On the Maximal Set of Feasible Coefficients in Interval Linear Systems

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Abstract

The paper deals with a system of linear equations Ax = b whose input data are described by intervals. By a set of feasible coefficients we mean a set F containing all matrices A and vectors b in given intervals for which the corresponding system Ax=b has a nonnegative solution.In this way, the set F is closely connected with an interval linear programming problem. A description of the set F is given which enables to construct a maximal set of feasible coefficients.

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1 Introduction and Notations

Let us consider a system of linear equations with an $m \times n$ matrix A and a right-hand sides vector b, i.e. the system

$$Ax = b. \tag{1}$$

Following practical problems we shall assume that all coefficients vary independently in given intervals. Thus the input data are described by an interval $m \times n$ matrix $[A]$ and an interval $m-$vector $[b]$, which are given by their lower and upper bounds $\underline{A}$, $\overline{A}$, $\underline{b}$ and $\overline{b}$:

$$[A] = \{A \in \mathrm{R}^{mn} : \underline{A} \leq A \leq \overline{A}\} \tag{2}$$
$$[b] = \{b \in \mathrm{R}^{m} : \underline{b} \leq b \leq \overline{b}\}. \tag{3}$$

The family of systems (1), where $A \in [A]$ and $b \in [b]$, will be called an *interval linear system* (abbr. ILS).

The center matrix of $[A]$ is given by $A_0 = (\overline{A} + \underline{A})/2$ and the radius matrix by $\Delta = (\overline{A} - \underline{A})/2$. The center vector b_0 and the radius vector δ are defined analogously.

For each $A \in [A]$ and $b \in [b]$ let us denote by $X(A,b)$ the set of nonnegative solutions of the system (1):

$$X(A,b) = \{x \in \mathrm{R}^n : Ax = b, x \geq 0\}. \tag{4}$$

Some results regarding the sets $X(A,b)$ and the set $X = \cup\{X(A,b) : A \in [A], b \in [b]\}$ are described in [3].

The paper deals with a *set of feasible coefficients* of a given ILS. The set is denoted by F and it is defined by

$$F = \{(A,b) \in [A] \times [b] : X(A,b) \neq \emptyset\}. \tag{5}$$

The set F is closely connected to a family of linear programming problems with interval coefficients as the set $X(A,b)$ contains all feasible solutions of a certain linear programming (abbr. LP) problem.

In [9] Rohn has studied so-called radius of feasibility. For $(A,b) \in \mathrm{R}^{mn} \times \mathrm{R}^m$, the radius of feasibility is defined as the value

$$f(A,b) = \inf \max\{\|A - A'\|_{1,\infty}, \|b - b'\|_\infty\} \tag{6}$$

subject to all systems $A'x = b'$ with $X(A',b') = \emptyset$. It was proved that computing $f(A,b)$ is NP-hard.

In this paper we give a description of the set F which shows a structure of the set. A main result consists in a theoretical background leading to a construction of a maximal set of feasible coefficients.

First we mention a solution of a special problem, whether $F = [A] \times [b]$. In such a case, all elements of given intervals are feasible coefficients, i.e. $X(A,b) \neq \emptyset$ for each $A \in [A], b \in [b]$.

To describe a result we consider two special kinds of problems (1). Let T^m denote the set of m-vectors t with components t_i satisfying $|t_i| \leq 1, i = 1, ..., m$ and let $T_t = \mathrm{diag}\{t_1, ..., t_m\}$ be the diagonal matrix with the vector t on the diagonal. A problem (1) with a matrix

$$A_t = A_c - T_t\Delta \qquad \text{and with a vector} \qquad b_t = b_c + T_t\delta, \quad t \in T^m \tag{7}$$

will be denoted by S_t and called a *t-system* of an ILS. Similarly, let H^m denote the set of m-vectors h with $|h_i| = 1, i = 1, ..., m$. A system S_h ,

$h \in H^m$ will be called an *extremal system*. Its i-th constraint has the form $(\overline{A}x)_i = \underline{b}_i$ if $h_i = -1$ and $(\underline{A}x)_i = \overline{b}_i$ if $h_i = +1$. These equations will be called the opposite extremal ones.

Theorem 1. *(Rohn [7]) The set F is equal to $[A] \times [b]$ if and only if each extremal system has a nonempty set $X(A, b)$.*

The algorithm INF2 given in [4] can be used to verify the assumption of Theorem 1. It is not necessary to solve 2^m extremal systems as an execution of 2^m steps of the Simplex method is sufficient.

