Separation of two convex polyhedral sets with parameters in one column of the constraint matrix

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Abstract

Separation is a famous principle and separation properties are important for optimization theory and various applications. In practice, input data are rarely known exactly and it is advisable to deal with parameters. In this article, we are concerned with the basic characteristics (existence, description, stability etc.) of separating hyperplanes of two convex polyhedral sets depending on parameters. We study the case, when parameters are situatedin one column of the constraint matrix from the description of the given convex polyhedral set.

Keywords: *separating hyperplane, parameters, convex polyhedra, solution set, stability set.*

1 Introduction

There are several kinds of separability of convex sets (cf. [9]). For the purpose of this paper it is convenient to introduce thefollowing one.

Definition 1. Sets $X, Y \subset \mathbb{R}^n$ are called *strongly separable* if $dim\, X = dim\, Y = n$ and there exists a hyperplane $\mathcal{R} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{r}^T\boldsymbol{x} = s\}$ such that $X \subseteq \overline{\mathcal{R}^-} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{r}^T\boldsymbol{x} \leq s\}$, and $Y \subseteq \overline{\mathcal{R}^+} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{r}^T\boldsymbol{x} \geq s\}$ hold. $\mathcal{R}$ is called *the separating hyperplane* of the sets X, Y.

We will use the following well known separation theorem (see e.g. [3, 8]):

Theorem 1. *Convex sets $X,\ Y \subset \mathbb{R}^n$ are strongly separable if and only if $dim\, X = dim\, Y = n$, and $int\, X \cap int\, Y = \emptyset$.*

In this paper we study the strong separability of two convex polyhedral sets ($\widetilde{\mathbf{A}} \in \mathbb{R}^{m\times n}$, $\widetilde{\mathbf{C}} \in \mathbb{R}^{l\times n}$, $\widetilde{\boldsymbol{b}} \in \mathbb{R}^m$, $\widetilde{\boldsymbol{d}} \in \mathbb{R}^l$):

$$\mathcal{M}_1 \equiv \{\boldsymbol{x} \in \mathbb{R}^n \mid \widetilde{\mathbf{A}}\boldsymbol{x} \leq \widetilde{\boldsymbol{b}}\}, \tag{1}$$

$$\mathcal{M}_2 \equiv \{\boldsymbol{x} \in \mathbb{R}^n \mid \widetilde{\mathbf{C}}\boldsymbol{x} \leq \widetilde{\boldsymbol{d}}\}, \tag{2}$$

In [7] we derived the basic separation properties of the sets (1), (2) with parameters in the right-hand side of inequalities. Some of the results obtained there, which we need in this paper, will be presented in section 2. In the following sections we study separation for the case there are parameters in one column of the matrix $\widetilde{\mathbf{A}}$. In dealing with parameters, we are inspired by [1, 2, 10]. We will define so called solution set and in the sequel so called stability sets. For definition of stability sets we use the explicit description of all separating hyperplanes of two fixed convex polyhedral sets from [4, 5].

Let us introduce some notation. Given a matrix $\mathbf{M}$, the expressions $\mathbf{M}_{i,\cdot}$, $\mathbf{M}_{\cdot,j}$ denote i-th row and j-th column of the matrix $\mathbf{M}$, respectively. Vector $\boldsymbol{e}_k$ denotes k-th unit vector. For given vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$, the expression $\boldsymbol{a} < \boldsymbol{b}$ means $a_i < b_i\ \forall i$. For any set $\mathcal{X}$ let us denote by $\overline{\mathcal{X}}$, $int\, \mathcal{X}$, $dim\, \mathcal{X}$, and $conv\, \mathcal{X}$ the closer, the interior, the dimension, and the convex hull of $\mathcal{X}$, respectively. A sign of a real number $r \in \mathbb{R}$ is defined as

$$sgn(r) = \begin{cases} 0 & r = 0, \\ 1 & r > 0, \\ -1 & r < 0. \end{cases}$$

Definition 2. The *basis* of a convex polyhedral set described by $\mathbf{M}\boldsymbol{x} = \boldsymbol{v},\ \boldsymbol{x} \geq \mathbf{0}$ ($\mathbf{M} \in \mathbb{R}^{m\times n}$, $\boldsymbol{v} \in \mathbb{R}^m$, $m \leq n$) is any vector $B \in \{1, \ldots, n\}^m$ for which $rank(\mathbf{M}_B) = m$ (where $\mathbf{M}_B$ means the restriction of the matrix $\mathbf{M}$ to the basic columns). A basis B is *feasible*, if $\mathbf{M}_B^{-1}\boldsymbol{v} \geq \mathbf{0}$.

The sub-basis of the convex polyhedral set described by $\mathbf{M}\boldsymbol{x} \leq \boldsymbol{v}$ ($\mathbf{M} \in \mathbb{R}^{m\times n}$, $\boldsymbol{v} \in \mathbb{R}^m$) is any vector $S \in \{1, \ldots, m\}^k$, $1 \leq k \leq n$, for which $rank(\mathbf{M}_S) = k$ holds (where $\mathbf{M}_S$ in this case means the restriction of the matrix $\mathbf{M}$ to the sub-basic rows). The sub-basis S is called *feasible*, if $\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{M}_S\boldsymbol{x} = \boldsymbol{v}_S,\ \mathbf{M}_N\boldsymbol{x} \leq \boldsymbol{v}_N\} \neq \emptyset$ for $N = \{1, \ldots, m\} \setminus S$. The basis of $\mathbf{M}\boldsymbol{x} \leq \boldsymbol{v}$ is any n-elemental sub-basis.

2 Some known results

Let us introduce

$$\mathcal{Q}^* \equiv \Big\{(\boldsymbol{u},\boldsymbol{v},v_{l+1}) \in \mathbb{R}^{m+l+1} \mid \begin{pmatrix} \widetilde{\mathbf{A}}^T & \widetilde{\mathbf{C}}^T & \mathbf{0} \\ \widetilde{\boldsymbol{b}}^T & \widetilde{\boldsymbol{d}}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix}, \quad (\boldsymbol{u},\boldsymbol{v},v_{l+1}) \geq \mathbf{0}\Big\}. \tag{3}$$

With help of the set $\mathcal{Q}^*$ we can describe all separating hyperplanes of $\mathcal{M}_1,\ \mathcal{M}_2$ from (1), (2). Theorem 2 comes from [7], Theorem 3 is a direct consequence of theorems from [5], [7].

Theorem 2. *Suppose that* $dim\,\mathcal{M}_1 = dim\,\mathcal{M}_2 = n,\ int\,\mathcal{M}_1 \cap int\,\mathcal{M}_2 = \emptyset$. *Let* $(\boldsymbol{u},\boldsymbol{v},v_{l+1}) \in \mathcal{Q}^*,\ \boldsymbol{u}^T\widetilde{\mathbf{A}} \neq \mathbf{0}^T$, *and* $\eta \in \langle 0, v_{l+1}\rangle$ *is arbitrary. Then*

$$\mathcal{R} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{u}^T(\widetilde{\mathbf{A}}\boldsymbol{x} - \widetilde{\boldsymbol{b}}) = \eta\} \tag{4}$$

represents a separating hyperplane of the convex polyhedral sets $\mathcal{M}_1$, $\mathcal{M}_2$. *Conversely, any separating hyperplane* $\mathcal{R}$ *of* $\mathcal{M}_1$, $\mathcal{M}_2$ *we can express in the form of* (4) *for a certain* $(\boldsymbol{u},\boldsymbol{v},v_{l+1}) \in \mathcal{Q}^*,\ \boldsymbol{u}^T\widetilde{\mathbf{A}} \neq \mathbf{0}^T$, *and* $\eta \in \langle 0, v_{l+1}\rangle$.

Theorem 3. *Let* $dim\,\mathcal{M}_1 = dim\,\mathcal{M}_2 = n$. *Then the convex sets* $\mathcal{M}_1$, $\mathcal{M}_2$ *are strongly separable if and only if* $\mathcal{Q}^* \neq \emptyset$.

The following theorem from [6] characterizes the nonemptiness of a convex polytope in the situation, when we add a constrain with parameters to its description.

Theorem 4. *Let* $\mathcal{M}$ *be a convex polytope described as follows*

$$\mathcal{M} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{M}\boldsymbol{x} = \boldsymbol{v},\ \boldsymbol{x} \geq \mathbf{0}\}$$

and for a fixed $(\boldsymbol{\lambda},\nu) \in \mathbb{R}^{n+1}$ *consider a convex polytope*

$$\mathcal{M}(\boldsymbol{\lambda},\nu) = \{\boldsymbol{x} \in \mathcal{M} \mid \boldsymbol{\lambda}^T\boldsymbol{x} = \nu\}.$$

Then the set of all $(\boldsymbol{\lambda},\nu) \in \mathbb{R}^{n+1}$ *for which* $\mathcal{M}(\boldsymbol{\lambda},\nu) \neq \emptyset$, *has the description*

$$\mathbb{R}^{n+1} \setminus (\mathcal{U}_1 \cup -\mathcal{U}_1),$$

where

$$\mathcal{U}_1 = \{(\boldsymbol{\lambda},\nu) \in \mathbb{R}^{n+1} \mid \boldsymbol{x}_i^T\boldsymbol{\lambda} > \nu \quad \forall i \in V\},$$

is the convex cone with dimension equal to n *and* $\boldsymbol{x}_i,\ i \in V$, *are all vertices of the convex polytope* $\mathcal{M}$.

3 Solution set

From now on, we study the situation, when there are parameters in one column of the matrix $\widetilde{\mathbf{A}}$ from (1) instead of fixed values. Without the loss of generality assume that parameters are situated in the last column of $\widetilde{\mathbf{A}}$, i.e. $\widetilde{\mathbf{A}} = (\mathbf{A}\ \boldsymbol{\delta})$ for fixed $\mathbf{A} \in \mathbb{R}^{m\times(n-1)}$ and vector of parameters $\boldsymbol{\delta} \in \mathbb{R}^m$. The problem will not be more complicated, if there are parameters in the last column of the matrix $\widetilde{\mathbf{C}}$ from (2) as well, i.e. $\widetilde{\mathbf{C}} = (\mathbf{C}\ \boldsymbol{\mu})$ for fixed $\mathbf{C} \in \mathbb{R}^{l\times(n-1)}$ and vector of parameters $\boldsymbol{\mu} \in \mathbb{R}^l$. Let us introduce a family of convex polyhedral sets

$$M_1(\boldsymbol{\delta}) \equiv \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} + \boldsymbol{\delta} x_n \leq \boldsymbol{b}\}, \tag{5}$$

$$M_2(\boldsymbol{\mu}) \equiv \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \mathbf{C}\boldsymbol{x} + \boldsymbol{\mu} x_n \leq \boldsymbol{d}\}, \tag{6}$$

where $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{d} \in \mathbb{R}^l$. Assume that matrices $(\mathbf{A}\ \boldsymbol{b})$, $(\mathbf{C}\ \boldsymbol{d})$ do not contain the zero row.

