swanepoel, the Ramsey degrees of posets of a fixed height was determined [10]. For other structures (including trees, bipartite graphs), see [9, 6]. The dynamical interpretation of these results will be discussed in [7].

It is an open problem whether every finite lattice has a Ramsey degree. In fact, even the simpler problem of finding all the Ramsey objects in the class of finite lattices is still open. In [17] it is shown that points and 2-chains are Ramsey objects in the class of finite lattices. It is tempting to guess that ordinal sums of antichains which are also lattices are Ramsey objects in this class. For distributive lattices, the problem is solved. One can deduce from the arguments in Prömel, Voigt[21] that, for a finite distributive lattice D, its Ramsey degree is given by $(S_n : A(D))$, where n is the smallest natural number such that D can be embedded in the Boolean algebra B_n.

In his doctoral thesis, Devlin [2] showed that the Ramsey degree t_k of the finite total order k in the dense order $(\mathbf{Q}, \leq)$ is equal to the $(2k+1)$st *tangent number* T_{2k+1} given by the formula $\tan z = \sum_{n=0}^{\infty} T_n z^n/n!$. Note that the existence of these Ramsey degrees was known to R. Laver before Devlin's work. The existence follows rather directly from results of K. Milliken [13]. For a recent (and very beautiful) proof of Devlin's result (which still is unpublished) see Vuksanovic [23]. For results about other Fraïssé limits such as for example the random graph, the reader is referred to Pouzet-Sauer [19] and Sauer [22]. For random partitions of Fraïssé limits, see Fouché, Potgieter [8]. In general, though, the determination (or even the existence!) of Ramsey degrees of finite substructures of Fraïssé limits is wide open.

The following result displays an unavoidable symmetry in words. This result was proved by the author in [3].

Theorem 2 *For given $n, r > 0$, there is some $N = N(n, r)$, such that any word w of length N over an alphabet of r elements will contain, for every permutation $\pi \in S_n$, factors $w_1, \cdots, w_n$ and Z, such that*

$$w_1 \ldots w_n Z w_{\pi(1)} \ldots w_{\pi(n)}$$

is another factor of the word w.

Even though the proof of this theorem is constructive, it yields an upper bound for $N(n, r)$ which just misses to be primitive recursive. (It is of the same recursive complexity as the well-known Ackermann function.) It is an interesting open problem whether one find primitive recursive upper bounds for N.

References

[1] F.G. Abramson and L.O. Harrington, Models without indiscernables, *J. Symb. Logic* **43** (1978), 572-600.

[2] D. Devlin, Some partition theorems and ultrafilters on ω, Ph.D Thesis, Dartmouth College, 1979.

[3] W.L. Fouché, Unavoidable regularities and factor permutations of words, *Proc. Royal Soc Edinburgh* **125A** (1995) 519-524.

[4] W.L. Fouché, Chain partitions of ordered sets, *Order* **13** (1996) 255-266.

[5] W.L. Fouché, Symmetry and the Ramsey degree of posets, *Discrete Math.* **167/168** (1997) 309-315

[6] W.L. Fouché, Symmetry and the Ramsey degree of finite relational structures, *J. Comb. Theory Ser A* **85** (1999) 135-147.

[7] W.L. Fouché, Some new extremely amenable groups, *in preparation.*

[8] W.L. Fouché and P.H. Potgieter, Kolmogorov complexity and symmetric relational structures, *J. Symb. Logic* (1998) **63** (1998) 1083-1094.

[9] W.L. Fouché, L.M. Pretorius and C.J. Swanepoel, The Ramsey degrees of bipartite graphs: a primitive recursive proof, *Discrete Math.* **293** (2005) 111-119.

[10] W.L. Fouché, L.M. Pretorius and C.J. Swanepoel, The Ramsey degrees of posets of a given height, *submitted*.

[11] R.L. Graham and B.L. Rothschild, Ramsey's theorem for n-parameter sets, *Trans. Amer. Math. Soc.* **159** (1971) 257-292.

[12] A.S. Kechris, V.G. Pestov and S. Todorcevic, Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups, *to appear.*

[13] K.R. Milliken, A Ramsey theorem for trees, *J. Comb. Theory Ser A* **26** (1979) 305-313.

[14] J. Nešetřil, Ramsey theory, in: *Handbook of Combinatorics Vol 2*, eds R.L. Graham, M. Grötschel and L. Lovász, (North Holland) (1995) pp. 1331-1403.

[15] J. Nešetřil and V. Rödl, Partitions of relational and set systems, *J. Comb. Theory Ser A* **83** (1977) (289-312.

[16] J. Nešetřil and V. Rödl, Ramsey classes of set systems, *J. Comb. Theory Ser A* **34** (1983) 183-201.

[17] J. Nešetřil and V. Rödl, Combinatorial partitions of finite posets and lattices– Ramsey lattices, *Algebra Universalis* **19** (1984) 106-119.

[18] J. Nešetřil and V. Rödl, The partite construction and Ramsey set systems, *Discrete Math.* **75** (1989) 327-334.

[19] M. Pouzet and N. Sauer, Edge partitions of the Rado graph, *Combinatorica* **16** (1996) 505-520.

[20] H.J. Prömel and B. Voigt, Recent results in partition (Ramsey) theory for finite lattices, *Discrete Math.* **35** (1985) 185-208.

[21] H.J. Prömel and B. Voigt, Canonical partition theorems for distributive lattices, *Supplemento ai Rendeconti del Circilo Mathematico di Palermo Serie ii* **2** (1982) 223-237.

[22] N. Sauer, Coloring finite substructures of countable structures, Paul Erdös and his Mathematics, II, *Bolyai Math. Studies* **11** (2002) 525-553.

[23] V. Vuksanov, A proof of a partition theorem for $[Q]^n$, *Proc. Amer. Math. Soc.* **130** (2002) 2857-2864.

On Empty Convex Hexagons In Planar Point Sets

Tobias Gerken
Technical University Munich, Garching, Germany

In 1935 Erdős and Szekeres [3] proved that for any positive integer k there exists a smallest positive integer $g(k)$ such that any planar set of at least $g(k)$ points in general position (that is, no three points are collinear) contains k points that are the vertices of a convex k-gon. The best known bounds for $g(k)$ are $2^{k-2}+1 \leq g(k) \leq \binom{2k-5}{k-2}+1$ due to Erdős and Szekeres [4] and Tóth and Valtr [9]. The lower bound is sharp for $k \leq 5$ and has been conjectured to be sharp for all k. Later Erdős [2] posed the problem of determining the smallest positive integer $h(k)$, if it exists, such that any set X of at least $h(k)$ points in general position in the plane contains k points which are the vertices of an *empty* convex polygon, that is, a convex k-gon whose interior does not contain any point of X. Trivially, $h(k) = k$ for $k \leq 3$. It is easy to see that $h(4) = 5$. In 1978 Harborth [5] proved that $h(5) = 10$, while Horton [6] showed in 1983 that for all $k \geq 7$, $h(k)$ does not exist. The problem of determining the existence of $h(6)$ has since been open. Based on computer experiments, Overmars [8] showed that $h(6) \geq 30$ (if it exists). In our talk, we present a promising new approach for a proof which would imply that every sufficiently large planar point set in general position contains the vertex set of an empty convex 6-gon:

Conjecture 1 $h(6) \leq g(9) < \infty$.

The above bounds yield $129 \leq g(9) \leq 1717$. Note that there exist sets of points without empty convex 6-gons that have eight points on the convex hull ([8]). For a survey of results related to the Erdős–Szekeres theorem, see [1, 7, 9].

The new approach goes as follows: Let X denote any finite planar set of points in general position that contains the vertex set of a convex 9-gon which by the Erdős–Szekeres theorem [3] is always the case if the cardinality of X is larger than or equal to $g(9)$. Let $H \subset X$ denote the vertex set of a convex 9-gon in X with the *smallest* possible $|X \cap \mathrm{conv}(H)|$, where $\mathrm{conv}(M)$ denotes the convex hull of the set M. Let $I := \mathrm{conv}(H) \cap (X \setminus H)$ be the points of X inside the convex hull of H. Note that $\mathrm{conv}(I)$ is a convex

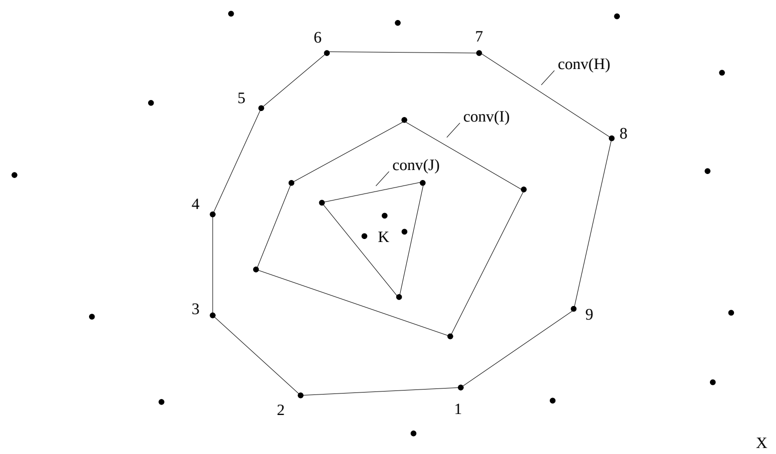

Figure 1: Basic notation

polygon and denote by ∂I its vertex set. If $|I| > 2$, let $J := \mathrm{conv}(I) \cap (X \setminus \partial I)$ be the points of X inside the convex hull of ∂I. Note that $\mathrm{conv}(J)$ is again a convex polygon and denote by ∂J its vertex set. Finally, set $K := \mathrm{conv}(J) \cap (X \setminus \partial J)$, confer Figure 1. Let $i := |\partial I|$ and $j := |\partial J|$ denote the cardinalities of the vertex sets of $\mathrm{conv}(I)$ and $\mathrm{conv}(J)$ respectively. Note that $0 \leq i, j \leq 8$ as otherwise there would be a 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$. This leaves us with the 57 cases $0 \leq i \leq 2$ and $(i, j) \in \{3, \ldots, 8\} \times \{0, \ldots, 8\}$. Now argue that in each case either the vertex set of an empty convex u-gon can be found ($u \geq 6$) or the vertex set of a convex 9-gon H' with smaller $|X \cap \mathrm{conv}(H')|$ is present which contradicts the minimality condition imposed on H.

References

[1] I. Bárány and Gy. Károlyi, *Problems and results around the Erdős-Szekeres convex polygon theorem*, Lect. Notes Comput. Sci. **2098** (2001), 91–105.

[2] P. Erdős, *Some more problems on elementary geometry*, Austral. Math. Soc. Gaz. **5** (1978), 52–54.

[3] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. **2** (1935), 463–470.

[4] ———, *On some extremum problems in elementary geometry*, Ann. Univ. Sci. Budapest, Eötvös, Sectio Mathematica **3/4** (1960–61), 53–62.

[5] H. Harborth, *Konvexe Fünfecke in ebenen Punktmengen*, Elem. Math. **33** (1978), 116–118.

[6] J.D. Horton, *Sets with no empty convex 7-gons*, Canad. Math. Bull. **26** (1983), 482–484.

[7] W. Morris and V. Soltan, *The Erdős-Szekeres problem on points in convex position – a survey*, Bull. Amer. Math. Soc. **37** (2000), 437–458.

[8] M. Overmars, *Finding sets of points without empty convex 6-gons*, Discrete Comp. Geom. **29** (2003), 153–158.

[9] G. Tóth and P. Valtr, *The Erdős-Szekeres theorem: upper bounds and related results*, submitted.

Nonrepetitive Colorings of Graphs

Jarek Grytczuk
University of Zielona Góra, Zielona Góra, Poland

A sequence $S = s_1 s_2 ... s_{2n}$ is called a *repetition* if $s_i = s_{n+i}$ for each $i = 1, ..., n$. A coloring of the vertices of a graph G is *nonrepetitive* if no simple path of G looks like a repetition. The minimum number of colors needed for a nonrepetitive coloring of G is denoted by $\pi(G)$ and is called the *Thue chromatic number* of G.

The celebrated 1906 theorem of Thue [5] asserts that $\pi(P_n) = 3$ for all $n \geq 4$, where P_n is a path with n vertices. Let $\pi(d)$ denote the supremum of $\pi(G)$ where G ranges over all graphs with $\Delta(G) \leq d$. In [1] it was proved by the probabilistic method that there are absolute positive constants c_1 and c_2 such that

$$c_1 \frac{d^2}{\log d} \leq \pi(d) \leq c_2 d^2.$$

Recently Kündgen and Pelsmajer [3] proved that $\pi(G) \leq 4^t$ for graphs of treewidth at most t. This implies, by the result of Robertson and Seymour [4], that any minor-closed class of graphs with unbounded Thue chromatic number must contain all planar graphs. This makes the following natural question even more intriguing:

Is the Thue chromatic number bounded for planar graphs?

In a seemingly weaker version of the problem we ask for the minimum number of colors $t = t(\mathcal{F})$ needed for a family of graphs $\mathcal{F}$ such that there exists, possibly huge, but finite k allowing for a t-coloring of every member of $\mathcal{F}$ with no k identical blocks on a path. For instance, for graphs of maximum degree at most d this threshold value of t is between $(d+1)/2$ and $d+1$, as shown in [2].

Is t finite for planar graphs?

References

[1] N. Alon, J. Grytczuk, M. Hałuszczak, O. Riordan, Nonrepetitive colorings of graphs, Random Struct. Alg. **21** (2002), 336-346.

[2] N. Alon, J. Grytczuk, Power-free colorings of graphs, manuscript.

[3] A. Kündgen, M.J. Pelsmajer, Nonrepetitive colorings of graphs of bounded treewidth, manuscript.