2 Description of the set of feasible coefficients

For a theoretical background we shall utilize the result of Grygarová published in [1]. Grygarová studied a system (1) with one added equation $\lambda^T x = \mu$. Let us denote by $M(\lambda, \mu)$ the set of nonnegative solutions of the new system, i.e.

$$M(\lambda, \mu) = \{x \in \mathrm{R}^n : Ax = b, \lambda^T x = \mu, x \geq 0\}. \tag{8}$$

A description of the set

$$M = \{(\lambda, \mu) \in \mathrm{R}^n \times \mathrm{R}^1 : M(\lambda, \mu) \neq \emptyset\} \tag{9}$$

is given in [1] . We shall generalize this result to solve the problem in a question. First, let us investigate a general case $[A] = \mathrm{R}^{mn}, [b] = \mathrm{R}^m$, i.e. the case with unbounded $[A]$ and $[b]$. Klatte in [2] has called such a problem a fully parametric problem and he has studied LP problems of this type, namely local stable regions and continuity of a solution function.

For $k = 1, 2, ..., m$ let us introduce sets

$$F^k = \{(A^k, b^k) \in \mathrm{R}^{kn} \times \mathrm{R}^k : X(A^k, b^k) \neq \emptyset\}. \tag{10}$$

It is obvious that F^k is a convex cone in the space $\mathrm{R}^{kn} \times \mathrm{R}^k$ with a vertex at the origin (O^k, o^k) and it holds

$$(A^k, b^k) \in F^k \Leftrightarrow (-A^k, -b^k) \in F^k. \tag{11}$$

To describe the sets F^k we use a following notation: If $A^k \in \mathrm{R}^{kn}$ then A^{k-1} denotes the submatrix containing the first $(k-1)$ rows of the matrix A^k and a vector a_k^T denotes the k-th row of the matrix A^k. In an analogous way, b^{k-1} and b_k denote a vector of the first $(k-1)$ components and the k-component of a given vector $b^k \in \mathrm{R}^k$, resp.

Suppose that $(A^{k-1}, b^{k-1}) \in F^{k-1}$. For any n-vector a_k let us consider the optimal values of two linear programming problems

$$\underline{g}(a_k) = \inf\{a_k^T x : x \in X(A^{k-1}, b^{k-1})\} \tag{12}$$

$$\overline{g}(a_k) = \sup\{a_k^T x : x \in X(A^{k-1}, b^{k-1})\}, \tag{13}$$

where we set $X(A^0, b^0) = \mathrm{R}^n_+ = \{x \in \mathrm{R}^n : x \geq 0\}$. For these values we have: $\underline{g}(a_k) \in \mathrm{R}^1$ or $\underline{g}(a_k) = -\infty$ and $\overline{g}(a_k) \in \mathrm{R}^1$ or $\overline{g}(a_k) = \infty$.

Theorem 2.

$$F^1 = \{(a_1, b_1) \in R^n \times R^1 : \underline{g}(a_1) \leq b_1 \leq \overline{g}(a_1)\} \tag{14}$$

and for k=2,...,m we have

$$F^k = \{(A^k, b^k) : (A^{k-1}, b^{k-1}) \in F^{k-1}, \underline{g}(a_k) \leq b_k \leq \overline{g}(a_k)\}. \tag{15}$$

Proof. It can be done in a similar way to the proof of Theorem 1 in [1]. □

Finally, let us introduce following notation:

$$C^k = \{(A^k, b^k) : (A^{k-1}, b^{k-1}) \in F^{k-1}\} \tag{16}$$

$$\underline{\mathcal{A}}^k = \{a_k \in \mathrm{R}^n : \text{LP problem (12) has an optimal solution}\} \tag{17}$$

$$\overline{\mathcal{A}}^k = \{a_k \in \mathrm{R}^n : \text{LP problem (13) has an optimal solution}\} \tag{18}$$

$$\underline{U}^k = \{(A^k, b^k) : a_k \in \underline{\mathcal{A}}^k,\ b_k < \underline{g}(a_k)\} \tag{19}$$

$$\overline{U}^k = \{(A^k, b^k) : a_k \in \overline{\mathcal{A}}^k,\ b_k > \overline{g}(a_k)\}. \tag{20}$$

Theorem 3. *The sets* $\underline{U}^k$,$\overline{U}^k$ *have following properties:*

(i) $\underline{U}^k \neq \emptyset$, $\overline{U}^k \neq \emptyset$.

(ii) $\underline{U}^k \cap \overline{U}^k = \emptyset$.

(iii) $(O^k, o^k) \notin \underline{U}^k \cup \overline{U}^k$.

(iv) $\underline{U}^k \cup (O^k, o^k)$ *and* $\overline{U}^k \cup (O^k, o^k)$ *are cones in* $R^{kn} \times R^k$ *with the vertex at the origin* (O^k, o^k).

(v) $(A^k, b^k) \in \underline{U}^k \Leftrightarrow (-A^k, -b^k) \in \overline{U}^k$.

Proof. (i) Let us denote by O^k a zero $(k \times n)$ matrix and by o^k a zero $m-$ vector. Then $X(O^{k-1}, o^{k-1}) = \mathrm{R}^n_+$ and $\underline{g}(o^n) = 0$. Thus $\underline{U}^k$ is not empty as it contains any element of the form (O^k, b^k) with $b^k_k < 0$ and $b^k_i = 0, i \neq k$. Analogously we have $\overline{U}^k \neq \emptyset$.