Furthermore let us introduce

$$M_1 \equiv \{\boldsymbol{x} \in \mathbb{R}^{n-1} \mid \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}\}, \tag{7}$$

$$M_2 \equiv \{\boldsymbol{x} \in \mathbb{R}^{n-1} \mid \mathbf{C}\boldsymbol{x} \leq \boldsymbol{d}\}. \tag{8}$$

The following observations hold trivially.

If $M_1 \neq \emptyset$, then $M_1(\boldsymbol{\delta}) \neq \emptyset\ \forall \boldsymbol{\delta} \in \mathbb{R}^m$ (since when $\boldsymbol{x} \in M_1$, then $(\boldsymbol{x}, 0) \in M_1(\boldsymbol{\delta})$). If $dim\, M_1 = n-1$, then $dim\, M_1(\boldsymbol{\delta}) = n\ \forall \boldsymbol{\delta} \in \mathbb{R}^m$ (since when $\boldsymbol{x} \in int\, M_1$, then $(\boldsymbol{x}, 0) \in int\, M_1(\boldsymbol{\delta})$). Analogically for the set M_2.

Definition 3. *The solution set* (for the strong separability of the convex polyhedral sets $M_1(\boldsymbol{\delta})$ from (5) and $M_2(\boldsymbol{\mu})$ from (6)) is the set of all $(\boldsymbol{\delta}, \boldsymbol{\mu}) \in \mathbb{R}^{m+l}$ such, that the convex polyhedral sets $M_1(\boldsymbol{\delta})$, $M_2(\boldsymbol{\mu})$ are strongly separable.

Let us introduce

$$\mathcal{P}_1 \equiv \{\boldsymbol{\delta} \in \mathbb{R}^m \mid dim\, M_1(\boldsymbol{\delta}) = n\}, \tag{9}$$

$$\mathcal{P}_2 \equiv \{\boldsymbol{\mu} \in \mathbb{R}^l \mid dim\, M_2(\boldsymbol{\mu}) = n\}. \tag{10}$$

Now we will be concerned with the description of the set $\mathcal{P}_1$. The description of $\mathcal{P}_2$ will be analogical.

Theorem 5. *The set $\mathcal{P}_1$ has the description*

$$\mathcal{P}_1 = \mathcal{V}_1 \cup -\mathcal{V}_1, \tag{11}$$

where

$$\mathcal{V}_1 = \{\boldsymbol{\delta} \in \mathbb{R}^m \mid \boldsymbol{h}_i^T \boldsymbol{\delta} > 0 \ \forall i \in I\} \tag{12}$$

and $\boldsymbol{h}_i$, $i \in I$ are extremal directions[1] *of the convex polyhedral cone*

$$\mathcal{N}_{\mathbf{A},\boldsymbol{b}} \equiv \{\boldsymbol{y} \in \mathbb{R}^m \mid \mathbf{A}^T \boldsymbol{y} = \mathbf{0},\ \boldsymbol{b}^T \boldsymbol{y} \leq 0,\ \boldsymbol{y} \geq \mathbf{0}\}. \tag{13}$$

Proof. $\mathcal{P}_1$ is the set of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which $int\, M_1(\boldsymbol{\delta}) \neq \emptyset$ or equivalently

$$\{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} + \boldsymbol{\delta} x_n \leq \boldsymbol{b} - \boldsymbol{\varepsilon}\} \neq \emptyset \tag{14}$$

for an infinitesimal vector $\boldsymbol{\varepsilon} > \mathbf{0}$. The situation (14) holds for a vector $\boldsymbol{\delta}$ if and only if the problem

$$\max\{\mathbf{0}^T \boldsymbol{x} + 0 x_n \mid \mathbf{A}\boldsymbol{x} + \boldsymbol{\delta} x_n \leq \boldsymbol{b} - \boldsymbol{\varepsilon}\}$$

has an optimal solution. It follows from the theory of duality in linear programming that this is equivalent to the condition, that the problem

$$\min\{(\boldsymbol{b} - \boldsymbol{\varepsilon})^T \boldsymbol{y} \mid \mathbf{A}^T \boldsymbol{y} = \mathbf{0},\ \boldsymbol{\delta}^T \boldsymbol{y} = 0,\ \boldsymbol{y} \geq \mathbf{0}\} \tag{15}$$

has an optimal solution. The set of feasible solutions to the problem (15) forms a convex polyhedral cone. Therefore the problem (15) has an optimal solution if and only if

$$\{\boldsymbol{y} \in \mathbb{R}^m \mid \mathbf{A}^T \boldsymbol{y} = \mathbf{0},\ \boldsymbol{\delta}^T \boldsymbol{y} = 0,\ \boldsymbol{y} \geq \mathbf{0},\ (\boldsymbol{b} - \boldsymbol{\varepsilon})^T \boldsymbol{y} < 0\} = \emptyset,$$

or, equivalently

$$\{\boldsymbol{y} \in \mathbb{R}^m \mid \mathbf{A}^T \boldsymbol{y} = \mathbf{0},\ \boldsymbol{\delta}^T \boldsymbol{y} = 0,\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{b}^T \boldsymbol{y} \leq 0,\ \boldsymbol{y} \neq \mathbf{0}\} = \emptyset.$$

Hence $\mathcal{P}_1$ is the set of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which

$$\{\boldsymbol{y} \in \mathcal{N}_{\mathbf{A},\boldsymbol{b}} \mid \boldsymbol{\delta}^T \boldsymbol{y} = 0\} = \{\mathbf{0}\} \tag{16}$$

holds. We claim that $\mathcal{P}_1 = \mathcal{V}_1 \cup -\mathcal{V}_1$.

[1] vectors in directions of unbounded edges

Let $\boldsymbol{\delta}^0 \in \mathcal{V}_1$. Then $\boldsymbol{h}_i^T \boldsymbol{\delta}^0 > 0 \ \forall i \in I$. Each nontrivial vector $\boldsymbol{y} \in \mathcal{N}_{\mathbf{A},\boldsymbol{b}}$ can be expressed as a linear combination $\boldsymbol{y} = \sum_{i \in I} \alpha_i \boldsymbol{h}_i$ for certain $\alpha_i \geq 0$, $\sum_{i \in I} \alpha_i > 0$. Therefore

$$\boldsymbol{y}^T \boldsymbol{\delta}^0 = \sum_{i \in I} \alpha_i \boldsymbol{h}_i^T \boldsymbol{\delta}^0 > 0$$

and the condition (16) holds. Analogically for $\boldsymbol{\delta}^0 \in -\mathcal{V}_1$.

Conversely, let $\boldsymbol{\delta}^0 \in \mathbb{R}^m$ and suppose that the condition (16) holds. Then either $\boldsymbol{y}^T \boldsymbol{\delta}^0 > 0$ for all nontrivial $\boldsymbol{y} \in \mathcal{N}_{\mathbf{A},\boldsymbol{b}}$ or $\boldsymbol{y}^T \boldsymbol{\delta}^0 < 0$ for all nontrivial $\boldsymbol{y} \in \mathcal{N}_{\mathbf{A},\boldsymbol{b}}$. In the first case we specially have $\boldsymbol{h}_i^T \boldsymbol{\delta}^0 > 0 \ \forall i \in I$ and thus $\boldsymbol{\delta}^0 \in \mathcal{V}_1$. In the second case we analogically have $\boldsymbol{\delta}^0 \in -\mathcal{V}_1$. □

Let us introduce

$$\begin{aligned} \mathcal{P}_1' &\equiv \{\boldsymbol{\delta} \in \mathbb{R}^m \mid M_1(\boldsymbol{\delta}) \neq \emptyset\}, \\ \mathcal{P}_2' &\equiv \{\boldsymbol{\mu} \in \mathbb{R}^l \mid M_2(\boldsymbol{\mu}) \neq \emptyset\}. \end{aligned}$$

Now we will be derive the description of the set $\mathcal{P}_1'$. The description of $\mathcal{P}_2'$ will be analogical.

Theorem 6. *Let us consider the convex polyhedral cone $\mathcal{N}_{\mathbf{A},\boldsymbol{b}}$ from* (13). *Let $\boldsymbol{g}_i$, $i \in I_1$ be extremal directions of $\mathcal{N}_{\mathbf{A},\boldsymbol{b}}$ with the property $\boldsymbol{g}_i^T \boldsymbol{b} < 0$ and let $\boldsymbol{h}_j$, $j \in I_2$ be extremal directions of $\mathcal{N}_{\mathbf{A},\boldsymbol{b}}$ with the property $\boldsymbol{h}_j^T \boldsymbol{b} = 0$. If $I_1 = \emptyset$, then $\mathcal{P}_1' = \mathbb{R}^m$. Otherwise the set $\mathcal{P}_1'$ has the description*

$$\mathcal{P}_1' = \mathcal{V}_1' \cup -\mathcal{V}_1',$$

where

$$\mathcal{V}_1' = \{\boldsymbol{\delta} \in \mathbb{R}^m \mid \boldsymbol{g}_i^T \boldsymbol{\delta} > 0 \ \forall i \in I_1, \quad \boldsymbol{h}_j^T \boldsymbol{\delta} \geq 0 \ \forall j \in I_2\}.$$

Proof. $\mathcal{P}_1'$ is the set of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which $M_1(\boldsymbol{\delta}) \neq \emptyset$, i.e. the problem

$$\max \{\mathbf{0}^T \boldsymbol{x} + 0 x_n \mid \mathbf{A}\boldsymbol{x} + \boldsymbol{\delta} x_n \leq \boldsymbol{b}\}$$

has an optimal solution. It follows from the theory of duality in linear programming that this is equivalent to the condition, that the problem

$$\min \{\boldsymbol{b}^T \boldsymbol{y} \mid \mathbf{A}^T \boldsymbol{y} = \mathbf{0}, \ \boldsymbol{\delta}^T \boldsymbol{y} = 0, \ \boldsymbol{y} \geq \mathbf{0}\} \tag{17}$$

has an optimal solution. The set of feasible solutions to the problem (17) forms a convex polyhedral cone. Therefore that the problem (17) has an optimal solution if and only if

$$\{\boldsymbol{y} \in \mathbb{R}^m \mid \mathbf{A}^T\boldsymbol{y} = \mathbf{0},\ \boldsymbol{\delta}^T\boldsymbol{y} = 0,\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{b}^T\boldsymbol{y} < 0\} = \emptyset. \tag{18}$$

Hence $\mathcal{P}_1'$ is the set of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which (18) holds. If $I_1 = \emptyset$, then $\boldsymbol{b}^T\boldsymbol{y} = 0$ for all $\boldsymbol{y} \in \mathcal{N}_{\mathbf{A},\boldsymbol{b}}$ and thus $\mathcal{P}_1' = \mathbb{R}^m$. Otherwise we assert that $\mathcal{P}_1' = \mathcal{V}_1' \cup -\mathcal{V}_1'$.