[4] N. Robertson, P.D. Seymour, Graph minors V: Excluding a planar graph, J. Combin. Theory Ser. B **41** (1986), 92-114.

[5] A. Thue, Über unendliche Zeichenreichen, Norske Vid Selsk. Skr. I. Mat. Nat. Kl. Christiana, (1906), 1-67.

Some Problems on Matchings and Toughness

Tomáš Kaiser
University of West Bohemia, Pilsen, Czech Republic

We present a couple of problems related to generalizations of the famous 1-factor theorem of Tutte [6]. For a graph H, let $\omega_{\text{odd}}(H)$ denote the number of components of H of odd order, and let $\omega(H)$ be the total number of components of H. Tutte's theorem says:

Theorem 1 (Tutte) *A graph G has a 1-factor if and only if for all $X \subset V(G)$,*

$$\omega_{\text{odd}}(G - X) \leq |X| .$$

There are several extensions of Theorem 1 dealing with systems of disjoint paths. The first such result was proved by Gallai [1]. Let (G, T) be a *graft*; that is, let G be a graph and $T \subset V(G)$ with $|T|$ even. A *T-path* is defined to be a path with ends in T.

Theorem 2 (Gallai) *The maximum number of vertex-disjoint T-paths is equal to*

$$\min_{X \subset V(G)} \Big(|X| + \sum_K \Big\lfloor \frac{|K \cap T|}{2} \Big\rfloor \Big),$$

where K ranges over components of $G - X$.

Recent work on [3] motivated us to examine systems of disjoint T-paths spanning all of T, which we called *T-path coverings* (in a given graft (G, T)). Specifically, we needed to show that if G is cubic and 3-connected, then every edge is contained in a T-path covering. This is somewhat reminiscent of a theorem of Plesník [5] for 1-factors:

Theorem 3 (Plesník) *Every edge of a k-regular $(k - 1)$-edge-connected graph is contained in a 1-factor.*

However, simple examples show that the direct analogue of Theorem 3 is not true for T-path coverings. On the other hand, we established [2] the following:

Theorem 4 *Every edge of a k-regular k-edge-connected graph is contained in a T-path covering.*

We obtained Theorem 4 as a corollary of Theorem 5 below, a result on tough graphs. Recall that a graph G is *tough* if for all $X \subset V(G)$,

$$\omega(G - X) \leq |X|\,.$$

Note that by a simple counting argument, every k-regular k-edge-connected graphs is tough.

Theorem 5 *Let (G, T) be a graft with G tough. An edge $uv \in E(G)$ is not contained in a T-path covering if and only if there is a set $X \subset V(G)$ such that:*

1. *$\{u, v\} \subset X \subset T$,*
2. *$G - X$ has precisely $|X|$ components, and*
3. *each of these components contains an odd number of vertices in T.*

It is natural to ask whether the above characterization could be extended to *pairs* of edges:

Problem 1 *For a graft (G, T) with G tough, characterize the pairs of edges that are not contained in a T-path covering.*

The line of proof of Theorem 5 does not seem to work for Problem 1. More specifically, the proof involves a reduction to the following result of Mader [4] that generalizes Theorem 2. Given a graph G and a system $\mathcal{S}$ of disjoint subsets of $V(G)$, an *$\mathcal{S}$-path* is defined to be a path with ends in distinct sets in $\mathcal{S}$.

Theorem 6 (Mader) *The maximum number of vertex-disjoint $\mathcal{S}$-paths equals*

$$\min_{X,F}\Big(|X| + \sum_{K} \Big\lfloor \frac{|K \cap (T \cup V(F))|}{2} \Big\rfloor\Big),$$

where the minimum is taken over all $X \subset V(G)$ and all $F \subset E(G - X)$ containing no $\mathcal{S}$-paths, and K ranges over components of $G - X - F$.

Problem 1, however, cannot be directly reduced to Mader's theorem. Instead, a result involving the structure of the $\mathcal{S}$-paths, in the following sense, would be useful. Let us define a subgraph of G to be $\mathcal{S}$*-acyclic* if the corresponding subgraph in $G/\mathcal{S}$ is a forest, where $G/\mathcal{S}$ is the graph obtained by contracting each set in $\mathcal{S}$ to a vertex.

Problem 2 *Is there a relation, similar to the one in Theorem 6, for the maximum number of disjoint $\mathcal{S}$-paths whose union is $\mathcal{S}$-acyclic?*

We conclude with a problem concerning a possible generalization of Theorem 1 along different lines:

Problem 3 *Is there a sufficient condition in the spirit of Tutte's theorem for the existence of two edge-disjoint 1-factors? How about the simplest candidate:*

$$\omega_{\text{odd}}(G - X) \leq \frac{|X|}{C},$$

for all $X \subset V(G)$, where C is a large constant?

References

[1] T. Gallai, Maximum-minimum Sätze und verallgemeinerte Faktoren von Graphen, *Acta Math. Acad. Sci. Hung.* **12** (1961), 131–173.

[2] T. Kaiser, Disjoint T-paths in tough graphs, manuscript.

[3] T. Kaiser and R. Škrekovski, T-joins intersecting small edge-cuts in graphs, submitted.

[4] W. Mader, Über die Maximalzahl kreuzungsfreier H-Wege, *Arch. Math. (Basel)* **31** (1978), 387–402.

[5] J. Plesník, Connectivity of regular graphs and the existence of 1-factors, *Mat. časopis* **22** (1972), 310–318.

[6] W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* **22** (1947), 107–111.

Random Planar Structures

Mihyun Kang
Humboldt University, Berlin, Germany

Recently random planar structures have received much attention [1, 3, 5, 6, 12, 13, 17, 18, 21, 24, 25, 27]. Typical questions one would ask about them are the following: how many of them are there, can we sample a random instance uniformly at random, and what properties does a random planar structure have ?

We first answer those questions about labeled *cubic planar* graphs, i.e., graphs that can be embedded in the plane, and in addition each of whose vertices has degree three [11]. We show that the number of labeled cubic planar graphs on n vertices is asymptotically $cn^{-7/2}\gamma^n n!$, where $\gamma \doteq 3.132595$ and c are analytic constants. Based on this we show that the number of components isomorphic to K_4 in a random cubic planar graph on n vertices has asymptotically poisson distribution with mean $\gamma^{-4}/4!$ and that a random cubic planar graph on n vertices contains at least linearly many triangles with probability tending to 1 as n converges to ∞. Using these facts, we derive that the chromatic number of a random cubic planar graph is four with probability tending to $1-e^{-\gamma^{-4}/4!}$, and is three with probability tending to $e^{-\gamma^{-4}/4!} \doteq 0.999568$.

For this we decompose the cubic planar graphs along their connectivity. For the asymptotic enumeration we interpret the decomposition in terms of generating funtions and derive the asymptotic numbers of cubic planar graphs, using *singularity analysis* [14]. For the exact enumeration and the uniform generation we use the so-called *recursive method* [15, 23]: We derive recursive counting formulas along the decomposition, which yields a deterministic polynomial time algorithm to sample a cubic planar graph that is uniformly distributed. This sampling procedure is implemented in [20], where several other empirical properties of a random cubic planar graph are discussed, e.g., the number of cut-edges and the diameter.

These methods have been successfully applied for several other planar structures: first for various kinds of planar *maps*, i.e., graphs that are embebbed in the plane, e.g., planar maps [3, 19, 27], 3-connected planar maps [2, 6, 22], triangulations [26], cubic planar maps [16], and then for *labeled* planar *graphs*, e.g., 2-connected planar graphs [4], outerplanar-graphs [5, 10], series-parallel graphs [5], planar graphs [8, 18]. However, concerning *unlabeled* planar structures, not much is known. Only very recently, uniform sampling algorithms that run in expected polynomial time are designed for outerplanar graphs [10], cubic planar graphs [7], and 2-connected planar graphs [9].

One interesting but challenging task is to design such a uniform sampler for unlabeled planar graphs. Another interesting task is to determine the asymptotic number of planar graphs with given degree sequences.

References

[1] C. Banderier, P. Flajolet, G. Schaeffer, and M. Soria. Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures and Algorithms*, 19:194–246, 2001.

[2] E. Bender and L.B.Richmond. The asymptotic enumeration of rooted convex polyhedra. *Journal of Combinatorial Theory*, (36):276–283, 1984.

[3] E. Bender and L.B.Richmond. A survey of the asymptotic behaviour of maps. *Journal of Combinatorial Theory*, (40):297–329, 1986.

[4] E. A. Bender, Z. Gao, and N. C. Wormald. The number of 2-connected labelled planar graph. *Electronic Journal of Combinatorics*, 9, 2002.

[5] M. Bodirsky, O. Giménez, M. Kang, and M. Noy. On the number of series-parallel and outerplanar graphs. To appear in the Proceedings of EUROCOMB2005.

[6] M. Bodirsky, C. Gröpl, D. Johannsen, and M. Kang. A direct decomposition of 3-connected planar maps. To appear in the Proceedings of FPSAC 2005.

[7] M. Bodirsky, C. Gröpl, and M. Kang. Decomposing, counting, and generating unlabeled cubic planar graphs uniformly at random. Submitted.

[8] M. Bodirsky, C. Gröpl, and M. Kang. Generating labeled planar graphs uniformly at random. In *Proceedings of Thirtieth International Colloquium on Automata*, volume Languages and Programming (ICALP'03), page 2003, 1095–1107. Journal version is accepted for publication in Theorical Computer Science.

[9] M. Bodirsky, C. Gröpl, and M. Kang. Sampling unlabeled biconnected planar graphs. In *Proceedings of the 16th Annual International Symposium on Algorithms and Computation*, 2005.

[10] M. Bodirsky and M. Kang. Generating random outerplanar graphs. *Combinatorics, Probability, and Computation*, accepted for publication.

[11] M. Bodirsky, M. Kang, M. Löffler, and C. McDiarmid. Random cubic planar graphs. Preprint, 2005.

[12] N. Bonichon, C. Gavoille, and N. Hanusse. Canonical decomposition of outerplanar maps and application to enumeration. Coding and Generation, *29 International Workshop, Graph-theoretic Concepts in Computer Science (WG)*, Springer-Verlag, LNCS 2880,81-92, 2003.

[13] A. Denise, M. Vasconcellos, and D. Welsh. Random graphs. *Congressus Numerantium*, 113:61–79, 1996.

[14] P. Flajolet and R. Sedgewick. *Analytic Combinatorics.* book in preparation, zeroth edition, November 23, 2004.

[15] P. Flajolet, P. Zimmerman, and B. V. Cutsem. A calculus for the random generation of labelled combinatorial structures. *Theoretical Computer Science*, 132:1–35, 1994.

[16] Z. Gao and N. C. Wormald. Enumeration of rooted cubic planar maps. *Annals of Combinatorics*, 6:313–325, 2002.

[17] S. Gerke and C. McDiarmid. On the number of edges in random planar graphs. *Combinatorics, Probability and Computin*, 13:165–18, 2004.

[18] O. Giménez and M. Noy. Asymptotic enumeration and limit laws of planar graphs. Preprint.

[19] V. A. Liskovets and T. Walsh. Ten steps to counting planar graphs. In *Proc. of Eighteenth Southeastern International Conference on Combinatoris, Graph Theory, and Computing, Congr. Numer.*, pages 269–277, 1987.

[20] M. Löffler. Uniform sampling and enumeration of planar structures. Diploma Thesis. Humboldt-Universität zu Berlin, 2005.

[21] C. McDiarmid, A. Steger, and D. Welsh. Random planar graphs. *Journal of Combinatorial Theory, Series B*, 93:187–205, 2005.

[22] R. C. Mullin and P. J. Schellenberg. On the mixing rate of the triangulation walk. *The Enumeration of c-Nets Via Quadrangulations*, 4:259–276, 1968.

[23] A. Nijenhuis and H. Wilf. *Combinatorial algorithms.* Academic Press Inc., 1979.

[24] D. Osthus, H. J. Prömel, and A. Taraz. On random planar graphs, the number of planar graphs and their triangulations. *J. Combin. Theory Ser. B*, 88:119–13, 2003.

[25] G. Schaeffer. Random sampling of large planar maps and convex polyhedra. In *Proceedings of the thirty-first annual ACM symposium on theory of computing (STOC'99)*, pages 760–769, 1999.

[26] W. Tutte. A census of planar triangulations. *Canad. J. Math.*, 14:21–38, 1962.

[27] W. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

On the Complexity of the Balanced Vertex Ordering Problem

Jan Kára
Charles University, Prague, Czech Republic

(Joint work with Jan Kratochvíl and David R. Wood)

1 Introduction

We study the complexity of finding a 'balanced' ordering of the vertices of the graph that is used by a several graph drawing algorithms as a starting point. Here balanced means that neighbours of each vertex v are as evenly distributed to the left and right of v as possible. The problem of determining such an ordering was recently studied by Biedl *et al.* [1]. We solve a number of open problems from [1] and study a few other related problems.

Let $G = (V, E)$ be a multigraph without loops. An *ordering* of G is a bijection $\sigma : V \to \{1, \ldots, |V|\}$. For $u, v \in V$ with $\sigma(u) < \sigma(v)$, we say that u is *to the left* of v and that v is *to the right* of u. The *imbalance* of $v \in V$ in σ, denoted by $B_\sigma(v)$, is

$$\big||\{e \in E : e = \{u, v\}, \sigma(u) < \sigma(v)\}| - |\{e \in E : e = \{u, v\}, \sigma(u) > \sigma(v)\}|\big|.$$

When the ordering σ is clear from the context we simply write $B(v)$ instead of $B_\sigma(v)$. The *imbalance* of ordering σ, denoted by $B_\sigma(G)$, is $\sum_{v \in V} B_\sigma(v)$. The minimum value of $B_\sigma(G)$, taken over all orderings σ of G, is denoted by $M(G)$. An ordering with imbalance $M(G)$ is called *minimum*. Clearly the following two facts hold for any ordering:

- Every vertex of odd degree has imbalance at least one.
- The two vertices at the beginning and at the end of any ordering have imbalance equal to their degrees.