(ii) Let us suppose that there is an element $(A^k, b^k) \in \underline{U}^k \cup \overline{U}^k$. Then we have $b^k_k < \underline{g}(a_k) \leq \overline{g}(a_k) < b^k_k$, a contradiction.

(iii) The assertion implies from the fact that an element (O^k, b^k) with $b^k_i = 0, i = 1, ..., k-1$ belongs in the set $\underline{U}^k$ if and only if $b^k_k < 0$. An element of the same form belongs in the set $\overline{U}^k$ if and only if $b^k_k > 0$, which completes the proof.

(iv) Let us consider a positive $\lambda \in \mathrm{R}^1$ and any element $(A^k, b^k) \in \underline{U}^k$. Then we have $b_k < \underline{g}(a_k)$ and $\lambda b_k < \lambda \underline{g}(a_k) = \inf\{\lambda a_k^T x : x \in X(\lambda A^{k-1}, \lambda b^{k-1})\}$.
It implies that $\lambda(A^k, b^k) \in \underline{U}^k$.
The assertion regarding $\overline{U}^k$ can be proved analogously.

(v) If $(A^k, b^k) \in \underline{U}^k$ then problem (12) has an optimal solution with the optimal value $\underline{g}(a_k) > b^k_k$. Then we have
$-b^k_k > -\underline{g}(a_k) = \overline{g}(-a_k) = \sup\{(-a_k)^T x : x \in X(-A^{k-1}, -b^{k-1})\}$. It implies that $(-A^k, -b^k) \in \overline{U}^k$. In the same way we prove that $(A^k, b^k) \in \underline{U}^k$ if $(-A^k, -b^k) \in \overline{U}^k$. $\square$

Using the above notation, a structure of the sets F^k can be described in a very simple form.

Theorem 4. *For $k = 2, ..., m$ we have*

$$F^k = C^k - (\underline{U}^k \cup \overline{U}^k). \tag{21}$$

Proof. Let us assume that an element $(A^k, b^k) \in F^k$. It implies that $X(A^{k-1}, b^{k-1}) \neq \emptyset$ and therefore $(A^k, b^k) \in C^k$. Further, the assumption $(A^k, b^k) \in F^k$ means that inequalities in (15) hold. There is exactly one of the following possibilities:

(a1) Both the values $\underline{g}(a_k)$, $\overline{g}(a_k)$ are finite.
(a2) $\overline{g}(a_k)$ is finite and $\underline{g}(a_k) = -\infty$.
(a3) $\overline{g}(a_k) = +\infty$ and $\underline{g}(a_k)$ is finite.
(a4) $\overline{g}(a_k) = +\infty$ and $\underline{g}(a_k) = -\infty$.

In each of these cases it can be verify similarly to the proof of Theorem 3 in [1] that $(A^k, b^k) \notin \underline{U}^k \cup \overline{U}^k$.

Let us assume that $(A^k, b^k) \in C^k$ and $(A^k, b^k) \notin \underline{U}^k \cup \overline{U}^k$. The first part of this assumption is valid if and only if $X(A^{k-1}, b^{k-1}) \neq \emptyset$. The second part of the assumption leads to the exactly one of the following cases.

(b1) $a_k \notin \underline{\mathcal{A}}^k \cap \overline{\mathcal{A}}^k$.

(b2) $a_k \notin \underline{\mathcal{A}}^k$ and $a_k \in \overline{\mathcal{A}}^k$ and $b_k^k \leq \overline{g}(a_k)$.

(b3) $a_k \in \underline{\mathcal{A}}^k$ and $b_k^k \geq \underline{g}(a_k)$ and $a_k \notin \overline{\mathcal{A}}^k$.

(b4) $a_k \in \underline{\mathcal{A}}^k \cap \overline{\mathcal{A}}^k$ and $\underline{g}(a_k) \leq b_k^k \leq \overline{g}(a_k)$.

In all the cases we prove analogously to the proof of Theorem 3 in [1] that $(A^k, b^k) \in F^k$. □

Going back to the original problem with the input data bounded by intervals $[A]$ and $[b]$ we have immediately

Theorem 5. *The set of feasible coefficients of a given ILS is described by the form*

$$F = ([A] \times [b]) \cap F^m. \tag{22}$$

The result seems to be a rather theoretical one. The following section gives, however, an application consisting in a construction of a maximal subset of the set of feasible coefficients.

3 Calculating a maximal set of feasible coefficients

For a given ILS we can use a description of the set F to construct an interval $G \subset F$ which is maximal in a following meaning: For any interval $C \supset G$ there is an element $(A, b) \in C$ such that $X(A, b) = \emptyset$, i.e. $(A, b) \notin F$.

The interval G will be constructed by induction.

3.1 Construction of interval G^1

An interval G^1 is constructed by the following theorem which is based on a description of the sets F^k.

Theorem 6. *An element* $(a_1, b_1) \in F^1$ *if and only if at least one of the following conditions is satisfied:*

(i) $b_1 = 0$.

