Labeling planar graphs with a condition at distance two*

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Abstract

An $L(2,1)$-labeling of a graph is a mapping $c : V(G) \to \{0, \ldots, K\}$ such that the labels assigned to neighboring vertices differ by at least 2 and the labels of vertices at distance two are different. The smallest K for which an $L(2,1)$-labeling of a graph G exists is denoted by $\lambda_{2,1}(G)$. Griggs and Yeh [SIAM J. Discrete Math. 5 (1992), 586–595] conjectured that $\lambda_{2,1}(G) \le \Delta^2$ for every graph G with maximum degree Δ. We prove the conjecture for planar graphs with maximum degree $\Delta \neq 3$. All our results also generalize to the list-coloring setting.

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1 Introduction

Special types of vertex-colorings found applications in the *frequency assignment problem* [14]. One of the most intensively studied types of such colorings is an $L(p,q)$-labeling. A vertex-labeling by non-negative integers of a graph G is called an $L(p,q)$*-labeling* if the labels of adjacentadjacent vertices differ by at least p and the labels of vertices at distance two differ by at least q. The *span* of an $L(p,q)$-labeling is the maximum label used by it. The smallest span of an $L(p,q)$-labeling of a graph G is denoted by $\lambda_{p,q}(G)$. Our work is motivated the conjecture of Griggs and Yeh [13] that asserts that $\lambda_{2,1}(G) \leq \Delta^2$ for every graph G with maximum degree $\Delta \geq 2$. We establish the conjecture for planar graphs with maximum degree $\Delta \neq 3$. The conjecture has also been proven for several other classes of graphs: graphs of maximum degree two, outer planar graphs [7], chordal graphs [25] (see also [6, 21]), Hamiltonian cubic graphs [16, 17], direct and strong products of graphs [18], etc. For general graphs, the original bound $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta$ of [13] was improved to $\lambda_{2,1}(G) \leq \Delta^2 + \Delta$ in [8]. A more general result contained in [20] yields $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 1$ and the present record of $\Delta^2 + \Delta - 2$ was proven by Gonçalves [12]. Algorithmic aspects of $L(2,1)$-labelings as well as $L(p,q)$-labelings are also widely investigated [1, 4, 10, 11, 19, 22] because of their potential applications in practice.

In this paper, we mainly focus on planar graphs. Let us briefly survey known results on $L(p,q)$-labelings of planar graphs: van den Heuvel et al. [15] showed that $\lambda_{p,q}(G) \leq (4q-2)\Delta + 10p - 38q - 23$, and Borodin et al. [5] provides the bound of $\lambda_{p,q}(G) \leq (2q-1)\lceil 9\Delta/5 \rceil + 8p - 8q + 1$ for $\Delta \geq 47$. The best asymptotic result $\lambda_{p,q}(G) \leq q\lceil 5\Delta/3 \rceil + 18p + 77q - 18$ is due to Molloy and Salavatipour [23, 24]. Better bounds are known for planar graphs without short cycles—Wang and Lih [26] showed the following:

- $\lambda_{p,q}(G) \leq (2q-1)\Delta + 4p + 4q - 4$ if G is a planar graph of girth at least seven,
- $\lambda_{p,q}(G) \leq (2q-1)\Delta + 6p + 12q - 9$ if G is a planar graph of girth at least six, and
- $\lambda_{p,q}(G) \leq (2q-1)\Delta + 6p + 24q - 15$ if G is a planar graph of girth at least five.

The bound for planar graphs with girth seven has recently been improved in [9] to $2p + q\Delta - 2$ under the assumption that the maximum degree Δ

is sufficiently large (this bound is best possible if $q = 1$ which includes the case of $L(2,1)$-labelings). Closely related results on coloring powers of planar graphs of higher order can be found in [2, 3].

The bound of van den Heuvel et al. [15] implies that the conjecture of Griggs and Yeh holds for planar graphs with maximum degree $\Delta \geq 7$. We consider a more general setting of list labelings and show that the conjecture holds (in the list version) for planar graphs with maximum degree $\Delta \neq 3$. Let us remark that our proof is computer-assisted in the case of planar graphs with maximum degree four.

We would also like to draw the attention of the reader to Lemmas 3.5 and 3.6. Results on distance constrained labelings are usually more difficult to prove than their counterparts for ordinary colorings because of a complex interaction between vertices at distance two, e.g., it is not known that the smallest counterexample to the conjecture of Griggs and Yeh is 2-connected. Since we needed a tool that would allow us to cope with this difficulty and that would also allow us to employ computers in our arguments, we developed a notion of degree configuration described in the next two section. Informally, we provide a condition on a graph H such that no minimal graph without an $L(2,1)$-labeling of a certain span (no minimal counterexample) contains a *locally injective homomorphic* image of H. Our technique does not apply only for $L(p,q)$-labelings, but counterparts of Lemmas 3.5 and 3.6 can be proved for ordinary coloring as well as for several other different notions of coloring.

2 Preliminaries

In this section, we introduce notation used throughout the paper. The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$ and the set of vertices at distance at most d from v by $N_G(v,d)$. We often deal with planar graphs, and we always assume they are simple and loopless. Whenever we say that we *add* an edge uv to a graph G, we mean that we add the edge if the vertices u and v are not adjacent, and we do nothing, otherwise. If W is a subset of vertices of a graph G, then the graph $G \setminus W$ is the graph obtained from G by removing the vertices of W with all their incident edges and the graph $G[W]$ is the subgraph of G induced by the vertices in W. If $W = \{v\}$, then we write $G \setminus v$ instead of $G \setminus \{v\}$. Finally, if φ is a mapping between two sets A and B, $\varphi(A')$ is the set of $\varphi(a)$ where a ranges through the elements of $A' \subseteq A$ and $\varphi^{-1}(B')$ is the set of the preimages of the elements of $B' \subseteq B$.

A *d-face* of a plane graph is a face of size d, i.e., d-face with a boundary walk of length d. More precisely, the size of a face is the number of edges incident with it counting bridges twice. A $\geq$*d-face* is a face of size at least d and a $\leq$*d-face* is a face of size at most d. A *d-vertex* is a vertex of degree d and we use a $\geq$ *d-vertex* and a $\leq$ *d-vertex* in the obvious meanings. An $(\ell_1, \dots, \ell_d)$-vertex is a d-vertex that is incident with faces of sizes $\ell_1, \dots, \ell_d$ (in this cyclic order around the vertex). We also use, e.g., a $(\geq 5, 4, 3)$-vertex in the natural meaning.

Proofs of our theorems (Theorems 4.8, 5.13, and 6.20) are based on the discharging method. First, each vertex and face of a plane graph is assigned a fixed amount of initial charge. The charge is then redistributed among the vertices and faces by certain rules. The rules describe when a face f sends charge to an incident vertex v or a vertex v sends charge to an incident face f. If the face f is incident several times with a vertex v (this happens when v is a cut-vertex), then f sends charge to v by several rules—each applies separately to each incidence of v and f and the charge sent to v is equal to the sum of the amounts of charge sent by all the rules that apply. The same applies for charge sent by vertices to the faces. We do not emphasize this important issue in the rest of the paper.

The key notion in our approach is the notion of *degree homomorphism*. A pair (H, d) where H is a graph and $d : V(H) \to \mathbb{N}$ is called a *degree configuration* (throughout the paper, $\mathbb{N}$ always denotes the set of all non-negative integers). A mapping $\varphi : V(H) \to V(G)$ is said to be a *degree homomorphism* from (H, d) to G if it is a degree preserving locally injective homomorphism, i.e., the following holds:

- the mapping φ is a homomorphism, i.e., $\varphi(u)\varphi(v) \in E(G)$ if $uv \in E(H)$,
- the mapping φ is locally injective, i.e., φ is injective when restricted to $N_H(v)$ for every $v \in V(H)$, and
- $\deg_G(\varphi(v)) = d(v)$ for every $v \in V(H)$.

We are sometimes vague when specifying the function d and if this is the case, then it holds that $d(v) = \deg_G(\varphi(v))$ whenever the value $d(v)$ is not specified.

Our argument that certain degree configurations cannot appear in a minimal counterexample is based on the results obtained for the channel assignment problem. An instance of the *channel assignment problem* is a graph G with a function $w : E(G) \to \mathbb{N}$ that assigns a positive integer to each edge of G. The value $w(e)$ is called the *weight* of an edge e. We will be interested in the *list channel assignment problem* in which, in addition, each vertex v of G is equipped with a *list* $L(v)$ of available labels, i.e., $L : V(G) \to 2^{\mathbb{N}}$. If $|L(v)| = k$ for every vertex v of G, L is called a *k-list assignment*. The goal is to find a labeling $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ and $|c(u) - c(v)| \geq w(uv)$ for every edge $uv \in E(G)$. Such a labeling is also called a *list labeling* or a *list $L(2,1)$-labeling* (if appropriate) for L and we often omit to emphasize the list assignment L if it is clear from the context. Finally, a *subproblem* of an instance of the channel assignment problem that is induced by a vertex set W is the problem with $G[W]$ and both the list of vertices and the edge-weights are the same as in the original problem.

$L(p,q)$-labelings can be viewed as instances of the channel assignment problem: if G is a graph, define $\mathcal{L}_{p,q}(G)$ to be the channel assignment problem with a graph that is the square G^2 of G and $w(e) = p$ if $e \in E(G)$ and $w(e) = q$, otherwise. Clearly, a graph G has an $L(2,1)$-labeling of span at most Λ if and only if there is a labeling c of $\mathcal{L}_{2,1}(G)$ for the lists $L(v) = \{0, \dots, \Lambda\}$, $v \in V(G)$.

3 Reduction tools

In all our proofs, we first identify certain configurations (subgraphs) that cannot appear in a minimal counterexample. The proofs of Lemmas 3.5 and 3.6 are based on a careful application of the following greedy algorithm, first used by McDiarmid [22] in the area of the channel assignment problem.

Algorithm 1.

Input: ordering of the vertices $v_1, \dots, v_n$
edge-weight function w
lists $L(v_1), \dots, L_(v_n)$ of labels available for vertices
Output: (partial) vertex labeling c

$X :=$ minimum label contained in $L(v_1) \cup \dots \cup L(v_n)$
maxcol $:=$ maximum label contained in $L(v_1) \cup \dots \cup L(v_n)$
`while` $X \leq$ maxcol `do`
  `for` $i := 1$ `to` n `do`

```
    if v_i is not labeled and X ∈ L(v_i)
    then
      if for all v_j ∈ N(v_i) that are labeled
        it holds that |c(v_j) − X| >= w(v_i v_j)
      then
        label v_i by setting c(v_i) := X
      fi
    fi
    X := X + 1
  endfor
endwhile
```

The following two lemmas can be found in [20]:

Lemma 3.1. *Let G be a graph with edge weights $w : E(G) \to \mathbb{N}$ and a list function $L : V(G) \to 2^{\mathbb{N}}$. Assume that Algorithm 1 is applied to G together with an ordering $v_1, \ldots, v_n$ of its vertices. If it holds that*

$$\sum_{i'<i, v_{i'}v_i \in E(G)} w(v_{i'}v_i) + \sum_{i'>i, v_{i'}v_i \in E(G)} (w(v_{i'}v_i) - 1) < |L(v_i)|,$$

then the vertex v_i is labeled by the algorithm.

Lemma 3.2. *Let G be a graph with edge weights $w : E(G) \to \mathbb{N}$ and a list function $L : V(G) \to 2^{\mathbb{N}}$. Assume that Algorithm 1 is applied to G together with an ordering $v_1, \ldots, v_n$ of its vertices and that for a vertex v_i the following equality holds:*

$$\sum_{j<i, v_j v_i \in E(G)} w(v_j v_i) + \sum_{j>i, v_j v_i \in E(G)} (w(v_j v_i) - 1) = |L(v_i)| \; .$$

If the vertex v_i is not labeled by the algorithm, then all its neighbors are labeled and the following holds:

$$\begin{aligned} L(v_i) \quad = \quad & \bigcup_{j<i, v_j v_i \in E(G)} [c(v_j), c(v_j) + w(v_i v_j) - 1] \cup \\ & \bigcup_{j>i, v_j v_i \in E(G)} [c(v_j) + 1, c(v_j) + w(v_i v_j) - 1] \; . \end{aligned}$$

Moreover, all the intervals in the above union are disjoint.

As the first step towards Lemma 3.5, we prove its version for non-induced subgraphs. Loosely speaking, we aim to prove a weaker version of it where locally injective homomorphisms are replaced by injective homomorphism.

Lemma 3.3. *Let G be graph with maximum degree Δ and H a subgraph of G (that is not necessarily induced). Let $V(H) = \{v_1, \ldots, v_n\}$. Assume that the following holds for some integer Λ and for every $i = 1, \ldots, n$:*

$$\begin{aligned}(\Delta+2)(\deg_G(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (\deg_G(v_j) - \deg_H(v_j)) + \\ \deg_H(v_i) + |\{v_1, \ldots, v_{i-1}\} \cap N_H(v_i, 2)| \quad \leq \quad \Lambda\ .\end{aligned}$$

Let L_0 be a $(\Lambda+1)$-list assignment of G. If $G \setminus V(H)$ has a list $L(2,1)$-labeling such that the labels of any two vertices at distance at most two in G are different, then G also has a list $L(2,1)$-labeling.

Proof. Fix a list $L(2,1)$-labeling of $G \setminus V(H)$ such that the labels of any two vertices at distance at most two in G are different. For every vertex $v_i \in V(H)$, let $L(v_i)$ be the labels of $L_0(v_i)$ that are not assigned to any vertex at distance at most two from v_i in G and that differ by at least two from the labels assigned to the neighbors of v_i. We verify that Algorithm 1, when applied to the channel assignment subproblem of $\mathcal{L}_{2,1}(G)$ induced by $V(H)$ together with lists L and the order $v_1, \ldots, v_n$ of its vertices, assigns each vertex of H a label. By Lemma 3.1, it is enough to verify that the following holds for every vertex $v_i \in V(H)$:

$$\begin{aligned}2|W_1 \cap \{v_1, \ldots, v_{i-1}\}| + |W_2 \cap \{v_1, \ldots, v_{i-1}\}| + |W_1 \cap \{v_{i+1}, \ldots, v_n\}| = \\ |W_1| + |W_1 \cap \{v_1, \ldots, v_{i-1}\}| + |W_2 \cap \{v_1, \ldots, v_{i-1}\}| < |L(v_i)|, \qquad (1)\end{aligned}$$

W_1 is the set of the vertices of H that are neighbors of v_i in $G[V(H)]$ and W_2 is the set of the vertices of H at distance two from v_i in G.

Let d_1 and d_2 be the numbers of vertices of $V(G) \setminus V(H)$ at distance one and two from v_i, respectively. Clearly, the following holds:

$$|L(v_i)| \geq \Lambda + 1 - 3d_1 - d_2 \qquad (2)$$

$$d_1 \leq \deg_G(v_i) - \deg_H(v_i) \qquad (3)$$

Note that the last inequality can be strict (since H need not be an induced subgraph of G). In such a case, $|W_1| = \deg_G(v_i) - d_1 > \deg_H(v_i)$. However, the following bounds hold:

$$\begin{aligned}|W_1| + |W_1 \cap \{v_1, \ldots, v_{i-1}\}| &\le \\ \deg_G(v_i) - d_1 + (\deg_G(v_i) - \deg_H(v_i)) - d_1 + |N_H(v_i) \cap \{v_1, \ldots, v_{i-1}\}| &\le \\ 2(\deg_G(v_i) - \deg_H(v_i) - d_1) + \deg_H(v_i) + |N_H(v_i) \cap \{v_1, \ldots, v_{i-1}\}| &. \quad (4)\end{aligned}$$

Similarly,

$$\begin{aligned}&|W_2 \cap \{v_1, \ldots, v_{i-1}\}| \le \\ &(\Delta - 1)\,(\deg_G(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (\deg_G(v_j) - \deg_H(v_j)) - d_2 + \\ &|\{v_1, \ldots, v_{i-1}\} \cap (N_H(v_i, 2) \setminus N_H(v_i))| \quad . \qquad (5)\end{aligned}$$

Note that the number $\deg_G(v_i) - \deg_H(v_i) - d_1$ is the number of neighbors of v_i in G that are not neighbors of v_i in H—such vertices may precede v_i in the order and thus we count them twice in (4). Similarly, the number $(\Delta - 1)\,(\deg_G(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (\deg_G(v_j) - \deg_H(v_j)) - d_2$ is an upper bound on the number of the vertices of $V(H)$ at distance two from v_i in G that are at distance three or more from v_i in H. Such vertices may precede v_i in the order. We combine (4) and (5):

$$\begin{aligned}&|W_1| + |W_1 \cap \{v_1, \ldots, v_{i-1}\}| + |W_2 \cap \{v_1, \ldots, v_{i-1}\}| \le \\ &((\Delta + 1)\,(\deg_G(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (\deg_G(v_j) - \deg_H(v_j)) - 2d_1 - d_2 + \\ &\deg_H(v_i) + |\{v_1, \ldots, v_{i-1}\} \cap N_H(v_i, 2)| \, . \qquad (6)\end{aligned}$$

We derive the following using (3) and (6):

$$\begin{aligned}&|W_1| + |W_1 \cap \{v_1, \ldots, v_{i-1}\}| + |W_2 \cap \{v_1, \ldots, v_{i-1}\}| \le \\ &((\Delta + 2)\,(\deg_G(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (\deg_G(v_j) - \deg_H(v_j)) - 3d_1 - d_2 + \\ &\deg_H(v_i) + |\{v_1, \ldots, v_{i-1}\} \cap N_H(v_i, 2)| \le \Lambda - 3d_1 - d_2 \, . \qquad (7)\end{aligned}$$

The condition (1) now follows from (2) and (7).