These two facts imply the following lower bound on the imbalance of an ordering. Let $\mathrm{odd}(A)$ denote the number of odd degree vertices among the vertices of $A \subseteq V$. Let $(d_1, \ldots, d_n)$ be the sequence of vertex degrees of G, where $d_i \leq d_{i+1}$ for all $1 \leq i \leq n-1$. Then

$$B_\sigma(G) \geq \mathrm{odd}(V) - (d_1 \bmod 2) - (d_2 \bmod 2) + d_1 + d_2.$$

An ordering σ is *perfect* if the above inequality holds with equality. PERFECT ORDERING is the decision problem whether a given multigraph G has a perfect ordering. This problem is clearly in $\mathcal{NP}$.

2 Results

Whether the balanced ordering problem is efficiently solvable for planar graphs with maximum degree four is of particular interest since a number of algorithms for producing orthogonal drawings of planar graphs with maximum degree four start with a balanced ordering of the vertices. We answer this question in the negative:

Theorem 1 *The* PERFECT ORDERING *problem is $\mathcal{NP}$-complete for planar graphs with maximum degree four.*

As the problem we reduce from we use the PLANAR 2–IN-4SAT. The $\mathcal{NP}$-completeness of this problem is also show in our paper. Next we study the case of regular graphs and prove:

Theorem 2 *The* PERFECT ORDERING *problem for 5-regular multigraphs is $\mathcal{NP}$-complete.*

Using a few lemmas we also show that:

Corollary 1 *It is $\mathcal{NP}$-hard to find a minimum ordering for 5-regular simple graphs.*

In the end we describe algorithms solving at least some special cases in a polynomial time. The algorithms are base on the following lemma:

Lemma 1 *There is an $O(n+m)$ time algorithm to test whether a multigraph G with n vertices and m edges has an ordering in which a given list of vertices imbalanced $= (v_1, \ldots, v_k)$ are the only imbalanced vertices, and $\sigma(v_i) < \sigma(v_{i+1})$ for all $1 \leq i \leq k-1$.*

The following theorem is a consequence of the previous lemma:

Theorem 3 *There is an algorithm that, given an n-vertex m-edge multigraph G, computes a minimum ordering of G with at most k imbalanced vertices (or answers that there is no such ordering) in time $O(n^k \cdot (m+n))$.*

Corollary 2 *There is a polynomial time algorithm to determine whether a given multigraph G has an ordering with imbalance less than a fixed constant c.*

Corollary 3 *The* PERFECT ORDERING *problem is solvable in $O(n^2(n+m))$ time for any n-vertex m-edge multigraph with all vertices of even degree.*

References

[1] Therese Biedl, Timothy Chan, Yashar Ganjali, Mohammad Taghi Hajiaghayi, David R. Wood, Balanced vertex-orderings of graphs, *Discrete Applied Mathematics* 148(1), pp. 27–48, 2005.

A Combinatorial Classification of 3-manifolds With Genera 0 and 1[1]

Ján Karabáš
Mathei Bel University, Banská Bystrica, Slovakia

(Joint work with Roman Nedela)

A *3-manifold* $\mathcal{M}$ is a topological space in which every point $x \in \mathcal{M}$ has a neighbourhood $O(x)$ homeomorphic to the Euclidean space E^3. Throughout this paper all 3-manifolds will be considered to be closed, connected and orientable. It is well-known that this class of 3-manifolds can be represented by means of *4-edge-coloured bipartite graphs*. This construction form a base of the *crystallisation theory* founded by Mario Pezzana in early 70-th of the last century. A classical invariant of closed 3-manifolds is the *Heegaard genus*. It is well known that the only 3-manifold with Heegaard genus zero is the 3-sphere S^3 and the 3-manifolds with Heegaard genus one are $S^1 \times S^2$ and lens-spaces. Classification of 3-manifolds with genera zero and one is well-known, see Hempel [4, p. 20-22] for the proof. Analysing "equivalence classes" of crystallisations we present an alternative approach to the classification based on combinatorial ideas.

Every closed, connected 3-manifold can be finitely triangulated [5]. Let us have a 3-manifold $\mathcal{M}$. One can triangulate it constructing the finite *simplicial complex* $S(\mathcal{M})$. Taking the dual (graph) $\Gamma(\mathcal{M})$ of the first barycentric subdivision of $S(\mathcal{M})$ we get a *combinatorial representation* of the manifold $\mathcal{M}$. Since $S(\mathcal{M})$ is of dimension 3 and boundary of $\mathcal{M}$ is empty, the degree of $\Gamma(\mathcal{M})$ is 4. Orientability of $\mathcal{M}$ forces $\Gamma(\mathcal{M})$ to be bipartite. The induced colouring of $\Gamma(\mathcal{M})$ can be easily constructed from the barycentric subdivision. It is proved [6] that this representation is "faithful" e.g. reconstruction of $\mathcal{M}$ from $\Gamma(\mathcal{M})$ is unique. Note that the verices of a crystallisations correspond to a 3-dimensional simplices of simplicial complex $S(\mathcal{M})$ and edges correspond to "gluing" of these simplices in 2-dimensional sub-simplices.

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A bipartite 4-edge-coloured graph $\Gamma(\mathcal{M})$ is called a *crystallisation* (of an orientable 3-manifold $\mathcal{M}$) if every subgraph induced by three colours is planar and connected [6]. Every crystallisation $\Gamma(\mathcal{M})$ can be naturally embedded into a surface S. There are 6 such embeddings for the fixed crystallisation $\Gamma(\mathcal{M})$. The *regular genus of the crystallisation* $\Gamma(\mathcal{M})$ is the minimal genus of surface taken through all these embeddings. A *regular genus of* $\mathcal{M}$ is the minimal genus of surfaces into which all crystallisations $\Gamma(\mathcal{M})$ embeds. It is proved [2] that the regular genus of 3-manifold $\mathcal{M}$ equals to the *Heegaard genus* of $\mathcal{M}$, well known in topology.

A 3-manifold $\mathcal{M}$ can be triangulated in infinitely many ways. Hence infinitely many crystallisations represents $\mathcal{M}$. Deciding whether two crystallisations Γ and Θ represent the same 3-manifold is the combinatorial equivalent of the *homeomorphism problem* – the most known problem of the algebraic topology. It is known [3] that the crystallisations Γ and Θ represents the same 3-manifold $\mathcal{M}$ if and only if Γ transforms into Θ in finitely many applications of *elementary dipole-moves*. Let us describe the elementary dipole move of type I. Let $u, v \in V(\Gamma(\mathcal{M}))$ and let u and v are incident only in one edge coloured by colour, say 0. If u and v belongs into different 1-2-3 connectivity components, cut the vertices u, v and the edge uv from $\Gamma(\mathcal{M})$. After removing this, glue the hanging edges coloured by 1,2 and 3. This operation is called *removing of dipole-move* (of type I). *Adding of dipole-move* (of type I) is the complementary operation and it can be done in the following way. Find 3-edge-cut coloured by the colours 1,2 and 3. Cut these edges and insert the dipole-move of type I. There are also dipole-moves of types II and III, as depicted on Figure 1 and one can use them in similar way as described.

Compositions of elementary dipole-moves induces the *dipole-move equivalence* on the set of crystallisations. The crystallisation $\Gamma(\mathcal{M})$ which does not allow the removing of dipole-move of any type is called a *reduced crystallisation*. Such crystallisation $\Gamma(\mathcal{M})$ can be taken as "representative" of 3-manifold $\mathcal{M}$.

As concerns crystallisations of 3-manifolds with regular genus zero we have the following claim.

Theorem 1 *Every bipartite 4-edge-coloured graph of regular genus zero is dipole-move equivalent to the 4-dipole, in other words, all planar crystallisations are dipole-move equivalent.*

The proof of Theorem 1 refers only to the properties of reduced crystallisations and Euler's equation.

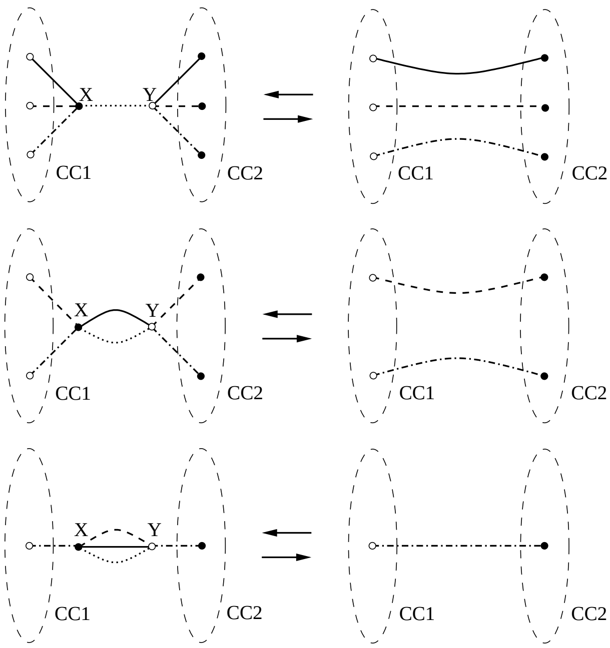

Figure 1: Elementary dipole moves

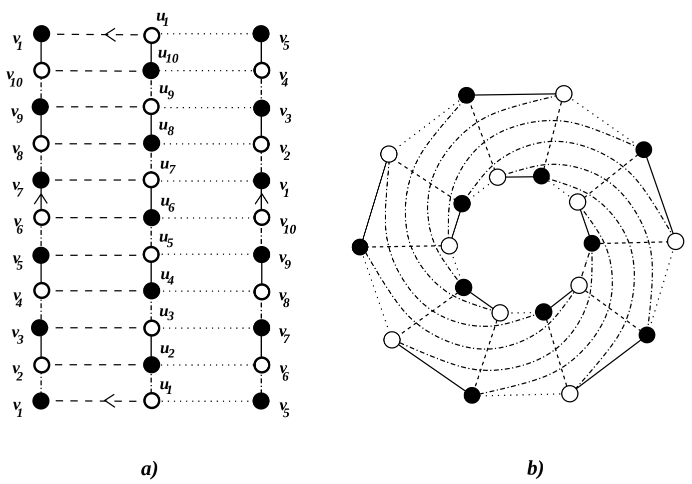

Figure 2: (a) Shifted rectangular grid of type $(2 \times 10; 4)$ and (b) its another drawing

A bit complicated is the classification of crystallisations of regular genus one. By a *shifted toroidal rectangular grid* of type $(k \times n; m)$ we mean a graph arising from a $k \times n$ rectangular grid by identifying the opposite horizontal an vertical sides. The vertical sides are identified with shift $m \geq 0$. Figure 2 shows the shifted (toroidal) rectangular grid of type $(2 \times 10; 4)$. By its definition every shifted rectangular grid forms a toroidal map of type $(4, 4)$. Altshuler and in a more general framework Thomassen have proved the converse implication [1, 7].

Theorem 2 *A reduced crystallisation of regular genus one is isomorphic either to a shifted rectangular grid of type* $(2 \times n, m)$*, for some* $m, n \in N$ *and* $(m, n) = 2$ *or it is dipole-move equivalent to the exceptional graph* $\mathcal{S}$ *depicted on Figure 3*

Note that the shifted rectangular grids of type $(2 \times n, m)$ determine crystallisations representing lens spaces $\mathcal{L}(p, q)$ where $p = \frac{m}{2}$ and $q = \frac{n}{2}$. The exceptional graph $\mathcal{S}$ determines a crystallisation of the space $S^1 \times S^2$. The classification of 3-manifolds with higher genera is intractable in this sense.

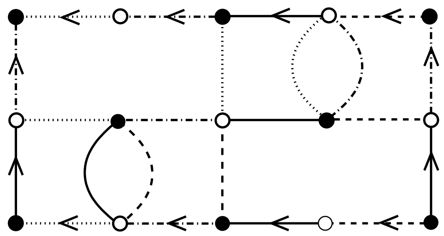

Figure 3: Non-simple crystallisation with regular genus one

Dipole-move equivalence gives us many possibilities to study 3-manifolds using combinatorial methods. The following problems are crucial in the topic.

Problem 1 *Is the problem to decide whether two 4-valent 4-edge-coloured bipartite graphs, say* Γ *and* Θ*, dipole-move equivalent algorithmically soluble, or not?*

Problem 2 *Can one find the finite sequence of dipole moves transforming* Γ *to* Θ *if they are dipole-move equivalent?*

References

[1] Amos Altshuler, *Construction and enumeration of regular maps on the torus*, Discrete Math. **4** (1973), 201–217.

[2] Paola Bandieri, Maria Rita Casali, and Carlo Gagliardi, *Representing manifolds by crystallization theory: foundations, improvements and related results*, Atti Sem. Mat. Fis. Univ. Modena **49** (2001), no. suppl., 283–337, Dedicated to the memory of Professor M. Pezzana (Italian).

[3] Massimo Ferri and Carlo Gagliardi, *Crystallisation moves*, Pacific J. Math. **100** (1982), no. 1, 85–103. MR661442 (83i:57011)

[4] John Hempel, 3-*Manifolds*, Princeton University Press, Princeton, N. J., 1976, Ann. of Math. Studies, No. 86. MR0415619 (54 #3702)

[5] Edwin E. Moise, *Affine structures in* 3-*manifolds. V. The triangulation theorem and Hauptvermutung*, Ann. of Math. (2) **56** (1952), 96–114.