Let $\boldsymbol{\delta}^0 \in \mathcal{V}_1'$. Then $\boldsymbol{g}_i^T\boldsymbol{\delta}^0 > 0\ \forall i \in I_1$ and $\boldsymbol{h}_j^T\boldsymbol{\delta}^0 \geq 0\ \forall j \in I_2$. Each point $\boldsymbol{y} \in \{\boldsymbol{y} \in \mathbb{R}^m \mid \mathbf{A}^T\boldsymbol{y} = \mathbf{0},\ \boldsymbol{b}^T\boldsymbol{y} < 0,\ \boldsymbol{y} \geq \mathbf{0}\}$ can be expressed as a linear combination $\boldsymbol{y} = \sum_{i\in I_1} \alpha_i \boldsymbol{g}_i + \sum_{j \in I_2} \beta_j \boldsymbol{h}_j$ for certain $\alpha_i, \beta_j \geq 0$, $\sum_{i\in I_1} \alpha_i > 0$. Therefore

$$\boldsymbol{y}^T\boldsymbol{\delta}^0 = \sum_{i\in I_1} \alpha_i \boldsymbol{g}_i^T\boldsymbol{\delta}^0 + \sum_{j\in I_2} \beta_j \boldsymbol{h}_j^T\boldsymbol{\delta}^0 > 0$$

and the condition (18) holds. Analogically for $\boldsymbol{\delta}^0 \in -\mathcal{V}_1'$.

Conversely assume, that $\boldsymbol{\delta}^0 \in \mathbb{R}^m$ and the condition (18) holds. Then either $\boldsymbol{y}^T\boldsymbol{\delta}^0 > 0$ for all $\{\boldsymbol{y} \in \mathbb{R}^m \mid \mathbf{A}^T\boldsymbol{y} = \mathbf{0},\ \boldsymbol{b}^T\boldsymbol{y} < 0,\ \boldsymbol{y} \geq \mathbf{0}\}$ or $\boldsymbol{y}^T\boldsymbol{\delta}^0 < 0$ for all $\{\boldsymbol{y} \in \mathbb{R}^m \mid \mathbf{A}^T\boldsymbol{y} = \mathbf{0},\ \boldsymbol{b}^T\boldsymbol{y} < 0,\ \boldsymbol{y} \geq \mathbf{0}\}$. In the first case we specially have $\boldsymbol{g}_i^T\boldsymbol{\delta}^0 > 0\ \forall i \in I_1$ and for infinitesimal $\varepsilon > 0$ also $(1-\varepsilon)\boldsymbol{h}_j^T\boldsymbol{\delta}^0 + \frac{\varepsilon}{|I_1|}\sum_{i\in I_1} \boldsymbol{g}_i^T\boldsymbol{\delta}^0 > 0\ \forall j \in I_2$. Hence $(1-\varepsilon)\boldsymbol{h}_j^T\boldsymbol{\delta}^0 \geq 0$ for infinitesimal $\varepsilon > 0$, and thus $\boldsymbol{h}_j^T\boldsymbol{\delta}^0 \geq 0\ \forall j \in I_2$. It follows that $\boldsymbol{\delta}^0 \in \mathcal{V}_1'$. In the second case we analogically have $\boldsymbol{\delta}^0 \in -\mathcal{V}_1'$. □

Let us introduce

$$\mathcal{Q}^*(\boldsymbol{\delta},\boldsymbol{\mu}) \equiv \left\{(\boldsymbol{u},\boldsymbol{v},v_{l+1}) \in \mathbb{R}^{m+l+1} \mid \mathbf{Z}(\boldsymbol{\delta},\boldsymbol{\mu})\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ v_{l+1} \end{pmatrix} = \boldsymbol{z},\ (\boldsymbol{u},\boldsymbol{v},v_{l+1}) \geq \mathbf{0}\right\}, \tag{19}$$

where

$$\mathbf{Z}(\boldsymbol{\delta},\boldsymbol{\mu}) \equiv \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T & \mathbf{0} \\ \boldsymbol{\delta}^T & \boldsymbol{\mu}^T & 0 \\ \boldsymbol{b}^T & \boldsymbol{d}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix}, \quad \boldsymbol{z} \equiv \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{20}$$

For the explicit description of all separating hyperplanes of the convex polyhedral sets $M_1(\boldsymbol{\delta})$, $M_2(\boldsymbol{\mu})$ with $\boldsymbol{\delta} \in \mathcal{P}_1$, $\boldsymbol{\mu} \in \mathcal{P}_2$ and $int\, M_1(\boldsymbol{\delta}) \cap int\, M_2(\boldsymbol{\mu}) = \emptyset$ we can directly use Theorem 2.

Assertion 1. *Let* $\boldsymbol{\delta} \in \mathcal{P}_1,\ \boldsymbol{\mu} \in \mathcal{P}_2,\ (\boldsymbol{u}, \boldsymbol{v}, v_{l+1}) \in \mathcal{Q}^*(\boldsymbol{\delta}, \boldsymbol{\mu})$. *Suppose that* $\left(\boldsymbol{u}^T \mathbf{A}, \boldsymbol{u}^T \boldsymbol{\delta}\right) \neq (\mathbf{0}^T, 0)$, *and* $\eta \in \langle 0, v_{l+1}\rangle$ *is arbitrary. Then*

$$\mathcal{R} = \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \boldsymbol{u}^T(\mathbf{A}\boldsymbol{x} + \boldsymbol{\delta} x_n - \boldsymbol{b}) = \eta\} \tag{21}$$

represents a separating hyperplane of the convex polyhedral sets $M_1(\boldsymbol{\delta}),\ M_2(\boldsymbol{\mu})$. *Conversely, any separating hyperplane* $\mathcal{R}$ *of convex polyhedral sets* $M_1(\boldsymbol{\delta}),\ M_2(\boldsymbol{\mu})$ *we can express in the form of* (21) *for a certain* $(\boldsymbol{u}, \boldsymbol{v}, v_{l+1}) \in \mathcal{Q}^*(\boldsymbol{\delta}, \boldsymbol{\mu})$, $\left(\boldsymbol{u}^T \mathbf{A}, \boldsymbol{u}^T \boldsymbol{\delta}\right) \neq (\mathbf{0}^T, 0)$, *and* $\eta \in \langle 0, v_{l+1}\rangle$.

Let $\boldsymbol{\delta} \in \mathcal{P}_1,\ \boldsymbol{\mu} \in \mathcal{P}_2$. Then according to Theorem 3 the convex polyhedral sets $M_1(\boldsymbol{\delta}),\ M_2(\boldsymbol{\mu})$ are strongly separable if and only if $\mathcal{Q}^*(\boldsymbol{\delta}, \boldsymbol{\mu}) \neq \emptyset$. Furthermore let us introduce

$$\mathcal{U} \equiv \{(\boldsymbol{\delta}, \boldsymbol{\mu}) \in \mathbb{R}^{m+l} \mid \mathcal{Q}^*(\boldsymbol{\delta}, \boldsymbol{\mu}) \neq \emptyset\}. \tag{22}$$

From Theorem 4 we have the relation $\mathcal{U} = \mathbb{R}^{m+l} \setminus (\mathcal{U}_1 \cup -\mathcal{U}_1)$, where

$$\mathcal{U}_1 = \{(\boldsymbol{\delta}, \boldsymbol{\mu}) \in \mathbb{R}^{m+l} \mid (\boldsymbol{\delta}^T, \boldsymbol{\mu}^T, 0)(\boldsymbol{u}, \boldsymbol{v}, v_{l+1})^i > 0\ \forall i \in V\}, \tag{23}$$

and $(\boldsymbol{u}, \boldsymbol{v}, v_{l+1})^i,\ i \in V$ are all vertices of the convex polytope

$$\mathcal{Q}_r^* \equiv \left\{(\boldsymbol{u}, \boldsymbol{v}, v_{l+1}) \in \mathbb{R}^{m+l+1} \mid \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T & \mathbf{0} \\ \boldsymbol{b}^T & \boldsymbol{d}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix}, \right.$$
$$\left. (\boldsymbol{u}, \boldsymbol{v}, v_{l+1}) \geq \mathbf{0} \right\}. \tag{24}$$

Hence we obtain (from Theorems 1 and 3) the following description of the solution set.

Assertion 2. *The solution set for* $M_1(\boldsymbol{\delta}),\ M_2(\boldsymbol{\mu})$ *has the description*

$$(\mathcal{P}_1 \times \mathcal{P}_2) \cap \mathcal{U} = (\mathcal{P}_1 \times \mathcal{P}_2) \setminus (\mathcal{U}_1 \cup -\mathcal{U}_1)\,. \tag{25}$$

Assertion 3. *If* $dim\, M_1 = dim\, M_2 = n-1$ *and the convex polyhedral sets* $M_1,\ M_2$ *from* (7), (8) *are not strongly separable, then the solution set for* $M_1(\boldsymbol{\delta}),\ M_2(\boldsymbol{\mu})$ *is empty.*

Proof. From the assumptions of the assertion it follows that there exists a point $\boldsymbol{x}^0 \in \mathbb{R}^{n-1}$ such that $\boldsymbol{x}^0 \in int\, M_1 \cap int\, M_2$. Hence for all $\boldsymbol{\delta} \in \mathcal{P}_1$, $\boldsymbol{\mu} \in \mathcal{P}_2$ the inclusion $(\boldsymbol{x}^0, 0) \in int\, M_1(\boldsymbol{\delta}) \cap int\, M_2(\boldsymbol{\mu})$ holds and therefore the convex polyhedral sets $M_1(\boldsymbol{\delta}),\ M_2(\boldsymbol{\mu})$ are not strongly separable. $\square$

Assertion 4. *We have* $(\mathcal{U}_1 \cup -\mathcal{U}_1) \subseteq (\mathcal{P}_1 \times \mathcal{P}_2)$.

Proof. We will prove that each point not belonging to the set $\mathcal{P}_1 \times \mathcal{P}_2$ does not belong to $\mathcal{U}_1 \cup -\mathcal{U}_1$ as well. Let $(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0) \in \mathbb{R}^{m+l} \setminus (\mathcal{P}_1 \times \mathcal{P}_2)$. Without the loss of generality assume that $\boldsymbol{\delta}^0 \notin \mathcal{P}_1$. Since $\boldsymbol{\delta}^0 \notin \mathcal{P}_1$, the problem (15) has not any optimal solution for the objective function $(\boldsymbol{b} - \boldsymbol{\varepsilon})^T \boldsymbol{y}$, where $\boldsymbol{\varepsilon} > \mathbf{0}$ is infinitesimal. Hence there exists $\boldsymbol{y}^0 \neq \mathbf{0}$, such that $\mathbf{A}^T \boldsymbol{y}^0 = \mathbf{0}$, $(\boldsymbol{\delta}^0)^T \boldsymbol{y}^0 = 0$, $\boldsymbol{y}^{\mathbf{0}} \geq \mathbf{0}$ and $(\boldsymbol{b} - \boldsymbol{\varepsilon})^T \boldsymbol{y}^0 < 0$ (i.e. $\boldsymbol{b}^T \boldsymbol{y}^0 \leq 0$). Define

$$(\boldsymbol{u}^0, \boldsymbol{v}^0, u_{l+1}^0) \equiv \frac{(\boldsymbol{y}^0, \mathbf{0}, -\boldsymbol{b}^T \boldsymbol{y}^0)}{\mathbf{1}^T \boldsymbol{y}^0} \in \mathcal{Q}^*(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0).$$

Hence $\mathcal{Q}^*(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0) \neq \emptyset$, $(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0) \in \mathcal{U}$ and eventually $(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0) \notin (\mathcal{U}_1 \cup -\mathcal{U}_1)$ holds. $\square$

4 Special cases

Let us consider some special cases of the convex polyhedral sets $M_1(\boldsymbol{\delta})$, $M_2(\boldsymbol{\mu})$ from (5), (6).