(ii) $b_1 \le 0$ *and there is* $i \in \{1, ..., n\}$ *with* $a_{1i} < 0$.

(iii) $b_1 \ge 0$ *and there is* $i \in \{1, ..., n\}$ *with* $a_{1i} > 0$.

(iv) There are $i, j \in \{1, ..., n\}$ *such that* $a_{1i} < 0, a_{1j} > 0$.

Proof. Relations (16) to (21) can be used to express a set F^1. The set $X(A_0, b_0)$ is unbounded and all its unbounded edges are halflines given by vectors of coordinate axes

$u^i = (0, ..., 0, u_i^i = 1, 0..., 0)$, $i = 1, 2, ..., n$.

Sets $\underline{\mathcal{A}}^1, \overline{\mathcal{A}}^1$ can be described by (3.4) of the paper [1]:

$\underline{\mathcal{A}}^1 = \{a_1 \in \mathrm{R}^n : (a_1)^T u^i \ge 0, \, i = 1, 2, ..., n\} = \{a_1 \in \mathrm{R}^n : a_1 \ge 0\}$.

In a similar way we have $\overline{\mathcal{A}}^1 = \{a_1 \in \mathrm{R}^n : a_1 \le 0\}$.

The sets $\underline{U}^1, \overline{U}^1$ can be expressed as follows:

$$\begin{aligned} \underline{U}^1 &= \{(a_1, b_1) \in \mathrm{R}^n \times \mathrm{R}^1 : a_1 \ge 0, \, b_1 < 0\}, \\ \overline{U}^1 &= \{(a_1, b_1) \in \mathrm{R}^n \times \mathrm{R}^1 : a_1 \le 0, \, b_1 > 0\}. \end{aligned} \tag{23}$$

Because of (21) we have: $(a_1, b_1) \in F^1 \iff (a_1, b_1) \notin (\underline{U}^1 \cup \overline{U}^1)$. It happens if and only if

($b_1 \ge 0$ or there is $i \in \{1, ..., n\}$ with $a_{1i} < 0$) and ($b_1 \le 0$ or there is $i \in \{1, ..., n\}$ with $a_{1i} > 0$). It completes the proof. □

Theorem 6 gives a characterization of elements (a_1, b_1) belonging to the set F^1. There is exactly one of the following three possibilities (a), (b) or (c) for a given interval $J^1 = [a_1] \times [b_1]$:

(a) $(\overline{a}_1, \underline{b}_1) \in F^1$ and $(\underline{a}_1, \overline{b}_1) \in F^1$.

Then $G^1 = [a_1] \times [b_1]$ because of Theorem 1.

(b) $(\overline{a}_1, \underline{b}_1) \in F^1$ and $(\underline{a}_1, \overline{b}_1) \notin F^1$.

The second part of this assumption is satisfied if and only if $(\underline{a}_1, \overline{b}_1) \in (\underline{U}^1 \cup \overline{U}^1)$. The first part $(\overline{a}_1, \underline{b}_1) \in F^1$ implies $(\underline{a}_1, \overline{b}_1) \notin \underline{U}^1$ and thus we have

$$\underline{a}_1 \le 0 \wedge \overline{b}_1 > 0. \tag{24}$$

The set F^1 is not empty in this case, but it is not equal to the interval J^1 because of Theorem 1.

By a proper increasing some component of the vector $\underline{a}_1$, or by decreasing value of $\overline{b}_1$ we can find an element $(\underline{a}_1^*, \overline{b}_1^*) \in J^1$, which belongs to the set F^1. The target should be received by changing exactly only one of the values $\underline{a}_{11}, ...\underline{a}_{1m}, \overline{b}_1$ with the smallest absolute value of the change.

In this way we reach an interval $[\underline{a}_1^*, \overline{a}_1] \times [\underline{b}_1 \times \overline{b}_1^*] \subset J^1$ of intended properties.

Let us describe such a procedure. First, we calculate the value

$$\min\{-\underline{a}_{11}, ... - \underline{a}_{1n}, \overline{b}_1\} = m. \tag{25}$$

Let $m = \overline{b}_1$. If $\underline{b}_1 \leq 0$ then we set $\overline{b}_1^* = 0$ which ensures that the element $(\underline{a}_1, \overline{b}_1^* = 0)$ satisfies condition (i) of Theorem 6. If $\underline{b}_1 > 0$ then $(\underline{a}_1, b_1) \notin F^1$ for each $b_1 \in \langle \underline{b}_1, \overline{b}_1 \rangle$. We return to the calculation of the value m in (25), but without the value $\overline{b}_1$.

Let $m = -\underline{a}_{1p}$. If $\overline{a}_{1p} > 0$ then we increase the value $\underline{a}_{1p}$ to satisfy condition (iii) of Theorem 6, i.e. $\underline{a}_{1p}^* := \epsilon > 0$, where $\epsilon < \overline{a}_{1p}$ is a small positive value. If $\overline{a}_{1p} \leq 0$ then no change of the value $\underline{a}_{1p}$ can ensure an element belonging to the set F^1. We return to the calculation of the value m in (25), but without the value $(-\underline{a}_{1p})$.