[6] Mario Pezzana, *Sulla struttura topologica delle varietà compatte*, Ati Sem. Mat. Fis. Univ. Modena **23** (1974), no. 1, 269–277 (1975).

[7] Carsten Thomassen, *Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface*, Trans. Amer. Math. Soc. **323** (1991), no. 2, 605–635.

Distribution of the Size of a Largest Planar Matching and Largest Planar Subgraph in Random Bipartite Graphs

Marcos Kiwi[1]
University of Chile, Santiago, Chile

(Joint work with Martion Loebl)

1 Introduction

Let U and V henceforth denote two disjoint totally ordered sets (both ordered relations will be referred to by $\preceq$). Typically, we will consider the case where $|U| = |V| = n$ and denote the elements of U and V by $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ respectively. Henceforth, we will always assume that the latter enumeration respects the ordered relation in U or V, i.e., $u_1 \preceq u_2 \preceq \ldots \preceq u_n$ and $v_1 \preceq v_2 \preceq \ldots \preceq v_n$.

Let $G = (U, V; E)$ denote a bipartite multi–graph with color classes U and V. Two distinct edges uv and $u'v'$ of G are said to be *noncrossing* if u and u' are in the same order as v and v'; in other words, if $u \prec u'$ and $v \prec v'$ or $u' \prec u$ and $v' \prec v$. A matching of G is called *planar* if every distinct pair of its edges is noncrossing. We let $L(G)$ denote the number of edges of a maximum size (largest) planar matching in G (note that $L(G)$ depends on the graph G *and* on the ordering of its color classes).

For the sake of simplicity we will concentrate solely in the case where $|E| = rn$ and G is r–regular.

When $r = 1$, an r–regular multi–graph with color classes U and V uniquely determines a permutation. A planar matching corresponds thus to an increasing sequence of the permutation, where an increasing sequence of length L of a permutation π of $\{1, \ldots, n\}$ is a sequence $1 \leq i_1 < i_2 <$

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$\ldots < i_L \leq n$ such that $\pi(i_1) < \pi(i_2) < \ldots < \pi(i_L)$. The Longest Increasing Sequence (LIS) problem concerns the determination of the asymptotic, on n, behavior of the LIS for a randomly and uniformly chosen permutation π. The LIS problem is also referred to as "Ulam's problem" (e.g., in [Kin73, BDJ99, Oko00]). Ulam is often credited for raising it in [Ula61] where he mentions (without reference) a "well–known theorem" asserting that given $n^2 + 1$ integers in any order, it is always possible to find among them a monotone subsequence of $n + 1$ (the theorem is due to Erdős and Szekeres [ES35]). Monte Carlo simulations are reported in [BB67], where it is observed that over the range $n \leq 100$, the limit of the LIS of $n^2 + 1$ randomly chosen elements, when normalized by n, approaches 2. Hammersley [Ham72] gave a rigorous proof of the existence of the limit and conjectured it was equal to 2. Later, Logan and Shepp [LS77], based on a result by Schensted [Sch61], proved that $\gamma \geq 2$; finally, Vershik and Kerov [VK77] obtained that $\gamma \leq 2$. In a major recent breakthrough due to Baik, Deift, Johansson [BDJ99] the asymptotic distribution of the LIS has been determined. For a detailed account of these results, history and related work see the surveys of Aldous and Diaconis [AD99] and Stanley [Sta02].

From the previous discussion, it follows that one way of generalizing Ulam's problem is to study the distribution of the size of the largest planar matching in randomly chosen r–regular bipartite multi–graphs (for a different generalization see [Ste77, BW88]). This line of research, originating in [KL02], turns out to be relevant for the study of several other issues like the Longest Common Subsequence problem (see [KLM05]), interacting particle systems [Sep77], digital boiling [GTW01], and is directly related to topics such as percolation theory [Ale94] and random matrix theory [Joh99].

1.1 Main Results

We establish combinatorial identities which express $g(n; d)$ — the number of r-regular bipartite multi–graphs with planar matchings with at most d edges — in terms of:

- The number of pairs of standard Young tableaux of the same shape and with a "descend-type" property.
- A signed sum of restricted lattice walks in $\mathbb{Z}^d$.

Our arguments can be extended in order to characterize the distribution of the largest size of planar subgraphs of randomly chosen r–regular bipartite multi–graphs.

References

[AD99] D. Aldous and P. Diaconis. Longest increasing subsequences: From patience sorting to the Baik–Deift–Johansson theorem. *Bull. of the AMS*, 36(4):413–432, 1999.

[Ale94] K. S. Alexander. The rate of convergence of the mean length of the longest common subsequence. *The Annals of Applied Probability*, 4(4):1074–1082, January 1994.

[BB67] R. Baer and P. Brock. Natural soritng over permutation spaces. *Mathematics of Computation*, pages 385–410, 1967.

[BDJ99] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutaions. *J. Amer. Math. Soc.*, 12:1119–1178, 1999.

[BW88] B. Bollobás and P. Winkler. The longest chin among random points in euclidean space. *Proceedings of the American Mathematical Society*, 103(2):347–353, 1988.

[ES35] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[GTW01] J. Gravner, C. Tracy, and H. Widom. Limit theorems for height fluctuations in a class of discrete space time growth models. *J. Stat. Phys.*, 102:1085–1132, 2001.

[Ham72] J. M. Hammersley. A few seedlings of research. In *Proc. Sixth Berkeley Sympos. Math. Stat. Prob.*, pages 345–394, Berkeley, Calif., 1972. Univ. of California Press.

[Joh99] K. Johansson. Shape fluctuations and random matrices. Technical Report math.CO/9903134, arXiv, October 1999.

[Kin73] J. F. C. Kingman. Subadditive ergodic theory. *The Annals of Prob.*, 1(6):883–909, 1973.

[KL02] M. Kiwi and M. Loebl. Largest planar matching in random bipartite graphs. *Random Structures and Algorithms*, 21(2):162–181, 2002.

[KLM05] M. Kiwi, M. Loebl, and J. Matoušek. Expected length of the longest common subsequence for large alphabets. *Advances in Mathematics*, 2005. In press.

[LS77] B. Logan and L. Shepp. A variational problem or random Young tableaux. *Adv. in Math.*, 26:206–222, 1977.

[Oko00] A. Okounkov. Random matrices and ramdom permutations. *International Mathematics Research Notices*, pages 1043–1095, 2000.

[Sch61] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.

[Sep77] T. Seppäläinen. Exact limiting shape for a simplified model of first–passage percolation on the plane. *The Annals of Probability*, 26(3):1232–1250, 1977.

[Sta02] R. Stanley. Recent progress in algebraic combinatorics. *Bull. of the AMS*, 40(1):55–68, 2002.

[Ste77] J. M. Steele. Limit properties of random variables associated with a partial ordering of $\mathbb{R}^d$. *The Annals of Probability*, 5(3):395–403, 1977.

[Ula61] S. Ulam. Monte Carlo calculations in problems of mathematical physics. In *Modern Mathematics for the Engineers*, pages 261–281. McGraw-Hill, 1961.

[VK77] A. Vershik and S. Kerov. Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux. *Dokl. Akad. Nauk SSSR*, 233:1024–1028, 1977.

Numbers of nonnested trees and a historical remark on numbers of noncrossing trees

Martin Klazar
Charles University, Prague, Czech Republic

A graph $G = ([n], E)$ on the vertex set $[n] = \{1, 2, \ldots, n\}$ is called *noncrossing* if no two distinct edges cross each other ($\min e < \min f < \max e < \max f$ for no two edges $e, f \in E$) and it is *nonnesting* if no two distinct edges are nested ($\min e < \min f < \max f < \max e$ for no two edges $e, f \in E$); in these definitions G is regarded as an ordered graph, with the standard total order on $[n]$.

Problem 1. What is the number t_n of nonnested trees $T = ([n], E)$ with n vertices? What about forests?

Problem 2. Can at least nonnested paths $P = ([n], E)$ with n vertices be enumerated?

Remarks and comments. 1. There is, of course, a trivial algorithm that after exponentially many steps (in n) outputs, for the input n, the value t_n, and similarly for the other problems. As an answer we accept only algorithms making polynomially many steps in n (or at least substantially fewer steps than is the cardinality of the enumerated set, which in all cases here is easily seen to be exponential in n), preferably in the form of a formula or a relation for the generating function, see Wilf [7].

2. Formulas for numbers (or corresponding generating functions) of noncrossing trees, forests, paths, graphs etc. are known and not too hard to obtain, see Flajolet and Noy [2] and Noy [5]. Also, nonnesting graphs can be enumerated, see Klazar [4].

3. I use this opportunity to point out a priority claim for the formula

$$\frac{1}{2n-1}\binom{3n-3}{n-1}$$

for the number of noncrossing trees $T = ([n], E)$ with n vertices. The paper [1] by Dulucq and Penaud is cited (e.g., Stanley [6, p. 138], Noy [5]) as the first explicit source for this result. However, the formula appears explicitly much earlier in the work of J. Jiřička [3, Theorem 3 on p. 59] and by the remark in [3, p. 59] it was obtained even earlier by a different method by L. Nebeský.

I want to thank Jiří Havelka for providing me a copy of [3].

References

[1] S. Dulucq and J.-G. Penaud, Cordes, arbres et permutations, *Discrete Math.* **117** (1993), 89–105.

[2] P. Flajolet and M. Noy, Analytic combinatorics of non-crossing configurations, *Discrete Math.* **204** (1999), 203–229.

[3] J. Jiřička, The number of projective trees with a given number of vertices, *Prague Bull. Math. Linguist.* **no. 24** (1975), 51–60; MR0432483.

[4] M. Klazar, Counting pattern-free set partitions. II. Noncrossing and other hypergraphs, *Electron. J. Combin.* **7** (2000), Research Paper 34, 25 pp.

[5] M. Noy, Enumeration of noncrossing trees on a circle, *Discrete Math.* **180** (1998), 301–313.

[6] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge University Press, Cambridge, 1999.

[7] H. S. Wilf, What is an answer?, *Amer. Math. Monthly* **89** (1982), 289–292.

Decomposition of Flow Polynomials[1]

Martin Kochol

Slovak Academy of Sciences, Bratislava, Slovakia

1 General formula

We consider finite undirected graphs with multiple edges and loops. If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. If e is an edge of G, then $G-e$ and G/e denote the graphs arising from G after deleting and contracting e, respectively.

Let A be an additive Abelian group of order k. Choose an arbitrary but fixed orientation of G. Then a nowhere-zero A-flow in G is a mapping $\varphi : E(G) \to A \setminus \{0\}$ so that for every vertex, the sum of the values of φ on the incoming edges equals the sum on the outgoing ones (see cf [1, 2]). The number of nowhere-zero A-flows in G does not depend on the chosen orientation and, by Tutte [5], neither on the structure of A, but only on its order k. Thus we can denote this number by $F_G(k)$. By Tutte [4], $F_G(k)$ is a polynomial function of k. This is called the *flow polynomial* of G.

Let $C = \{e_1, \dots, e_n\}$ be an edge cut of a graph G. Then $G-G$ is disjoint union of two subgraphs I and I' so that every edge of C has exactly one end in each of I and I'. Let v_i and v_i' $(i = 1, \dots, n)$ denote the ends of e_i in I and I', respectively. For $i = 1, \dots, n$, apply the following construction: delete e_i and add vertices u_i, u_i' and edges u_iv_i, $u_i'v_i'$. Let H (H') be the graph arising from I (I') after adding edges $u_1v_1, \dots, u_nv_n$ $(u_1'v_1', \dots, u_n'v_n')$.

Let $P = \{Q_1, \dots, Q_r\}$ be a partition of the set $\{1, \dots, n\}$. For $j = 1, \dots, r$, identify the set of vertices $\{u_i; i \in Q_j\}$ $(\{u_i'; i \in Q_j\})$ to a new vertex x_j (x_j') and denote the resulting graph by $[H, P]$ $([H', P])$. If one of $Q_1, \dots, Q_r$ is a singleton, P is called *nonproper*, otherwise it is called *proper*. Let $\mathcal{P}_n = \{P_{n,1}, \dots, P_{n,p_n}\}$, $p_n = |\mathcal{P}_n|$, denote the set of proper partitions of $\{1, \dots, n\}$. We write $[H, i]$, $[H', i]$, $F_{H,i}(k)$ and $F_{H',i}(k)$ instead of $[H, P_{n,i}]$, $[H', P_{n,i}]$, $F_{[H,P_{n,i}]}(k)$ and $F_{[H',P_{n,i}]}(k)$, respectively. Furthermore, $p_2 =$

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$p_3 = 1$ and for every $n \geq 4$, we have

$$p_n = 1 + \sum_{i=2}^{n-2} \binom{n-1}{i-1} p_{n-i}.$$

Take n isolated edges $w_1w_1', \ldots, w_nw_n'$ and $P, P' \in \mathcal{P}_n$, where $P = \{Q_1, \ldots, Q_r\}$, $P' = \{Q_1', \ldots, Q_{r'}'\}$. For $j = 1, \ldots, r$ and $j' = 1, \ldots, r'$, identify the sets of vertices $\{w_i; i \in Q_j\}$ and $\{w_i'; i \in Q_{j'}'\}$ to new vertices y_j and $y_{j'}'$, respectively, and denote the resulting graph by $[P, P']$. We write $[i, j]$ and $F_{i,j}(k)$ instead of $[P_{n,i}, P_{n,j}]$ and $F_{[P_{n,i}, P_{n,j}]}(k)$, respectively.

