A NEW PROOF THAT ALTERNATING LINKS ARE NON-TRIVIAL

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ABSTRACT. We use a simple geometric argument and small cancellation properties of link groups to prove that alternating links are non-trivial. This proof uses only classic results in topology and combinatorial group theory.

1. Statement of Results

There are several approaches in the literature for showing that alternating links are non-trivial. Most of these approaches rely upon the use of a powerful knot invariant: the determinant in [2]; the Alexander polynomial in [3] and [10]; the Jones polynomial in [6] and the Q-polynomial in [7], although a purely geometric proof was given in [9]. These proofs provide different perspectives as to why the result holds. The argument presented in this paper differs from these proofs as it uses Dehn's lemma and a solution to the word problem for link groups to show in a very direct way that spanning disks for the link can not exist. This approach therefore uses only classic topology and combinatorial group theory to provide a direct and intuitive proof for the non-triviality of alternating links.

To state our main result we need to recall some notation. A link projection D divides the plane into *regions.* D is said to be *reduced* if at each crossing four distinct regions of the plane meet. A reduced projection is said to be *prime* if it is connected, contains at least one crossing and there does not exist a simple closed curve in the plane intersecting D transversally in exactly two points on different arcs of D. We prove the following.

Theorem 1. *If L is a link admitting a reduced, alternating projection, then L is non-trivial.*

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Our method of proof is to solve the word problem for the link groups to prove that the longitudes, and hence the link, are non-trivial: If L is an oriented link with components $L_1, \dots, L_n$ and N_i is a tubular neighbourhood of L_i, a *meridian* μ_i of L_i is a non-separating simple closed curve in ∂N_i that bounds a disc in N_i and a longitude λ_i is a simple closed curve in ∂N_i that is homologous to L_i in N_i and null-homologous in the exterior $S^3 - L_i$.

A standard and well known consequence of Dehn's lemma and the loop theorem is that a link is trivial if and only if all of its longitudes are trivial in the link group. This reduces Theorem 1 to solving the word problem for the longitudes of the link. We do this by using a simple geometric argument to rewrite the longitudes of the link in a certain normal form with respect to the checker-board colouring of a link projection. Specifically, we write the longitude as a curve which intersects black regions of the checker-board colouring of the link projection before any white regions. This normal form allows us to apply some basic results in small cancellation theory and solve the word problem for the link groups, concluding that the longitudes are non-trivial.

2. A Normal Form for the Longitudes

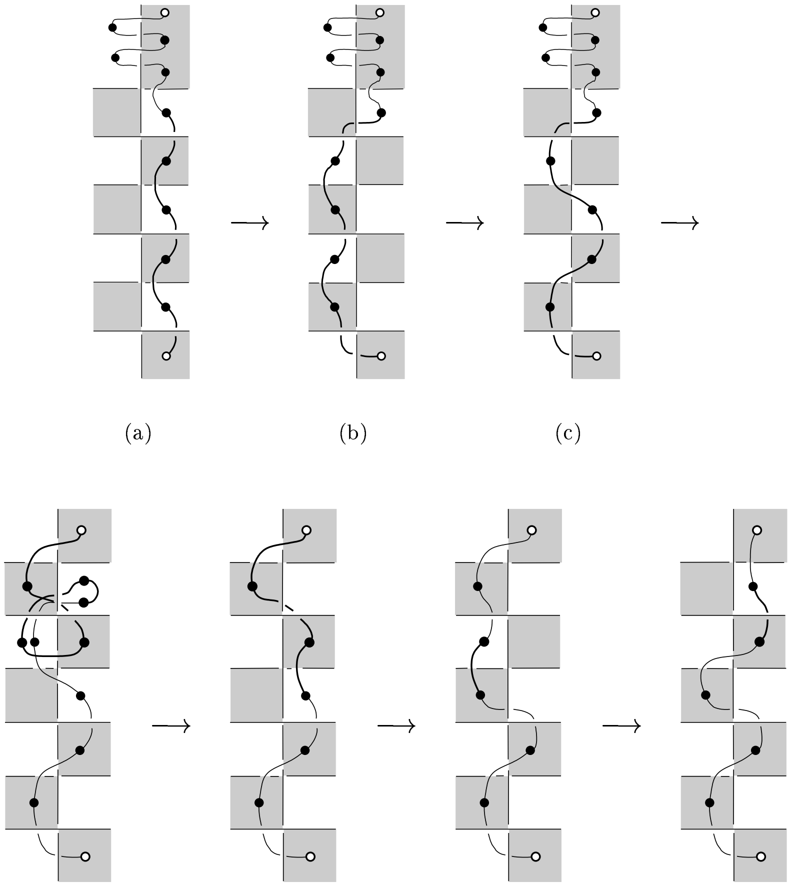
Recall that the *checker-board colouring* of a link projection is an assignment of a colour black or white to each of the regions of the projection in such a way that adjacent regions are assigned a different colour.