It remains to verify that the labeling obtained by combining the labeling of $G \setminus V(H)$ and the labeling of H is an $L(2,1)$-labeling of G. The labels assigned to the neighboring vertices differ by at least two by the choice of the lists L and the fact that we apply Algorithm 1 to the channel assignment subproblem of $\mathcal{L}_{2,1}(G)$. Let u and v be two vertices at distance two. If both u and v are contained in $V(G) \setminus V(H)$, then their labels are different by our assumption on the $L(2,1)$-labeling of $G \setminus V(H)$. If both u and v are contained in $V(H)$, then their labels are different because Algorithm 1 was applied to the channel assignment subproblem of $\mathcal{L}_{2,1}(G)$. Finally, if $u \in V(H)$ and $v \notin V(H)$ (or vice versa), their labels are different by the choice of the list $L(u)$. $\square$

A careful inspection of the proof of Lemma 3.3 allows us to extend it to the following lemma that will also be needed in our considerations. In our applications, we will usually not be able to compute the numbers α_i and β_i precisely, but we will be able to establish some lower bounds on them (therefore, we formulate the lemma with conditions that the numbers α_i and β_i are at most the described quantities).

Lemma 3.4. *Let G be graph with maximum degree Δ and H a subgraph of G (that is not necessarily induced). Let $V(H) = \{v_1, \dots, v_n\}$. Assume that the following holds for some integer Λ and for every $i = 1, \dots, n$:*

$$(\Delta+2)(\deg_G(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (\deg_G(v_j) - \deg_H(v_j)) + \deg_H(v_i) + |\{v_1, \dots, v_{i-1}\} \cap N_H(v_i, 2)| - 2\alpha_i - \beta_i \quad \leq \quad \Lambda \ ,$$

where α_i is (at most) the number of edges e between the neighbors of v_i such that $e \notin E(H)$ and β_i is (at most) the number of vertices of $V(G) \setminus V(H)$ at distance two from v_i in G that are neighbors of two distinct neighbors of v_i. Let L_0 be a $(\Lambda+1)$-list assignment of G. If $G \setminus V(H)$ has a list $L(2,1)$-labeling such that the labels of any two vertices with distance at most two in G are different, then G also has a list $L(2,1)$-labeling.

We are ready to prove our main reduction lemma:

Lemma 3.5. *Let G be a graph with maximum degree Δ and (H, d) a degree configuration with $V(H) = \{v_1, \dots, v_n\}$. Assume that the following holds for every $i = 1, \dots, n$:*

$$(\Delta+2)(d(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (d(v_j) - \deg_H(v_j)) + \deg_H(v_i) + |\{v_1, \dots, v_{i-1}\} \cap N_H(v_i, 2)| \quad \leq \quad \Lambda \ .$$

Let L_0 be a $(\Lambda+1)$-list assignment of G. If φ is a degree homomorphism from H to G and $G \setminus \varphi(V(H))$ has a list $L(2,1)$-labeling such that any two *vertices of $G \setminus \varphi(V(H))$ at distance at most two in G are assigned different labels, then G has a list $L(2,1)$-labeling.*

Proof. Fix a degree homomorphism φ from H to G. Let $W = \varphi(V(H))$ and let $H_0 = G[W]$. For every vertex $w \in W$, let $\alpha(w)$ be the largest index i such that $\varphi(v_i) = w$. Let $w_1, \ldots, w_{n_0}$ be the vertices of W listed in the increasing order determined by the numbers $\alpha(w)$. We verify that H_0 with the order $w_1, \ldots, w_{n_0}$ satisfy the conditions of Lemma 3.3.

Fix an integer $i_0 \in \{1, \ldots, n_0\}$ and set $i = \alpha(w_{i_0})$. Let $N_2 = \{\varphi(v) | v \in N_H(v_i, 2)\}$, i.e., the set of images of the vertices that are distance at most 2 from v_i in H. By the definition of the order $w_1, \ldots, w_{n_0}$, if $j < i_0$, then $\alpha(w_j) < \alpha(w_{i_0})$ and thus the following holds:

$$|\{w_1, \ldots, w_{i_0-1}\} \cap N_2| \leq |\{v_1, \ldots, v_{i-1}\} \cap N_H(v_i, 2))| \tag{8}$$

Let A_1 be the neighbors $w \in V(H_0)$ of w_{i_0} in H_0 such that $\varphi^{-1}(\{w\}) \cap N_H(v_i) = \emptyset$, i.e., the set of neighbors of w_{i_0} whose preimages are not neighbors of v_i. It is easy to check that the following equality holds:

$$|A_1| = \deg_{H_0}(w_{i_0}) - \deg_H(v_i) \tag{9}$$

Let A_2 be the vertices $w \in V(H_0)$ at distance two from w_{i_0} in H_0 such that $\varphi^{-1}(w) \cap N_H(w_{i_0}, 2) = \emptyset$ and $A_2' \subseteq A_2$ those vertices whose neighbor is contained in A_1. The following two bounds are straightforward:

$$|A_2'| + \sum_{w \in A_1} \left(\deg_G(w) - \deg_{H_0}(w)\right) \leq (\Delta - 1)|A_1| \tag{10}$$

$$|A_2 \setminus A_2'| \leq \sum_{v \in N_H(v_i)} (\deg_{H_0}(\varphi(v)) - \deg_H(v)) \; . \tag{11}$$

We combine (8)–(11) to get the condition of Lemma 3.3:

$$
\begin{aligned}
&(\Delta+2)(\deg_G(w_{i_0})-\deg_{H_0}(w_{i_0}))+\sum_{w\in N_{H_0}(w_{i_0})}\big(\deg_G(w)-\deg_{H_0}(w)\big)+\\
&\deg_{H_0}(w_{i_0})+|\{w_1,\dots,w_{i_0-1}\}\cap N_{H_0}(w_{i_0},2)| \quad \underset{\text{by (9)}}{\le}\\
&(\Delta+2)(\deg_G(w_{i_0})-\deg_{H_0}(w_{i_0}))+\\
&\sum_{v\in N_H(v_i)}\big(\deg_G(\varphi(v))-\deg_{H_0}(\varphi(v))\big)+\sum_{w\in A_1}\big(\deg_G(w)-\deg_{H_0}(w)\big)+\\
&\deg_H(v_i)+|A_1|+|\{w_1,\dots,w_{i_0-1}\}\cap(N_2\cup A_1\cup A_2)| \quad \le\\
&(\Delta+2)(\deg_G(w_{i_0})-\deg_{H_0}(w_{i_0}))+\\
&\sum_{v\in N_H(v_i)}\big(\deg_G(\varphi(v))-\deg_{H_0}(\varphi(v))\big)+\sum_{w\in A_1}\big(\deg_G(w)-\deg_{H_0}(w)\big)+\\
&\deg_H(v_i)+|A_1|+|\{w_1,\dots,w_{i_0-1}\}\cap N_2|+|A_1|+|A_2'|+|A_2\setminus A_2'| \quad \underset{\text{by (10)}}{\le}\\
&(\Delta+2)(\deg_G(w_{i_0})-\deg_{H_0}(w_{i_0}))+\sum_{v\in N_H(v_i)}\big(\deg_G(\varphi(v))-\deg_{H_0}(\varphi(v))\big)+\\
&\deg_H(v_i)+(\Delta+1)|A_1|+|\{w_1,\dots,w_{i_0-1}\}\cap N_2|+|A_2\setminus A_2'| \quad \underset{\text{by (11)}}{\le}\\
&(\Delta+2)(\deg_G(w_{i_0})-\deg_{H_0}(w_{i_0}))+(\Delta+1)|A_1|+\sum_{v\in N_H(v_i)}\big(\deg_G(\varphi(v))-\deg_{H_0}(\varphi(v))\big)+\\
&\sum_{v\in N_H(v_i)}(\deg_{H_0}(\varphi(v))-\deg_H(v))+\deg_H(v_i)+|\{w_1,\dots,w_{i_0-1}\}\cap N_2| \quad \underset{\text{by (8)}}{\le}\\
&(\Delta+2)(\deg_G(w_{i_0})-\deg_{H_0}(w_{i_0})))+(\Delta+1)|A_1|+\sum_{v\in N_H(v_i)}(\deg_G(\varphi(v))-\deg_H(v))\\
&\deg_H(v_i)+|\{v_1,\dots,v_{i-1}\}\cap N_H(v_i,2)| \quad \underset{\text{by (9)}}{\le}\\
&(\Delta+2)(d(v_i)-\deg_{H_0}(w_{i_0})))+(\Delta+2)(\deg_{H_0}(w_{i_0})-\deg_H(v_i))\\
&\sum_{v\in N_H(v_i)}(d(v)-\deg_H(v))+\deg_H(v_i)+|\{v_1,\dots,v_{i-1}\}\cap N_H(v_i,2)| \quad \le\\
&(\Delta+2)(d(v_i)-\deg_H(v_i)))+\\
&\sum_{v\in N_H(v_i)}(d(v)-\deg_H(v))+\deg_H(v_i)+|\{v_1,\dots,v_{i-1}\}\cap N_H(v_i,2)| \quad \le\Lambda
\end{aligned}
$$

The statement of the lemma now follows by Lemma 3.3. □

If we apply Lemma 3.4 instead of Lemma 3.3, we can prove the following generalization of Lemma 3.6.

Lemma 3.6. *Let G be a graph with maximum degree Δ and (H_0, d) a degree configuration. Let $H \subseteq H_0$ and $V(H) = \{v_1, \ldots, v_n\}$. Assume that the following holds for every $i = 1, \ldots, n$:*

$$(\Delta+2)(\deg_G(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (\deg_G(v_j) - \deg_H(v_j)) +$$
$$\deg_H(v_i) + |\{v_1, \ldots, v_{i-1}\} \cap N_H(v_i, 2)| - 2\alpha_i - \beta_i \quad \leq \quad \Lambda\,,$$

where α_i is the number of edges e between the neighbors of v_i in H_0 such that $e \notin E(H)$ and β_i is the number of vertices of $V(H_0) \setminus V(H)$ at distance two from v_i in H_0 that are neighbors of two distinct neighbors of v_i. Let L_0 be a $(\Lambda+1)$-list assignment of G. If φ is a degree homomorphism from H_0 to G and $G \setminus \varphi(V(H))$ has a list $L(2,1)$-labeling for L_0 such that any two vertices of $G \setminus \varphi(V(H))$ with distance at most two in G are assigned different labels, then G has a list $L(2,1)$-labeling for L_0, too.

In our applications of Lemma 3.6, both H and H_0 will be plane graphs and we will estimate α_i as the number of 3-faces of H_0 incident with v_i that are not contained in H and β_i as the number of 4-faces of H_0 incident with v_i that are not contained in H.

At the end of this section, we state an auxiliary lemma. The proof of the lemma uses the methods described in this section, in particular Lemmas 3.1 and 3.2. The lemma itself is later used in the proof of Lemma 6.8.

Lemma 3.7. *The channel assignment problem depicted in Figure 1 can be labeled from any list assignment $L : V \to 2^{\mathbb{N}}$ such that $|L(v_1)| = |L(v_2)| = 3$, $|L(v_3)| = 5$ and $|L(v_4)| = |L(v_5)| = 8$.*

Proof. Apply Algorithm 1 for the vertex sequence $v_1, \ldots, v_5$. Let $c : V \to \mathbb{N}$ be the (partial) labeling constructed by the algorithm. By Lemma 3.1, each of the vertices v_1, v_3, v_4 and v_5 receives a label. If the vertex v_2 is also labeled, we have a labeling of the channel assignment problem. Assume that the vertex v_2 is not labeled. By Lemma 3.2, $L(v_2) = \{c(v_1), c(v_1)+1, c(v_4)+1\}$. In particular, $c(v_1) \neq c(v_4)+1$. Let $x = c(v_4)$. Consider a modified list assignment L':

$$L'(v) = \begin{cases} L(v_4) \setminus \{x\} & \text{if } v = v_4, \\ L(v) & \text{otherwise.} \end{cases}$$

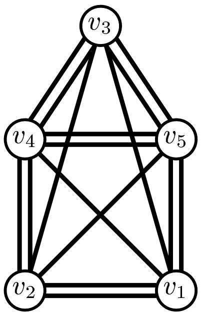

Figure 1: The list channel assignment problem from Lemma 3.7; the edges of weight one are depicted by single lines, those of weight two by double lines.

Apply Algorithm 1 with the lists L'. The algorithm proceeds in the same way until the point when the vertex v_4 was assigned the label x. In particular, the label x has not been assigned to v_1 or v_3. Since $x \notin L'(v_4)$, the vertex v_4 remains unlabeled.

Assume that x is not assigned to the vertex v_5 either. The label $x+1$ cannot be assigned to v_1 since the algorithm during the first run would have assigned the label $x+1$ to v_1, too. Next, the label $x+1$ is assigned to the vertex v_2: none of the vertices joined to v_2 by an edge of weight two is assigned the label x or $x+1$ and none of the vertices joined to v_2 by an edge weight one is assigned the label $x+1$. Hence, the vertex v_2 is labeled. The remaining vertices are labeled by Lemma 3.1.

If the label x is assigned to the vertex v_5, then v_1 cannot be assigned the label $x+1$ because of the edge v_1v_5. Hence, v_2 is labeled $x+1$. The remaining vertices are again labeled by Lemma 3.1. This completes the proof. $\square$

4 Planar graphs with maximum degree six

In this section, we prove the conjecture of Griggs and Yeh for planar graphs with maximum degree six. In fact we establish a stronger result that each planar graph with maximum degree six has a list $L(2,1)$-labeling for any 33-list assignment. For the sake of simplicity, we state our arguments only for the coloring setting but the reader can easily verify that the arguments smoothly translate to the list labelings. Throughout the section, we say that a planar graph G is 6-*minimal* if G has maximum degree six, $\lambda_{2,1}(G) > 32$ and every planar graph with maximum degree six and fewer vertices than G has an $L(2,1)$-labeling of span at most 32.

4.1 Reducible configurations

We first identify several configurations that cannot appear in a 6-minimal graph.

Lemma 4.1. *Every* 6*-minimal graph G has minimum degree at least three.*

Proof. Consider a 6-minimal graph G with minimum degree one or two. If G has a vertex v of degree one, let $G' = G \setminus v$. If G has a vertex v of degree two, let G' be the graph obtained by suppressing the vertex v. By the 6-minimality of G, G' has an $L(2,1)$-labeling with span at most 32. The $L(2,1)$-labeling can be extended to the vertex v: at most $2 \cdot 3$ labels cannot be assigned to v because they differ by at most one from a label assigned to a neighbor of v and additional at most $2 \cdot (\Delta - 1) = 10$ labels cannot be assigned to v because they are assigned to a vertex at distance two from v. Altogether, there are 16 such labels. Hence, the $L(2,1)$-labeling of G' can be extended to v. This yields an $L(2,1)$-labeling of G with span at most 32 and contradicts our assumption that G is 6-minimal. □

Let us remark that in the remaining proofs of this and the following sections, we will not provide a detailed counting of the labels that cannot be assigned to removed vertices as in the proof of Lemma 4.1, but we just state the number of labels that cannot be assigned to the removed vertices and let the reader verify it him/herself.