Let $M_n(k)$ be the $p_n \times p_n$ a symmetric matrix so that the (i, j)-entry is $F_{i,j}(k)$, $i, j \in \{1, \ldots, p_n\}$. Let $M_{G,C}(k)$ be the matrix arising from $M_n(k)$ after adding (p_n+1)st row and (p_n+1)st column so that the (p_n+1, p_n+1)-entry is $F_G(k)$, the $(p_n + 1, i)$-entry is $F_{H,i}(k)$ and the $(i, p_n + 1)$-entry is $F_{H',i}(k)$, $i \in \{1, \ldots, p_n\}$. Let $M_{G,C}'(k)$ arises from $M_{G,C}(k)$ after replacing the (p_n+1, p_n+1)-entry by 0. Let $M_{n,i,j}(k)$ denote the matrix arising from $M_n(k)$ after deleting the ith row and the jth column, $i, j \in \{1, \ldots, p_n\}$. If M is a matrix, then $|M|$ denotes the determinant of M.

Theorem 1 *Let G be a graph and $C = \{e_1, \ldots, e_n\}$, $n \geq 2$, be an edge cut of G. Then $|M_n(k)|$ is a nonzero polynomial of degree $\sum_{i=1}^{p_n}(n - r_i)$,*

$$|M_{G,C}(k)| \;=\; F_G(k) \cdot |M_n(k)| \; +$$

$$\sum_{i=1}^{p_n}\sum_{j=1}^{p_n}(-1)^{2p_n+1+i+j} \cdot F_{H,i}(k) \cdot F_{H',j}(k) \cdot |M_{n,i,j}(k)| \;=\; 0,$$

and

$$F_G(k) = -|M_{G,C}'(k)|/|M_n(k)| =$$

$$\left(\sum_{i=1}^{p_n}\sum_{j=1}^{p_n}(-1)^{2p_n+i+j} \cdot F_{H,i}(k) \cdot F_{H',j}(k) \cdot |M_{n,i,j}(k)|\right) |M_n(k)|^{-1}.$$

In particular, if $n = 2$ (resp. $n = 3$), then $\mathcal{P}_n$ has exactly one partition, $|M_n(k)|$ is equal to $k - 1$ (resp. $k^2 - 3k + 2$), and from Theorem 1 we get the following (see also Sekine and Zhang [3]).

Corollary 1 *Let G be a graph with an edge cut C of cardinality $n = 2$ or 3. Then $F_G(k) = F_{H,1}(k) \cdot F_{H',1}(k)/|M_n(k)|$, where $|M_2(k)| = k - 1$ and $|M_3(k)| = k^2 - 3k + 2$.*

2 Planar case

We identify a planar graph with its embedding in the plane. An edge cut C of a planar graph G is called *planar* if there exists a closed Jordan curve in the plane which intersects each edge of C exactly once and does not intersect any other edge or vertex of G.

Let $P = \{Q_1, \ldots, Q_r\}$ be a partition of the set $\{1, \ldots, n\}$. Take a circuit C_n having vertices $v_1, \ldots, v_n$ and add vertices $u_1, \ldots, u_n$ and edges $v_1u_1, \ldots, v_nu_n$. For $j = 1, \ldots, r$, identify the set of vertices $\{u_i; i \in Q_j\}$ to a new vertex x_j. Suppose that the resulting graph has an embedding in the plane so that no vertex x_j, $j = 1, \ldots, r$, is inside the circuit C_n and no two edges intersect. Then P is called *planar*. Let $\overline{\mathcal{P}}_n$ be the set of nontrivial planar partitions of $\{1, \ldots, n\}$ and $\overline{p}_n = |\overline{\mathcal{P}}_n|$. Then $\overline{p}_1 = 0$, $\overline{p}_2 = \overline{p}_3 = \overline{p}_0 = 1$, and for every $n \geq 4$,

$$\overline{p}_n = \sum_{i=1}^{n-1} \overline{p}_{i-1}(\overline{p}_{n-i-1} + \overline{p}_{n-i}).$$

Let $\overline{M}_n(k)$, $\overline{M}_{G,C}(k)$ and $\overline{M}'_{G,C}(k)$ be the matrices arising from $M_n(k)$, $M_{G,C}(k)$ and $M'_{G,C}(k)$ after deleting the rows and columns corresponding to nonplanar partitions of $\mathcal{P}_n$, respectively. Let $\overline{M}_{n,i,j}(k)$ denote the matrix arising from $\overline{M}_n(k)$ after deleting the ith row and the jth column, $i, j \in \{1, \ldots, \overline{p}_n\}$.

Theorem 2 *Let G be a planar graph and $C = \{e_1, \ldots, e_n\}$, $n \geq 2$, be a planar edge cut of G. Then $|\overline{M}_n(k)|$ is a nonzero polynomial of degree $\sum_{i=1}^{\overline{p}_n}(n - r_i)$,*

$$|\overline{M}_{G,C}(k)| = F_G(k) \cdot |\overline{M}_n(k)| + \sum_{i=1}^{\overline{p}_n}\sum_{j=1}^{\overline{p}_n}(-1)^{2\overline{p}_n+1+i+j} \cdot F_{H,i}(k) \cdot F_{H',j}(k) \cdot |\overline{M}_{n,i,j}(k)| = 0,$$

and

$$F_G(k) = -|\overline{M}'_{G,C}(k)|/|\overline{M}_n(k)| = \left(\sum_{i=1}^{\overline{p}_n}\sum_{j=1}^{\overline{p}_n}(-1)^{2\overline{p}_n+i+j} \cdot F_{H,i}(k) \cdot F_{H',j}(k) \cdot |\overline{M}_{n,i,j}(k)|\right) |\overline{M}_n(k)|^{-1}.$$

References

[1] F. Jaeger, Nowhere-zero flow problems, in L.W. Beineke, R.J. Wilson (Eds.), *Selected Topics in Graph Theory 3*, Academic Press, New York, 1988, pp. 71–95.

[2] M. Kochol, Superposition and constructions of graphs without nowhere-zero k-flows, *European J. Combin.* **23** (2002) 281–306.

[3] K. Sekine and C.-Q. Zhang, Decomposition of the flow polynomial, *Graphs Combin.* **13** (1997) 189–196.

[4] W.T. Tutte, A contribution to the theory of chromatic polynomials, *Canad. J. Math.* **6** (1954) 80–91.

[5] W.T. Tutte, A class of Abelian groups, *Canad. J. Math.* **8** (1956) 13–28.

Hamilton Cycles in Tough Chordal Graphs

Daniel Kráľ'
Charles University, Prague, Czech Republic

(Joint work with Tomáš Kaiser and Ladislav Stacho)

The notion of toughness is a well-established notion closely related to hamiltonian graphs. A graph G is *β-tough* if for every set A of its vertices, $G \setminus A$ is connected or the number $\kappa(G \setminus A)$ of its components does not exceed $|A|/\beta$. Clearly, if G contains a Hamilton cycle, then G is 1-tough. A famous conjecture of Chvátal asserts that there exists a constant β such that every β-tough graph is hamiltonian. It is known that the conjecture is not true with $\beta < 9/4$.

The conjecture was verified for several special classes of graphs, among those interval graphs, split graphs and chordal graphs. Interval graphs coincide with the intersection graphs of subpaths of a path, split graphs with the intersection graphs of subtrees of a star and chordal graphs with the intersection graphs of subtrees of a tree. It is known that 1-tough interval graphs and 3/2-tough split graphs are hamiltonian and the bounds are tight. On the other hand, it is known that 18-tough chordal graphs are hamiltonian but there exist $(7/4 - \varepsilon)$-tough chordal graphs that are not hamiltonian. Motivated by these results, we study the existence of Hamilton cycles in spider graphs, intersection graphs of subtrees of a subdivision of a star. We show using the matroid intersection theorem that 3/2-tough spider graphs are hamiltonian. Since the class of spider graphs includes both interval and split graphs, our result is tight.

Edge-coloring of Multigraphs

Naďa Krivoňáková
University of Žilina, Žilina, Slovakia

(Joint work with Martin Kochol and Silvia Smejová)

We introduce a monotone invariant $\pi(G)$ on graphs and show that it is an upper bound of the chromatic index of graphs. Moreover, there exist polynomial time algorithms for computing $\pi(G)$ and for coloring edges of a multigraph G by $\pi(G)$ colors. This generalizes the classical edge-coloring theorems of Shannon and Vizing.

Operations With Structures

Ralph McKenzie
Vanderbilt University, Nashville, USA

Consider a family $\mathcal{F}$ of similar structures closed under direct (Cartesian) products — whenever $\mathbf{A}, \mathbf{B} \in \mathcal{F}$ then $\mathbf{A} \times \mathbf{B} \in \mathcal{F}$. Thus we have an algebra $\langle \mathcal{F}, \times \rangle$. The relation of isomorphism, $\cong$, induces a congruence on this algebra; and $\mathcal{F}, \times \rangle / \cong$ is a commutative semigroup. If $\mathcal{G}$ is, for example, the class of all finite groups, then $\langle \mathcal{G}, \times \rangle / \cong$ is a denumerably infinite, commutative, monoid. It was shown by R. Remak and J. H. M. Wedderburn, early in this century, that this monoid is isomorphic to the positive integers under multiplication. Actually, they proved the fundamental property of direct product, considered as an abstract operation over finite groups: namely, that every finite group $\mathbf{G}$ can be expressed as a direct product of directly indecomposable groups, $\mathbf{G} \cong \mathbf{H}_1 \times \cdots \times \mathbf{H}_n$, in essentially only one way, if we admit re-ordering the list of factors and replacing factors by isomorphic groups.

The class $\mathcal{P}$ of all finite ordered (i.e., partially ordered) sets under the operation of direct product does not have this unique factorization property. However, if we consider $\mathcal{P}$ under the two operations $\mathbf{A} \times \mathbf{B}$ (direct product) and $\mathbf{A}+\mathbf{B}$ (disjoint union), then there is a nice result available, analogous to the Wedderburn-Remak theorem. It was proved by J. Hashimoto in 1951, and states that $\langle \mathcal{P}, \times, + \rangle / \cong$ is isomorphic to the semiring $Z^+[\bar{x}]$ consisting of all non-zero polynomials $f(\bar{x})$ with non-negative integral coefficients in the indeterminates $\bar{x} = (x_0, x_1, \ldots, x_n \ldots)$, and where the operations are ordinary multiplication and addition of polynomials. The Hashimoto theorem encompasses two results: every finite ordered set has an essentially unique expression as isomorphic to a disjoint union of connected ordered sets; and every finite, connected, ordered set has an essentially unique expression as isomorphic to a direct product of directly indecomposable (and connected) ordered sets.

Alfred Tarski, along with C.C. Chang and Bjarni Jónsson and several other collaborators, proved in the late 50's and early 60's many deep results about direct product considered, up to isomorphism, as an operation on various classes of structures. He asked if it is true for any two finite structures or algebras of the same signature, $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A} \not\cong \mathbf{B}$, that for all positive integers n, $\mathbf{A}^n \not\cong \mathbf{B}^n$. Laszlo Lovasz proved that this is true with an elegant counting argument. Lovasz' entire argument occupies only seven pages in his 1967 paper, *Operations with structures.* He actually proves the following: Let $\mathcal{F}$ be the class of all finite structures of some fixed signature. Then $\langle \mathbf{F}, \times \rangle / \cong$ is embeddable into the monoid $\langle N, \circ \rangle^{\omega}$ (the denumerable direct power of the non-negative integers under multiplication). He does this by showing that if for all finite $\mathbf{C}$ similar to $\mathbf{A}$ and to $\mathbf{B}$, the sets $\mathrm{hom}(\mathbf{C}, \mathbf{A})$ and $\mathrm{hom}(\mathbf{C}, \mathbf{B})$ are of the same size, then $\mathbf{A} \cong \mathbf{B}$ follows.

G. Birkhoff introduced a third binary operation on ordered sets, the exponent. For ordered sets, $\mathbf{A}$ and $\mathbf{B} = \langle B, \leq \rangle$, $\mathbf{A}^{\mathbf{B}}$ is an ordered set defined as the set $\mathrm{hom}(\mathbf{B}, \mathbf{A})$ of all monotone mappings from $\mathbf{B}$ to $\mathbf{A}$, with the order induced by the inclusion of the set $\mathrm{hom}(\mathbf{B}, \mathbf{A})$ into the direct power $\mathbf{A}^{B}$. Birkhoff asked, in 1941, if $\mathbf{A}^{\mathbf{C}} \cong \mathbf{B}^{\mathbf{C}}$ implies $\mathbf{A} \cong \mathbf{B}$ where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are arbitrary finite ordered sets. Lovasz' result mentioned above proves that Birkhoff cancellation does hold in the case that $\mathbf{C}$ is discrete (an antichain). I proved Birkhoff's conjecture in 2000. I would like to present some ideas from the proof in this talk, if there is time. I was able to solve the word problem for the algebra of isomorphism types of finite ordered sets under direct product, disjoint union, and exponent$^{(1)}$, where exponent$^{(1)}$ is an operation closely related to Birkhoff's exponent, but differing from it for some ordered sets that do not possess least or greatest elements. I will conjecture that Birkhoff cancellation also holds for graphs (with some suitable restrictions) if we adapt his definition of $\mathbf{A}^{\mathbf{B}}$ in a natural way to the case that $\mathbf{A}, \mathbf{B}$ are graphs.