4.1 One parameter

Let us consider only one parameter in the description (5) of the convex polyhedral set $M_1(\boldsymbol{\delta})$, without the loss of generality in the last row. Thus

$$\begin{aligned} M_1(\delta) &\equiv \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} + \begin{pmatrix} \boldsymbol{a} \\ \delta \end{pmatrix} x_n \leq \boldsymbol{b}\}, \\ M_2 &\equiv \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \mathbf{C}\boldsymbol{x} + \boldsymbol{c}x_n \leq \boldsymbol{d}\}, \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m \times (n-1)}$, $\mathbf{C} \in \mathbb{R}^{l \times (n-1)}$, $\boldsymbol{a} \in \mathbb{R}^{m-1}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^l$ are known and $\delta \in \mathbb{R}$ is a parameter. Then the sets $\mathcal{V}_1$, $-\mathcal{V}_1$ from (12) is described as follows

$$\begin{aligned} \mathcal{V}_1 &= \{\delta \in \mathbb{R} \mid \boldsymbol{h}_i^T \begin{pmatrix} \boldsymbol{a} \\ \delta \end{pmatrix} > 0 \ \forall i \in I\} = \begin{cases} \emptyset, \\ \mathbb{R}, \\ \{\delta \in \mathbb{R} \mid \delta > \delta^0\}, \end{cases} \\ -\mathcal{V}_1 &= \{\delta \in \mathbb{R} \mid \boldsymbol{h}_i^T \begin{pmatrix} \boldsymbol{a} \\ \delta \end{pmatrix} < 0 \ \forall i \in I\} = \begin{cases} \emptyset, \\ \mathbb{R}, \\ \{\delta \in \mathbb{R} \mid \delta < \delta^1\} \end{cases} \end{aligned}$$

for certain $\delta^0, \delta^1 \in \mathbb{R}$. Hence the set $\mathcal{P}_1$ from (11) represents $\emptyset$, $\mathbb{R}$ or a union of two intervals $\{\delta \in \mathbb{R} \mid \delta > \delta^0\} \cup \{\delta \in \mathbb{R} \mid \delta < \delta^1\}$. Analogously the set $\mathcal{U}_1 \cup -\mathcal{U}_1$ from (23) represents $\emptyset$, $\mathbb{R}$ or a union of two intervals $\{\delta \in \mathbb{R} \mid \delta > \delta^2\} \cup \{\delta \in \mathbb{R} \mid \delta < \delta^3\}$ for certain $\delta^2, \delta^3 \in \mathbb{R}$. Hence it follows that the solution set represents a union of at most two real intervals (not necessary bounded).

4.2 Linear structure with one parameter

Let us consider one parameter with a fixed direction in this way

$$
\begin{aligned}
M_1(\delta) &\equiv \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} + (\boldsymbol{a} + \delta \boldsymbol{a}')x_n \leq \boldsymbol{b}\},\\
M_2(\delta) &\equiv \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \mathbf{C}\boldsymbol{x} + (\boldsymbol{c} + \delta \boldsymbol{c}')x_n \leq \boldsymbol{d}\},
\end{aligned}
$$

where $\mathbf{A} \in \mathbb{R}^{m\times(n-1)}$, $\mathbf{C} \in \mathbb{R}^{l\times(n-1)}$, $\boldsymbol{a}, \boldsymbol{a}', \boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c}, \boldsymbol{c}', \boldsymbol{d} \in \mathbb{R}^l$, are known and $\delta \in \mathbb{R}$ is a parameter. Then the set $\mathcal{V}_1$ from (12) represents an open real interval (not necessary bounded) and therefore the set $\mathcal{P}_1$ from (11) represents a union of at most two open real intervals. $\mathcal{P}_1 \times \mathcal{P}_1$ consisis of at most three open real intervals. The set $\mathcal{U}_1 \cup -\mathcal{U}_1$ from (23) represents a union of at most two open real intervals. Hence it follows that the solution set represents a union of at most five real intervals.

4.3 A permanent separating hyperplane

Let us consider the convex polyhedral sets $M_1(\boldsymbol{\delta})$, $M_2(\boldsymbol{\mu})$ from (5), (6) with the property $\boldsymbol{\delta} \in \mathcal{Z}_1$, $\boldsymbol{\mu} \in \mathcal{Z}_2$, where $\mathcal{Z}_1 \subset \mathbb{R}^m$, $\mathcal{Z}_2 \subset \mathbb{R}^l$ are convex polytopes. Without the loss of generality assume that $\mathcal{Z}_1 \subset \mathcal{P}_1'$ and $\mathcal{Z}_2 \subset \mathcal{P}_2'$ (otherwise we restrict the sets to $\mathcal{Z}_1 \cap \mathcal{P}_1'$, $\mathcal{Z}_2 \cap \mathcal{P}_2'$ respectively). Moreover, we will assume for the sake of simplicity, that all the convex polyhedral sets $M_1(\boldsymbol{\delta})$, $\boldsymbol{\delta} \in \mathcal{Z}_1$, $M_2(\boldsymbol{\mu})$, $\boldsymbol{\mu} \in \mathcal{Z}_2$ contain at least one vertex. The question is, whether there exists a fixed hyperplane $\mathcal{R}$ such that:

$$
M_1(\boldsymbol{\delta}) \subseteq \overline{\mathcal{R}^-} \;\forall \boldsymbol{\delta} \in \mathcal{Z}_1, \quad M_2(\boldsymbol{\mu}) \subseteq \overline{\mathcal{R}^+} \;\forall \boldsymbol{\mu} \in \mathcal{Z}_2.
$$

Such a hyperplane $\mathcal{R}$ is called *a permanent separating hyperplane*. Note, that a permanent separating hyperplane need not exist even if $M_1(\boldsymbol{\delta})$, $M_2(\boldsymbol{\mu})$ are separable for all $\boldsymbol{\delta} \in \mathcal{Z}_1$, $\boldsymbol{\mu} \in \mathcal{Z}_2$ (see Example 1). We check the existence of a permanent separating hyperplane by the following process: Compute the convex hulls *conv* $\left(\cup_{\boldsymbol{\delta}\in\mathcal{Z}_1} M_1(\boldsymbol{\delta})\right)$ and *conv* $\left(\cup_{\boldsymbol{\mu}\in\mathcal{Z}_2} M_2(\boldsymbol{\mu})\right)$ and check separability of these convex hulls.

Lemma 1. *Let B_1 be a sub-basis of the convex polyhedral set $M_1(\boldsymbol{\delta})$. Let us consider the following convex polyhedral cone*

$$\mathcal{N}_{B_1} \equiv \{(\boldsymbol{y},\boldsymbol{z}) \in \mathbb{R}^m \mid \mathbf{A}_{B_1}^T\boldsymbol{y} + \mathbf{A}_{N_1}^T\boldsymbol{z} = \mathbf{0},\ \boldsymbol{b}_{B_1}^T\boldsymbol{y} + \boldsymbol{b}_{N_1}^T\boldsymbol{z} \leq 0,\ \boldsymbol{z} \geq \mathbf{0}\}, \tag{26}$$

where $N_1 \equiv \{1,\ldots,m\} \setminus B_1$. Let us denote by $(\boldsymbol{g}_i^{\boldsymbol{y}},\boldsymbol{g}_i^{\boldsymbol{z}})$, $i \in I_1$ extremal directions of (26) *with the property $(\boldsymbol{g}_i^{\boldsymbol{y}},\boldsymbol{g}_i^{\boldsymbol{z}})^T(\boldsymbol{b}_{B_1},\boldsymbol{b}_{N_1}) < 0$ and denote by $(\boldsymbol{h}_j^{\boldsymbol{y}},\boldsymbol{h}_j^{\boldsymbol{z}})$, $j \in I_2$ extremal directions of* (26) *with the property $(\boldsymbol{h}_j^{\boldsymbol{y}},\boldsymbol{h}_j^{\boldsymbol{z}})^T(\boldsymbol{b}_{B_1},\boldsymbol{b}_{N_1}) = 0$. The set $\mathcal{S}_{B_1}$ of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which is sub-basis B_1 feasible for $M_1(\boldsymbol{\delta})$ has the following description:*

If $I_1 = \emptyset$, then $\mathcal{S}_{B_1} = \mathbb{R}^m$. Otherwise

$$\mathcal{S}_{B_1} = \mathcal{S}_{B_1}^* \cup -\mathcal{S}_{B_1}^*,$$

where

$$\begin{aligned}\mathcal{S}_{B_1}^* = \{\boldsymbol{\delta} \in \mathbb{R}^m \mid\ & (\boldsymbol{g}_i^{\boldsymbol{y}},\boldsymbol{g}_i^{\boldsymbol{z}})^T(\boldsymbol{\delta}_{B_1},\boldsymbol{\delta}_{N_1}) > 0\ \forall i \in I_1,\\ & (\boldsymbol{h}_j^{\boldsymbol{y}},\boldsymbol{h}_j^{\boldsymbol{z}})^T(\boldsymbol{\delta}_{B_1},\boldsymbol{\delta}_{N_1}) \geq 0\ \forall j \in I_2\}.\end{aligned}$$

Proof. Feasibility of the sub-basis B_1 preserves for the values $\boldsymbol{\delta} \in \mathbb{R}^m$ for which the set

$$\begin{aligned}&\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}_{B_1}\boldsymbol{x} + \boldsymbol{\delta}_{B_1}x_n = \boldsymbol{b}_{B_1},\ \mathbf{A}_{N_1}\boldsymbol{x} + \boldsymbol{\delta}_{N_1}x_n \leq \boldsymbol{b}_{N_1}\} =\\ &\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}_{B_1}\boldsymbol{x} + \boldsymbol{\delta}_{B_1}x_n \leq \boldsymbol{b}_{B_1},\ -\mathbf{A}_{B_1}\boldsymbol{x} - \boldsymbol{\delta}_{B_1}x_n \leq -\boldsymbol{b}_{B_1},\\ &\qquad \mathbf{A}_{N_1}\boldsymbol{x} + \boldsymbol{\delta}_{N_1}x_n \leq \boldsymbol{b}_{N_1}\}\end{aligned}$$

is nonempty. Consider the convex polyhedral cone

$$\begin{aligned}\{(\boldsymbol{y}^1,\boldsymbol{y}^2,\boldsymbol{z}) \in \mathbb{R}^{m+|B_1|}\ |&\mathbf{A}_{B_1}^T\boldsymbol{y}^1 - \mathbf{A}_{B_1}^T\boldsymbol{y}^2 + \mathbf{A}_{N_1}^T\boldsymbol{z} = \mathbf{0},\\ &\boldsymbol{b}_{B_1}^T\boldsymbol{y}^1 - \boldsymbol{b}_{B_1}^T\boldsymbol{y}^2 + \boldsymbol{b}_{N_1}^T\boldsymbol{z} \leq 0,\ \boldsymbol{y}^1,\boldsymbol{y}^2,\boldsymbol{z} \geq \mathbf{0}\}.\end{aligned} \tag{27}$$