After $n+1$ steps (am latest) we find a value $\underline{a}_1^*$ or $\overline{b}_1^*$ such that $(\underline{a}_1^*, \overline{b}_1^*) \in J^1$ satisfies some of conditions of Theorem 6.

(c) $(\overline{a}_1, \underline{b}_1) \notin F^1$.

It implies that $(\overline{a}_1, \underline{b}_1) \in (\underline{U}^1 \cup \overline{U}^1)$ and due to (23) we have

$$\overline{a}_1 \leq 0 \wedge \underline{b}_1 > 0 \tag{26}$$

or

$$\overline{a}_1 \geq 0 \wedge \underline{b}_1 < 0. \tag{27}$$

Both the situations are described in following theorems.

Theorem 7. *Let (26) hold. Then the set of feasible coefficients F is empty.*

Proof. For each $(a_1, b_1) \in J^1$ it holds $a_1 \leq \overline{a}_1 \leq 0$, $b_1 \geq \underline{b}_1 > 0$. Theorem 6 implies that $(a_1, b_1) \notin F^1$ and thus the set $X(a_1, b_1)$ is empty which should be to prove. □

Theorem 8. *Let (27) hold. Then the set F^1 is empty if and only if*

$$\underline{a}_1 \geq 0 \wedge \overline{b}_1 < 0. \tag{28}$$

Proof. If (28) holds then for each $(a_1, b_1) \in J^1$ we have $a_1 \geq \underline{a}_1 \geq 0$ and $b_1 \leq \overline{b}_1 < 0$. Then $(a_1, b_1) \in \underline{U}^1$ due to (23) and thus $X(a_1, b_1) = \emptyset$. To prove the second part let us suppose that (27) holds but (28) does not hold. First, let $\overline{b}_1 \geq 0$. If $\overline{a}_1 = 0$ then an element $(o^n, 0) \in G^1$ as the set $X(o^n, 0)$ is not empty. If $\overline{a}_{1j} > 0$ for some index j then an element $(\overline{a}_1, \overline{b}_1)$ satisfies condition (iii) of Theorem 6 and the set G^1 is not empty. On the contrary, let (28) does not hold because of $\underline{a}_{1j} < 0$ for some index j. It implies that $\underline{b}_1 < 0$ due to (27). Then the element $(\underline{a}_1, \underline{b}_1)$ satisfies condition (ii) of Theorem 6 and thus $(\underline{a}_1, \underline{b}_1) \in G^1$ which completes the proof. □

If both (27) and (28) hold then G^1 is an empty set due to the previous Theorem. It implies that a feasible set F is empty.

If (27) holds and (28) does not hold then the set G^1 is not empty due to Theorem 8. It is not equal, however, to the interval J^1. In this case we find the set G^1 in an analogous way to the procedure of the case (b).

We calculate the value

$$\min\{\overline{a}_{11}, ...\overline{a}_{1n}, -\underline{b}_1\} = m. \tag{29}$$

Let $m = -\underline{b}_1$. If $\overline{b}_1 \geq 0$ then we set $\underline{b}_1^* = 0$ which ensures that the element $(\overline{a}_1, \underline{b}_1^* = 0)$ satisfies condition (i) of Theorem 6. If $\overline{b}_1 < 0$ then we return to the calculation of the value m in (29), but without the value $(-\underline{b}_1)$.

Let $m = \overline{a}_{1p} \geq 0$. If $\underline{a}_{1p} < 0$ then we set $\overline{a}_{1p}^* := -\epsilon < 0$, where $(-\epsilon) > \underline{a}_{1p}$. If $\underline{a}_{1p} \geq 0$ then we return to the calculation of the value m in (29), but without the value $\overline{a}_{1p}$.

Finishing the first step we either conclude that the feasible set F is empty or we use the above procedure to calculate an interval G^1 of the intended property.

3.2 Construction of interval G^k

Let us suppose that an interval G^{k-1} was constructed. To calculate an interval G^k we should find elements $(A^k, b^k) \in J^k = [A^k] \times [b^k]$ with nonempty set $X(A^k, b^k)$, i.e. satisfying $(A^k, b^k) \in F^k$.

The previous construction of G^{k-1} ensures that each element (A^k, b^k) of the interval J^k belongs to the set C^k. The assertion of Theorem 4 implies that $(A^k, b^k) \in F^k$ if and only if $(A^k, b^k) \notin (\underline{U}^k \cup \overline{U}^k)$. Because of (12), (13), (19) and (20) it is equivalent to the following condition:

$$(a_k \notin \underline{\mathcal{A}}^k \text{ or } b_k \geq \underline{g}^k(a_k)) \text{ and } (a_k \notin \overline{\mathcal{A}}^k \text{ or } b_k \leq \overline{g}^k(a_k)). \tag{30}$$

If (30) is satisfied for each element $(A_h^k, b_h^k), h \in H^k$ then G^k is equal to the interval J^k because of theorem 1.