On the Existence of Tied Perfect Matchings in $G(2n, p)$

Enrique García Moreno
Cinvestav, Mexico

(Joint work with Tommy R. Jensen and Oleg Pikhurko)

We report on the existence with high probability of perfect matchings which contract to a complete graph in graphs in $G(2n, p)$, the binomial model for random graphs with $2n$ vertices, when the edge probability p is derived from the bound obtained from the Erdős, Suen, and Winkler restricted random graph process which generates complements of edge-minimal graphs of stability $\alpha(G) = 2$. This problem is of interest in connection to the Hadwiger conjecture for graphs of stability 2. Our findings give overwhelmingly positive evidence for Hadiwger's conjecture. In particular we show with second moment methods that perfect matchings which contract to a complete graph exist asymptotically almost surely in graphs of $G(2n, p)$ when $p = 1 - o(n^{2/7})$.

Circulants and the Chromatic Index of Steiner Triple

Roman Nedela
Mathei Bel University, Banská Bystrica, Slovakia

(Joint work with Mariusz Meszka and Alex Rosa)

A *Steiner triple system* $STS(v)$ is a pair (V, B), where V is a set of v points and B is a collection of sets of cardinality 3, called triples or blocks, satisfying the following condition: each pair x, y of points is contained in exactly one triple. It is well known that a Steiner triple system on v points exists if and only if $v \equiv 1$ or 3 mod 6.

A *block-color class* is a system of pairwise disjoint triples. An m-block-coloring is a partitioning of the set B into m color classes. The *chromatic index* $\chi'(S)$ of a Steiner triple system S is the least m for which an m-block-coloring exists. The *block intersection graph* of a Steiner triple system $S = (V, B)$ is a graph with the vertex set B; the vertices are adjacent if and only if the respective triples intersect. Since the degree of a block intersection graph equals $3(v-3)/2$, Brooks' theorem gives an upper bound $\chi'(S) \leq 3(v-3)/2$ for $v > 7$. An obvious lower bound is $\chi'(S) \geq (v-1)/2$ if $v \equiv 3$ mod 6, and $\geq (v+1)/2$ for $v \equiv 1$ mod 6. Hence we have $(v-1)/2 \leq \chi'(S) \leq 3(v-3)/2$ if $v \equiv 3$ mod 6 and $(v+1)/2 \leq \chi'(S) \leq 3(v-3)/2$ if $v \equiv 1$ mod 6. The lower bound $\chi'(S) = (v-1)/2$ is reached if and only if the Steiner triple system is resolvable.

The upper bound $\chi'(S) \leq 3(v-3)/2$ seems to be weak in general. In fact, using probabilistic methods Pippenger and Spencer in [5] proved that $\chi'(STS(v))$ is asymptotic to $v/2$. Also no examples of STS with $v > 7$ exceeding the above lower bounds by more than two are known. For more information on the chromatic index of Steiner triple systems the reader is referred to Chapter 18 of [2].

For some classes of STS the upper bound was improved. In particular, Colbourn and Colbourn [1] improved it for cyclic $STS(v)$ by proving $\chi'(STS(v)) \leq v$. A Steiner triple system $STS(v)$ is called *cyclic* if it is isomorphic to one whose points are $0, 1, \ldots, v-1$ and the mapping $i \mapsto i+1$ mod v is an automorphism. The result in [1] is based on the following idea. Let $S = STS(v)$ be a cyclic Steiner triple system. The block intersection graph has $v(v-1)/6$ vertices and it admits an induced action of the cyclic group of order v. Then the orbits of the induced action decompose the intersection graph into $(v-1)/6$ six-valent circulants of order v if $v \equiv 1 \mod 6$, and into $(v-3)/6$ six-valent circulants of order v and one short orbit if $v \equiv 3 \mod 6$. By Brooks' theorem each of the circulants (with one exception) can be colored by 6 colors. Taking different sets of colors for different orbits one gets a v-block coloring of any cyclic STS.

It is not difficult to see that each of the six-valent circulants of order v is determined by a triple of positive integers of the form $\{a, b, a+b\}$. Note that $\{a, b, a+b\}$ need not generate the cyclic group Z_v, hence the graph may be a disjoint union of isomorphic circulants of smaller order. Employing results of Yeh and Zhu [6], and Collins and Hutchinson [3] we complete the determination of the chromatic number of 6-valent circulants of the form $C(n; a, b, a+b)$ as follows.

Theorem 1 *Let $G = C(n; a, b, c)$ be a connected 6-valent circulant, where $n \geq 7$, $c = a+b$ or $n-c = a+b$ are pairwise distinct positive integers different from $n/2$. Let $\chi(G)$ be the chromatic number of G. Then*

1. $\chi(G) = 7$ *if and only if* $G \cong K_7 \cong C(7; 1, 2, 3)$,
2. $\chi(G) = 6$ *if and only if* $G \cong T_{11} \cong C(11; 1, 2, 3)$,
3. $\chi(G) = 5$ *if and only if* $G \cong C(n; 1, 2, 3)$ *and* $n \neq 7, 11$ *is not divisible by* 4, *or* G *is isomorphic to one of the following circulants:* $C(13; 1, 3, 4)$, $C(17; 1, 3, 4)$, $C(18; 1, 3, 4)$, $C(19; 1, 7, 8)$, $C(25; 1, 3, 4)$, $C(26; 1, 7, 8)$, $C(33; 1, 6, 7)$, $C(37; 1, 10, 11)$.
4. $\chi(G) = 3$ *if and only if* $3|n$ *and none of* a, b, c *is divisible by 3.*
5. $\chi(G) = 4$ *in all the remaining cases.*

Using Theorem 1 we get the following improved upper bounds on the chromatic index of cyclic Steiner triple systems.

Corollary 1 *Let $S = STS(v)$ be a cyclic Steiner triple system. Then*

(a) $\chi'(S) \leq \frac{5v+13}{6}$ *if* $v \equiv 1 \mod 6$,

(b) $\chi'(S) \leq \frac{5v+9}{6}$ *if* $v \equiv 3 \mod 6$.

Let us call circulants of type (1), (2) or (3) from Theorem 1 *exceptional*.

Corollary 2 *Let $S = STS(v)$ be a cyclic Steiner triple system of order v, and assume that no circulant induced on an orbit of the block intersection graph contains as a component an exceptional circulant. Then $\chi'(S) \leq \frac{2}{3}(v-1)$ if $v \equiv 1$ $(mod\ 6)$, and $\chi'(S) \leq \frac{2}{3}v - 1$ if $v \equiv 3$ $(mod\ 6)$.*

Let $min_v = \frac{v-1}{2}$ if $v \equiv 3$ $(mod\ 6)$ and $min_v = \frac{v+1}{2}$ if $v \equiv 1$ $(mod\ 6)$. We checked on a computer that for all *cyclic* Steiner triple systems of order $v \leq 43$ (note that the number of cyclic Steiner triple systems of order 43 is known to be 9508, not a small number [2]) , we have for the chromatic index the bounds $min_v \leq \chi' \leq min_v + 2$. This suggests that the following could be true:

Conjecture 1 *The chromatic index of every cyclic STS(v), $v > 7$, takes on one of three values: min_v, $min_v + 1$, $min_v + 2$.*

It would be tempting to conjecture that the same holds for *any* Steiner triple system but admittedly the evidence for such a claim is scarce.

Complete version of the paper which includes proofs of the above results and some more related material can be found in [4].

References

[1] C. Colbourn and M. Colbourn: The chromatic index of cyclic Steiner 2-designs, *Internat. J. Math. Math. Sci.*, volume 5, 1982, pp. 823–825.

[2] C. Colbourn and A. Rosa: *Triple Systems*, Clarendon Press, Oxford, 1999.

[3] K. Collins and J. Hutchinson: Four-coloring six-regular graphs on the torus, *CRM Proceedings and Lecture Notes*, volume 23, 1999, pp. 21–33.

[4] M. Meszka, R. Nedela, A. Rosa: Circulants and the chromatic index of Steiner triple systems, accepted for publication in Mathematica Slovaca.

[5] N. Pippenger and J. Spencer: Asymptotic behavior of the chromatic index for hypergraphs, *J. Combin. Theory (A)*, volume 51, 1989, pp. 24–42.

[6] H.-G. Yeh, X. Zhu: 4-colorable 6-regular toroidal graphs, *Discrete Math.*, volume 273, 2003, pp. 261–274.

Weight choosability of graphs and Combinatorial Nullstellensatz

Stanisław Niwczyk
University of Zielona Góra, Zielona Góra, Poland

(Joint work with Tomasz Bartnicki and Jaroslaw Grytczuk)

Let S be a subset of the integers. We say G is *S-weight colorable* if there is an edge weighting $w : E \to S$ such that for any two adjacent vertices u, v of G the sum of weights of the edges incident to u is different than the sum of weights of the edges incident to v.

Conjecture 1 (Karoński, Łuczak, Thomason [2]) *Each connected graph with more than one edge is* $\{1,2,3\}$*-weight colorable.*

Recently Addario-Berry et al. [1] have proved that each such graph is $\{1,2,\ldots,16\}$-weight colorable, and that for any fixed $p \in (0,1)$ the random graph $G_{n,p}$ is asymptotically almost surely 2-weight colorable. We attempt to attack the conjecture by applying Alon-Tarsi Combinatorial Nullstellensatz. Roughly, we assign to a given graph G a polynomial in $m = |E(G)|$ variables $f(x_1, ..., x_m)$ which encodes our problem, and try to prove that there must be a nonvanishing monomial in f each of whose exponents is at most 2. If this would be true the conjecture would follow in the following stronger sense.

Let $S_1, ..., S_m$ be a collection of integer subsets assigned to the edges of G. We say that G is *d-weight choosable* if for any such assignment with $|S_1| = ... = |S_m| = d$, there is an edge weighting $w : E \to \mathbb{Z}$ such that $w(e_i) \in S_i$, and for any two adjacent vertices u, v the sum of weights of the edges incident to u is different than the sum of weights of the edges incident to v.

Conjecture 2 *Every connected graph with at least two edges is* 3*-weight choosable.*

Maybe this conjecture is too optimistic, as we even do not know if there is any finite bound. But why not to state optimistic conjectures?

References

[1] L. Addario-Berry, R. E. L. Aldred, K. Dalal, B. A. Reed, Vertex colouring edge partitions, J. Combin. Theory Ser. B 94 (2005), 237-244 .

[2] M. Karoński, T. Łuczak, A. Thomason, Edge weights and vertex colours, J. Combin. Theory Ser. B 91 (2004), 151-157.

On Ramsey Infinite Graphs

Mark Siggers
Thompson Rivers University, Kamloops, Canada

Low Tree-depth Partitions of Classes With Bounded Expansion

Patrice Ossona de Mendez
École des Hautes Études en Sciences Sociales, Paris, France

(Joint work with Jaroslav Nešetřil)

1 Tree-width, Tree-depth and Coloring

A *k-tree* is a graph which is either a single vertex, or is obtained from a smaller k-tree by adding a vertex joined to the vertices of a clique of size at most k. The *tree-width* $\mathrm{tw}(G)$ of a graph G is the smallest k such that G is a subgraph of a k-tree. Tree-width has been proved to be a fundamental parameter, especially in the study of minor closed classes of graphs.

In a paper motivated by a question of R. Thomas [9], DeVos, Ding, Oporowski, Sanders, Reed, Seymour and Vertigan [1] proved that for every graph K and integer $j \geq 1$ there is an integer $i_V = i_V(K, j)$, such that every graph with no K-minor has a vertex partition into i_V graphs, so that any $j' \leq j$ parts form a graph with tree-width at most $j' - 1$. This proof relies on an important result of Robertson and Seymour on the structure of graphs without a particular graph as a minor [8].

In a previous work [6], we introduced the *tree-depth* $\mathrm{td}(G)$ of graph G as the minimum height of a tree which closure contains G as a subgraph. This minor monotone invariant $\mathrm{td}(G)$ is obviously at least $\mathrm{tw}(G) + 1$ but $\mathrm{td}(G)$ fails to be bounded by a function of $\mathrm{tw}(G)$ (a path of length n has tree-width 1 and tree-depth $\lceil \log_2(n + 1) \rceil$).

Using the above mentioned result [1], we proved in [6] that for any proper minor closed class of graphs $\mathcal{C}$ (that is: any minor closed class of graphs excluding at least one minor) and integer $j \geq 1$ there is an integer $F_j(\mathcal{C})$, such that every graph in $\mathcal{C}$ has a vertex partition into $F_j(\mathcal{C})$ graphs, so that any $j' \leq j$ parts form a graph with tree-depth at most j', and that the tree-depth is the maximum graph invariant for which such a statement holds.

For graph G and integer $j \geq 1$, define $\chi_j(G)$ has the smallest integer N, such that G may be N-colored in such a way that, for any $H \subseteq G$, H gets at least $\min(j, \mathrm{td}(H))$ colors. Then the previous statement may be restated as:

Theorem 1 *For any proper minor closed class of graphs $\mathcal{C}$ and any integer $j \geq 1$, $\chi_j(G)$ is bounded on $\mathcal{C}$.*

Notice that for any graph G of order n, $\chi_1(G) = 1$, $\chi_2(G)$ is the usual chromatic number $\chi(G)$. We have then

$$\chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_n(G) = \mathrm{td}(G).$$

2 Homomorphism Duality

The previous result has then been applied to prove that any proper minor closed class $\mathcal{C}$ of graphs has a *restricted homomorphism duality* for any connected graph F, that is: For any connected graph F, there exists a graph $D_F^{\mathcal{C}}$ so that:

- F has no homomorphism to $D_F^{\mathcal{C}}$:

$$F \not\longrightarrow D_F^{\mathcal{C}}$$

- any graph $G \in \mathcal{C}$ has no homomorphism **from** F if and only it has a homomorphism **to** $D_F^{\mathcal{C}}$:

$$\forall G \in \mathcal{C}, \quad (F \not\longrightarrow G) \iff (G \longrightarrow D_F^{\mathcal{C}})$$

Such a restricted duality exists for proper minor closed classes of graphs [6] and for classes of bounded degree graphs [7, 2].