The proofs of all the remaining lemmas of this subsection proceed in the same way. We assume that G is 6-minimal, remove one vertex v_1 from G and add some new edges in such a way that the distance of any pair u and v of vertices in the new graph G' is less or equal than the distance between u and v in G in such a way that G' is still planar with maximum degree at most six. By the 6-minimality of G, G' has an $L(2,1)$-labeling c'. By counting the number of labels that cannot be assigned to the vertex v because they are assigned to some of the vertices of $N_G(v,2)$, we establish that the $L(2,1)$-labeling c' can be extended to G. This contradicts the 6-minimality of G.

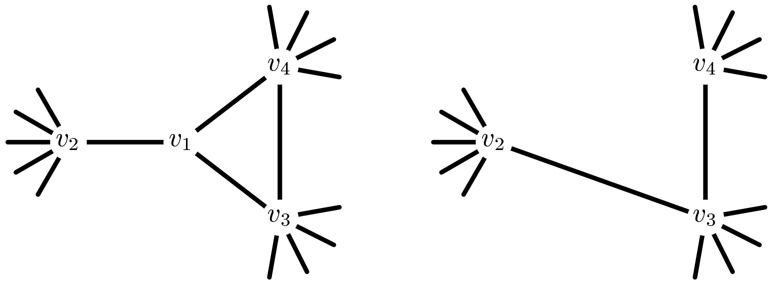

Figure 2: The configuration from Lemma 4.2 and its replacement.

Lemma 4.2. *No 6-minimal graph G contains a 3-vertex incident to a 3-face.*

Proof. Let v_1 be a 3-vertex of G incident to a 3-face, v_3 and v_4 the other two vertices of the 3-face, and v_2 the remaining neighbor of v_1 (see Figure 2). Remove v_1 from G and add the edge v_2v_3. By the 6-minimality of G, there exists an $L(2,1)$-labeling c' of the graph G' with span at most 32.

A similar calculation as in Lemma 4.1 yields that there are at most $3 \cdot (3+5) = 24$ labels that cannot be assigned to v_1. Since the distances among vertices in G' are less or equal than in G, c' can be extended to an $L(2,1)$-labeling of the entire graph G. $\square$

Lemma 4.3. *No 6-minimal graph G contains a 3-vertex incident with two 4-faces.*

Proof. Let v_1 be a 3-vertex incident with two 4-faces, v_4 the neighbor of v_1 incident with the two 4 faces, and v_2 and v_6 the remaining two neighbors of v_1 (see Figure 3). Remove v_1 from G and add the edge v_6v_2. The resulting graph G' an $L(2,1)$-labeling c' with span at most 32 by the 6-minimality of G. Since the number of labels that cannot be assigned to v_1 is at most $3 \cdot (3+5) = 24$, the labeling c' can be extended to the entire graph G. $\square$

Lemma 4.4. *If G contains a 3-vertex adjacent to a ≤ 5-vertex, then G is not 6-minimal.*

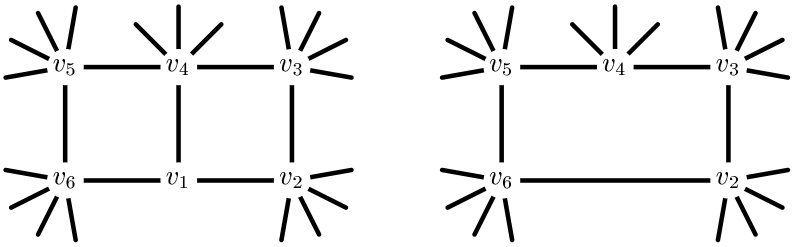

Figure 3: The configuration from Lemma 4.3 and its replacement.

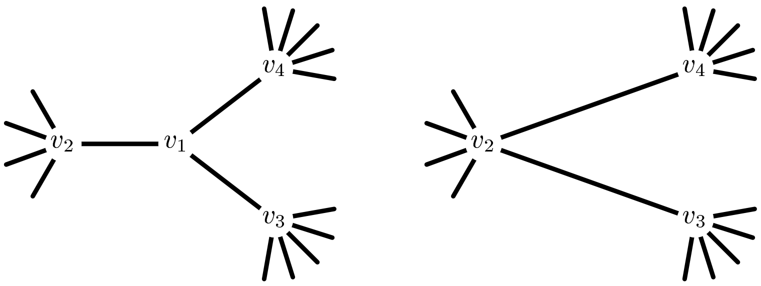

Figure 4: The configuration from Lemma 4.4 and its replacement.

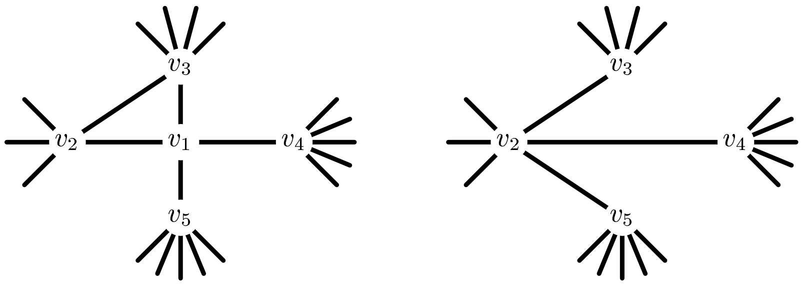

Figure 5: The configuration from Lemma 4.5 and its replacement.

Proof. Let v_1 be a 3-vertex of G, v_2 a ≤ 5-neighbor of v_1, and v_3 and v_4 the remaining neighbors of v_1 (see Figure 4). Remove v_1 and add two new edges: v_2v_3 and v_2v_4. Let G' be the obtained graph. Since v_2 is a ≤ 5-vertex in G, the degree of v_2 in G' does not exceed six. Hence, G' has an $L(2,1)$-labeling c' with span at most 32 by the 6-minimality of G. Since the number of labels that cannot be assigned to v_1 is at most $3 \cdot (3+5) = 24$, the labeling c' can be extended to an $L(2,1)$-labeling of the graph G. $\square$

Lemma 4.5. *No 6-minimal graph G contains a 4-vertex incident with a 3-face that contains another ≤ 5-vertex.*

Proof. Let v_1 be a 4-vertex of G, v_2 a ≤ 5-vertex adjacent to v_1 contained in the 3-face incident to v_1, and v_3 be the remaining vertex of the 3-face. Also let v_4 and v_5 be the remaining neighbors of v_1 (see Figure 5). Remove v_1 and add new edges v_2v_4 and v_2v_5. Since the obtained graph G' is planar and its maximum degree is at most 6, it has an $L(2,1)$-labeling c' with span at most 32.

The vertex v_1 cannot be assigned at most $4 \cdot 3 + 5 + 5 + 4 + 4 = 30$ labels because of distance constraints. Hence, the $L(2,1)$-labeling c' can be extended to G. $\square$

Lemma 4.6. *No 6-minimal graph G contains a $(3,3,\geq 3,\geq 3)$-vertex (see Figure 6).*

Proof. Let v_1 be a 4-vertex of the graph G incident to two adjacent 3-faces, v_2 the neighbor of v_1 incident to both the 3-faces and v_3 and v_5 the two

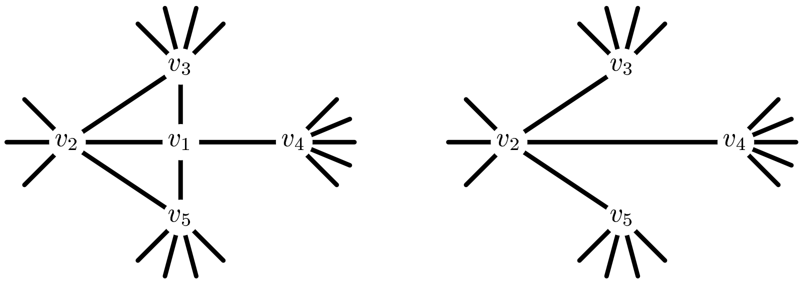

Figure 6: The configuration from the Lemma 4.6 and its replacement.

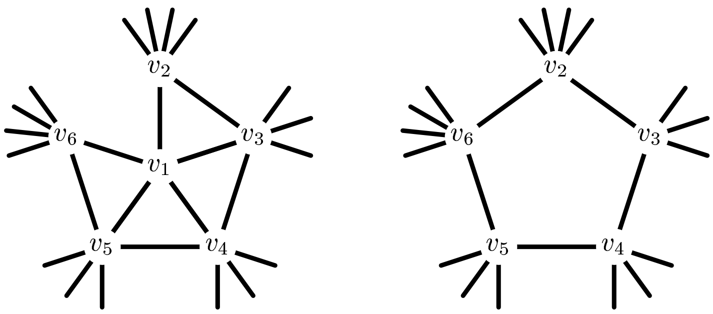

Figure 7: The configuration from Lemma 4.7 and its replacement.

neighbors of v_1 incident to one of the 3-faces. Finally, let v_4 be the remaining neighbor of v_1. As before, we remove the vertex v_1 and add an edge v_2v_4. Since the obtained graph G' is planar and its maximum degree is at most 6, there exists an $L(2,1)$-labeling c' of G' with span at most 32.

The number of labels that cannot be assigned to the vertex v_1 is $4 \cdot 3 + (3 + 4 + 5 + 4) = 28$. Since the distances in G' are less or equal than the distances between the corresponding vertices in G, we can extend the labeling c' to a proper labeling of the graph G. $\square$

Lemma 4.7. *No* 6*-minimal graph* G *contains a* 5*-vertex incident to four* 3*-faces (see Figure 7).*

Proof. Let v_1 be a 5-vertex of the graph G incident to four 3-faces and $v_2, \ldots, v_6$ the neighbors of v_1 as drawn in Figure 7. Remove the vertex v_1 and add an edge v_2v_6. Since the resulting graph G' is planar and its maximum degree is at most 6, there exists an $L(2,1)$-labeling c' of G' with span at most 32. Since the number of labels that cannot be assigned to v_1 is at most $5 \cdot 3 + (4+3+3+3+4) = 32 < 33$, the labeling c' can be extended to an $L(2,1)$-labeling of G with span at most 32. $\square$

4.2 Discharging Procedure

In this section, we describe the amount of initial charge of the vertices and the faces, the rules used for redistributing charge and show that after applying the rules to a 6-minimal graph, the final amount of charge of each vertex and of each face is non-negative. Since the sum of the amounts of initial charge assigned to the vertices and faces is negative, there is no 6-minimal graph that yields the desired result.

Theorem 4.8. *Every planar graph G with maximum degree at most 6 has an $L(2,1)$-labeling with span at most 32, i.e., there is no 6-minimal graph.*

Proof. Assume that there exists a 6-minimal graph G. Clearly, G is connected and has at least two vertices. Assign each vertex v $\deg_G(v) - 4$ units of charge and each face f $\deg_G(f) - 4$ units of charge. It is easy to verify that the sum of initial charge of all the vertices and faces is negative (-8).

Initial charge is redistributed based on the following set of rules:

Rule A Each ≥ 5-face sends each incident 3-vertex $1/2$ units of charge.

Rule B Each ≥ 5-vertex sends each incident 3-face $1/3$ units of charge.

Rule C If uv is and edge contained in a 3-face f and ≥ 4-face and v is a 6-vertex, then v sends $1/6$ units of charge to f.

Note that the total amount of charge is preserved.

We determine the final amount of charge of each vertex and face. First, we verify that each vertex has non-negative final charge. Observe that G contains only ≥ 3-vertices by Lemma 4.1. Let us consider a 3-vertex v. By Lemmas 4.2 and 4.3, v is not incident to a 3-face or two 4-faces. Hence, v is contained in at least two ≥ 5-faces. Each of the incident ≥ 5-faces sends $1/2$ units of charge to v and v receives altogether at least one unit of charge and its final charge is non-negative.

Since a 4-vertex neither receives nor sends out any charge, its final charge is zero. Each 5-vertex v of G is contained in at most 3-faces by Lemma 4.7. Hence, it sends out at most unit charge by Rule B in total. It remains to analyze final charge of 6-vertices. Each 6-vertex v has initial charge 2. Let t be the number of 3-faces that contains v. Observe that Rule C can apply at most $2 \cdot (6-t)$ times. We infer that v sends out at most $t/3$ units of charge by Rule B and at most $2(6-t)/6 = (6-t)/3$ units of charge by Rule C. We conclude that the final amount of charge of v is non-negative.

We have verified that the final amount of charge of each vertex is non-negative. Next, we consider final charge of the faces of G.

3-faces: Consider a 3-face f of G. Lemma 4.2 implies that f is incident with no 3-vertex. By Lemma 4.5, the face f is incident with either one 4-vertex and two 6-vertices or three $\geq$5-vertices.

- In the former case, let u be the 4-vertex incident to f and v_1 and v_2 the remaining 6-vertices incident to it. The face f receives charge of 1/3 units of charge from each of the vertices v_1 and v_2 by Rule B. By Lemma 4.6, the 4-vertex u is not a $(3,3,\geq 3,\geq 3)$-vertex. In particular, the other face containing the edge uv_1 is a $\geq$4-face. Hence, v_1 sends additional 1/6 units of charge to f by Rule C. Similarly, v_2 sends additional 1/6 units of charge. We conclude that final charge of f is zero.
- In the latter case, each of the three incident vertices sends f 1/3 units of charge by Rule B.

We conclude that the final amount of charge of f is at least 0.

4-faces: Since no 4-face receives or sends out any charge, its final charge is zero.

$\geq$5-faces: Consider a $\geq$5-face f. By Lemma 4.4, G contains no adjacent 3-vertices. Hence, f is incident with at most $\lfloor k/2 \rfloor$ 3-vertices. In particular, it sends at most $\lfloor k/2 \rfloor \cdot 1/2$ units of charge by Rule A. We conclude that final charge of f is non-negative.

Since the final amount of charge of each vertex and each face of G is non-negative, we obtain a contradiction and conclude that there is no 6-minimal graph. $\square$

5 Planar graphs with maximum degree five

Throughout this section, we say that a planar graph G is 5-*minimal* if G has maximum degree five, $\lambda_{2,1}(G) > 25$ and every planar graph with maximum degree five and with fewer vertices, or with the same number of vertices but fewer edges has an $L(2,1)$-labeling with span at most 25. When applying Lemmas 3.5 or 3.6 for a degree configuration H (with at least one edge) that appears in G via a degree homomorphism φ, we first remove from G one edge contained in the image of H. By the 5-minimality of G, the resulting graph has an $L(2,1)$-labeling with span at most 25. It can be verified that any two vertices that are not contained in $\varphi(V(H))$ and that are at distance two in G receive different labels. We then verify other conditions of Lemmas 3.5 and 3.6. In the rest of this section, we do further not emphasize in which way we obtain the $L(2,1)$-labeling of $G\varphi(V(H))$ and kindly ask the reader to recall the just described procedure.

As in the previous section, all our arguments translate to the list labelings but we leave the reader to verify the details.

5.1 Reducible Configurations

We first establish a simple lower bound on the sum of degrees of adjacent vertices in a 5-minimal graph.