Denote by $(\boldsymbol{g}_i^1,\boldsymbol{g}_i^2,\boldsymbol{g}_i^{\boldsymbol{z}})$, $i \in I_1'$ extremal directions of the convex polyhedral cone (27) with the property $(\boldsymbol{g}_i^1,\boldsymbol{g}_i^2,\boldsymbol{g}_i^{\boldsymbol{z}})^T(\boldsymbol{b}_{B_1},-\boldsymbol{b}_{B_1},\boldsymbol{b}_{N_1}) < 0$ and denote by $(\boldsymbol{h}_j^1,\boldsymbol{h}_j^2,\boldsymbol{h}_j^{\boldsymbol{z}})$, $j \in I_2'$ extremal directions of (27) with the property $(\boldsymbol{h}_j^1,\boldsymbol{h}_j^2,\boldsymbol{h}_j^{\boldsymbol{z}})^T(\boldsymbol{b}_{B_1},-\boldsymbol{b}_{B_1},\boldsymbol{b}_{N_1}) = 0$. After substitution $\boldsymbol{y} \equiv \boldsymbol{y}^1 - \boldsymbol{y}^2$, $\boldsymbol{g}_i^{\boldsymbol{y}} \equiv \boldsymbol{g}_i^1 - \boldsymbol{g}_i^2$, $\boldsymbol{h}_j^{\boldsymbol{y}} \equiv \boldsymbol{h}_j^1 - \boldsymbol{h}_j^2$ we obtain according to Theorem 6 the statement of Lemmma 1, since if a vector $(\boldsymbol{g}_i^1,\boldsymbol{g}_i^2,\boldsymbol{g}_i^{\boldsymbol{z}})$ or $(\boldsymbol{h}_j^1,\boldsymbol{h}_j^2,\boldsymbol{h}_j^{\boldsymbol{z}})$ represents an extremal direction of (27), then the vector $(\boldsymbol{g}_i^{\boldsymbol{y}},\boldsymbol{g}_i^{\boldsymbol{z}})$ or $(\boldsymbol{h}_j^{\boldsymbol{y}},\boldsymbol{h}_j^{\boldsymbol{z}})$ represents an extremal direction of (26) and conversely. $\square$

Let $\boldsymbol{\delta}^0 \in \mathbb{R}^m$ and B_1 any feasible sub-basis of $M_1(\boldsymbol{\delta}^0)$. Let us introduce

$$\mathcal{S}_{B_1}^{\boldsymbol{\delta}^0} \equiv \begin{cases} \mathbb{R}^m & \text{if } I_1 = \emptyset \text{ (from Lemma 1)}, \\ \mathcal{S}_{B_1}^* & \text{if } \boldsymbol{\delta}^0 \in \mathcal{S}_{B_1}^*, \\ -\mathcal{S}_{B_1}^* & \text{if } \boldsymbol{\delta}^0 \in -\mathcal{S}_{B_1}^*. \end{cases} \tag{28}$$

Lemma 2. *Let $\boldsymbol{\delta}^0 \in \mathbb{R}^m$ and S an arbitrary $(n-1)$-elemental sub-basis of the convex polyhedral set $M_1(\boldsymbol{\delta}^0)$. Let us assume that the basis S determines an edge of $M_1(\boldsymbol{\delta}^0)$ unbounded in the direction of $(\boldsymbol{h}^0, h_n^0) \neq (\mathbf{0}, 0)$ and this edge originates from the vertex corresponding to the basis $S \cup \{i\}$ for a certain $i \in \{1, \dots, m\} \setminus S$. Then the set $\mathcal{H}_S^i$ of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which the edge, corresponding to the sub-basis S, represents an unbounded edge of $M_1(\boldsymbol{\delta})$ originating from the vertex determined by the basis $S \cup \{i\}$, has the following description:*

If $h_n^0 = 0$, then $\mathcal{H}_S^i = \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$. Otherwise

$$\begin{aligned} \mathcal{H}_S^i = \Big\{ & \boldsymbol{\delta} \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0} \mid \\ & \left(\delta_j - \mathbf{A}_{j,\cdot}^T \mathbf{A}_S^{-1} \boldsymbol{\delta}_S\right) sgn(h_n^0) \leq 0 \ \forall j \in \{1, \dots, m\} \setminus (S \cup \{i\}) \Big\}. \end{aligned} \tag{29}$$

Proof. The unbounded edge of the convex polyhedral set $M_1(\boldsymbol{\delta})$ corresponding to the sub-basis S and originating from the vertex determined by the basis $S \cup \{i\}$ is described by the system

$$\mathbf{A}_S \boldsymbol{x} + \boldsymbol{\delta}_S x_n = \boldsymbol{b}_S, \ \mathbf{A}_{i,\cdot} \boldsymbol{x} + \delta_i x_n \leq b_i$$

and is unbounded in direction which represents (according to [12]) a nontrivial solution to

$$\mathbf{A}_S \boldsymbol{x} + \boldsymbol{\delta}_S x_n = \mathbf{0}, \ \mathbf{A}_{i,\cdot} \boldsymbol{x} + \delta_i x_n \leq 0, \tag{30}$$

whereas the inequalities $\mathbf{A}_{j,\cdot} \boldsymbol{x} + \delta_j x_n \leq 0$ with $j \in \{1, \dots, m\} \setminus (S \cup \{i\})$ if added to (30) are redundant. Specially, for $(\boldsymbol{x}, x_n) = (\boldsymbol{h}^0, h_n^0)$, $\boldsymbol{\delta} = \boldsymbol{\delta}^0$ we have

$$\mathbf{A}_S \boldsymbol{h}^0 + \boldsymbol{\delta}_S^0 h_n^0 = \mathbf{0}, \ \mathbf{A}_{i,\cdot} \boldsymbol{h}^0 + \delta_i^0 h_n^0 \leq 0. \tag{31}$$

If $h_n^0 = 0$, then the vector $(\boldsymbol{h}^0, h_n^0)$ is obviously an extremal direction of $M_1(\boldsymbol{\delta})$ for all $\boldsymbol{\delta} \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$. Otherwise, the matrix $\mathbf{A}_S$ must be nonsingular (since $rank\left(\mathbf{A}_S\ \boldsymbol{\delta}_S^0\right) = n-1$). From (30) we have $\boldsymbol{x} = -\mathbf{A}_S^{-1}\boldsymbol{\delta}_S x_n$ and consequently

$$\left(\delta_i - \mathbf{A}_{i,\cdot}\mathbf{A}_S^{-1}\boldsymbol{\delta}_S\right)x_n \leq 0.$$

The equation

$$\delta_i - \mathbf{A}_{i,\cdot}\mathbf{A}_S^{-1}\boldsymbol{\delta}_S = det\,(\mathbf{A}_S^{-1}).det\begin{pmatrix} \mathbf{A}_S & \boldsymbol{\delta}_S \\ \mathbf{A}_{i,\cdot} & \delta_i \end{pmatrix}$$

holds. The determinant $det\begin{pmatrix} \mathbf{A}_S & \boldsymbol{\delta}_S \\ \mathbf{A}_{i,\cdot} & \delta_i \end{pmatrix}$ has a constant sign for all $\boldsymbol{\delta} \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$, since if for certain $\boldsymbol{\delta}^1, \boldsymbol{\delta}^2 \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$

$$d_1 = det\begin{pmatrix} \mathbf{A}_S & \boldsymbol{\delta}_S^1 \\ \mathbf{A}_{i,\cdot} & \delta_i^1 \end{pmatrix} > 0 \quad \text{and} \quad d_2 = det\begin{pmatrix} \mathbf{A}_S & \boldsymbol{\delta}_S^2 \\ \mathbf{A}_{i,\cdot} & \delta_i^2 \end{pmatrix} < 0,$$

hold, then for the convex combination $\boldsymbol{\delta}^3 \equiv \frac{1}{|d_1|+|d_2|}(|d_2|\boldsymbol{\delta}^1 + |d_1|\boldsymbol{\delta}^2) \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$ we have

$$det\begin{pmatrix} \mathbf{A}_S & \boldsymbol{\delta}_S^3 \\ \mathbf{A}_{i,\cdot} & \delta_i^3 \end{pmatrix} = 0,$$

which contradicts the feasibility of the basis $S\cup\{i\}$. Hence an element x_n from (30) has a constant sign for all $\boldsymbol{\delta} \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$, namely $sgn(h_n^0)$. To be $(\boldsymbol{x}, x_n)$ a vector of the unbounded edge of $M_1(\boldsymbol{\delta})$, it must $\mathbf{A}_{j,\cdot}\boldsymbol{x} + \delta_j x_n \leq 0$ hold for all $j \in \{1,\ldots,m\}\setminus(S\cup\{i\})$. Hence

$$-\mathbf{A}_{j,\cdot}^T\mathbf{A}_S^{-1}\boldsymbol{\delta}_S x_n + \delta_j x_n \leq 0\ \forall j \in \{1,\ldots,m\}\setminus(S\cup\{i\}),$$

or, equivalently

$$\left(\delta_j - \mathbf{A}_{j,\cdot}^T\mathbf{A}_S^{-1}\boldsymbol{\delta}_S\right)sgn(h_n^0) \leq 0\ \forall j \in \{1,\ldots,m\}\setminus(S\cup\{i\}).$$

□

Let $\boldsymbol{\delta}^0 \in \mathcal{Z}_1$. Denote by $\mathfrak{S}$ a family of all feasible bases of the convex polyhedral set $M_1(\boldsymbol{\delta}^0)$ and $\mathfrak{H}$ a family of pairs (S,i), where S is a feasible sub-basis of $M_1(\boldsymbol{\delta}^0)$ to which it corresponds an unbounded edge originated from a vertex determined by the basis $S\cup\{i\}$. Let us introduce

$$\mathcal{Z}_1(\mathfrak{S},\mathfrak{H}) \equiv \mathcal{Z}_1 \cap \left(\cap_{B\in\mathfrak{S}}\,\mathcal{S}_B^{\boldsymbol{\delta}^0}\right) \cap \left(\cap_{(S,i)\in\mathfrak{H}}\,\mathcal{H}_S^i\right).$$

The set $\mathcal{Z}_1(\mathfrak{S},\mathfrak{H})$ represents a set of all $\boldsymbol{\delta} \in \mathcal{Z}_1$ such that all bases from $\mathfrak{S}$ are (according to Lemma 1) feasible for $M_1(\boldsymbol{\delta})$ and the family of unbounded edges is preserved (Lemma 2). In this way we can devide the set $\mathcal{Z}_1$ into into the sets $\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)$, $k \in K$, where K is a finite index set. Each set $\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)$, $k \in K$, represents a convex set, a closure of which is a convex polytope.