3 Grad of a Graph and Classes with Bounded Expansion

The *greatest reduced average density (grad) of G with rank r*: $\nabla_r(G)$ is related to the maximum average degree of a minor obtained by contracting a family of disjoints connected subgraphs, each having radius bounded by

r. Precisely, a *connected partition* of G is a partition $\mathcal{P} = (V_1, \dots, V_p)$ of G such that each subgraph $G[V_i]$ of G induced by V_i is connected. The *radius* $\rho(\mathcal{P})$ of a connected partition $\mathcal{P}$ is

$$\rho(\mathcal{P}) = \max_{X \in \mathcal{P}} \min_{x \in X} \max_{y \in X} \mathrm{dist}_{G[X]}(x, y)$$

and the *grad of rank* r of G is

$$\nabla_r(G) = \max_{\rho(\mathcal{P}) \leq r} \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|}$$

where the maximum is taken over all the connected partitions $\mathcal{P}$ of G having radius at most r and where $G/\mathcal{P}$ stands for the **simple** graph obtained by contracting each of the $G[V_i], V_i \in \mathcal{P}$ to a single vertex.

For any graph G of order n, $\nabla_0(G)$ is half of the maximum average degree (mad) of G and

$$\frac{\mathrm{mad}(G)}{2} = \nabla_0(G) \leq \nabla_1(G) \leq \dots \leq \nabla_n(G) \leq \mathrm{td}(G) - 1$$

The last inequality is straightforward: any minor of order n of a graph with tree-depth at most k has tree-depth at most k hence size at most $(k-1)n$. Moreover, $\nabla_r(G)$ is obviously bounded by a constant for any proper minor closed class of graphs and by $O(\Delta^r)$ for any graph with maximum degree Δ.

A class $\mathcal{C}$ has a *bounded expansion* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\forall r \in \mathbb{N}, \forall G \in \mathcal{C}, \quad \nabla_r(G) \leq f(r).$$

4 Transitive Fraternal Orientation and Augmentation

It is well known that a graph G is a k-tree (for some k) if and only if it admits an acyclic *fraternal orientation*, that is an orientation of its edges such that when (x, z) and (y, z) are both arcs of G then either (x, y) or (y, x) is an arc of G.

We will relax the properties of acyclic fraternal orientations to build some locally fraternal orientations. In the following, directed graphs may have, for distinct vertices x and y, one arc (at most) from x to y **and** one arc (at most) from y to x.

Let $\vec{G}$ be a directed graph. A 1-*transitive fraternal augmentation* of $\vec{G}$ is a directed graph $\vec{H}$ with the same vertex set, including all the arcs of $\vec{G}$ and such that, for any vertices x, y, z,

- if (x, z) and (z, y) are arcs of $\vec{G}$ then (x, y) is an arc of $\vec{H}$ (*transitivity*),
- if (x, z) and (y, z) are arcs of $\vec{G}$ then (x, y) or (y, x) or both are arcs of $\vec{H}$ (*fraternity*).

A *transitive fraternal augmentation* of a directed graph $\vec{G}$ is a sequence $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \vec{G}_{i+1} \subseteq \cdots$, such that $\vec{G}_{i+1}$ is a 1-transitive fraternal augmentation of $\vec{G}_i$ for any $i \geq 1$.

Theorem 2 *Let $\mathcal{C}$ be a class with bounded expansion. There exists a function $F : \mathbb{N}^2 \to \mathbb{N}$ such that for any integer Δ^-, there exists a sequence $\mathcal{C} = \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots \subseteq \mathcal{C}_i \subseteq \cdots$ of classes with bounded expansion, so that any orientation $\vec{G}$ of a graph $G \in \mathcal{C}$ with $\Delta^-(\vec{G}) \leq \Delta^-$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where $\vec{G}_i$ is an orientation with maximum indegree $\Delta^-(\vec{G}_i) \leq F(\Delta^-, i)$ of some $G_i \in \mathcal{C}_i$.*

5 Main Results

Our first main result stands in a characterization of classes with bounded expansion, while the second one expresses the existence of restricted dualities for such classes [3].

Theorem 3 *Let $\mathcal{C}$ be a class of graphs. The following conditions are equivalent:*

- *$\mathcal{C}$ has bounded expansion,*
- *for any integer c, the class $\mathcal{C}[K_c] = \{G[K_c] : G \in \mathcal{C}\}$ has bounded expansion, where $G[K_c]$ denotes the lexicographic product of G and K_c;*
- *there exists a function $F : \mathbb{N}^2 \to \mathbb{N}$ such that any orientation $\vec{G}$ of a graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where $\Delta^-(\vec{G}_i) \leq F(\Delta^-(\vec{G}), i)$;*
- *for any integer i, $\chi_i(G)$ is bounded on $\mathcal{C}$.*

¿From a computational point of view, it has to be noticed that, for any fixed class $\mathcal{C}$ with bounded expansion and any fixed integer j, the proof of the theorems yields a linear time algorithm to compute, for any graph $G \in \mathcal{C}$, a coloring using at most $N(\mathcal{C}, j)$ colors so that any $j' \leq j$ parts not only induce a subgraph of tree-depth at most j', but actually induce a subgraph which connected components have the property that some color appears exactly once in them, which is a stronger statement. Further algorithmic consequences are discussed in [5]

¿From the previous theorem, we further prove that classes with bounded expansion also admits restricted dualities [4]:

Theorem 4 *Any class of graphs with bounded expansion has a restricted duality for any connected graph: for every connected graph F, there exists a graph $D_F^{\mathcal{C}}$ so that $F \not\longrightarrow D_F^{\mathcal{C}}$ and*

$$\forall G \in \mathcal{C}, \quad (F \not\longrightarrow G) \iff (G \longrightarrow D_F^{\mathcal{C}})$$

References

[1] DeVos, M., G. Ding, B. Oporowski, D. Sanders, B. Reed, P. Seymour and D. Vertigan, *Exluding any graph as a minor allows a low tree-width 2-coloring*, Journal of Combinatorial Theory, Series B **91** (2004), pp. 25–41.

[2] Häggkvist, R. and P. Hell, *Universality of A-mote graphs*, Europ. J. Combinatorics **14** (1993), pp. 23–27.

[3] Nešetřil, J. and P. Ossona de Mendez, *Grad and classes with bounded expansion I. decompositions*, European Journal of Combinatorics (2005), (submitted).

[4] Nešetřil, J. and P. Ossona de Mendez, *Grad and classes with bounded expansion II. algorithmic aspects*, European Journal of Combinatorics (2005), (submitted).

[5] Nešetřil, J. and P. Ossona de Mendez, *Grad and classes with bounded expansion III. restricted dualities*, European Journal of Combinatorics (2005), (submitted).

[6] Nešetřil, J. and P. Ossona de Mendez, *Tree depth, subgraph coloring and homomorphism bounds*, European Journal of Combinatorics (2005), (in press).

[7] P., D., M. Ch. and J. Nešetřil, *Universal h-colorable graphs without a given configuration*, Discrete Math. **250** (2002), pp. 245–25.

[8] Robertson, N. and P. Seymour, *Graph Minors. XVI. Excluding a non-planar graph*, Journal of Combinatorial Theory, Series B **89** (2003), pp. 43–76.

[9] Thomas, R., *Problem session of the Third Slovene Conference on Graph Theory, Bled, Slovenia* (1995).

A graph G is *k-ramsey with respect to H* if any k-colouring of its edges yields a monochromatic copy of H. It is *k-ramsey-minimal with respect to* H if it is k-ramsey, but none of its proper subgraphs are.

In 1976, Burr, Erd[o]s, and Lovász showed that for any integer $d \geq 3$ there are infinitely many non-isomorphic graphs that are 2-ramsey-minimal with respect to the complete graph K_d on d vertices. In 1985, Burr, Nešetřil, and Rödl extended this result to show that for any integer $d \geq 3$ there exists a constant $c > 1$ such that for all n large enough, there exist at least c^{nlogn} non-isomorphic graphs on at most n vertices that are 2-ramsey-minimal with respect to K_d.

In this talk, we present a construction that proves, for any integers $d \geq 3$ and $k \geq 2$, that there exists a constant $c > 1$ such that for all n large enough,there exist at least c^{n^2} non-isomorphic graphs on at most n vertices that are k-ramsey-minimal with respect to K_d.

Problems related to Pfaffian orientations of graphs

Robin Thomas
Georgia Institute of Technology, Atlanta, USA

An orientation of a graph G is Pfaffian if every even cycle C such that $G \setminus V(C)$ has a perfect matching has an odd number of edges directed in either direction of the cycle. The significance of Pfaffian orientations is that if a graph has one, then the number of perfect matchings (a.k.a. the dimer problem) can be computed in polynomial time.

The question of what bipartite graphs have Pfaffian orientations is equivalent to many other problems of interest, such as a permanent problem of Polya, the even directed cycle problem, or the sign-nonsingular matrix problem for square matrices. These problems are now reasonably well-understood.

In order to find a linear-time algorithm to solve the above problems it would be nice to have a linear-time algorithm to find all 4-shredders in a graph. (A 4-shredder is a set X of vertices of G of size four such that $G \setminus X$ has at least three components.)

There are other interesting open problems related to Pfaffian orientations. Many of them can be found in the Ph.D. thesis of my student Serguei Norine or in his/our papers. For instance, is it true that every 2-connected cubic Pfaffian graph is 3-edge-colorable? If true, this would generalize the Four-Color Theorem.

Finally, here is an unrelated conjecture. For every integer t there exists an integer N such that every t-connected graph on at least N vertices with no K_t minor has a set X of at most $t-5$ vertices such that $G \setminus X$ is planar.

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Chain Intersecting Set Families

Dániel Gerbner
Rényi Institute, Budapest, Hungary

(Joint work with Attila Bernáth)

Let $\mathcal{F}$ be a family of subsets of an n-element set. $\mathcal{F}$ is *(p,q)-chain intersecting* if it does not contain $A_1 \subset \cdots \subset A_p$, $B_1 \subset \cdots \subset B_q$ such that $A_p \cap B_q = \emptyset$. We determined the maximum size of these families. The optimum families are also characterized.

A Polynomial Proof of the Erdős-Ko-Rado Theorem

Zoltán Füredi
Rényi Institute, Budapest, Hungary

(Joint work with Kyung-Won Hwang and Paul M. Weichsel)

In 1961, Erdős, Ko, and Rado proved that if $\mathcal{F}$ is a k-uniform family of subsets of a set of n elements with $k \leq \frac{1}{2}n$, and every pair of members of $\mathcal{F}$ intersect, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. They also showed that for $n > 2k$ here equality holds if $\mathcal{F}$ consists of all k sets containing a given element of the underlying set.

Beside their remarkable proof (induction on k, and for a given k left-shifting and induction on n), there are many interesting new proofs. In 1972, Katona used a simple and an elegant argument, the permutation method. Daykin obtained Erdős-Ko-Rado from the Kruskal-Katona theorem. Hajnal and Rothschild proved it for $n > n_0(k)$ by an early version of the kernel (or delta-system) method, developed and used very successfully by Frankl (the first full description of the method was published in Deza, Erdős, and Frankl). Most remarkable, Lovász in his ground-breaking paper used its geometric representation method to prove that the Shannon capacity of the Kneser graph $K(n, k)$ is at most $\binom{}{n-1k-1}$, $(n \geq 2k)$, thus yielding another proof (and generalization). Wilson gave an ingenious proof, using Delsarte's linear programming bound. (Actually, he proved much more concerning t-intersecting families.)

The aim of this paper is to exhibit a true, short, polynomial proof for the Erdős-Ko-Rado Theorem.

How Bipartite Is a Graph with No Cycle of Length 2k+1?

Ervin Győri
Rényi Institute, Budapest, Hungary

In this talk, different parameters "measuring" bipartiteness are studied for graphs without (2k+1)-cycles, sometimes with extra degree conditions: how difficult is to make them bipartite, how many triangles are in them, etc.

Excluded subposets in the Boolean lattice

Gyula O.H. Katona
Rényi Institute, Budapest, Hungary

Let $[n] = \{1, 2, \dots, n\}$ be a finite set, families $\mathcal{F}$ of its subsets will be investigated. $\binom{[n]}{k}$ denotes the family of all k-element subsets of $[n]$. Let P be a poset. The goal of the present investigations is to determine the maximum size of a family $\mathcal{F} \subset 2^{[n]}$ which does not contain P as a (non-necessarily induced) subposet. This maximum is denoted by $\mathrm{La}(n, P)$.

There are some Ps for which $\mathrm{La}(n, P)$ has been exactly determined. The easiest example is the case when P consist of two comparable elements. Then we are actually looking for the largest family without inclusion that is without two distinct members $F, G \in \mathcal{F}$ such that $F \subset G$. The well-known Sperner theorem gives the answer, the maximum is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

In most cases, however $\mathrm{La}(n, P)$ is only asymptotically determined in the sence that the main term is the size of the largest level (sets of size $\lfloor \frac{n}{2} \rfloor$) while the second term is $\frac{c}{n}$ times the second largest level where the lower and upper estimates contain diffrent constants c.

Let the poset N consist of 4 elements illustrated here with 4 distinct sets satisfying $A \subset B, C \subset B, C \subset D$. We were not able to determine $\mathrm{La}(n, N)$ for a long time. Recently, a new method jointly developed by J.R. Griggs, helped us to prove the following theorem.