Lemma 5.1. *No 5-minimal graph G contains adjacent vertices v_1 and v_2 with degrees d_1 and d_2 such that $d_1 + d_2 \leq 7$. In particular, the minimum degree of G is at least three.*

Proof. By symmetry, we can assume that $d_1 \leq d_2$ and thus $d_1 \leq 3$. Contract the edge v_1v_2 to a vertex w. Since the obtained graph is a planar graph with maximum degree at most five, it has an $L(2,1)$-labeling of span at most 25 by the 5-minimality of G. Assign the vertex v_2 the label of w. The remaining vertices keep their labels. Since there are at most $d_1 \cdot 7 \leq 21$ labels that cannot be assigned to v_1, the $L(2,1)$-labeling can be extended to v_1. $\square$

In the next two lemmas, we study the structure of neighborhoods of 3-vertices in 5-minimal graphs:

Lemma 5.2. *If v is a 3-vertex contained in a 5-minimal graph G, then v is $(\geq 4, \geq 5, \geq 5)$-vertex and all the neighbors of v are 5-vertices.*

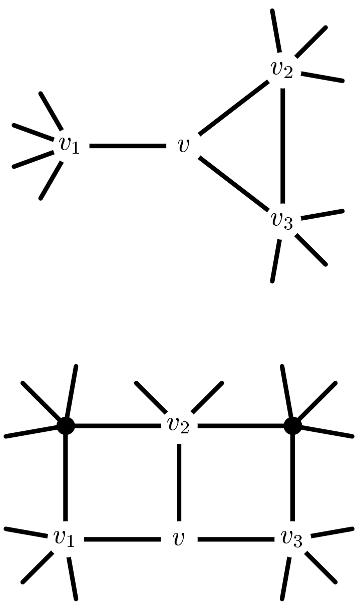

Figure 8: The configurations from Lemma 5.2.

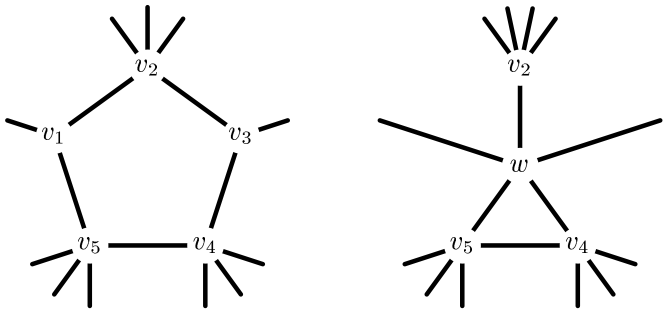

Figure 9: The configuration from Lemma 5.3 and its replacement.

Proof. All the neighbors of v are 5-vertices by Lemma 5.1. The statement of the lemma is violated if v is either a $(3, \geq 3, \geq 3)$-vertex or a $(4, 4, \geq 4)$-vertex. Let v_1, v_2 and v_3 be the neighbors of v as drawn if Figure 8. Let G' be the graph obtained from $G \setminus v$ by adding the edge v_1v_3. By the 5-minimality of G, G' has an $L(2,1)$-labeling of span at most 25. Label the vertices of G with the labels of their counterparts. Since there are at most $3 \cdot 7 = 21$ labels that cannot be assigned to v, the $L(2,1)$-labeling can be extended to v. □

Lemma 5.3. *If G is a 5-minimal graph, then each 5-face f of G is incident with at most one 3-vertex.*

Proof. Let $v_1, \ldots, v_5$ be the vertices incident with f in the cyclic order along f. Assume that v_1 and v_3 are 3-vertices. Note that by Lemma 5.1, no 3-vertices can be adjacent. Add the edge v_1v_3 to G and contract it. Let G' be the obtained graph and w the new vertex (see Figure 9). By the 5-minimality of G, the graph G' has an $L(2,1)$-labeling with span at most 25. We keep the labels of all the vertices of G' except for w to get a labeling of $G \setminus v_1, v_3$. Since each of the vertices v_1 and v_3 cannot be assigned at most $3 \cdot 7 = 21$ labels, this labeling can be extended to v_1 and v_3 assigning v_1 and v_3 distinct colors. We conclude that G has an $L(2,1)$-labeling with span at most 25. □

Next, we focus on the structure around 3-faces.

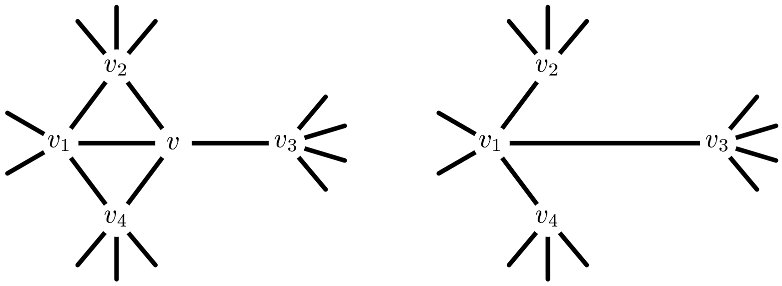

Figure 10: The configuration from Lemma 5.5 and its reduction.

Lemma 5.4. *Each* 3*-face of a* 5*-minimal graph* G *is incident with at most one* ≤ 4*-vertex.*

Proof. G does not contain a 3-vertex incident with a 3-face by Lemma 5.2. Assume that G contains two 4-vertices v_1 and v_2 that are both incident with the same 3-face of G. By the 5-minimality of G, the graph G without the edge v_1v_2 has an $L(2,1)$-labeling with span at most 25. Remove now the labels of the vertices v_1 and v_2. We claim that the labeling can be extended to the vertices v_1 and v_2 in the original graph G. Once the vertex v_2 is assigned a label, the neighbors of v_1 prevent v_1 from assigning at most $7+7+6+5=25$ labels. We obtain an $L(2,1)$-labeling of G with span at most 25. □

Lemma 5.5. *A* 5*-minimal graph* G *does not contain a* $(3,3,\geq 3,\geq 3)$*-vertex.*

Proof. Assume the opposite and let v be a $(3,3,\geq 3,\geq 3)$-vertex of G and v_1, $\ldots$, v_4 its neighbors as depicted in Figure 10. Remove the vertex v from G and add an edge v_1v_3. Let G' be the resulting graph. By the 5-minimality of G, G' has an $L(2,1)$-labeling with span at most 25. We claim that the labeling can be extended to the vertex v: the neighbors and the vertices at distance two prevent v from assigning at most $7+2\cdot 6+5=24$ labels. Since the labels of the vertices v_1, v_2, v_3 and v_4 are also mutually distinct, we have obtained an $L(2,1)$-labeling of G with span at most 25. □

In the proof of the next lemma, we first apply one of our reduction lemmas proven in Section 3:

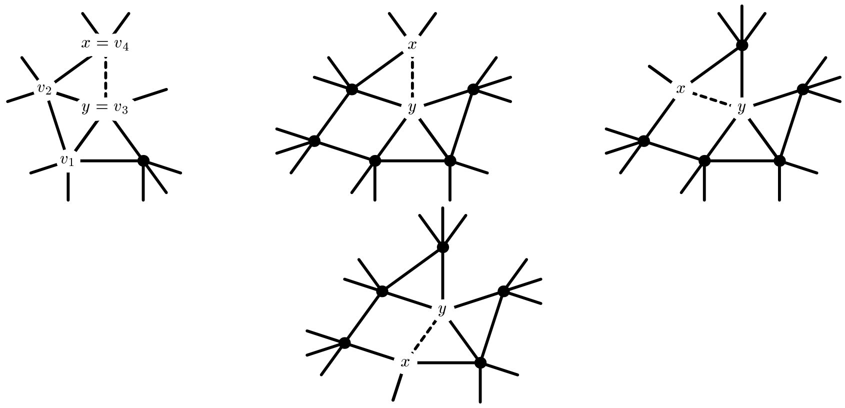

Figure 11: The degree configurations from Lemma 5.6.

Lemma 5.6. *Let G be a 5-minimal graph and xy an edge of G contained in a 3-face. If x is a 4-vertex, then y is neither a $(3,3,3,\geq 3,\geq 3)$-vertex nor a $(3,4,3,3,\geq 3)$-vertex.*

Proof. By Lemma 5.5, the vertex x incident with at most one 3-face. If y is a $(3,3,3,\geq 3,\geq 3)$-vertex or a $(3,4,3,3,\geq 3)$-vertex, then G contains one of the four degree configurations depicted in Figure 11. Let $W = \{v_1, v_2, v_3, v_4\}$ if y is a $(3,3,3,\geq 3,\geq 3)$-vertex, and let $W = \{x, y\}$, otherwise.

We consider an $L(2,1)$-labeling of the graph G with the edge xy removed (as explained in the beginning of this section) and verify the conditions of Lemma 3.6. In the case of the first configuration, the four inequalities are the following:

$$\begin{array}{lcl}
v_1\colon\ 7\cdot 3 - 2 + (2+2) + 2 + 0 & = & 25 \leq 25 \\
v_2\colon\ 7\cdot 2 - 0 + (3+2+2) + 3 + 1 & = & 25 \leq 25 \\
v_3\colon\ 7\cdot 2 - 2 + (3+2+2) + 3 + 2 & = & 24 \leq 25 \\
v_4\colon\ 7\cdot 2 - 0 + (2+2) + 2 + 3 & = & 23 \leq 25
\end{array}$$

The two inequalities for the second configuration are the following:

$$\begin{array}{lcl}
x\colon\ 7\cdot 4 - 3\cdot 2 - 1 + 3 + 1 + 0 & = & 25 \leq 25 \\
y\colon\ 7\cdot 3 - 2 + 4 + 1 + 1 & = & 25 \leq 25
\end{array}$$

In the third case and the fourth cases, the two inequalities are the same and they are the following:

$$\begin{array}{lcl} x\colon\ 7\cdot 4-3\cdot 2-1+3+1+0 & = & 25\leq 25 \\ y\colon\ 7\cdot 3-2-1+4+1+1 & = & 24\leq 25 \end{array}$$

In all the cases, Lemma 3.6 implies that the graph G is not 5-minimal. □

In the two final lemmas of this subsection, we study 5-vertices that are incident to four or five 3-faces:

Lemma 5.7. *No* 5*-minimal graph* G *contains a* $(3,3,3,3,3)$*-vertex* v*.*

Proof. By the 5-minimality of G, $G\backslash v$ has an $L(2,1)$-labeling of span at most 25. This labeling can be extended to v since there are at most $5\cdot 3+10=25$ labels that cannot be assigned to v. Observe that the labels of all the neighbors of v are different. Hence, we obtain a contradiction with our assumption that G is 5-minimal. □

Lemma 5.8. *Let* G *be a* 5*-minimal graph and* v *a* $(3,3,3,3,\geq 3)$*-vertex of* G*. If* f' *is a face that is not incident with* v *and that shares an edge with a* 3*-face incident with* v*, then* f' *is a* $\geq$5*-face.*

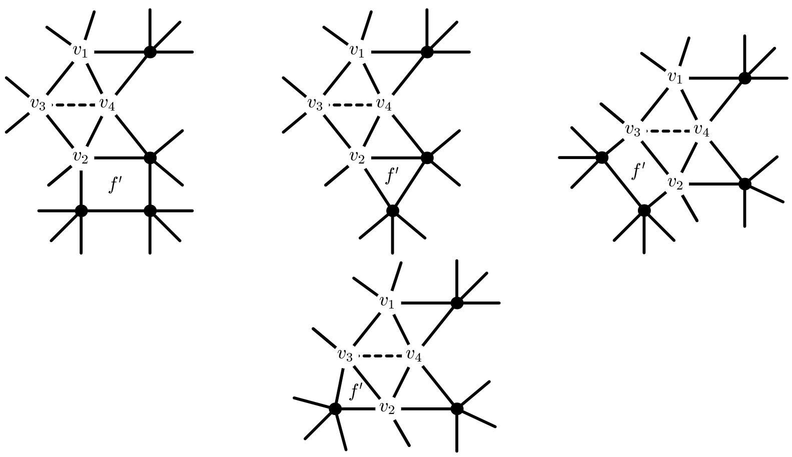

Figure 12: The degree configurations from Lemma 5.8.

Proof. If f' is a 3-face or 4-face, then G contains one of the four degree configurations shown in Figure 12 where $v_4 = v$. Let $W = \{v_1, v_2, v_3, v_4\}$. As explained in the beginning of this section, we remove one edge of G, say v_3v_4, consider an $L(2,1)$-labeling of the resulting graph and apply Lemma 3.6. If f' is 4-face and it is not incident with v_3, then the inequalities that we have to verify are the following:

$$\begin{array}{lcl} v_1\colon\ 7\cdot 3 - 2 + (2+2) + 2 + 0 & = & 25 \\ v_2\colon\ 7\cdot 3 - 3 + (2+2) + 2 + 1 & = & 25 \\ v_3\colon\ 7\cdot 2 - 0 + (3+3+2-2) + 3 + 2 & = & 25 \\ v_4\colon\ 7\cdot 2 - 4 + (3+3+2) + 3 + 3 & = & 24 \end{array}$$

If f' is a 3-face, then we should subtract -2 instead of -1 in the inequality for v_2. Similarly, if f' is incident with v_3, then we should subtract -1 or -2, depending whether f' is a 3-face or a 4-face, in the inequality for v_3. In all the cases, the conditions of Lemma 3.6 are satisfied and G is not a 5-minimal graph. $\square$

5.2 Discharging Procedure

Assign each vertex v $\deg_G(v)-4$ units of charge and each face f $\deg_G(f)-4$ units of charge, It is easy to verify that the sum of initial charge of all the vertices and faces is negative (-8) if G is a connected planar graph. The charge assigned to the vertices and faces is redistributed by the following rules:

Rule F.1 Each $\geq$5-face sends each incident 3-vertex 1/2 unit of charge.

Rule F.2 Each $\geq$5-face f sends 1/6 units of charge to each 3-face f' such that the faces f and f' share an edge.

Rule V.1 Each 5-vertex incident with at most two 3-faces sends 1/2 units of charge to each incident 3-face.

Rule V.2 Each 5-vertex incident with precisely three 3-faces sends 1/3 units of charge to each incident 3-face.

Rule V.3 Each $(3,3,3,3,\geq 4)$-vertex v incident with the 3-faces f_1, f_2, f_3 and f_4 (in the order around v as depicted in Figure 13) sends 1/3 units of charge to each of f_1 and f_4, and 1/6 units of charge to each of the faces f_2 and f_3.

In the next four lemmas, we analyze the final charge of vertices and faces.

Lemma 5.9. *If G is a 5-minimal graph, then the final charge of each vertex is non-negative.*

Proof. If v is a 3-vertex, then it is incident with at least two ≥ 5-faces by Lemma 5.2. Each of these faces sends $1/2$ units of charge to v by Rule F.1 and the final charge of v is non-negative. If v is a 4-vertex, then it does not send out or receive any charge and thus its final charge is zero. If v is a 5-vertex, then either Rule V.1, V.2 or V.3 applies. In each of the cases, it sends out at most one unit of charge in total and thus its final charge is non-negative. □

Lemma 5.10. *If G is a 5-minimal graph, then the final charge of any 3-face f incident with a 4-vertex v is non-negative.*

Proof. By Lemma 5.4, the remaining two vertices v_1 and v_2 incident with f are 5-vertices. For each $i = 1, 2$, we claim that f receives from v_i and from the other face f_i incident with the edge vv_i $1/2$ units of charge in total. By Lemma 5.6, the vertex v_i is either incident with at most two 3-faces or it is incident with three 3-faces and f_i is ≥ 5-face. In the former case, the vertex v_i sends f $1/2$ units of charge by Rule V.1. In the latter case, v_i sends f $1/3$ units of charge by Rule V.2 and f_i sends f $1/6$ units of charge. We conclude that f receives $1/2$ units of charge from v_i and f_i. In total, f receives at least one unit of charge and its final charge is non-negative. □

Lemma 5.11. *If G is a 5-minimal graph, then the final charge of any 3-face f is non-negative.*

Proof. The face f cannot be incident with a 3-vertex by Lemma 5.2. If f is incident with a 4-vertex, then its final charge is non-negative by Lemma 5.10. Hence, f is incident only with 5-vertices. Note that G does not contain a $(3,3,3,3,3)$-vertex by Lemma 5.7. If f is not incident with a $(3,3,3,3,\geq 4)$-vertex, then it receives charge of at least $1/3$ from each of the three incident 5-vertices by Rules V.1 and V.2. Hence, its final charge is non-negative.

It remains to consider the case when the face f is incident with a $(3,3,3,3,\geq 4)$-vertex v. Let f_1, f_2, f_3 and f_4 be the 3-faces incident with f in the cyclic order around v and v_1, v_2, v_3, v_4 and v_5 be the neighbors of v (see Figure 13). By Lemma 5.8, each of the other four faces that contain

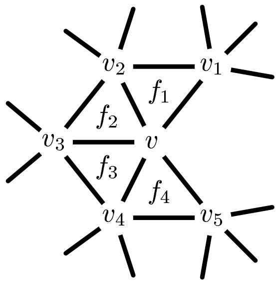

Figure 13: The notation used in the definition of Rule V.3 and in the proof of Lemma 5.11.

one of the edges v_1v_2, v_2v_3, v_3v_4 and v_4v_5 is a $\geq$5-face. In particular, each of the vertices v_2, v_3 and v_4 is incident with at most three 3-faces.