Let us suppose that an element (A_h^k, b_h^k) does not belong to the set F^k for $h = (h_1, ..., h_{k-1}, 1) \in H^k$. It happens if and only if

$$\begin{array}{r} \text{LP problem: } \min\{\overline{a}_k^T x : x \in X(A_h^{k-1}, b_h^{k-1})\} = m_1 \\ \text{has an optimal solution with } \underline{b}_k < m_1 \end{array} \tag{31}$$

or

$$\begin{array}{r} \text{LP problem: } \max\{\overline{a}_k^T x : x \in X(A_h^{k-1}, b_h^{k-1})\} = M_1 \\ \text{has an optimal solution with } \underline{b}_k > M_1. \end{array} \tag{32}$$

Let us consider next LP problems

$$\min\{\underline{a}_k^T x : x \in X(A_h^{k-1}, b_h^{k-1})\} = m_2 \tag{33}$$

$$\max\{\underline{a}_k^T x : x \in X(A_h^{k-1}, b_h^{k-1})\} = M_2 \tag{34}$$

and the following optimization problems

$$\min\{\underline{g}(\overline{a}_k) : (A^{k-1}, b^{k-1}) \in G^{k-1}\} = m_3 \tag{35}$$

$$\min\{\underline{g}(\underline{a}_k) : (A^{k-1}, b^{k-1}) \in G^{k-1}\} = m_4 \tag{36}$$

$$\max\{\overline{g}(\overline{a}_k) : (A^{k-1}, b^{k-1}) \in G^{k-1}\} = M_3 \tag{37}$$

$$\max\{\overline{g}(\underline{a}_k) : (A^{k-1}, b^{k-1}) \in G^{k-1}\} = M_4. \tag{38}$$

These four LP problems with interval coefficients can be solved effectively by using an algorithm which is described in [4].

Theorem 9. *Let (31) hold. If $\overline{b}_k < m_4$ then the set F is empty. If $\overline{b}_k \geq m_4$ then there is an element $(A^k, b^k) \in J^k$ which belongs into the set F^k.*

Proof. If $\overline{b}_k < m_4$ then for any element $(A^k, b^k) \in J^k$ it holds
$b_k \leq \overline{b}_k < m_4 \leq \underline{g}(\underline{a}_k) \leq \underline{g}(a_k)$.

Then $(A^k, b^k) \in \underline{U}^k$ because of (19). It implies that the set $X(A^k, b^k)$ is empty and thus the set F is empty, too.

If $\overline{b}_k \geq m_4$ then exactly one of the following cases (a) to (f) appears. In each of these cases a new constructed element (A^k_h, b^k_h) satisfies condition (30) and thus (A^k_h, b^k_h) belongs into the set F^k.

(a) If $\overline{b}_k > m_1$ then the k-component of vector $\underline{b}$ is increased by setting $\underline{b}_k := m_1$.

(b) If $m_1 \geq \overline{b}_k > \underline{b}_k \geq m_2$ then a continuous dependency of the solution function $\underline{g}_k$ on a parametr a_k (see [2] or [6]) implies an existence of a feasible value a^*_k whith $\underline{b}_k = \underline{g}_k(a^*_k)$. In this case we change the k-th row of the matrix A^k_h by setting $\overline{a}_k := a^*_k$.

(c) If $m_1 \geq \overline{b}_k \geq m_2 > \underline{b}_k$ then a continuous dependency of the solution function $\underline{g}_k$ on a parametr a_k implies an existence of feasible values a^*_k, b^*_k, for which $b^*_k = \underline{g}_k(a^*_k)$. In this case we change the k-th row of the matrix A^k_h by setting $\overline{a}_k := a^*_k$ and the k-th component of vector b^k_h is changed by setting $\underline{b}_k := b^*_k$.

(d) If $m_2 > \overline{b}_k > \underline{b}_k \geq m_3$ then a continuous dependency of the solution function $\underline{g}_k$ implies an existence of feasible values $((A^{k-1})^*, (b^{k-1})^*)$ for which $\underline{b}_k = \underline{g}_k(((A^{k-1})^*, (b^{k-1})^*), \overline{a}_k)$. In this case we set $(A^{k-1}_h, b^{k-1}_h) := ((A^{k-1})^*, (b^{k-1})^*)$.

(e) If $m_2 > \overline{b}_k > m_3 > \underline{b}_k$ then the k-th component of vector b^k_h is changed by setting $\underline{b}_k = \underline{g}_k(((A^{k-1})^*, (b^{k-1})^*), \overline{a}_k)$.