Theorem

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n} + o(\frac{1}{n})\right) \le \mathrm{La}(n, N) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{4}{n} + o(\frac{1}{n})\right).$$

Distance Constrained Labelings of Graphs of Bounded Treewidth

Jan Kratochvíl
Charles University, Prague, Czech Republic

(Joint work with Jiří Fiala and Petr A. Golovach)

We prove that the L(2,1)-labeling problem is NP-complete for graphs of treewidth two, thus adding a natural and well studied problem to the short list of problems whose computational complexity separates treewidth one from treewidth two. We prove similar results for other variants of the distance constrained graph labeling problem.

Towards a Theory of Frustration and Degeneracy

Martin Loebl
Charles University, Prague, Czech Republic

(Joint work with Jan Vondrák)

Several open problems related to multiplicity of groundstates in finite and infinite square grids are presented.

On Vertex-sets of the Hypercube Whose Span Avoids Given Hyperplanes

Dezső Miklós
Rényi Institute, Budapest, Hungary

Let C_n denote the vertices of the n dimensional hypercube and let $M \subseteq C_n$ be a subset of it (or a subset of it consisting of vertices of weight k, where $1 \leq k \leq n$). We will investigate and a present a few results about the maximum size of M assuming that the span of the vertices in M completely avoids (or does not contain) the hyperplane consisting of the vertices of the hypercube of weight m. (where the weight of a vertex is the number of 1 coordinates of it).

The distance of $\mathcal{F}$-free families

Balázs Patkós
Rényi Institute, Budapest, Hungary

If $\mathcal{F}$ is a fixed hypergraph, then for two $\mathcal{F}$-free hypergraphs $\mathcal{H}_1 = (V, E_1)$ and $\mathcal{H}_2 = (V, E_2)$ we define their $\mathcal{F}$-free distance by the number of copies of $\mathcal{F}$ in $\mathcal{H}_1 \cup \mathcal{H}_2 = (V, E_1 \cup E_2)$ (and denote it by $D_{\mathcal{F}}(\mathcal{H}_1; \mathcal{H}_2)$). For a collection $\mathcal{C}$ of hypergraphs the $\mathcal{C}$-free distance of two $\mathcal{C}$-free hypergraphs (that is $\mathcal{F}$-free for all $\mathcal{F} \in \mathcal{C}$) is $D_{\mathcal{C}}(\mathcal{H}_1; \mathcal{H}_2) = \sum_{\mathcal{F} \in \mathcal{C}} D_{\mathcal{F}}(\mathcal{H}_1; \mathcal{H}_2)$. In the talk we will consider several collections of forbidden hypergraphs. For some of them we will obtain exact results on the maximum distance of pairs of $\mathcal{C}$-free hypergraphs while for others we will give upper and lower bounds on the maximum distance.

On the Empty Hexagon Theorem

Pavel Valtr
Charles University, Prague, Czech Republic

Tobias Gerken has very recently proved that any sufficiently large set P of points in general position in the plane contains the six vertices of a convex hexagon with no point of P in the interior. We explain the main parts of an alternative proof of this result.

Extremal Functions for Rooted Minors

Paul Wollan
Georgia Institute of Technology, Atlanta, USA

The graph G contains a graph H as a *minor* if there exist pair-wise disjoint sets $\{S_i \subseteq V(G) | i = 1, \ldots, |V(H)|\}$ such that for every i, $G[S_i]$ is a connected subgraph and for every edge uv in H, there exists an edge of G with one end in S_u and the other end in S_v. A *rooted* H minor in G is a minor where each S_i of minor contains a predetermined $x_i \in V(G)$. We prove that if the constant c is such that every graph on n vertices with cn edges contains a H minor, then every $|V(H)|$-connected graph G with $(18c + 1236|V(H)|)|V(G)|$ edges contains a rooted H minor for every choice of vertices $\{x_1, \ldots, x_{|V(H)|}\} \subseteq V(G)$. The proof methodology is sufficiently robust to find the exact extremal function for an infinite family of rooted bipartite minors previously studied by Jorgensen, Kawarabayashi, and Böhme and Mohar.

Appendix

List of Participants

Martin Bálek balek@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Imre Bárány barany@renyi.hu
Rényi Institute, Budapest, Hungary

Sergey Bereg besp@utdallas.edu
University of Texas, Dallas, USA

Sarah Bleiler bleilesa@shu.edu
Seton Hall University, South Orange, USA

Manuel Bodirsky bodirsky@informatik.hu-berlin.de
Humboldt University, Berlin, Germany

Anthony Bonato abonato@rogers.com
Wilfrid Laurier University, Waterloo, Canada

Julia Böttcher boettche@ma.tum.de
Technical University Munich, Garching, Germany

Craig Bowles dcb27@cornell.edu
Cornell University, Ithaca, USA

Kamila Bumbová kamila@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Sergio Cabello sergio.cabello@imfm.uni-lj.si
Institute for Mathematics, Physics and Mechanics, Ljubljana, Slovenia

Josef Cibulka josef.cibulka@seznam.cz
Charles University, Prague, Czech Republic

Maurice Cochard cochard@math.ethz.ch
Swiss Federal Institute of Technology, Zurich, Switzerland

Jakub Černý kuba@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Matt DeVos mjaredd@yahoo.com
Princeton University, Princeton, USA

Babak Farzad babak@cs.toronto.edu
University of Toronto, Toronto, Canada

Jiří Fiala fiala@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Jan Foniok foniok@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Willem Fouché fouchwl@unisa.ac.za
University of South Africa, Pretoria, South Africa

Zoltán Füredi z-furedi@math.uiuc.edu
Rényi Institute, Budapest, Hungary

Dániel Gerbner gerbnerd@freemail.hu
Rényi Institute, Budapest, Hungary

Tobias Gerken gerken@ma.tum.de
Technical University Munich, Garching, Germany

Jarek Grytczuk j.grytczuk@wmie.uz.zgora.pl
University of Zielona Góra, Zielona Góra, Poland

András Gyárfás gyarfas@sctaki.hu
Computer and Automation Research Institute, Hungarian Academy of Sciences, Budapest, Hungary

Ervin Győri gyori@renyi.hu
Rényi Institute, Budapest, Hungary

David Hartman hartman@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Jan Herman hermitko@email.cz
Charles University, Prague, Czech Republic

Christian Heup cheup@fargonauten.de
Bielefeld University, Bielefeld, Germany

Jan Hladký hladk@seznam.cz
Charles University, Prague, Czech Republic

Jan Hubička hubicka@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Vít Jelínek jelinek@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Tomáš Kaiser kaisert@kma.zcu.cz
University of West Bohemia, Pilsen, Czech Republic

Jeong Hyun Kang j-kang5@math.uiuc.edu
University of Central Florida, Orlando, USA

MIHYUN KANG kang@informatik.hu-berlin.de
Humboldt University, Berlin, Germany

JAN KÁRA kara@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

JÁN KARABÁŠ karabas@savvb.sk
Mathei Bel University, Banská Bystrica, Slovakia

GYULA KATONA ohkatona@renyi.hu
Rényi Institute, Budapest, Hungary

MARCOS KIWI mkiwi@dim.uchile.cl
University of Chile, Santiago, Chile

MARTIN KLAZAR klazar@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

MARTIN KOCHOL kochol@savba.sk
Slovak Academy of Sciences, Bratislava, Slovakia

PETR KOLMAN kolman@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

JÁNOS KOMLÓS komlos@math.rutgers.edu
Rutgers University, Piscataway, USA

DANIEL KRÁL' kral@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

JAN KRATOCHVÍL honza@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Marek Krčál marek.krcal@seznam.cz
Charles University, Prague, Czech Republic

Naďa Krivoňáková nada.krivonakova@fpv.utc.sk
University of Žilina, Žilina, Slovakia

Brenda Latka latka@dimacs.rutgers.edu
Rutgers University, Piscataway, USA

Martin Loebl loebl@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Ralph McKenzie ralph.n.mckenzie@vanderbilt.edu
Vanderbilt University, Nashville, USA

Dezső Miklós dezso@renyi.hu
Rényi Institute, Budapest, Hungary

Melissa Mitchell mistwin@aol.com
University of Detroit, Detroid, USA

Enrique García Moreno enrique@math.ohio-state.edu
Cinvestav, Mexico

Olga Myndyuk omynduk@stevens.edu
Stevens Institute of Technology, Hoboken, USA

Roman Nedela nedela@savbb.sk
Mathei Bel University, Banská Bystrica, Slovakia

Samuel Nelson snelson@bucknell.edu
Bucknell University, Bucknell, USA

Jaroslav Nešetřil nesetril@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Yared Nigussie yared@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Stanislav Niwczyk s.niwczyk@wmie.uz.zgora.pl
University of Zielona Góra, Zielona Góra, Poland

Eva Ondráčková efa@matfyz.cz
Charles University, Prague, Czech Republic

Patrice Ossona de Mendez pom@ehess.fr
École des Hautes Études en Sciences Sociales, Paris, France

Ondřej Pangrác pangrac@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Balázs Patkós patkos@renyi.hu
Rényi Institute, Budapest, Hungary

Martin Pergel perm@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Diana Piguet diana@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Aleš Přívětivý privetivy@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Pavel Pudlák pudlak@math.cas.cz
Academy of Sciences, Prague, Czech Republic

Lara Pudwell   lpudwell@math.rutgers.edu
Rutgers University, Piscataway, USA

Mark Siggers   msigger@emory.edu
Thompson Rivers University, Kamloops, Canada

Vera Sos   sos@renyi.hu
Rényi Institute, Budapest, Hungary

Robert Šámal   samal@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Arjun Talwar   arjunt@stanford.edu
Stanford University, Stanford, USA

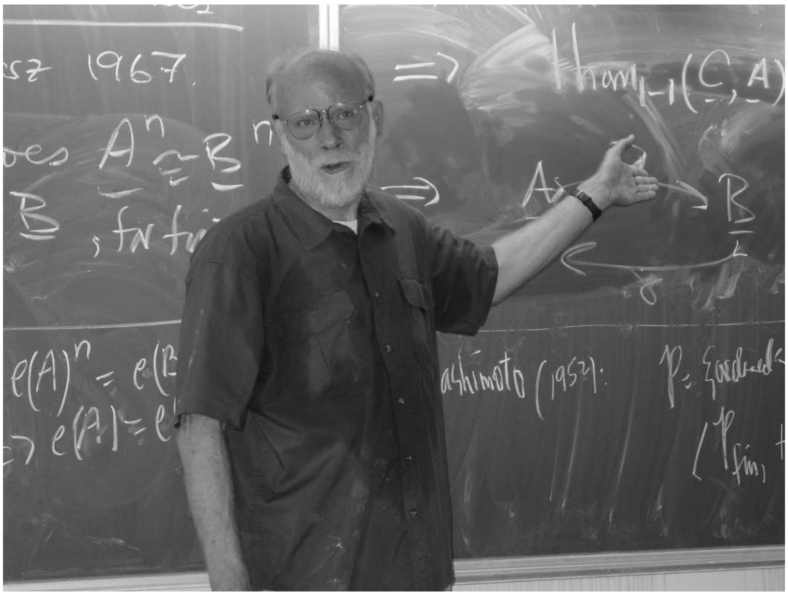
Robin Thomas   thomas@math.gatech.edu
Georgia Institute of Technology, Atlanta, USA

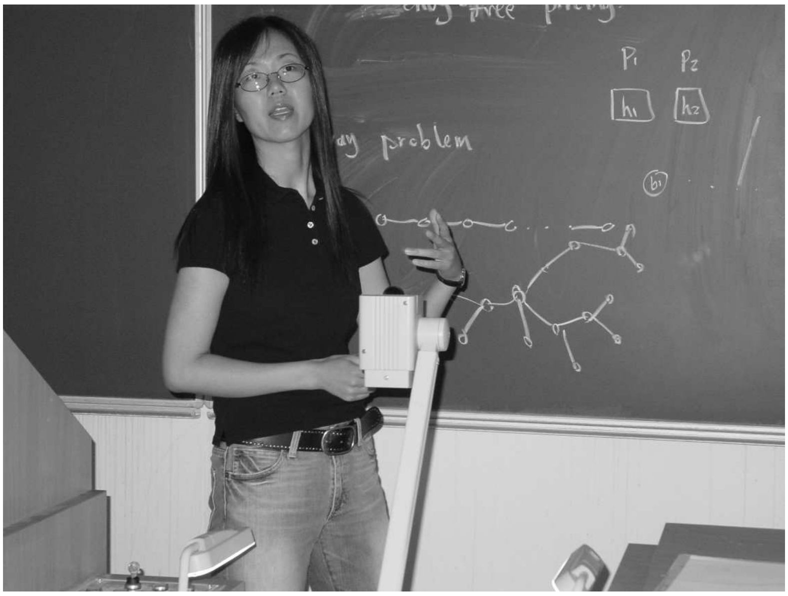
Tomáš Valla   valla@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

Pavel Valtr   valtr@kam.mff.cuni.cz
Charles University, Prague, Czech Republic

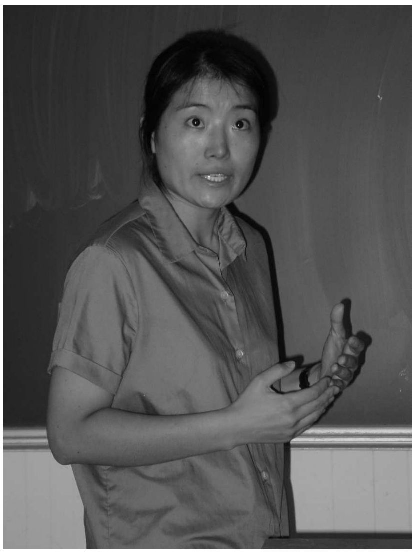
Paul Wollan   wollan@math.gatech.edu
Georgia Institute of Technology, Atlanta, USA

Photos




problem


problem – Ulam's problem
LIS(π) = max

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