By symmetry, we can assume that the face f is either f_1 or f_2. If $f = f_1$, then f receives charge of $1/3$ units from v by Rule V.3, at least $1/6$ from v_1 (by Rule V.1, V.2 or V.3), at least $1/3$ from v_2 (by Rule V.1 or V.2) and $1/6$ from the $\geq$5-face containing the edge v_1v_2 by Rule F.2. If $f = f_2$, then f receives charge of $1/6$ units from v by Rule V.3, at least $1/3$ from each of the vertices v_2 and v_3 (by Rule V.1 or V.2), and charge of $1/6$ units from the $\geq$5-face containing the edge v_2v_3 by Rule F.2. We conclude that f receives at least one unit of charge and its final charge is non-negative. $\square$

We finish the analysis of final charge of faces:

Lemma 5.12. *If G is a 5-minimal graph, then the final charge of each face f of G is non-negative.*

Proof. If f is a 3-face, its final charge is non-negative by Lemma 5.11. If it is a 4-face, it neither receives nor sends out any charge and its final charge is zero. We now consider the case that f is a 5-face. By Lemma 5.3, f is incident with at most one 3-vertex. If f is incident with no 3-vertices, then it sends out at most $5/6$ units of charge by Rule F.2. If f is incident with one 3-vertex, then it shares an edge with at most three 3-faces (note that a 3-vertex cannot be incident to a 3-face by Lemma 5.2). Hence, f sends out $1/2$ units of charge by Rule F.1 and at most $1/2$ units of charge in total by Rule F.2.

It remains to consider the case that f is an ℓ-face with $\ell \geq 6$. The initial amount of charge of f is $\ell - 4$. Let k be the number of 3-vertices incident with f. Since no two 3-vertices of G can be adjacent by Lemma 5.1, $k \leq \ell/2$. Let k' be the number of 3-faces that share an edge with f. Since no 3-face can be incident with a 3-vertex by Lemma 5.2, $k' \leq \ell - 2k$. By Rules F.1 and F.2, the face f sends out the following amount of charge:

$$k/2 + k'/6 \leq k/2 + \frac{\ell - 2k}{6} = \ell/6 + k/6 \leq \ell/6 + \ell/12 = \ell/4 \; .$$

Since $\ell \geq 6$, charge sent out of the face f does not exceed $\ell - 4$ and the final charge of f is non-negative. □

We immediately infer from Lemmas 5.9 and 5.12 that the following holds:

Theorem 5.13. *Every planar graph G with maximum degree five has an $L(2,1)$-labeling with span at most* 25.

6 Planar graphs with maximum degree four

As in the previous two section, we prove our theorem for ordinary $L(2,1)$-labelings and the reader is welcomed yourself to verify that the same proof applies for list $L(2,1)$-labelings. We say that a graph G is 5-minimal if G is a planar graph with maximum degree at most four and $\lambda_{2,1}(G) > 16$ and every planar graph G' with maximum degree at most four and with fewer vertices than G has an $L(2,1)$-labeling with span at most 16. Two special types of 4-vertices and 3-faces in 5-minimal graphs will require our special attention. A 4-vertex is *red* if it is a $(3,4,\leq 4,4)$-vertex, and it is *blue* if it is a $(3,4,3,\geq 5)$-vertex. A 3-face is *red* if it is incident with a red vertex and it is *blue* if it is incident with a blue vertex. We later show that no 3-face is both red and blue (Lemma 6.7).

Before we proceed with showing that some (degree) configurations cannot appear in a 4-minimal graph, we restate Lemmas 3.5 and 3.6 to the case of 4-minimal graphs. Note that in both Lemmas 6.1 and 6.2, the assumption on the existence of an $L(2,1)$-labeling of $G \setminus \varphi(V(H))$ that assigns different labels to vertices of $G \setminus \varphi(V(H))$ at distance at most two in G is dismissed. This allows us more straightforward applications of both the lemmas in our proofs since we do not have to construct a suitable graph (smaller than G) to apply induction for in each of our proofs separately.

Lemma 6.1. *Let (H, d) be a degree configuration and $V(H) = \{v_1, \ldots, v_n\}$. Assume that the following holds for every $i = 1, \ldots, n$:*

$$\begin{aligned}6(d(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (d(v_j) - \deg_H(v_j)) +& \\ \deg_H(v_i) + |\{v_1, \ldots, v_{i-1}\} \cap N_H(v_i, 2)| \quad &\leq \quad 16 .\end{aligned}$$

If there is a degree homomorphism from H to a graph G, then the graph G is not 4-minimal.

Proof. Let φ be the degree homomorphism from H to a 4-minimal graph G. Note that each vertex $v \in \varphi(V(H))$ has at most two neighbors not contained in $\varphi(H)$ since $d(v_i) - \deg_H(v_i) \leq 2$ by the condition of the lemma. Let G' be the graph obtained from G by removing the vertices of the set $\varphi(V(H))$, and adding an edge $v'v''$ for each vertex $v \in \varphi(V(H))$ with two neighbors $v', v'' \notin \varphi(V(H))$. Observe that this does not increase the degree of the vertices v' and v''. In particular, G' is a planar graph with maximum degree four. By the 4-minimality of G, G' has an $L(2,1)$-labeling with span at most 16. Since any two vertices contained in G' with distance at most two in G are at distance at most two in G', G is not 4-minimal by Lemma 3.5. $\square$

Lemma 6.2. *Let G be a plane graph with maximum degree four and (H_0, d) a plane degree configuration, i.e., H_0 is a plane graph. Let $H \subseteq H_0$ and $V(H) = \{v_1, \ldots, v_n\}$. Assume that the following holds for every $i = 1, \ldots, n$:*

$$\begin{aligned}6(\deg_G(v_i) - \deg_H(v_i)) + \sum_{v_j \in N_H(v_i)} (\deg_G(v_j) - \deg_H(v_j)) +& \\ \deg_H(v_i) + |\{v_1, \ldots, v_{i-1}\} \cap N_H(v_i, 2)| - 2\alpha_i - \beta_i \quad &\leq \quad 16 ,\end{aligned}$$

where α_i is the number 3-faces that contains v_i, are contained in H but not in H_0, and β_i is the number of such 4-faces. Assume in addition that $\deg_G(v_i) - \deg_H(v_i) \leq 2$. If φ is a degree homomorphism from H_0 to G, then G is not 4-minimal.

The proof of Lemma 6.2 is analogous to the proof of Lemma 6.1. We apply Lemma 3.6 instead of Lemma 3.5 and use the 3-faces and 4-faces contained in H but not in H_0 to estimate the numbers α_i and β_i from the statement of Lemma 3.6. We leave further details to the reader.

6.1 Reducible configurations

In this subsection, we identify substructures that cannot appear in a 4-minimal graph.

Lemma 6.3. *The minimum degree of a* 4*-minimal graph* G *is at least three.*

Proof. Assume that G contains a ≤ 2-vertex v_1 and let v_2 be any of its neighbors. Contract the edge v_1v_2 in G to a vertex w. By the 4-minimality of G, the obtained graph has an $L(2,1)$-labeling of span at most 16. Assign the vertex v_2 the label of w and let the remaining vertices keep their labels. Since there are at most $2 \cdot 6 = 12$ labels that cannot be assigned to v_1, the $L(2,1)$-labeling can be extended to v_1. □

In the next two lemmas, we focus on 3-vertices in 4-minimal graphs:

Lemma 6.4. *No* 4*-minimal graph* G *contains two adjacent* 3*-vertices.*

Proof. Assume that G contains the degree configuration formed by an edge v_1v_2 with $d(v_1) = d(v_2) = 3$. We aim to apply Lemma 6.1 and verify the two inequalities of the statement of the lemma:

$$\begin{aligned} v_1\colon\ 6 \cdot 2 + 2 + 1 + 0 &= 15 \leq 16 \\ v_2\colon\ 6 \cdot 2 + 2 + 1 + 1 &= 16 \leq 16 \end{aligned}$$

We conclude that this degree configuration cannot appear in a 4-minimal graph. □

Lemma 6.5. *If* v *is a* 3*-vertex of a* 4*-minimal graph* G*, then* v *is a* $(\geq 4, \geq 5, \geq 5)$*-vertex.*

Proof. All the neighbors of v are 4-vertices by Lemma 6.4. The statement of the lemma is violated if v is either a $(3, \geq 3, \geq 3)$-vertex or a $(4, 4, \geq 4)$-vertex. Let v_1, v_2 and v_3 be the neighbors of v as drawn if Figure 14. Let G' be the graph obtained from $G \setminus v$ by adding an edge v_1v_3. Observe that G' is a graph with maximum degree at most four. By the 4-minimality of G, G' has an $L(2,1)$-labeling of span at most 16. Label the vertices of G as their counterparts are labeled in G' and extend the labeling to v: since there are at most $3 \cdot 6 - 2 = 16$ labels that cannot be assigned to v, this is possible. Note that by the construction of G', all the vertices v_1, v_2 and v_3 have different labels. □

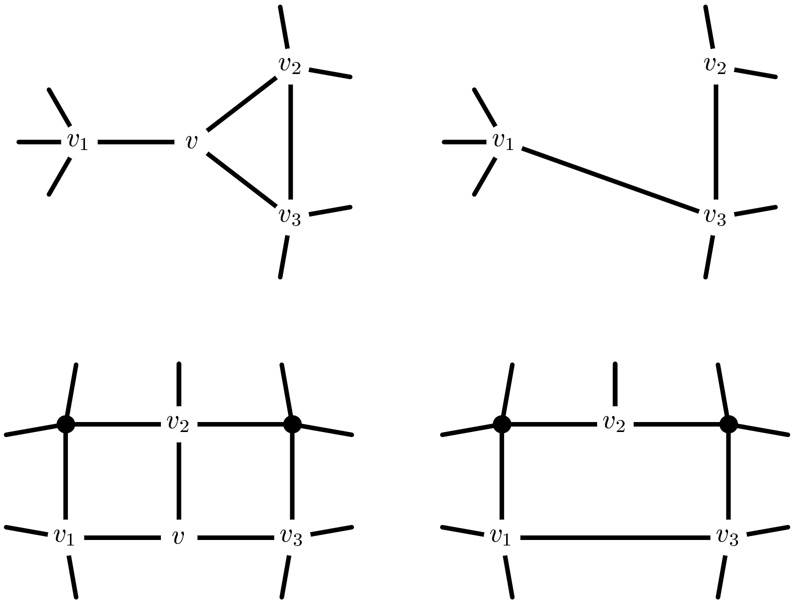

Figure 14: The configurations from Lemma 6.5 and their reductions.

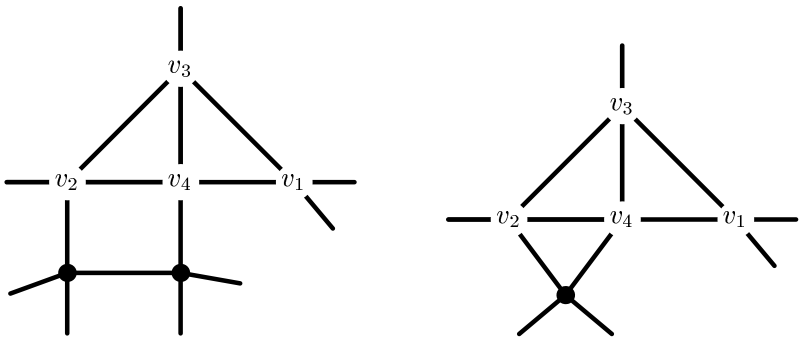

Figure 15: The degree configurations from the proof of Lemma 6.6.

The next lemma states that if two 3-faces share an edge in a 4-minimal graph, then any other face that shares an edge with either of them is a ≥ 5-face.

Lemma 6.6. *A* 4*-minimal graph does not contain a* $(3, 3, \leq 4, \geq 3)$*-vertex.*

Proof. Assume that a 4-minimal graph G contains a $(3, 3, \leq 4, \geq 3)$-vertex. It follows that G contains one of the two degree configurations depicted in Figure 15 with $d(v_i) = 4$ for $i = 1, 2, 3, 4$ (note that no 3-face can be incident with a 3-vertex by Lemma 6.5). We aim to apply Lemma 6.2. The inequalities for the degree configuration depicted in the left part of Figure 15 are the following:

$$\begin{aligned} v_1\colon\ 6 \cdot 2 - 0 + 2 + 2 + 0 &= 16 \leq 16 \\ v_2\colon\ 6 \cdot 2 - 1 + 2 + 2 + 1 &= 16 \leq 16 \\ v_3\colon\ 6 \cdot 1 - 0 + 5 + 3 + 2 &= 16 \leq 16 \\ v_4\colon\ 6 \cdot 1 - 1 + 5 + 3 + 3 &= 16 \leq 16 \end{aligned}$$

If the degree configuration in the right part appears in G, then there is -2 subtracted instead of -1 in the second and the fourth inequality. $\square$

In the next two lemmas, we turn our attention to red faces and red vertices:

Lemma 6.7. *If* f *is a red* 3*-face of a* 4*-minimal graph* G*, then* f *is incident with a red vertex* v *and two* $(3, \geq 4, \geq 4, \geq 4)$*-vertices. In particular, no red*

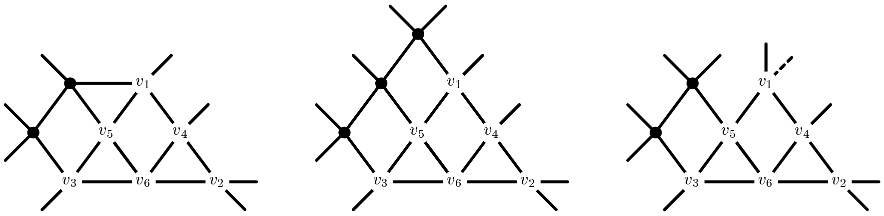

Figure 16: The degree configurations from the proof of Lemma 6.7.

face is incident with a blue vertex. In addition, if f is a blue 3-face, then it shares an edge with exactly one 4-face of G.

Proof. Assume that u is a red vertex incident with a face f, and that f is incident with a vertex $w \neq u$ such that w is incident with two 3-faces. By Lemma 6.6, w is a $(3, 4, 3, \geq 4)$-vertex and G contains one of the two degree configurations depicted in the left part of Figure 16 with $u = v_5$ and $w = v_6$. Note that $d(v_i) = 4$ for $i = 1, \ldots, 6$ since all the vertices $v_1, \ldots, v_6$ have degree four by Lemma 6.5.

If a blue 3-face shares an edge with two 4-faces, then the degree configuration depicted in the right part of Figure 16 appears in G. In this case, $d(v_i) = 4$ for $i = 2, \ldots, 6$ (by Lemma 6.5) and $d(v_1)$ is either 3 or 4.

We aim to apply Lemma 6.2. We verify the conditions for the rightmost degree configuration depicted in Figure 16 (with $d(v_1) = 4$):

$$\begin{aligned}
v_1 &: 6 \cdot 2 - 0 + 2 + 2 + 0 &= 16 \\
v_2 &: 6 \cdot 2 - 0 + 1 + 2 + 1 &= 16 \\
v_3 &: 6 \cdot 2 - 1 + 1 + 2 + 2 &= 16 \\
v_4 &: 6 \cdot 1 - 0 + 4 + 3 + 3 &= 16 \\
v_5 &: 6 \cdot 1 - 1 + 4 + 3 + 4 &= 16 \\
v_6 &: 6 \cdot 0 - 0 + 6 + 4 + 5 &= 15
\end{aligned}$$

The conditions get weaker in the remaining cases. $\square$

Lemma 6.8. *Each red 3-face f of a 4-minimal graph G is incident with exactly one red vertex.*

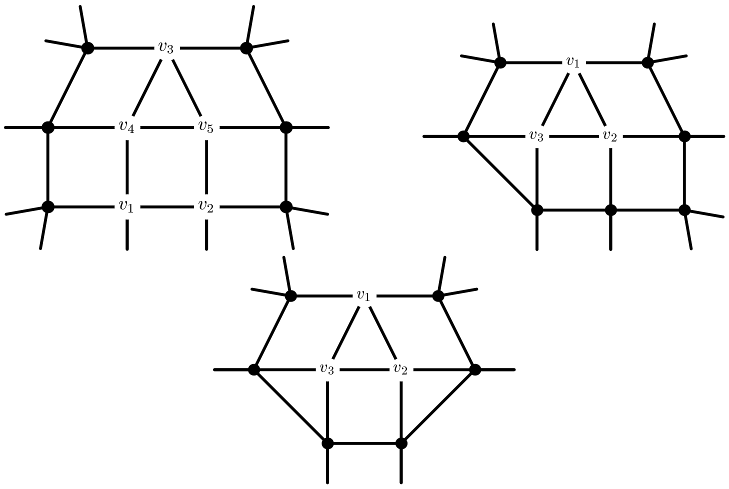

Figure 17: The degree configurations from the proof of Lemma 6.8.