Lemma 1. *The i-th longitude λ_i of a link L is homotopic to a simple closed curve $J \subset S^3 - L$ such that, in terms of the projection, all intersections of J with white regions of the checker-board colouring occur before any intersections with black regions, with respect to a chosen base point and orientation.*

The reader may find it helpful to refer to figure 1 while reading the following proof.

Proof. Begin by fixing a projection D of L. For convenience assign a label $x_1, \dots, x_{2n}$ to each of the regions of D. Then, up to homotopy, a based oriented loop in the link complement can be described by a word in the alphabet $\mathcal{A} = \{x_i, x_i^{-1} | i = 1, \dots, 2n\}$ by assigning the letter x_i whenever the loop passes downwards through the region x_i and x_i^{-1} whenever the loop passes upwards through the region x_i. Notice that the checker-board colouring induces a colour on each letter in $\mathcal{A}$.

We need to choose a representative of the longitude. To do this we define the *i-th double* $\Delta_i(L)$, of a link L to be the curve determined by a parallel



(a) (b) (c)

(d) (e) (f) (g)

FIGURE 1

copy of the i-th component of its projection (so this is the curve determined by the black-board framing). It is then a standard fact (see eg. [11]) that the longitude λ_i can be represented by $\Delta_i(L)\cdot m_i^{-\,\mathrm{lk}(i,i)}$, where m_i represents the i-th meridian (we will specify a representative shortly) and $\mathrm{lk}(i,i)$ is the self-linking number of L_i. Since there are two choices for the double $\Delta_i(L)$ (either side of L_i), we may choose m_i and $\Delta_i(L)$ such that the longitude λ_i is represented by a word of the form $w = (l_1^{\pm 1} l_2^{\mp 1} l_3^{\pm 1} \cdots l_{2n}^{\mp 1}) \cdot (l_1^{\pm 1} a^{\mp 1})^k$, where $k = |\,\mathrm{lk}(i,i)|$. Notice that w alternates in colour and sign.

Without loss of generality we may assume that l_1 is black. We will also assume that l_1 appears with a positive sign, *ie.* the first letter of w is l_1^{+1}. A similar argument deals with the l_1^{-1} case. We now describe how to deform this representative to the form required in the statement of the lemma. We split the argument into three steps.

Step 1. Begin by wrapping $\Delta_i(L)$ around the component L_i. To do this fix the first and last intersection points, l_1 and l_{2n}^{-1}, of $\Delta_i(L)$ and slide the arc $l_2^{-1} \cdots l_{2n-1}$ underneath L_i as in figure 1(b). Now, moving inwards along this arc from both ends, fix the next two intersection points and slide the rest of the arc over L_i as in figure 1(c). Continue this process of sliding the arc under and over for as long as is possible. This procedure gives a curve $(\ell_1 \ell_2^{-1} \cdots \ell_{2n}^{-1})(l_1^{\pm 1} a^{\mp 1})^k$, such that $\ell_1, \ldots, \ell_n^{-1}$ are coloured black and $\ell_{n+1}, \ldots, \ell_{2n}^{-1}$ are white. Again this is shown in figure 1(c).

Step 2. Next pull the curve $(l_1^{\pm 1} a^{\mp 1})^k$ along the component L_i so that the white intersections follow the deformed i-th double for as long as possible, and the remaining intersection points lie in regions on the opposite side of the curve L_i. Thus, since $2|\,\mathrm{lk}(i,i)| \leq |\ \Delta_i(L)|$, we obtain a curve $(\ell_1 \ell_2^{-1} \cdots \ell_{2n}^{-1})(\ell_{2n} \cdots \ell_{2n-k}^{\pm 1} a_1^{\mp 1} \cdots a_k$, where $a_1^{\mp 1}, \ldots, a_k)$ are black. This is indicated in figure 1(d).

This representative of the longitude doubles back upon itself and so we can remove some of the white pairs of intersection points, giving the isotopic curve $\ell_1 \ell_2^{-1} \cdots \ell_{2n-k-1}^{\mp 1} a_1^{\mp 1} \cdots a_k$, as in figure 1(e).

Now if $2k = |\ \Delta_i(L)|$ we are done, otherwise move on to step three.

Step 3. All that remains is to move the remaining white intersections $\ell_{n+1}, \ldots \ell_{2n-k-1}^{\mp 1}$ to the end of the arc. Clearly this can be done by a sequence of the moves shown in figure 2. These moves are indicated in figure 1 (e), (f) and (g).

$\square$

Remark 1. Stopping after step 2 in the proof shows that the longitude λ_i is conjugate to a curve with the properties of J in the lemma. In actual fact this is enough to prove the main theorem, however we prefer the stronger form of the lemma.