(f) If $m_3 \geq \overline{b}_k$ then we consider a solution function $\underline{g}_k$ of a parametric LP problem

$$\begin{aligned} \min\{a^T_k x : x \in X(A^{k-1}, b^{k-1})\} &= \underline{g}_k(a_k) \\ \underline{a}_k \leq a_k &\leq \overline{a}_k. \end{aligned} \tag{39}$$

Then a continuous dependency of the solution function $\underline{g}_k$ implies an existence of feasible values $((A^k)^*, (b^k)^*) \in J^k$ for which $= \underline{g}_k(a^*_k) = b^*_k$. In this case we set $(A^k_h, b^k_h) := ((A^k)^*, (b^k)^*)$. □

The above proof leads to the construction of a changed element (A^k_h, b^k_h) describing a new interval J^k. In case (a) a calculation of the element (A^k_h, b^k_h) is given explicitly. In cases (b) and (c) let us consider a parametric LP problem

$$\begin{aligned} \min\{a_k^T x : x \in X(A_h^{k-1}, b_h^{k-1})\} = \underline{g}_k(a_k) \\ \underline{a}_k \leq a_k \leq \overline{a}_k. \end{aligned} \tag{40}$$

It is a problem with a fixed set of feasible solutions and with parameters in the objective function. In the chapter 11.4 of [6] there is a description of an algorithm for solving such a problem inclusive of a calculation of the solution function $\underline{g}_k(a_k)$.

In case (d) we start to solve interval LP problem (35) by using an algorithm ILP in [4] with a modification for minimum. During a calculation we find optimal solutions x^k, x^l of extremal subproblems differing in exactly one equation. Let it be the q-th equation and let x^k, x^l satisfy a condition

$$c^T x^k \geq \underline{b}_k \geq c^T x^l. \tag{41}$$

If one of these inequalities is satisfied as an equation then the corresponding constraint gives an element $((A^{k-1})^*, (b^{k-1})^*)$. If both of inequalities are sharp then a procedure is more complicated. If $l = k+1$ then an element $((A^{k-1})^*, (b^{k-1})^*)$ is created by coefficients of constraint corresponding to x^k except of the q-th equation. Its coefficients (a_q^*, b_q^*) are given by an equation

$$\frac{c^T x^k - \underline{b}_k}{\underline{b}_k - c^T x^{k+1}} = \frac{\overline{a}_q^T \underline{b}_q - (a_q^*)^T b_q^*}{(a_q^*)^T b_q^* - \underline{a}_q^T \overline{b}_q}, \tag{42}$$

as the objective function $\overline{a}_k^T x$ decreases linearly between the points x^k, x^{k+1}.

If $l > k+1$ then we find neighbouring points x^s, x^{s+1} satisfying a condition

$$c^T x^k \geq c^T x^s \geq \underline{b}_k \geq c^T x^{s+1} \geq c^T x^l. \tag{43}$$

We shall calculate values $t^{'}, t^{''}$ corresponding to the points x^s, x^{s+1} resp., i.e. satisfying equations

$$\begin{aligned} (A_{t'} x^s)_q = (b_{t'})_q, \\ (A_{t''} x^{s+1})_q = (b_{t''})_q. \end{aligned} \tag{44}$$

The objective function decreases linearly between the points x^s, x^{s+1} and therefore an equation

$$\frac{c^T x^s - \underline{b}_k}{\underline{b}_k - c^T x^{s+1}} = \frac{t^{'} - t_q}{t_q - t^{''}} \tag{45}$$

leads to the value t_q which determines the q-th row $((A_t)_q, (b_t)_q)$ of the intended constraint.

In the case (e) we find $((A^{k-1})^*, (b^{k-1})^*)$ by solving problem (35). The intended element $((A^{k-1})^*, (b^{k-1})^*)$ is given by coefficients of the constraint, which corresponds to the optimal solution of the problem (35). We use the algorithm ILP in [4] again to solve the problem.

In case (f), the problem (39) is again a parametric LP problem with a fixed set of feasible solutions. A solution function g^k can be determined by the procedure described in [6]. Thus an element a_k^* is calculated. An intended element $((A^{k-1})^*, (b^{k-1})^*)$ can be found in an analogous way to the case (d), i.e. by solving an interval LP problem for calculating minimum of the objective function $(a_k^*)^T x$.

Theorem 10. *Let (32) hold. If $\underline{b}_k > M_3$ then the set F is empty. If $\underline{b}_k \le M_3$ then there is an element $(A^k, b^k) \in J^k$ which belongs into the set F^k.*

Proof. If $\underline{b}_k > M_3$ then for any element $(A^k, b^k) \in J^k$ it holds
$b_k \ge \underline{b}_k > M_3 \ge \overline{g}(\overline{a}_k) \ge \overline{g}(a_k)$.

Then $(A^k, b^k) \in \overline{U}^k$ because of (20). It implies that the set $X(A^k, b^k)$ is empty and thus the set F is empty, too.