Proof. If the lemma does not hold, then G contains one of the three degree configurations depicted in Figure 17. Note that $d(v_i) = 4$ for all v_i by Lemma 6.5. The middle configuration depicted in the figure cannot appear in G by Lemma 6.2:

$$\begin{array}{lcl} v_1\colon\ 6 \cdot 2 - 2 + 4 + 2 + 0 & = & 16 \leq 16 \\ v_2\colon\ 6 \cdot 2 - 3 + 4 + 2 + 1 & = & 16 \leq 16 \\ v_3\colon\ 6 \cdot 2 - 4 + 4 + 2 + 2 & = & 16 \leq 16 \end{array}$$

In the case of the rightmost configuration, we subtract -4 instead of -3 in the condition for v_2, thus Lemma 6.2 applies again.

If the degree configuration that is depicted leftmost in Figure 17 appears in G, we proceed as follows. First, observe that v_1, v_2, v_3, v_4 and v_5 are mutually distinct. If v_2 is adjacent to v_3, the degree configuration depicted in Figure 17 with the edge v_2v_3 added cannot be contained in a 4-minimal graph by Lemma 6.1 applied to the ordering v_1, v_2, v_3, v_4 and v_5 of the

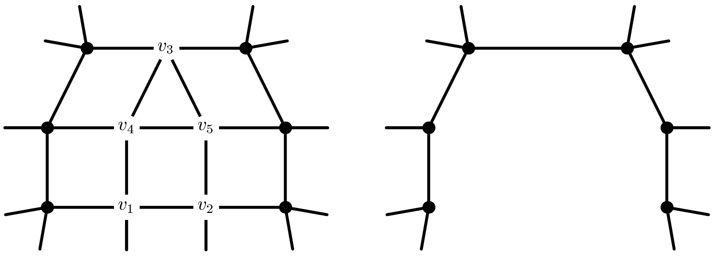

Figure 18: The replacement of the leftmost degree configuration from Figure 17 in the last case considered in the proof of Lemma 6.8.

vertices:

$$\begin{array}{lcl} v_1\colon\ 6\cdot 2+2+2+0 & = & 16\leq 16 \\ v_2\colon\ 6\cdot 1+4+3+1 & = & 14\leq 16 \\ v_3\colon\ 6\cdot 1+3+3+2 & = & 14\leq 16 \\ v_4\colon\ 6\cdot 1+4+3+3 & = & 16\leq 16 \\ v_5\colon\ 6\cdot 1+3+3+4 & = & 16\leq 16 \end{array}$$

Similarly, if v_2 and v_4 are adjacent, then the degree configuration depicted in Figure 17 with the added edge v_2v_4 cannot be contained in a 4-minimal graph by Lemma 6.1 applied to the order v_1, v_3, v_2, v_5, v_4:

$$\begin{array}{lcl} v_1\colon\ 6\cdot 2+1+2+0 & = & 15\leq 16 \\ v_3\colon\ 6\cdot 2+1+2+1 & = & 16\leq 16 \\ v_2\colon\ 6\cdot 1+3+3+2 & = & 14\leq 16 \\ v_5\colon\ 6\cdot 1+3+3+3 & = & 15\leq 16 \\ v_4\colon\ 6\cdot 0+6+4+4 & = & 14\leq 16 \end{array}$$

The cases when v_1 is adjacent to v_3 or v_5 are symmetric. Hence, we may assume in the rest that v_1 is adjacent to neither v_3 nor v_5 and v_2 to neither v_3 nor v_4.

Neither Lemma 6.1 nor Lemma 6.2 can be used to handle with this final case. A finer argument, using Lemma 3.7, is needed. We first remove the degree configuration from G and add the edges as shown in Figure 18. Let G' be the resulting planar graph. The graph G' has an $L(2,1)$-labeling of span at most 16 by the 4-minimality of G. Since any two vertices of $V(G')$ at distance at most two in G are also at distance at most two in G', it is enough to show that the $L(2,1)$-labeling of G' can be extended to the

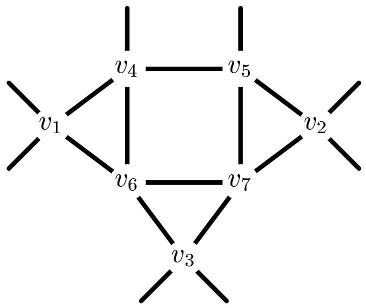

Figure 19: The degree configuration from the proof of Lemma 6.9.

vertices v_1, v_2, v_3, v_4 and v_5. It is easy to verify that there at least 3 labels that do not conflict with labels of the vertices at distance at most two from v_1. Similarly, there are 3 labels that can be assigned to v_2, 5 labels that can be assigned to v_3 and 8 labels that can be assigned to each of the vertices v_4 and v_5. By Lemma 3.7, the $L(2,1)$-labeling can be extended to the subgraph induced by v_1, v_2, v_3, v_4 and v_5 (the ordering of the vertices matches that of Lemma 3.7). We have obtained an $L(2,1)$-labeling of G with span at most 16 which contradicts our assumption that G is 4-minimal. $\square$

Similarly to the red faces, each blue face can be incident only with one blue vertex:

Lemma 6.9. *Each blue* 3*-face* f *of a* 4*-minimal graph* G *is incident with exactly one blue vertex.*

Proof. Assume the opposite. Since f shares an edge with at most one 4-face by Lemma 6.7, G must contain the degree configuration depicted in Figure 19. Note that $d(v_i) = 4$ for $i = 1, \dots, 7$ by Lemma 6.5. We now verify the conditions of Lemma 6.1 and eventually obtain a contradiction

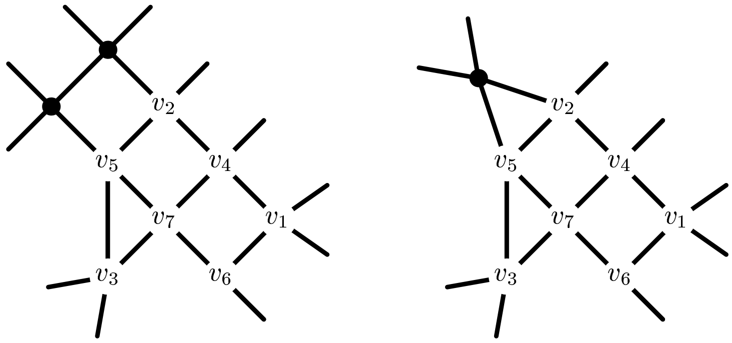

Figure 20: The degree configurations from Lemma 6.10.

with the assumption that G is 4-minimal:

$$\begin{aligned}
v_1\colon\ 6\cdot 2+1+2+0 &= 15\leq 16\\
v_2\colon\ 6\cdot 2+1+2+0 &= 15\leq 16\\
v_3\colon\ 6\cdot 2+0+2+2 &= 16\leq 16\\
v_4\colon\ 6\cdot 1+3+3+3 &= 15\leq 16\\
v_5\colon\ 6\cdot 1+3+3+4 &= 16\leq 16\\
v_6\colon\ 6\cdot 0+5+4+5 &= 14\leq 16\\
v_7\colon\ 6\cdot 0+5+4+6 &= 15\leq 16
\end{aligned}$$

□

We end this subsection with five auxiliary lemmas that we will need in the proof of Lemma 6.17. Each of the next five lemmas states that certain pairs of vertices cannot be adjacent on a boundary of a ≥ 5-face.

Lemma 6.10. *Let G be a 4-minimal graph and u and v be two vertices consecutive on a boundary of a ≥ 5-face. If u is a $(3,4,4,\geq 5)$-vertex contained in a red or a blue 3-face, then v is not a 3-vertex.*

Proof. Since a red face is incident with exactly one red vertex by Lemma 6.8, G contains one of the two degree configurations depicted in Figure 20. We assume that $d(v_i)=4$ for $i=1,2,3,4,5,7$ and $d(v_6)=3$. If $d(v_i)$ is also equal to 3 for $i\neq 6$, our arguments smoothly translate to this case, too. We

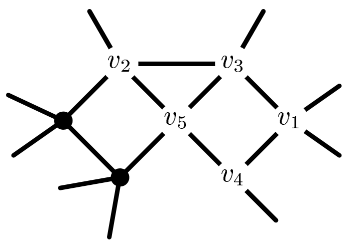

Figure 21: The degree configuration from Lemma 6.11.

apply Lemma 6.2:

$$
\begin{aligned}
v_1&\colon\ 6\cdot 2-0+2+2+0 &=& \quad 16\le 16\\
v_2&\colon\ 6\cdot 2-1+2+2+0 &=& \quad 15\le 16\\
v_3&\colon\ 6\cdot 2-0+1+2+1 &=& \quad 16\le 16\\
v_4&\colon\ 6\cdot 1-0+4+3+3 &=& \quad 16\le 16\\
v_5&\colon\ 6\cdot 1-1+4+3+3 &=& \quad 15\le 16\\
v_6&\colon\ 6\cdot 1-0+2+2+4 &=& \quad 14\le 16\\
v_7&\colon\ 6\cdot 0-0+5+4+6 &=& \quad 15\le 16
\end{aligned}
$$

In the case when the edge v_2v_5 is contained in a 3-face, we subtract -2 instead of -1 in the conditions for the vertices v_2 and v_5. □

Lemma 6.11. *Let G be a 4-minimal graph and u and v be two vertices consecutive on a boundary of a ≥ 5-face. If u is a $(4,3,4,\geq 5)$-vertex, then v is not a 3-vertex.*

Proof. Assume the opposite, i.e., G contains the degree configuration depicted in Figure 21 with $d(v_i)=4$ for $i=1,2,3,5$ and $d(v_4)=3$ (note that degrees of the vertices v_1, v_2, v_3 and v_5 are four by Lemmas 6.4 and 6.5). We apply Lemma 6.2:

$$
\begin{aligned}
v_1&\colon\ 6\cdot 2-0+2+2+0 &=& \quad 16\le 16\\
v_2&\colon\ 6\cdot 2-1+2+2+1 &=& \quad 16\le 16\\
v_3&\colon\ 6\cdot 1-0+5+3+2 &=& \quad 16\le 16\\
v_4&\colon\ 6\cdot 1-0+3+2+3 &=& \quad 14\le 16\\
v_5&\colon\ 6\cdot 1-1+4+3+4 &=& \quad 16\le 16
\end{aligned}
$$

□

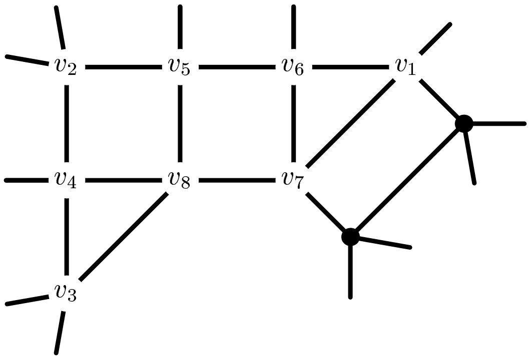

Figure 22: The degree configuration from Lemma 6.12.

Lemma 6.12. *Let G be a 4-minimal graph and u and v be two vertices consecutive on a boundary of a ≥ 5-face. If u is a $(3,4,4,\geq 5)$-vertex, then v is not a $(4,3,4,\geq 5)$-vertex.*

Proof. Assume the opposite, i.e., G contains the degree configuration depicted in Figure 22. Assume $d(v_i) = 4$ for all i—our arguments translate smoothly to the case when some of v_i's are 3-vertices. We apply Lemma 6.2:

$$\begin{aligned}
v_1\colon\ 6\cdot 2 - 1 + 2 + 2 + 0 &= 15 \leq 16\\
v_2\colon\ 6\cdot 2 - 0 + 2 + 2 + 0 &= 16 \leq 16\\
v_3\colon\ 6\cdot 2 - 0 + 1 + 2 + 1 &= 16 \leq 16\\
v_4\colon\ 6\cdot 1 - 0 + 4 + 3 + 2 &= 15 \leq 16\\
v_5\colon\ 6\cdot 1 - 0 + 3 + 3 + 4 &= 16 \leq 16\\
v_6\colon\ 6\cdot 1 - 0 + 4 + 3 + 3 &= 16 \leq 16\\
v_7\colon\ 6\cdot 1 - 1 + 3 + 3 + 5 &= 16 \leq 16\\
v_8\colon\ 6\cdot 0 - 0 + 5 + 4 + 7 &= 16 \leq 16
\end{aligned}$$

□

Lemma 6.13. *Let G be a 4-minimal graph and u and v be two vertices consecutive on a boundary of a ≥ 5-face. If the vertex u is a $(3,4,4,\geq 5)$-vertex and the edge uv is contained in a 4-face, then v is not a $(4,4,3,\geq 5)$-vertex.*

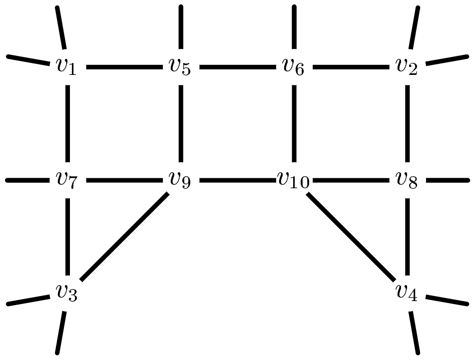

Figure 23: The degree configuration from Lemma 6.13.