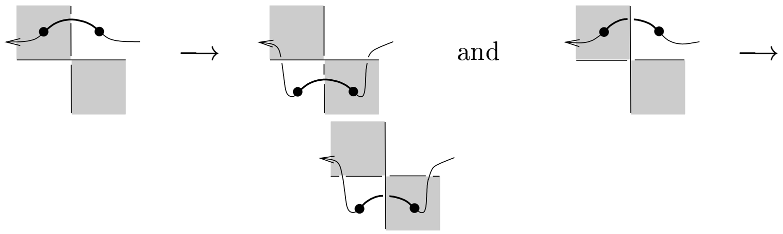


FIGURE 2

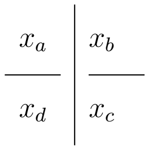


FIGURE 3

3. The Proof of the Theorem

As in the proof of the lemma, label the regions of the projection D of L with $x_1, \ldots, x_n$. Now form a group presentation by taking the set of labels $\{x_1, \ldots, x_n\}$ as the set of generators and deriving a relator $x_a x_b^{-1} x_c x_d^{-1}$ from each crossing according to the scheme shown in figure 3. This gives a presentation which after *symmetrization*, *ie.* adjoining all cyclic permutations and inverses of the relators to the presentation, we call the *augmented Dehn presentation* of L.

We also define the *augmented link* obtained from D to be the link obtained by adding an extra unknot component bounding the projection, and forming a link by regarding $\mathbb{R}^2$ as the x-y axis of $\mathbb{R}^3 \cup \infty = S^3$ and "pulling the overcrossings up a little".

If we choose a base point above the plane, a generator x_i of the augmented Dehn presentation is realized in the complement of the augmented link by a loop which passes downwards through the region $x_i \subset \mathbb{R}^2 \subset \mathbb{R}^3 \cup \infty = S^3$ and passes back up through the unbounded region of the augmented link, to the base point. Notice that up to homotopy there is a clear correspondence between words in the augmented Dehn presentation and words arising from based oriented loops as in the proof of lemma 1.

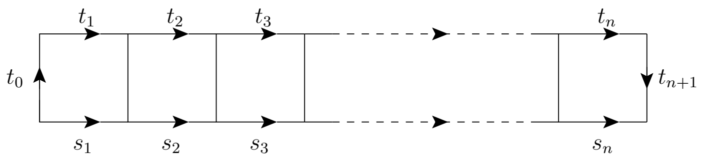


FIGURE 4

Clearly the augmented Dehn presentation is the symmetrization of the Dehn presentation (see eg. [8]) of the group of the augmented link. Consequently the augmented Dehn presentation is a presentation of the free product of the infinite cyclic group and the link group. Therefore solving the word problem for the augmented Dehn presentation of L solves it for the link group of L.

Weinbaum proved the following lemma for knots, but his proof also works for links.

Lemma 2 (Weinbaum [12])**.** *The augmented Dehn presentation read from a reduced, prime, alternating projection of a link is a* $C''(4) - T(4)$ *small cancellation group.*

As we are only interested in the characterization of geodesics, we exclude the definition of a small cancellation group. This can be found in [8].

A *chain* is a Van Kampen diagram having the form shown in figure 4, where $n \geq 1$. We call the word $t_0 t_1 t_2 \cdots t_{n+1}$ a *chain word.*

Given an arbitrary finite group presentation, the set of lengths of all words representing an element of the group has a minimum. We call any word which attains this minimum a *geodesic*. Geodesics in a $C''(4) - T(4)$ presentation are characterized by the absence of chain words.

Geodesic Characterization Theorem ([1, 4, 5])**.** *A word in a* $C''(4) - T(4)$ *presentation is geodesic if and only if it is freely reduced and contains no chain subwords.*

Using the checker-board colouring we can assign a *parity*, black or white, to each generator-inverse pair according to the colour of the region that generator corresponds to. Notice that the relators of the augmented Dehn presentation are words which alternate in parity and therefore the horizontal and vertical edges of a chain correspond to letters of different parity.

Putting all this together, we can prove our main result.

Proof of Theorem 1. Since the sum of two non-trivial links is non-trivial (eg. by the additivity of genus), it is enough to prove the theorem for prime links. In this case, by Lemma 1 and the geometric interpretation of the generators of the augmented Dehn presentation, the longitude can be represented by a non-empty word w which changes parity exactly once. Since the projection is reduced, w is freely reduced. A word of this form can not contain a chain word (as these change parity twice) and since the augmented Dehn presentation is a $C''(4) - T(4)$ small cancellation group, the geodesic characterization theorem tells us that the longitudes are non-trivial and therefore the link itself is non-trivial. □

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