If $\underline{b}_k \le M_3$ then a continuous dependency of the solution function $\overline{g}_k$ implies an existence of feasible values $((A^{k-1})^*, (b^{k-1})^*)$ for which
$\underline{b}_k = \overline{g}_k(((A^{k-1})^*, (b^{k-1})^*), \overline{a}_k)$. In this case we set
$(A_h^{k-1}, b_h^{k-1}) := ((A^{k-1})^*, (b^{k-1})^*)$ and then a corresponding element (A_h^k, b_h^k) satisfies the condition (30). ☐

It remains to describe the algorithm in the case $(A^k, b^k) \notin F^k$ for $h = (h_1, ..., h_{k-1}, -1) \in H^k$. It happens if and only if

$$\text{LP problem (33) has an optimal solution with } \overline{b}_k < m_2 \qquad (46)$$

or

$$\text{LP problem (34) has an optimal solution with } \overline{b}_k > M_2. \qquad (47)$$

Theorem 11. *Let (46) hold. If $\overline{b}_k < m_4$ then the set F is empty. If $\overline{b}_k \ge m_4$ then there is an element $(A^k, b^k) \in J^k$ which belongs into the set F^k.*

Proof. If $\overline{b}_k < m_4$ then we prove an assertion analogously to the proof of Theorem 9. If $\overline{b}_k \geq m_4$ then a continuous dependency of the solution function $\underline{g}_k$ implies an existence of feasible values $((A^{k-1})^*, (b^{k-1})^*)$ for which $\overline{b}_k = \underline{g}_k(((A^{k-1})^*, (b^{k-1})^*), \underline{a}_k)$. In this case we set $(A_h^{k-1}, b_h^{k-1}) := ((A^{k-1})^*, (b^{k-1})^*)$ and then a corresponding element (A_h^k, b_h^k) satisfies the condition (30). □

Remark. The element $((A^{k-1})^*, (b^{k-1})^*)$ from the proof of Theorem 10 and Theorem 11 can be determine by solving problem (37), (36) resp., in an analogous way to the procedure (d) of the proof of Theorem 9.

Theorem 12. *Let (47) hold. If $\underline{b}_k > M_3$ then the set F is empty. If $\underline{b}_k \leq M_3$ then there is an element $(A^k, b^k) \in J^k$ which belongs into the set F^k.*

Proof. The first part is the same as in Theorem 10. If $\underline{b}_k \leq M_3$ then we have exactly one of the following cases, similarly to Theorem 9.

(a) If $\underline{b}_k < M_2$ then the k-component of vector $\overline{b}$ is decreased by setting $\overline{b}_k := M_2$.

(b) If $M_2 \leq \underline{b}_k < \overline{b}_k \leq M_1$ then there is a feasible value a_k^* for which $\overline{b}_k = \overline{g}_k(a_k^*)$. In this case we change the k-th row of the matrix A_h^k by a setting $\underline{a}_k := a_k^*$.

(c) If $M_2 \leq \underline{b}_k \leq M_1 < \overline{b}_k$ then there are feasible values a_k^*, b_k^*, for which $b_k^* = \overline{g}_k(a_k^*)$. In this case we set $\underline{a}_k := a_k^*$ and $\overline{b}_k := b_k^*$.

(d) If $M_1 < \underline{b}_k < \overline{b}_k \leq M_4$ then there is an element $((A^{k-1})^*, (b^{k-1})^*)$ for which $\overline{b}_k = \overline{g}_k(((A^{k-1})^*, (b^{k-1})^*), \underline{a}_k)$. In this case we set $(A_h^{k-1}, b_h^{k-1}) := ((A^{k-1})^*, (b^{k-1})^*)$.

(e) If $M_1 < \underline{b}_k < M_4 < \overline{b}_k$ then the k-th component of vector b_h^k is changed by setting $\overline{b}_k = \overline{g}_k(((A^{k-1})^*, (b^{k-1})^*), \underline{a}_k)$.

(f) If $M_4 \leq \underline{b}_k$ then we consider a solution function $\overline{g}_k$ of a parametric LP problem

$$\begin{aligned} \max\{a_k^T x : x \in X(A^{k-1}, b^{k-1})\} &= \overline{g}_k(a_k), \\ \underline{a}_k \leq a_k &\leq \overline{a}_k. \end{aligned} \tag{48}$$

Then there is a feasible element $((A^k)^*, (b^k)^*) \in J^k$, for which $= \overline{g}_k(a_k^*) = b_k^*$. In this case we set $(A_h^k, b_h^k) := ((A^k)^*, (b^k)^*)$.

In each of the cases (a) to (f) a new constructed element (A_h^k, b_h^k) satisfies condition (30) and thus (A_h^k, b_h^k) belongs into the set F^k. □

Remark. An element (A_h^k, b_h^k) of the previous Theorem, which determines a new interval J^k can be found analogously to the procedure described behind Theorem 9 with keeping the following differences. In cases (b) and (c) we consider a parametric LP problem

$$\begin{aligned} \max\{a_k^T x : x \in X(A_h^{k-1}, b_h^{k-1})\} = \overline{g}_k(a_k) \\ \underline{a}_k \leq a_k \leq \overline{a}_k. \end{aligned} \tag{49}$$

In cases (d) and (e) we solve the problem (38), i.e. an interval LP problem with an objective function $\underline{a}_k^T x$.

In this way we have discussed all possibilities. The algorithm results either with the conclusion that the feasible set F is empty or with calculating interval G^k of the intended properties.

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