Proof. Assume the opposite, i.e., G contains the degree configuration depicted in Figure 23. Assume $d(v_i) = 4$ for all i—our arguments translate smoothly to the case when some of v_i's are 3-vertices. We apply Lemma 6.1:

$$\begin{array}{lcl}
v_1\colon\ 6\cdot 2+2+2+0 & = & 16 \le 16 \\
v_2\colon\ 6\cdot 2+2+2+0 & = & 16 \le 16 \\
v_3\colon\ 6\cdot 2+1+2+1 & = & 16 \le 16 \\
v_4\colon\ 6\cdot 2+1+2+1 & = & 16 \le 16 \\
v_5\colon\ 6\cdot 1+3+3+3 & = & 15 \le 16 \\
v_6\colon\ 6\cdot 1+3+3+4 & = & 16 \le 16 \\
v_7\colon\ 6\cdot 1+4+3+3 & = & 16 \le 16 \\
v_8\colon\ 6\cdot 1+4+3+3 & = & 16 \le 16 \\
v_9\colon\ 6\cdot 0+4+4+7 & = & 15 \le 16 \\
v_{10}\colon\ 6\cdot 0+4+4+8 & = & 16 \le 16
\end{array}$$

□

Lemma 6.14. *Let G be a 4-minimal graph and u and v be two vertices consecutive on a boundary of a ≥ 5-face. If the vertex u is a $(4,3,4,\geq 5)$-vertex contained in a red 3-face, then v is not a $(4,3,4,\geq 5)$-vertex.*

Proof. Assume the opposite, i.e., G contains of the four degree configurations depicted in Figure 24. Assume $d(v_i) = 4$ for all i—our arguments

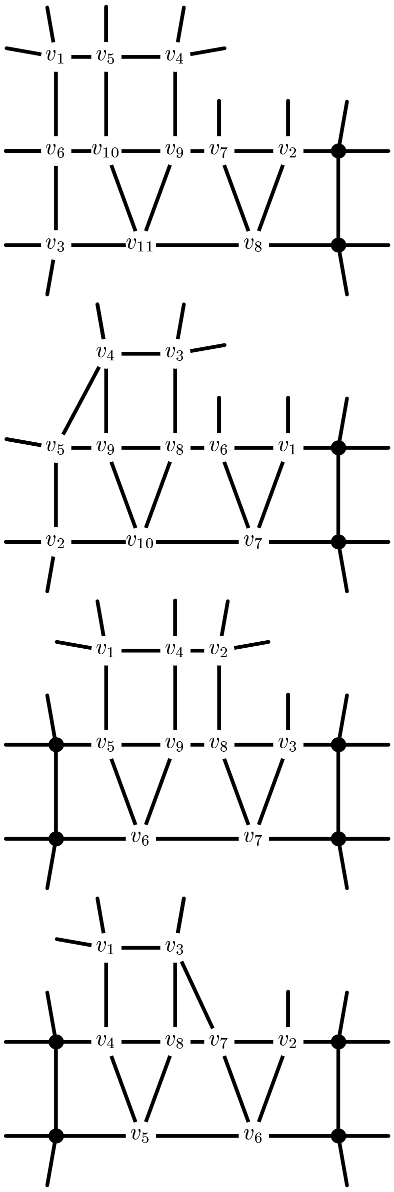

Figure 24: The degree configurations from Lemma 6.14.

translate smoothly to the case when some of v_i's are 3-vertices. We apply Lemma 6.2. In the case of the first degree configuration, we have the following:

$$\begin{array}{lcl}
v_1\colon\ 6\cdot 2-0+2+2+0 & = & 16\leq 16\\
v_2\colon\ 6\cdot 2-1+2+2+0 & = & 15\leq 16\\
v_3\colon\ 6\cdot 2-0+1+2+1 & = & 16\leq 16\\
v_4\colon\ 6\cdot 2-0+1+2+1 & = & 16\leq 16\\
v_5\colon\ 6\cdot 1-0+4+3+2 & = & 15\leq 16\\
v_6\colon\ 6\cdot 1-0+4+3+3 & = & 16\leq 16\\
v_7\colon\ 6\cdot 1-0+3+3+2 & = & 14\leq 16\\
v_8\colon\ 6\cdot 1-1+3+3+3 & = & 14\leq 16\\
v_9\colon\ 6\cdot 0-0+3+4+7 & = & 14\leq 16\\
v_{10}\colon\ 6\cdot 0-0+2+4+8 & = & 14\leq 16\\
v_{11}\colon\ 6\cdot 0-0+3+4+9 & = & 16\leq 16
\end{array}$$

The second one yields the following system of conditions:

$$\begin{array}{lcl}
v_1\colon\ 6\cdot 2-1+2+2+0 & = & 15\leq 16\\
v_2\colon\ 6\cdot 2-0+1+2+0 & = & 15\leq 16\\
v_3\colon\ 6\cdot 2-0+1+2+0 & = & 15\leq 16\\
v_4\colon\ 6\cdot 1-0+3+3+2 & = & 14\leq 16\\
v_5\colon\ 6\cdot 1-0+3+3+3 & = & 15\leq 16\\
v_6\colon\ 6\cdot 1-0+3+3+2 & = & 14\leq 16\\
v_7\colon\ 6\cdot 1-1+3+3+3 & = & 14\leq 16\\
v_8\colon\ 6\cdot 0-0+3+4+7 & = & 14\leq 16\\
v_9\colon\ 6\cdot 0-0+2+4+7 & = & 13\leq 16\\
v_{10}\colon\ 6\cdot 0-0+3+4+9 & = & 15\leq 16
\end{array}$$

For the third degree configuration, we verify the following conditions:

$$\begin{array}{lcl}
v_1\colon\ 6\cdot 2-0+2+2+0 & = & 16\leq 16\\
v_2\colon\ 6\cdot 2-0+1+2+1 & = & 16\leq 16\\
v_3\colon\ 6\cdot 2-1+1+2+1 & = & 15\leq 16\\
v_4\colon\ 6\cdot 1-0+4+3+2 & = & 15\leq 16\\
v_5\colon\ 6\cdot 1-1+3+3+2 & = & 13\leq 16\\
v_6\colon\ 6\cdot 1-1+3+3+3 & = & 14\leq 16\\
v_7\colon\ 6\cdot 1-1+2+3+5 & = & 15\leq 16\\
v_8\colon\ 6\cdot 0-0+5+4+6 & = & 16\leq 16\\
v_9\colon\ 6\cdot 0-0+3+4+8 & = & 15\leq 16
\end{array}$$

Finally, the next set of conditions appears in the fourth case:

$$\begin{aligned}
v_1\colon\ 6\cdot 2-0+2+2+0 &= 16\le 16\\
v_2\colon\ 6\cdot 2-1+1+2+0 &= 14\le 16\\
v_3\colon\ 6\cdot 1-0+2+3+2 &= 13\le 16\\
v_4\colon\ 6\cdot 1-1+3+3+2 &= 13\le 16\\
v_5\colon\ 6\cdot 1-1+3+3+3 &= 14\le 16\\
v_6\colon\ 6\cdot 1-1+2+3+5 &= 15\le 16\\
v_7\colon\ 6\cdot 0-0+4+4+5 &= 13\le 16\\
v_8\colon\ 6\cdot 0-0+3+4+7 &= 14\le 16
\end{aligned}$$

□

6.2 Discharging phase

In this subsection, we describe the discharging phase of the proof. First, each vertex v is assigned $\deg_G(v)-4$ units of charge and each face f is assigned $\deg_G(f)-4$ units of charge. The sum of initial charge of all the vertices and faces is negative (-8). Charge assigned to the vertices and faces is redistributed by the following set of rules (cf. also Figure 25):

Rule F.r A red 3-face receives $1/2$ units of charge from each incident vertex that is not red.

Rule F.b1 A blue 3-face receives $1/2$ units of charge from the incident $[3,4,\geq 4,\geq 5]$-vertex (note such a vertex is unique by Lemma 6.9).

Rule F.b2 A blue 3-face receives $1/3$ units of charge from the incident $[3,\geq 5,\geq 3,\geq 5]$-vertex (note such a vertex is unique by Lemma 6.9).

Rule F.b3 A blue 3-face receives $1/6$ units of charge from the incident $[3,4,3,\geq 5]$-vertex (note such a vertex is unique by Lemma 6.9).

Rule F A 3-face that is neither red nor blue receives $1/3$ units of charge from each incident vertex.

Rule 3.1 A (≥ 5)-face sends $1/2$ to every incident $[4,\geq 5,\geq 5]$-vertex.

Rule 3.2 A (≥ 5)-face sends $1/3$ to every incident $[\geq 5,\geq 5,\geq 5]$-vertex.

Rule 4.1 A (≥ 5)-face sends $1/2$ to every incident $[3,4,4,\geq 5]$-vertex that is contained in a blue 3-face.

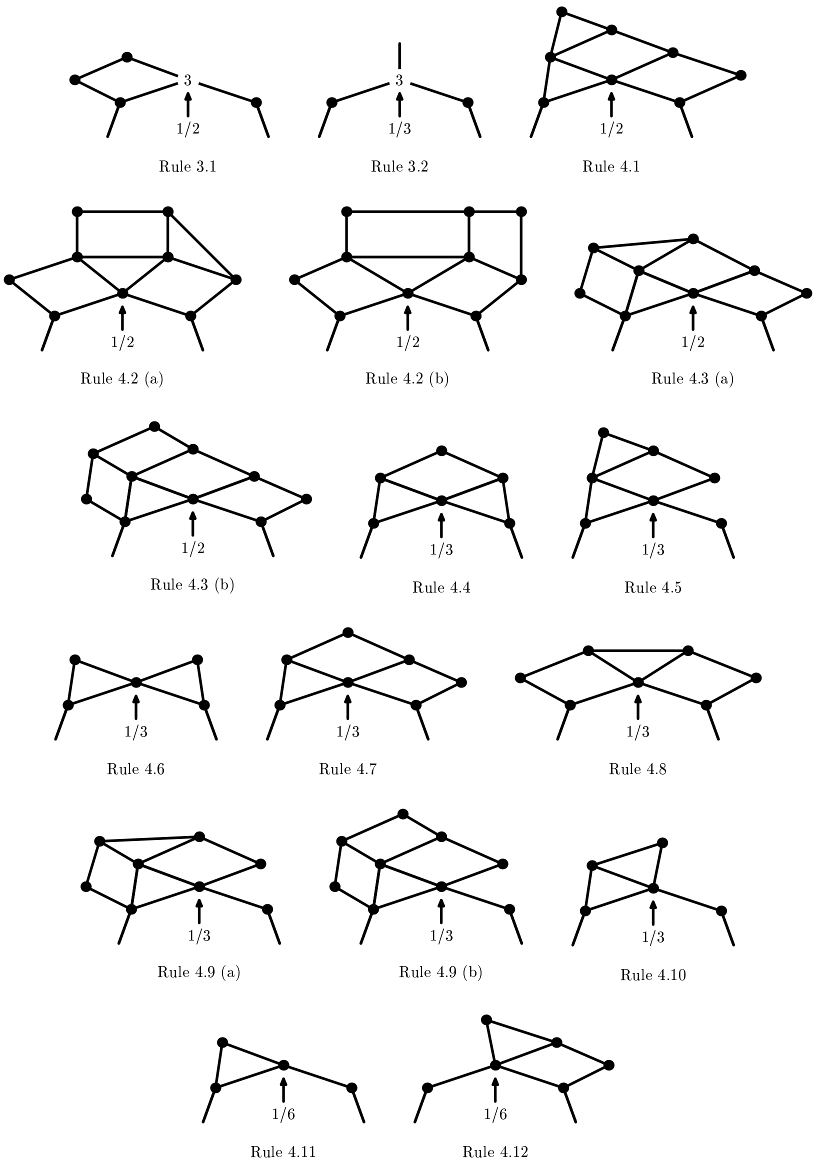

Figure 25: Rules 3.1–4.12 used in Section 6.

Rule 4.2 A (≥ 5)-face sends $1/2$ to every incident $[4, 3, 4, \geq 5]$-vertex that is contained in a red 3-face.

Rule 4.3 A (≥ 5)-face sends $1/2$ to every incident $[3, 4, 4, \geq 5]$-vertex that is contained in a red 3-face.

Rule 4.4 A (≥ 5)-face sends $1/3$ to every incident blue vertex, i.e., a $[3, 4, 3, \geq 5]$-vertex.

Rule 4.5 A (≥ 5)-face sends $1/3$ to every incident $[3, 4, \geq 5, \geq 5]$-vertex that is contained in a blue 3-face.

Rule 4.6 A (≥ 5)-face sends $1/3$ to every incident $[3, \geq 5, 3, \geq 5]$-vertex.

Rule 4.7 A (≥ 5)-face sends $1/3$ to every incident $[3, 4, 4, \geq 5]$-vertex that is not contained in a red 3-face.

Rule 4.8 A (≥ 5)-face sends $1/3$ to every incident $[4, 3, 4, \geq 5]$-vertex that is not contained in a red 3-face.

Rule 4.9 A (≥ 5)-face f sends $1/3$ to every incident $[3, 4, \geq 5, \geq 5]$-vertex v that is contained in a red 3-face that shares an edge with the face f.

Rule 4.10 A (≥ 5)-face sends $1/3$ to every incident $[3, 3, \geq 5, \geq 5]$-vertex.

Rule 4.11 A (≥ 5)-face f sends $1/6$ to every incident $[3, \geq 4, \geq 4, \geq 4]$-vertex v such that the 3-face containing v shares an edge with f and f sends v charge by none of Rules 4.1–4.10.

Rule 4.12 A (≥ 5)-face f sends $1/6$ to every incident $[\geq 5, 3, 4, \geq 5]$-vertex v if neither Rule 4.9 nor Rule 4.11 apply (note that in such a case the face f shares an edge with the 4-face containing v).

Next, we analyze final charge of vertices and faces of a 4-minimal graph after applying the described set of rules. We start with determining final charge of the vertices:

Lemma 6.15. *If G is a 4-minimal graph, then the amount of final charge of every vertex is zero.*

Proof. G contains no ≤ 2-vertices by Lemma 6.3. Let v be a vertex of G. If v is a 3-vertex, then v is either a $(4, \geq 5, \geq 5)$-vertex or a $(\geq 5, \geq 5, \geq 5)$-vertex by Lemma 6.5. In the former case, v receives $1/2$ units of charge from every incident ≥ 5-face by Rule 3.1. In the latter case, it receives $1/3$ units of charge from each of the three incident ≥ 5-faces by Rule 3.2. In both the cases, the final charge of v is zero.

We now focus on the case that v is a 4-vertex. The vertex v is incident with at most two 3-faces by Lemma 6.6. If v is incident with no 3-face, then it neither receives nor sends out any charge and its final charge is zero.

We first consider the case that v is incident with two 3-faces, say f_1 and f_2. By Lemma 6.6, v is a $(3, \geq 4, 3, \geq 4)$-vertex or a $(3, 3, \geq 5, \geq 5)$-vertex. Suppose first that v is a $(3, \geq 4, 3, \geq 4)$-vertex. If v is a $(3, 4, 3, 4)$-vertex, then v neither receives nor sends out any charge and its final charge is zero. Otherwise, neither of the two faces f_1 and f_2 is red by Lemma 6.7. If v is a $(3, 4, 3, \geq 5)$-vertex, then both faces f_1 and f_2 are blue. Hence, v receives $1/3$ units of charge by Rule 4.4 and it sends to each incident blue face $1/6$ units of charge by Rule F.b3. We conclude that its final charge is zero. If v is a $(3, \geq 5, 3, \geq 5)$-vertex, then it receives $1/3$ units of charge from each incident ≥ 5-face by Rule 4.6 and sends $1/3$ units of charge to each incident 3-face by Rule F or F.b2. Hence, its final charge is zero.

The other case is that v is a $(3, 3, \geq 5, \geq 5)$-vertex. Note that neither f_1 nor f_2 is red or blue by Lemma 6.6. The vertex v receives charge of $1/3$ units from each incident ≥ 5-face by Rule 4.10 and it sends each of the faces f_1 and f_2 $1/3$ units of charge by Rule F.

It remains to consider the case when v is incident with a single 3-face f, i.e., v is a $(3, \geq 4, \geq 4, \geq 4)$-vertex. We distinguish three cases: the face f is red, f is blue or f is neither red nor blue. If f is red and v is a $(3, 4, 4, 4)$-vertex, then v neither receives nor sends out any charge. If f is red and v is a $(3, 4, \geq 5, 4)$-vertex, v receives $1/2$ units of charge by Rule 4.2 and sends out $1/2$ units of charge by Rule F.r. If f is red and v is a $(3, 4, 4, \geq 5)$-vertex, v receives $1/2$ units of charge by Rule 4.3 and sends out $1/2$ units of charge by Rule F.r. Finally, if f is red and v is a $(3, 4, \geq 5, \geq 5)$-vertex, v receives $1/3$ units of charge by Rule 4.9 and $1/6$ units of charge by Rule 4.12, and v sends out $1/2$ units of charge to f by Rule F.r.

Next, assume that the face f is blue. Since v is incident with only one 3-face, v is not blue. By Lemma 6.7, v is either a $(3, 4, \geq 4, \geq 5)$-vertex or a $(3, \geq 5, \geq 3, \geq 5)$-vertex. If v is a $(3, 4, 4, \geq 5)$-vertex, then v receives $1/2$ units of charge by Rule 4.1 and it sends f charge of $1/2$ units by Rule 3.b1. If v is a $(3, 4, \geq 5, \geq 5)$-vertex, then v receives $1/3$ units of charge by

Rule 4.5 and $1/6$ units of charge by Rule 4.11, and v sends f $1/2$ units of charge by Rule 3.b1. The vertex v cannot be a $(3, \geq 5, 3, \geq 5)$-vertex since it is incident with a single 3-vertex. If v is a $(3, \geq 5, \geq 4, \geq 5)$-vertex, then it receives $1/6$ units of charge from the two incident ≥ 5-faces and it sends f $1/3$ units of charge by Rule 3.b2. We conclude that in all the cases, the final charge of v is zero.