Assertion 5. *Let $k \in K$ and $\mathfrak{S}_k \neq \emptyset$. Let us assume that the set $\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)$ is closed. Then the set conv $\left(\cup_{\boldsymbol{\delta}\in\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)} M_1(\boldsymbol{\delta})\right)$ represents a convex polyhedral set and*

$$conv\left(\bigcup_{\boldsymbol{\delta}\in\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)} M_1(\boldsymbol{\delta})\right) = conv\left(\bigcup_{\delta \text{ vertices of } \mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)} M_1(\boldsymbol{\delta})\right).$$

Proof. We will prove that for an arbitrary $\boldsymbol{\delta}^1, \boldsymbol{\delta}^2 \in \mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)$ and an arbitrary convex combination $\boldsymbol{\delta}^c \equiv (1-c)\boldsymbol{\delta}^1 + c\boldsymbol{\delta}^2$, $c \in (0,1)$ we have

$$M_1(\boldsymbol{\delta}^c) \subseteq conv\left(M_1(\boldsymbol{\delta}^1) \cup M_1(\boldsymbol{\delta}^2)\right).$$

To prove this it is sufficient to show that all vertices of the convex polyhedral set $M_1(\boldsymbol{\delta}^c)$ can be expressed as a convex combination of vertices $M_1(\boldsymbol{\delta}^1)$, $M_1(\boldsymbol{\delta}^2)$ and all extremal directions of $M_1(\boldsymbol{\delta}^c)$ can be expressed as nonnegative linear combinations of extremal directions of $M_1(\boldsymbol{\delta}^1)$, $M_1(\boldsymbol{\delta}^2)$. Let $B \in \mathfrak{S}_i$ be a basis of $M_1(\boldsymbol{\delta})$ and denote by $\boldsymbol{v}^1$, $\boldsymbol{v}^2$, and $\boldsymbol{v}^c$ the vertex corresponding to the basis B of the convex polyhedral set $M_1(\boldsymbol{\delta}^1)$, $M_1(\boldsymbol{\delta}^2)$, and $M_1(\boldsymbol{\delta}^c)$, respectively. Next denote $\mathbf{M} \equiv \left(\mathbf{A}_B \ \boldsymbol{\delta}_B^1\right)$. According to the familiar Sherman–Morrison formula, we get

$$\begin{aligned}\boldsymbol{v}^c &= \left(\mathbf{M} + c(\boldsymbol{\delta}_B^2 - \boldsymbol{\delta}_B^1)\boldsymbol{e}_n^T\right)^{-1}\boldsymbol{b}_B \\ &= \left(\mathbf{M}^{-1} - \frac{c\mathbf{M}^{-1}(\boldsymbol{\delta}_B^2 - \boldsymbol{\delta}_B^1)\boldsymbol{e}_n^T\mathbf{M}^{-1}}{1 + c\boldsymbol{e}_n^T\mathbf{M}^{-1}(\boldsymbol{\delta}_B^2 - \boldsymbol{\delta}_B^1)}\right)\boldsymbol{b}_B. \end{aligned} \tag{32}$$

Since B is a feasible basis of $M_1(\boldsymbol{\delta}^c)$ for all $c \in \langle 0,1\rangle$, the denominator of expression (32) has a constant sign for all $c \in \langle 0,1\rangle$. Hence the expression (32) represents a monotone function of a variable c with $c \in \langle 0,1\rangle$ and therefore the vertex $\boldsymbol{v}^c$ is a convex combination of $\boldsymbol{v}^1$, $\boldsymbol{v}^2$.

Let $(S,i) \in \mathfrak{H}_k$, where S is a $(n-1)$-elemental sub-basis determining unbounded edge of $M_1(\boldsymbol{\delta}^c)$. A vector in direction of this unbounded edge for $M_1(\boldsymbol{\delta}^1)$, $M_1(\boldsymbol{\delta}^2)$, and $M_1(\boldsymbol{\delta}^c)$ denote as $(\boldsymbol{h}^1, h_n^1)$, $(\boldsymbol{h}^2, h_n^2)$, and $(\boldsymbol{h}^c, h_n^c)$, respectively. From the proof of Lemma 2 we have $sgn(h_n^1) = sgn(h_n^2) = sgn(h_n^c)$. Consider the following three cases. If $h_n^1 = h_n^2 = h_n^c = -1$, then

$$\boldsymbol{\delta}^1 = \mathbf{A}_S \boldsymbol{h}^1, \ \boldsymbol{\delta}^2 = \mathbf{A}_S \boldsymbol{h}^2, \ \boldsymbol{\delta}^c = \mathbf{A}_S \boldsymbol{h}^c.$$

Hence we obtain $\mathbf{0} = \mathbf{A}_S\big(\boldsymbol{h}^c - (1-c)\boldsymbol{h}^1 - c\boldsymbol{h}^2\big)$. From the nonsingularity of the matrix $\mathbf{A}_S$ it follows that $(\boldsymbol{h}^c, h_n^c) = (1-c)(\boldsymbol{h}^1, h_n^1) + c(\boldsymbol{h}^2, h_n^2)$. The case $h_n^1 = h_n^2 = h_n^c = 1$ is quite analogical. In the case $h_n^1 = h_n^2 = h_n^c = 0$ all the vectors $\boldsymbol{h}^1$, $\boldsymbol{h}^2$, $\boldsymbol{h}^c$ determines the same direction. □

According to the Proposition 5 we can under some assumption reduce a computation of convex hull of infinite number of convex polyhedral sets to finite number (for explicit description of convex hulls see [5]). In this way we can the whole computation of $conv\left(\cup_{\boldsymbol{\delta}\in\mathcal{Z}_1} M_1(\boldsymbol{\delta})\right)$ reduce to computation of convex hull of finite number of convex polyhedral sets, since

$$conv\left(\bigcup_{\boldsymbol{\delta}\in\mathcal{Z}_1} M_1(\boldsymbol{\delta})\right) = conv\left(\bigcup_{k\in K} \ \bigcup_{\boldsymbol{\delta} \text{ vertices of } \mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)} M_1(\boldsymbol{\delta})\right).$$

Example 1. Given

$$\mathbf{A} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 12 \\ 15 \end{pmatrix},$$
$$\mathcal{Z}_1 = \left\{\boldsymbol{\delta} \in \mathbb{R}^3 \mid \boldsymbol{\delta} = (1,1,2)^T + t(1,-1,2)^T, \ t \in \langle 0, 6\rangle\right\}$$

and $M_2(\boldsymbol{\mu}) = \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}\boldsymbol{x} \le \begin{pmatrix} 14 \\ -12 \end{pmatrix}\right\}$ is fixed. We will compute the convex hull $conv\big(\cup_{\boldsymbol{\delta}\in\mathcal{Z}_1} M_1(\boldsymbol{\delta})\big)$ and check the existence of a permanent separating hyperplane.

1. Choose $\boldsymbol{\delta}^1 \in \mathcal{Z}_1$, e.g. as $\boldsymbol{\delta}^1 = (1,1,2)^T$. The family of all feasible bases of the convex polyhedral set $M_1(\boldsymbol{\delta}^1)$ is $\mathfrak{S}_1 = \{(1,2)\}$. The convex polyhedral cone $\mathcal{N}_{(1,2)}$ from (26) contains two unbounded edges, which are unbounded in directions of $\boldsymbol{g}_1 = (-1,-1,0)^T$, $\boldsymbol{h}_1 = (-9,-5,4)^T$. Hence

$$\mathcal{S}^{\boldsymbol{\delta}^1}_{(1,2)} = \{\boldsymbol{\delta} \in \mathbb{R}^3 \mid -\delta_1 - \delta_2 < 0, \ -9\delta_1 - 5\delta_2 + 4\delta_3 \le 0\}.$$

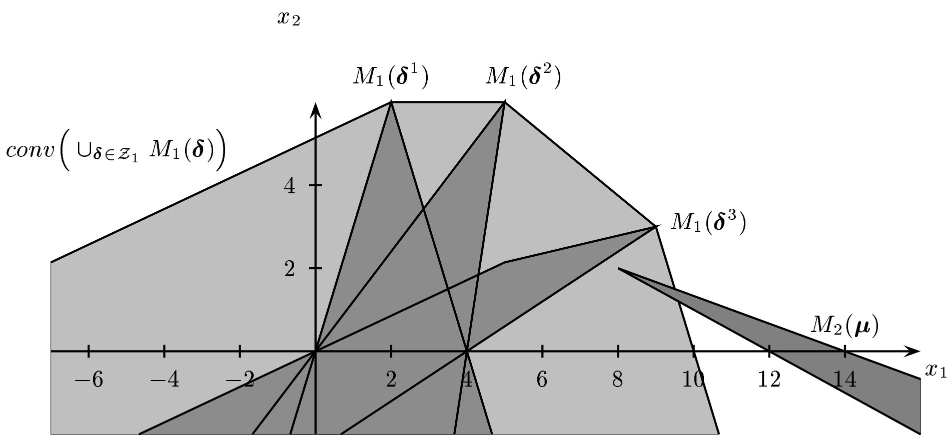

Figure 1: Illustration to Example 1.

The convex polyhedral set $M_1(\boldsymbol{\delta}^1)$ contains two unbounded edges, which are unbounded in the direction of $(-1,-3)$ (it corresponds to the sub-basis (1) originating from a vertex determined by the basis $(1,2)$) and in the direction of $(1,-3)$ (it corresponds to the sub-basis (2) originating from a vertex determined by the basis $(1,2)$). Hence $\mathfrak{H}_1 = \{((1),2),((2),1)\}$. Sets $\mathcal{H}^2_{(1)}$, $\mathcal{H}^1_{(2)}$ have according to (29) the description

$$\mathcal{H}^2_{(1)} = \{\boldsymbol{\delta} \in \mathcal{S}^{\boldsymbol{\delta}^1}_{(1,2)} \mid -\delta_1 + \delta_3 \geq 0\},$$
$$\mathcal{H}^1_{(2)} = \{\boldsymbol{\delta} \in \mathcal{S}^{\boldsymbol{\delta}^1}_{(1,2)} \mid \delta_2 + \delta_3 \geq 0\}.$$

The convex polytope $\mathcal{Z}_1(\mathfrak{S}_1,\mathfrak{H}_1)$ is equal to

$$\begin{aligned}\mathcal{Z}_1(\mathfrak{S}_1,\mathfrak{H}_1) &= \mathcal{Z}_1 \cap \mathcal{S}^{\boldsymbol{\delta}^1}_{(1,2)} \cap \mathcal{H}^2_{(1)} \cap \mathcal{H}^2_{(1)} \\ &= \big\{\boldsymbol{\delta} \in \mathbb{R}^3 \mid \boldsymbol{\delta} = (1,1,2)^T + t(1,-1,2)^T,\ t \in \langle 0, \tfrac{3}{2}\rangle\big\}\end{aligned}$$

and consists of two vertices $\boldsymbol{\delta}^1$ and $\boldsymbol{\delta}^2 = (\frac{5}{2},-\frac{1}{2},5)^T$.

2. Choose $\boldsymbol{\delta}^3 \in \mathcal{Z}_1 \setminus \mathcal{Z}_1(\mathfrak{S}_1,\mathfrak{H}_1)$, e.g. as $\boldsymbol{\delta}^3 = (7,-5,14)^T$. The family of all feasible bases of the convex polyhedral set $M_1(\boldsymbol{\delta}^3)$ is $\mathfrak{S}_2 =$

$\{(1,3),(2,3)\}$. The convex polyhedral cone $\mathcal{N}_{(1,3)}$ from (26) has two extremal directions $\boldsymbol{g}_2=(1,-1,0)^T$, $\boldsymbol{h}_2=(9,-4,5)^T$. Hence

$$\mathcal{S}^{\boldsymbol{\delta}^3}_{(1,3)}=\{\boldsymbol{\delta}\in\mathbb{R}^3 \mid \delta_1-\delta_3>0,\ 9\delta_1+5\delta_2-4\delta_3\geq 0\}.$$

The convex polyhedral cone $\mathcal{N}_{(2,3)}$ has two extremal directions $\boldsymbol{g}_3=(-1,-1,0)^T$, $\boldsymbol{h}_3=(5,-4,9)^T$ and hence

$$\mathcal{S}^{\boldsymbol{\delta}^3}_{(2,3)}=\{\boldsymbol{\delta}\in\mathbb{R}^3 \mid -\delta_2-\delta_3>0,\ 9\delta_1+5\delta_2-4\delta_3\geq 0\}.$$

The convex polyhedral set $M_1(\boldsymbol{\delta}^3)$ has two extremal directions: $(-7,-3)$ (it corresponds to the sub-basis (1) originating from a vertex determined by the basis $(1,3)$) and in the direction of $(-5,-3)$ (it corresponds to the sub-basis (2) originating from a vertex determined by the basis $(2,3)$). Hence $\mathfrak{H}_2=\{((1),3),((2),3)\}$. Sets $\mathcal{H}^3_{(1)}$, $\mathcal{H}^3_{(2)}$ have according to (29) the description

$$\begin{aligned}\mathcal{H}^3_{(1)}&=\{\boldsymbol{\delta}\in\mathcal{S}^{\boldsymbol{\delta}^3}_{(1,3)} \mid \delta_1+\delta_2\geq 0\},\\ \mathcal{H}^3_{(2)}&=\{\boldsymbol{\delta}\in\mathcal{S}^{\boldsymbol{\delta}^3}_{(2,3)} \mid \delta_1+\delta_2\geq 0\}.\end{aligned}$$

The convex polytope $\mathcal{Z}_1(\mathfrak{S}_2,\mathfrak{H}_2)$ is equal to

$$\begin{aligned}\mathcal{Z}_1(\mathfrak{S}_2,\mathfrak{H}_2)&=\mathcal{Z}_1\cap\mathcal{S}^{\boldsymbol{\delta}^3}_{(1,3)}\cap\mathcal{S}^{\boldsymbol{\delta}^3}_{(2,3)}\cap\mathcal{H}^3_{(1)}\cap\mathcal{H}^3_{(2)}\\ &=\big\{\boldsymbol{\delta}\in\mathbb{R}^3 \mid \boldsymbol{\delta}=(1,1,2)^T+t(1,-1,2)^T,\ t\in\langle\tfrac{3}{2},6\rangle\big\}\end{aligned}$$

and consists of two vertices $\boldsymbol{\delta}^2$ a $\boldsymbol{\delta}^3$.

On the whole we obtain

$$\begin{aligned}conv\Big(\bigcup_{\boldsymbol{\delta}\in\mathcal{Z}_1}M_1(\boldsymbol{\delta})\Big)&=conv\big(M_1(\boldsymbol{\delta}^1)\cup M_1(\boldsymbol{\delta}^2)\cup M_1(\boldsymbol{\delta}^3)\big)\\ &=\left\{\boldsymbol{x}\in\mathbb{R}^n \mid \begin{pmatrix}-3&7\\0&1\\3&4\\3&1\end{pmatrix}\boldsymbol{x}\leq\begin{pmatrix}36\\6\\39\\30\end{pmatrix}\right\}.\end{aligned}$$

There it does not exist any permanent separating hyperplane (since the sets $conv\big(\cup_{\boldsymbol{\delta}\in\mathcal{Z}_1}M_1(\boldsymbol{\delta})\big)$ and $M_2(\boldsymbol{\mu})$ are not separable), even though $M_1(\boldsymbol{\delta})$, $M_2(\boldsymbol{\mu})$ are strongly separable for all $\boldsymbol{\delta}\in\mathcal{Z}_1$.

5 Stability sets

In this section we deal with so called stability sets. Stablity sets are defined in the similar way in [10, 7]. It is natural to define stability sets as sets of all $(\boldsymbol{\delta}, \boldsymbol{\mu})$ such that all the sets $\mathcal{Q}^*(\boldsymbol{\delta}, \boldsymbol{\mu})$ from (19) have the same system of feasible bases.

Definition 4. Let an arbitrary vector $(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$ from the solution set (25) be given and suppose that the set $\mathcal{Q}^*(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$ is nonempty. Denote by $\mathcal{S}$ the system of all feasible bases of the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$. *The stability set* corresponding to the system $\mathcal{S}$ is the closure of the set of all $(\boldsymbol{\delta}, \boldsymbol{\mu}) \in (\mathcal{P}_1 \times \mathcal{P}_2) \cap \mathcal{U}$ under which all feasible bases from $\mathcal{S}$ remain feasible for $\mathcal{Q}^*(\boldsymbol{\delta}, \boldsymbol{\mu})$.

Without the loss of generality let us assume that

$$rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{pmatrix} = n. \tag{33}$$

Otherwise it would occur any of the following possibilities:

(i) If $rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{pmatrix} = rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \end{pmatrix}$, then $\mathcal{Q}^*_r = \emptyset$ and the solution set is empty.

(ii) If $rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{pmatrix} > rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \end{pmatrix}$, then $rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \end{pmatrix} < n-1$ and in the description of $\mathcal{Q}^*(\boldsymbol{\delta}, \boldsymbol{\mu})$ there are linear dependent equations, which we can remove.

Now we derive the description of stability sets. Let $(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$ be from the solution set and B a feasible basis of the convex polytope $\mathcal{Q}^*(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$. Denote $\mathbf{D}(\boldsymbol{\delta}, \boldsymbol{\mu}) \equiv \mathbf{Z}_B(\boldsymbol{\delta}, \boldsymbol{\mu})$, $\mathbf{D} \equiv \mathbf{D}(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$. The basis B remain feasible for all values of parameters $\boldsymbol{\delta}$, $\boldsymbol{\mu}$ satisfying the relation

$$\mathbf{D}^{-1}(\boldsymbol{\delta}, \boldsymbol{\mu})\boldsymbol{z} \geq \mathbf{0}. \tag{34}$$

Define vectors $\boldsymbol{p} \in \mathbb{R}^{n+2}$, $\widetilde{\boldsymbol{q}}, \boldsymbol{q} \in \mathbb{R}^{m+l+1}$:

$$\boldsymbol{p} \equiv \boldsymbol{e}_n = \begin{pmatrix} \mathbf{0} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widetilde{\boldsymbol{q}} \equiv \begin{pmatrix} \boldsymbol{\delta} - \boldsymbol{\delta}^0 \\ \boldsymbol{\mu} - \boldsymbol{\mu}^0 \\ 0 \end{pmatrix}, \quad \boldsymbol{q} \equiv \begin{pmatrix} \boldsymbol{\delta} \\ \boldsymbol{\mu} \\ 0 \end{pmatrix}. \tag{35}$$

From the assumption $1 + \widetilde{\boldsymbol{q}}_B^T \mathbf{D}^{-1} \boldsymbol{p} \neq 0$ and the well-known Sherman–Morrison formula it follows

$$\mathbf{D}^{-1}(\boldsymbol{\delta}, \boldsymbol{\mu}) = (\mathbf{D} + \boldsymbol{p}\widetilde{\boldsymbol{q}}_B^T)^{-1} = \mathbf{D}^{-1} - \frac{\mathbf{D}^{-1}\boldsymbol{p}\widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}}{1 + \widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}\boldsymbol{p}}.$$

Since for the choice $\boldsymbol{\delta} = \boldsymbol{\delta}^0$, $\boldsymbol{\mu} = \boldsymbol{\mu}^0$ is the denominator $1 + \widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}\boldsymbol{p} = 1$ (i.e. positive), assume moreover the following constraint

$$1 + \widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}\boldsymbol{p} > 0. \tag{36}$$

Let us rearrange the expression (34):

$$\begin{aligned}
\mathbf{D}^{-1}(\boldsymbol{\delta}, \boldsymbol{\mu})\boldsymbol{z} &\geq \mathbf{0}, \\
\left(\mathbf{D}^{-1} - \frac{\mathbf{D}^{-1}\boldsymbol{e}_n\widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}}{1 + \widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}\boldsymbol{e}_n}\right)\boldsymbol{e}_{n+2} &\geq \mathbf{0}, \\
\mathbf{D}^{-1}_{\cdot,n+2} - \frac{\mathbf{D}^{-1}_{\cdot,n}\widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}_{\cdot,n+2}}{1 + \widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}_{\cdot,n}} &\geq \mathbf{0}.
\end{aligned}$$

From the assumption (36) is this inequality equivalent to

$$\mathbf{D}^{-1}_{\cdot,n+2} + \mathbf{D}^{-1}_{\cdot,n+2}(\widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}_{\cdot,n}) - \mathbf{D}^{-1}_{\cdot,n}(\widetilde{\boldsymbol{q}}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) \geq \mathbf{0}. \tag{37}$$

For the vector $\widetilde{\boldsymbol{q}}_B^T$ it $\widetilde{\boldsymbol{q}}_B^T = \boldsymbol{q}_B^T - \mathbf{D}_{n,\cdot}$ holds, and hence the expression (37) is equivalent to

$$\begin{aligned}
\mathbf{D}^{-1}_{\cdot,n+2} + \mathbf{D}^{-1}_{\cdot,n+2}\left((\boldsymbol{q}_B^T - \mathbf{D}_{n,\cdot})\mathbf{D}^{-1}_{\cdot,n}\right) - \mathbf{D}^{-1}_{\cdot,n}\left((\boldsymbol{q}_B^T - \mathbf{D}_{n,\cdot})\mathbf{D}^{-1}_{\cdot,n+2}\right) &\geq \mathbf{0}, \\
\mathbf{D}^{-1}_{\cdot,n+2}(\boldsymbol{q}_B^T\mathbf{D}^{-1}_{\cdot,n}) - \mathbf{D}^{-1}_{\cdot,n}(\boldsymbol{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) &\geq \mathbf{0}.
\end{aligned} \tag{38}$$

The expression (38) represents a system of linear inequalities with respect to the variables $\boldsymbol{\delta}$, $\boldsymbol{\mu}$.