Finally, assume that the face f is neither red or blue. In particular, at least one face incident with v is a ≥ 5-face. If v is a $(3, 4, 4, \geq 5)$-vertex or a $(3, 4, \geq 5, 4)$-vertex, then it receives charge of $1/3$ units from the incident ≥ 5-face by Rule 4.7 or Rule 4.8. If v is a $(3, 4, \geq 5, \geq 5)$-vertex, v receives charge of $1/6$ units from each of the incident ≥ 5-faces by Rule 4.11 and 4.12. Finally, if v is a $(3, \geq 5, \geq 4, \geq 5)$-vertex, then it receives charge of $1/6$ units from each of the incident ≥ 5-faces by Rule 4.11. Since v sends out $1/3$ units of charge to f by Rule F, its final charge is zero. $\square$

It is rather easy to determine the final amount of charge of 3-faces and 4-faces.

Lemma 6.16. *If G is a 4-minimal graph, then the amount of final charge of every 3-face and every 4-face is zero.*

Proof. Since a 4-face does not send out or receive any charge, its final charge is zero. Let us consider a 3-face f. It is incident only with 4-vertices by Lemma 6.5. If f is red, then it is incident exactly with one red vertex by Lemma 6.8. Hence, f receives charge of $1/2$ units from each of the incident non-red vertices by Rule F.r and its final charge is zero.

If f is blue, then it shares an edge with exactly one 4-face and it is incident with exactly one blue vertex by Lemma 6.9. In particular, f is incident with a $(3, 4, 3, \geq 5)$-vertex (the blue vertex), a $(3, 4, \geq 4, \geq 5)$-vertex and a $(3, \geq 5, \geq 3, \geq 5)$-vertex. It receives charge of $1/2$, $1/3$ and $1/6$ units from the incident vertices by Rules F.b1, F.b2 and F.b3. Hence, the final charge of f is zero.

The last case is that f is neither red nor blue. In this case, f receives charge of $1/3$ units from each incident vertex by Rule F and thus its final charge is zero. $\square$

The next lemma will be needed in the analysis of the final charge of ≥ 7-faces:

Lemma 6.17. *Let u and v be two vertices consecutive on a boundary of a ≥ 5-face f. The face sends u and v together at most $5/6$ units of charge.*

Proof. Since the f face sends each incident vertex at most $1/2$ units of charge, we have to exclude the case when both u and v receive $1/2$ from f. By Lemma 6.4, at most one of the vertices u and v is a 3-vertex. Assume that u is a 4-vertex and v is a 3-vertex. Note that v cannot be incident to a 3-face by Lemma 6.5. If u receives $1/2$ units of charge by Rule 4.1 or 4.3, then v cannot be a 3-vertex by Lemma 6.10. If u receives $1/2$ units of charge by Rule 4.2, then v is not a 3-vertex by Lemma 6.11. The case when u is a 3-vertex and v is a 4-vertex is symmetric.

We now consider the case when both u and v are 4-vertices. We distinguish six cases based on which rules apply to u and v:

- **Rule 4.1 applies to both u and v.** By structural reasons, the other face containing the edge uv is a 4-face and both u and v are $(3, 4, 4, \geq 5)$-vertices. However, Lemma 6.13 excludes this case.

- **Rule 4.2 applies to both u and v.** Hence, u and v are $(4, 3, 4, \geq 5)$-vertices and the edge uv is contained in a 4-face. However, this is excluded by Lemma 6.14.

- **Rule 4.3 applies to both u and v.** Since each red face is incident with one red vertex by Lemma 6.8, the other face containing the edge uv is a 4-face and both u and v are $(3, 4, 4, \geq 5)$-vertices. However, Lemma 6.13 excludes this case.

- **Rule 4.1 applies to u and Rule 4.2 applies to v.** By structural reasons, the edge uv is contained in a 4-face, u is a $(3, 4, 4, \geq 5)$-vertex and v is a $(4, 3, 4, \geq 5)$-vertex. However, this is impossible by Lemma 6.12.

- **Rule 4.1 applies to u and Rule 4.3 applies to v.** Since the 3-face incident to u is blue and the 3-face incident to v is red, the edge uv must be contained in a 4-face. Note that both u and v are $(3, 4, 4, \geq 5)$-vertices. However, this case is excluded by Lemma 6.13.

- **Rule 4.2 applies to u and Rule 4.3 applies to v.** By structural reasons, the edge uv is contained in a 4-face, u is a $(4, 3, 4, \geq 5)$-vertex and v is a $(3, 4, 4, \geq 5)$-vertex. However, this is impossible by Lemma 6.12.

We can now conclude that no two consecutive vertices on a boundary of a ≥ 5-face can together receive 1 unit of charge. $\square$

We now analyze the final charge of ≥ 7-faces.

Lemma 6.18. *Let G be a 4-minimal graph. The final charge of every ≥ 7-face f of G is non-negative.*

Proof. By Lemma 6.17, the face f sends each pair of consecutive vertices on its boundary at most $5/6$ units of charge. Hence, if f is an ℓ-face, then it sends at most $5\ell/12$ units of charge to all its incident vertices. Since the initial charge of f is $\ell - 4$ and $\ell \geq 7$, the final charge of f is non-negative. □

The next lemma has been verified using a computer program. We provide a detailed explanation of the procedure in the next subsection.

Lemma 6.19. *Let G be a 4-minimal graph. The final charge of every 5- and every 6-face is non-negative.*

Since the sum of the initial amounts of charge of all the vertices and faces of a 4-minimal graph is -8 (note that each 4-minimal graph G is connected) and the final charge of all the vertices and faces after redistributing by the described set of rules is non-negative (see Lemmas 6.15, 6.16, 6.18 and 6.19), we conclude that there is no 4-minimal graph G.

Theorem 6.20. *Every planar graph with maximum degree four has an $L(2,1)$-labeling with span at most 16.*

Since all our arguments are based either on simple counting forbidden labels or on Lemmas 3.5 and 3.6 that hold for list labelings, Theorem 6.20 translates to this setting:

Theorem 6.21. *Every planar graph with maximum degree four has a list $L(2,1)$-labeling for any 17-list assignment.*

6.3 Computer-checked cases

In this subsection, we describe our approach for verification that Lemma 6.19 holds. Two of the authors independently wrote computer programs that proceeded in the following way:

1. The program first generates the description of a "neighborhood" of a 5-face or a 6-face. This determines which discharging rules apply to each of the vertices.

2. The program then computes the charge sent out by the face.

3. If the amount of charge was larger than 1 in the case of a 5-face and larger than 2 units in the case of a 6-face, the program verified using Lemma 6.2 that the corresponding configuration cannot appear in a 5-minimal graph.

Next, we describe in more detail each of the three steps.

Fix size $\ell \in \{5,6\}$ of the face. Each configuration that can appear around an ℓ-face is encoded by a sequence of 2ℓ integers α_i ($i = 1,\ldots,2\ell$) between 1 and 5. Let us consider an ℓ-face f_0 of a 4-minimal graph that is incident with vertices $v_1,\ldots,v_\ell$. The number α_{2i-1} is defined as follows, where f is the face that contains the edge $v_{i-1}v_i$ and is different from f_0 (indices of vertices are modulo ℓ and of α's modulo 2ℓ where appropriate):

- $\alpha_{2i-1} = 1$ if f is a 3-face and the 4-vertex w incident with f that is different from v_{i-1} and v_i is contained in a 3-face that contains neither the edge $v_{i-1}w$ nor v_iw.
- $\alpha_{2i-1} = 2$ if f is a 3-face and the 4-vertex w incident with f that is different from v_{i-1} and v_i is contained in a 4-face that contains neither the edge $v_{i-1}w$ nor v_iw.
- $\alpha_{2i-1} = 3$ if f is a 3-face and neither of the previous two rules apply.
- $\alpha_{2i-1} = 4$ if f is a 4-face.
- $\alpha_{2i-1} = 5$ if f is a $\geq$5-face.

The number α_{2i} is equal to 1 if v_i is a 3-vertex. Otherwise, v_i is a 4-vertex and α_{2i} is defined as follows where f is the face incident with v_i that contains neither the edge $v_{i-1}v_i$ nor v_iv_{i+1}:

- $\alpha_{2i} = 2$ if f is a 3-face and all the three vertices incident with f are $(4,3,4,\geq 3)$-vertices (note that this implies that $\alpha_{2i-1} = \alpha_{2i+1} = 4$ as we discuss later).
- $\alpha_{2i} = 3$ if f is a 3-face and the previous rule does not apply.
- $\alpha_{2i} = 4$ if f is a 4-face.
- $\alpha_{2i} = 5$ if f is a $\geq$5-face.

An example of encoding is described in Figure 26.

In order to avoid multiple tests for the same configuration, a sequence of 2ℓ integers is tested to be the lexicographically minimal one among all

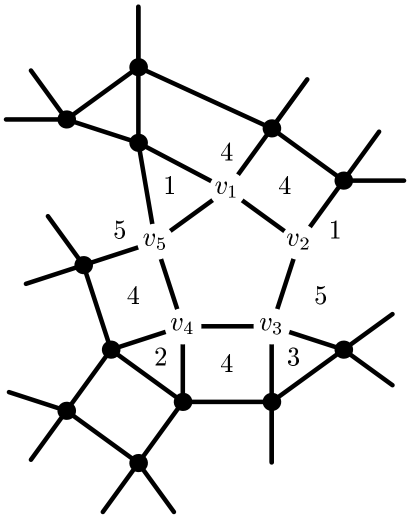

Figure 26: The configuration encoded by the sequence 1441534245 (the numbers α_i are written inside the corresponding faces neighboring with the central 5-face). Note that v_2 has degree three. The charge sent out by the face is computed as follows: $1/2 + 1/2 + 1/6 + 1/2 + 1/6 = 11/6$. The set A found by the program is the set $A = \{v_1, \ldots, v_5\}$ and the order of the vertices is v_4, v_5, v_3, v_1, v_2.

its cyclic rotations (by an even number of positions) and reflections. If the sequence is not lexicographically minimal, the sequence is not further tested (this reduces time required by the computation). The sequence is also not further tested if it contains one of the following subsequences $\alpha_{2i-1}\alpha_{2i}\alpha_{2i+1}$ (the sign * stands for any number between 1 and 5) since the corresponding configurations cannot appear in a 4-minimal graph.

11*,21*,31*,*11,*12,*13 The configurations corresponding to such a subsequence are excluded by Lemma 6.5 since no 3-vertex can be incident with a 3-face in a 4-minimal graph.

414 The configuration corresponding to this subsequence is excluded by Lemma 6.5 since a 4-minimal graph cannot contain a $(4, 4, \ell)$-vertex.

12*,13*,*21,*31,333,334,433 The configurations corresponding to this subsequences are excluded by Lemma 6.6 since a 4-minimal graph cannot contain a $(3, 3, \leq 4, \geq 3)$-vertex.

141,142,143,241,242,243,341,342 The configurations corresponding to such subsequences are excluded by Lemmas 6.7 and 6.9 or it is the same as 343.

444,445,544,525,535,545 If we replace $\alpha_{2i} \in \{1, 2, 3, 4\}$ with 5, we get a configuration that receives the same amount of charge as the original one and the original one is a subconfiguration of the new one. Hence, it is enough only to test the new one.

$\alpha_{2i-1}2\alpha_{2i+1}$ **with** $\alpha_{2i-1} \neq 4$ **or** $\alpha_{2i+1} \neq 4$ If we replace $\alpha_{2i-1}2\alpha_{2i+1}$ with $\alpha_{2i-1}3\alpha_{2i+1}$, then the vertex v_i receives the same amount of charge in the new configuration as in the original one. Since the new one is a subconfiguration of the original one, it is enough to test the new one only.

The following subsequences $\alpha_{2i-1}\alpha_{2i}\alpha_{2i+1}\alpha_{2i+2}$ are also excluded.

*****$\alpha_{2i}2\alpha_{2i+2}$ **with** $\alpha_{2i} \neq 4$ **or** $\alpha_{2i+2} \neq 4$ Similarly to the last item of the previous list, it is enough to test the configuration corresponding to the sequence where $\alpha_{2i}2\alpha_{2i+2}$ is replaced with $\alpha_{2i}3\alpha_{2i+2}$.

*****515 Similarly to the previous case, it is enough to test the corresponding configuration with the sequence replaced with *535.

Charge	The subsequence $\alpha_{2i-1}\alpha_i\alpha_{2i+1}$
1/2	415, 514 (Rule 3.1) 144, 441 (Rule 4.1 or Rule 4.3) 424 (Rule 4.2) 244, 442 (Rule 4.3)
1/3	515 (Rule 3.2) 343 (Rule 4.4) 145, 541 (Rule 4.5 or Rule 4.9) 151, 153, 351, 353 (Rule 4.6) 344, 443 (Rule 4.7) 434 (Rule 4.8) 245, 542 (Rule 4.9) 335, 533 (Rule 4.10)
1/6	154, 155, 345, 354, 355, 451, 551, 453, 543, 553 (Rule 4.11) 435, 534 (Rule 4.12)

Table 1: The amount of charge sent to a vertex v_i.

***1*1** The configuration corresponding to such a subsequence cannot appear in a 4-minimal graph, since no two 3-vertices can be adjacent in a 4-minimal graph by Lemma 6.4.

If the encoding does not contain any of the subsequences that are listed above, the charge sent out to the neighboring vertices by the face f_0 is computed. The amount of charge sent to the vertex v_i is determined by the subsequence $\alpha_{2i-1}\alpha_{2i}\alpha_{2i+1}$ (see Table 1). Note that in the case when $\alpha_{2i-1}\alpha_{2i}\alpha_{2i+1} = 424$, the vertex v_i may receive only $1/3$ units of charge, but the computed charge is always an upper bound on the amount of charge sent out by the face.

If the total charge sent out by the face is more than $\ell - 4$, the degree configuration H corresponding to the configuration around the ℓ-face is constructed (see Figure 26 for an example). Let $n = |V(H)|$.

The program then generates all the $2^n - 1$ non-empty subsets A of the vertices of H together with all $|A|!$ orders of their vertices and for each such subset A, the program checks whether the following holds:

- each vertex of A has at most two neighbors out of the set A, and
- the condition of Lemma 6.2 is satisfied for each vertex of A.

Once such a subset A satisfying these conditions is found, the testing of the configuration is stopped because we can infer from Lemma 6.2 that the configuration H cannot appear in a 4-minimal graph. If no such subset A had existed, the program would have reported it. This has not happened, i.e., either each generated configuration causes that the face sends out at most $\ell - 4$ units of charge or it cannot appear in a 4-minimal graph by Lemma 6.2. The reader can find all 3032 configurations tested for $\ell = 5$ together with the found subsets A and all 2409 configurations tested for $\ell = 6$ together with the sets A witnessing that they cannot appear in a 4-minimal graph on `http://kam.mff.cuni.cz/~kral/l21-planar.html`.

7 Conclusion

As we have already noted all our results apply to the list setting as well. Moreover, it is not hard to check that the conjecture of Griggs and Yeh also holds for planar graphs with maximum degree $\Delta > 6$ in the list setting. Hence, we may state the following theorem:

Theorem 7.1. *Let G be a planar graph with maximum degree $\Delta \neq 3$ and let $L : V(G) \to 2^{\mathbb{N}}$ be a $(\Delta^2 + 1)$-list assignment. There exists a list $L(2,1)$-labeling c of G for L.*

We finish with a remark on L(2,1)-labelings of subcubic planar graphs, the remaining open case for planar graphs. The conjecture of Griggs and Yeh remains open for planar graphs with maximum degree three. Note that Kang [16] showed that the conjecture holds for cubic hamiltonian graphs. It is also not difficult to observe that the conjecture holds for subcubic bipartite planar graphs: let A and B be the parts of the bipartite graph G. Let G_A be the graph with vertex set A such that two vertices are adjacent in G_A if and only if their distance in G is two, i.e., they have a common neighbor in G. Similarly, let us define a graph G_B. Observe that both graphs G_A and G_B are planar. By the Four Color Theorem, G_A can be labeled with labels 0, 1, 2 and 3 and G_B with labels 5, 6, 7 and 8 in such a way that no two vertices with the same label are adjacent in G_A or G_B. These two labelings form an $L(2,1)$-labeling of G with span at most eight.

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