Remark 1. Let us investigate the expression (36). It is equivalent to

$$\begin{aligned}
1 + (\boldsymbol{q}_B^T - \mathbf{D}_{n,\cdot})\mathbf{D}^{-1}_{\cdot,n} &> 0, \\
\boldsymbol{q}_B^T\mathbf{D}^{-1}_{\cdot,n} &> 0.
\end{aligned} \tag{39}$$

When we multiply the system (38) by the vector $\mathbf{D}_{n+2,\cdot} \geq \mathbf{0}$, then we obtain

$$
\begin{aligned}
(\mathbf{D}_{n+2,\cdot}\mathbf{D}^{-1}_{\cdot,n+2})(\boldsymbol{q}_B^T\mathbf{D}^{-1}_{\cdot,n}) - (\mathbf{D}_{n+2,\cdot}\mathbf{D}^{-1}_{\cdot,n})(\boldsymbol{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) &\geq 0, \\
\boldsymbol{q}_B^T\mathbf{D}^{-1}_{\cdot,n} &\geq 0.
\end{aligned}
$$

Since the stability set is defined as a closed set, the constraint (39) is redundant.

Remark 2. (The description of stability sets) Given $\boldsymbol{\delta}^0$, $\boldsymbol{\mu}^0$ from the solution set (25). The stability set (corresponding to $\boldsymbol{\delta}^0$, $\boldsymbol{\mu}^0$) is the set of all ($\boldsymbol{\delta}$, $\boldsymbol{\mu}$) $\in (\mathcal{P}_1 \times \mathcal{P}_2) \cap \mathcal{U}$ satisfying the following systems of inequalities

$$
\mathbf{D}^{-1}_{\cdot,n+2}(\boldsymbol{q}_B^T\mathbf{D}^{-1}_{\cdot,n}) - \mathbf{D}^{-1}_{\cdot,n}(\boldsymbol{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) \geq \mathbf{0}.
$$

for all feasible bases B of the convex polytope $\mathcal{Q}^*(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$ from (19). There is always a finite number of stability sets.

Example 2. Let

$$
\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} -1 \end{pmatrix}, \ \boldsymbol{d} = \begin{pmatrix} -2 \end{pmatrix}.
$$

be given. We provide the description of the solution set and all stability sets.

Since the convex polyhedral cone $\mathcal{N}_{\mathbf{A},\boldsymbol{b}}$ from (13) contains only one edge and this edge is in the direction of $\boldsymbol{h}_1 = (1,1)^T$, the set $\mathcal{P}_1$ (according to Theorem 5) is described as follows

$$
\mathcal{P}_1 = \{\boldsymbol{\delta} \in \mathbb{R}^2 \mid \delta_1 + \delta_2 > 0\} \cup \{\boldsymbol{\delta} \in \mathbb{R}^2 \mid \delta_1 + \delta_2 < 0\}.
$$

The set $\mathcal{P}_2$ is equal to $\mathbb{R}$, since the convex cone $\mathcal{N}_{\mathbf{C},\boldsymbol{d}} = \{\boldsymbol{y} \in \mathbb{R}^l \mid \mathbf{C}^T\boldsymbol{y} = \mathbf{0},\ \boldsymbol{d}^T\boldsymbol{y} \leq 0,\ \boldsymbol{y} \geq \mathbf{0}\} = \{\mathbf{0}\}$ has no edge. The convex polytope $\mathcal{Q}_r^*$ from (24) has the description

$$
\left\{(u_1,u_2,v_1,v_2) \in \mathbb{R}^4 \mid \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ u_1,u_2,v_1,v_2 \geq 0\right\}
$$

and contains the following two vertices $\boldsymbol{w}^1 = \frac{1}{2}(1,1,0,1)^T$, $\boldsymbol{w}^2 = \frac{1}{2}(1,0,1,2)^T$. Hence the convex polyhedral set $\mathcal{U}_1$ from (23) is described as follows

$$
\mathcal{U}_1 = \{(\delta_1,\delta_2,\mu_1) \in \mathbb{R}^3 \mid \delta_1 + \delta_2 > 0,\ \delta_1 + \mu_1 > 0\}.
$$

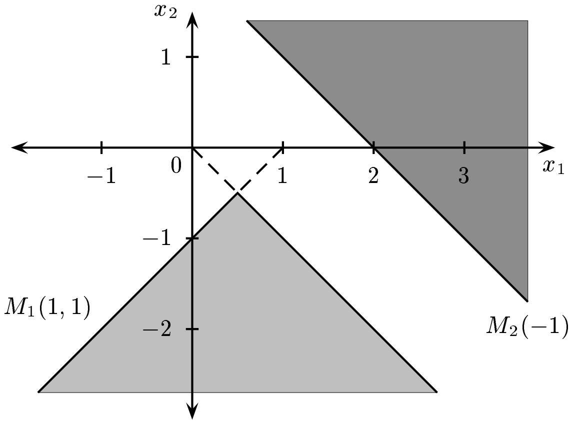

Figure 2: Illustration to Example 2 for values $\boldsymbol{\delta} = (1,1)^T,\ \boldsymbol{\mu} = (-1)$.

The solution set is (according to (25)) described by

$$\{(\delta_1, \delta_2, \mu_1) \in \mathbb{R}^3 \mid \delta_1 + \delta_2 > 0,\ \delta_1 + \mu_1 \leq 0\} \cup$$
$$\{(\delta_1, \delta_2, \mu_1) \in \mathbb{R}^3 \mid \delta_1 + \delta_2 < 0,\ \delta_1 + \mu_1 \geq 0\}.$$

Now we compute all stability sets according to Remark 2.

1. Choose $(\delta_1^1, \delta_2^1, \mu_1^1)$ from the solution set, e.g. in this way: $(\delta_1^1, \delta_2^1, \mu_1^1) = (1, 1, -1)$. The convex polytope $\mathcal{Q}^*(\delta_1^1, \delta_2^1, \mu^1)$ has only one feasible basis, $B = (1, 2, 3, 4)$, and the first stability set is described by the following system of inequalities

$$\delta_1 + \delta_2 > 0,\ \delta_1 + \mu_1 \leq 0.$$

2. Choose $(\delta_1^2, \delta_2^2, \mu_1^2)$ from the solution set, but not from the first stability set, e.g. in this way: $(\delta_1^2, \delta_2^2, \mu_1^2) = (-1, -1, 1)$. The convex polytope $\mathcal{Q}^*(\delta_1^2, \delta_2^2, \mu^2)$ has only one feasible basis, $B = (1, 2, 3, 4)$, and the first stability set is described by the following system of inequalities

$$\delta_1 + \delta_2 < 0,\ \delta_1 + \mu_1 \geq 0.$$

We have obtained two stability sets (except degenerated stability sets, which have a dimension less than n) the solution set consists of.

Tables 1 – 2 contain further results obtained on `PC (x86), Pentium 4, 2.6 GHz, 512 MB RAM, Gentoo Linux`. Our source code was written in MATLAB 6.5. In each of the mentioned tables, there are for given matrix **A**, vector ***b***, matrix **C** and vector ***d***, written down the number of stability sets and the computing time (in minutes and seconds). The input data of **A**, **C**, ***b***, ***d*** were generated pseudorandomly. With the increase of m, l, n the number of stability sets and the computing time increases very rapidly.

Table 1: Examples in $\mathbb{R}^2$.

matrix $\mathbf{A}$	vector $\boldsymbol{b}$	matrix $\mathbf{C}$	vector $\boldsymbol{d}$	number of stability sets	computing time
$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} -3 \\ 9 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$	12	4 s
$\begin{pmatrix} -6 \\ -7 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 12 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -5 \\ 9 \\ 1 \end{pmatrix}$	44	15 s
$\begin{pmatrix} -4 \\ -2 \\ -6 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 9 \\ -4 \end{pmatrix}$	$\begin{pmatrix} 8 \\ 1 \\ -5 \end{pmatrix}$	$\begin{pmatrix} 8 \\ -6 \\ 2 \end{pmatrix}$	90	1 min 19 s
$\begin{pmatrix} -2 \\ -2 \\ 7 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -8 \\ -6 \\ -9 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 11 \\ 3 \\ -5 \\ 12 \end{pmatrix}$	206	4 min 8 s
$\begin{pmatrix} 0 \\ 8 \\ -6 \\ -9 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 0 \\ -3 \\ 9 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ -9 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 6 \\ 10 \\ -5 \end{pmatrix}$	968	29 min 51 s

Table 2: Examples in $\mathbb{R}^3$, $\mathbb{R}^4$.

matrix $\mathbf{A}$	vector $\boldsymbol{b}$	matrix $\mathbf{C}$	vector $\boldsymbol{d}$	number of stability sets	computing time
$\begin{pmatrix} 5 & 5 \\ 7 & 1 \\ -6 & -6 \end{pmatrix}$	$\begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix}$	$\begin{pmatrix} -8 & 9 \\ 4 & 8 \\ 2 & 2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 6 \\ 11 \end{pmatrix}$	41	25 s
$\begin{pmatrix} -7 & 8 \\ 3 & 7 \\ 8 & -9 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -5 & -3 \\ -3 & 1 \\ 9 & 8 \\ 1 & 7 \end{pmatrix}$	$\begin{pmatrix} 7 \\ -4 \\ -6 \\ -3 \end{pmatrix}$	429	10 min 45 s
$\begin{pmatrix} 0 & -3 \\ 5 & -3 \\ 0 & 8 \\ 9 & 9 \\ -7 & -7 \end{pmatrix}$	$\begin{pmatrix} 7 \\ 6 \\ 11 \\ 0 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 7 & -1 \\ -4 & 5 \\ -5 & 6 \end{pmatrix}$	$\begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$	608	29 min 56 s
$\begin{pmatrix} 8 & 5 & 3 \\ -7 & 1 & 1 \\ 2 & 8 & 4 \end{pmatrix}$	$\begin{pmatrix} 7 \\ -4 \\ 7 \end{pmatrix}$	$\begin{pmatrix} 4 & 5 & -2 \\ 6 & 8 & -2 \\ -4 & -7 & -2 \\ 2 & 6 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -3 \\ -6 \\ -2 \end{pmatrix}$	44	27 s
$\begin{pmatrix} 7 & 6 & -2 \\ -6 & 3 & 3 \\ -9 & 0 & -2 \\ 8 & -4 & 7 \end{pmatrix}$	$\begin{pmatrix} -3 \\ 2 \\ -3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 5 & -7 \\ -4 & -1 & -4 \\ -2 & -5 & -4 \\ -7 & -3 & 7 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \\ 12 \\ -1 \end{pmatrix}$	131	3 min 56 s

6 Conclusion

In this article, we were concerned with the separation properties of two convex polyhedral sets $\mathcal{M}_1$, $\mathcal{M}_2$, which depended on parameters. Parameters were situated in one column of the constraint matrix from the description of these convex polyhedral sets. The situation, when there are parameters in the right-hand side of inequalities was dealt with in [7]. The situation, when there are parameters in one row of the constraint matrix will be followed in the next paper. We defined so called solution set (a set of parameters for which $\mathcal{M}_1$, $\mathcal{M}_2$ are strongly separable) and stability sets (a set of parameters for which separability of $\mathcal{M}_1$, $\mathcal{M}_2$ has the same characteristics). On stability sets, there could be applied various kinds of postoptimality analyses (parametric analysis, sensitivity analysis or tolerance analysis – see e.g. [2]), but it was not the subject of this paper. We produced also a lot of examples, which were carried out on a computer.

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