COLOURINGS OF THE CARTESIAN PRODUCT OF GRAPHS AND MULTIPLICATIVE SIDON SETS

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ABSTRACT. Let $\mathcal{F}$ be a family of connected bipartite graphs, each with at least three vertices. A proper vertex colouring of a graph G with no bichromatic subgraph in $\mathcal{F}$ is $\mathcal{F}$*-free*. The $\mathcal{F}$*-free chromatic number* $\chi(G, \mathcal{F})$ of a graph G is the minimum number of colours in an $\mathcal{F}$-free colouring of G. For appropriate choices of $\mathcal{F}$, several well-known types of colourings fit into this framework, including acyclic colourings, star colourings, and distance-2 colourings. This paper studies $\mathcal{F}$-free colourings of the cartesian product of graphs.

Let H be the cartesian product of the graphs $G_1, G_2, \ldots, G_d$. Our main result establishes an upper bound on the $\mathcal{F}$-free chromatic number of H in terms of the maximum $\mathcal{F}$-free chromatic number of the G_i and the following number-theoretic concept. A set S of natural numbers is k*-multiplicative Sidon* if $ax = by$ implies $a = b$ and $x = y$ whenever $x, y \in S$ and $1 \leq a, b \leq k$. Suppose that $\chi(G_i, \mathcal{F}) \leq k$ and S is a k-multiplicative Sidon set of cardinality d. We prove that $\chi(H, \mathcal{F}) \leq 1 + 2k \cdot \max S$. We then prove that the maximum density of a k-multiplicative Sidon set is $\Theta(1/\log k)$. It follows that $\chi(H, \mathcal{F}) \leq \mathcal{O}(dk \log k)$. We illustrate the method with numerous examples, some of which generalise or improve upon existing results in the literature.

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1. Introduction

Sabidussi [30] proved that the chromatic number of the cartesian product of a set of graphs equals the maximum chromatic number of a graph in the set. No such result is known for more restrictive colourings (such as acyclic, star, and distance-2 colourings). This paper investigates such colourings of cartesian products under a general model of restriction, in which arbitrary bichromatic subgraphs are excluded. Our study leads to a number-theoretic problem regarding multiplicative Sidon sets that is of independent interest. This problem is then solved using a combination of number-theoretic and graph-theoretic approaches.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. (All graphs considered are undirected, simple, and finite.) A *colouring* of G is a function $c : V(G) \to \mathbb{Z}$ such that $c(v) \neq c(w)$ for every edge $vw \in E(G)$. A colouring c with $|\{c(v) : v \in V(G)\}| \leq k$ is a *k-colouring*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum integer k for which there is a k-colouring of G.

Let $\mathcal{F}$ be a family of connected bipartite graphs, each with at least three vertices, called a *forbidden family*. A colouring c of a graph G is *$\mathcal{F}$-free* if it contains no bichromatic subgraph in $\mathcal{F}$; that is, $|\{c(v) : v \in V(H)\}| \geq 3$ for every subgraph H of G that is isomorphic to a graph in $\mathcal{F}$. The *$\mathcal{F}$-free chromatic number* of G, denoted by $\chi(G, \mathcal{F})$, is the minimum integer k for which there is an $\mathcal{F}$-free k-colouring of G. When $\mathcal{F} = \{H\}$ is a singleton, we write *H-free* instead of $\mathcal{F}$-free, and refer to the *H-free chromatic number* $\chi(G, H)$. The framework was introduced by Albertson *et al.* [5]; an even more general model of restrictive graph colourings is considered by Nešetřil and Ossona de Mendez [25].

$\mathcal{F}$-free colourings correspond to many well-studied types of colourings. Let P_n and C_n respectively be the path and cycle on n vertices. Let $\mathcal{C} := \{C_n : n \text{ even}\}$. Then $\mathcal{C}$-free colourings are the *acyclic colourings* [7, 9–13, 37]. Here each bichromatic subgraph is a forest. By a further restriction we obtain the P_4-free colourings, which are called *star colourings*, since each bichromatic subgraph is a collection of disjoint stars [5, 7, 8, 12, 19, 24, 37]. A colouring is P_3-free if and only if every pair of vertices at distance at most two receive distinct colours (called a *distance*-2 colouring). That is, $\chi(G, P_3) = \chi(G^2)$. Here G^k is the k-th *power* of G, the graph with vertex set $V(G)$, where two vertices are adjacent in G^k whenever they are at distance at most k in G. Often motivated by applications in frequency assignment, colourings of graph powers has recently attracted much attention [1–4, 22, 23, 35]. By definition,

$$\chi(G) \leq \chi(G,\mathcal{C}) \leq \chi(G,P_4) \leq \chi(G,P_3) \ .$$

Let G_1 and G_2 be graphs. The *cartesian product* of G_1 and G_2, denoted by $G_1 \square G_2$, is the graph with vertex set

$$V(G_1 \square G_2) := V(G_1) \times V(G_2) := \{(a,v) : a \in V(G_1), v \in V(G_2)\} \ ,$$

where $(a,v)(b,w)$ is an edge of $G_1 \square G_2$ if and only if $ab \in E(G_1)$ and $v = w$, or $a = b$ and $vw \in E(G_2)$. Assuming isomorphic graphs are equal, the cartesian product is associative, and $G_1 \square G_2 \square \cdots \square G_d$ is well-defined. Sabidussi [30] proved that

$$\chi(G_1 \square G_2 \square \cdots \square G_d) = \max\{\chi(G_i) : 1 \leq i \leq d\} \ .$$

This paper studies $\mathcal{F}$-free colourings of cartesian products. The following upper bound on the $\mathcal{F}$-free chromatic number of a cartesian product is our main result. Here and throughout the paper, $\gamma = 0.5772\ldots$ is Euler's constant, and logarithms are base $\boldsymbol{e} = 2.718\ldots$ unless stated otherwise.

Theorem 1. *Let $\mathcal{F}$ be a forbidden family. Let $G_1, G_2, \ldots, G_d$ be graphs, each with $\mathcal{F}$-free chromatic number $\chi(G_i,\mathcal{F}) \leq k+1$. Then*

$$\chi(G_1 \square G_2 \square \cdots \square G_d, \mathcal{F}) \leq 2k(kd - k + 1) + 1 \ .$$

Moreover, for all $\epsilon > 0$ and for large $d > d(k,\epsilon)$,

$$\chi(G_1 \square G_2 \square \cdots \square G_d, \mathcal{F}) \leq 1 + \frac{2\,\boldsymbol{e}^{\gamma}}{1-\epsilon}\, dk \log k \ .$$

We actually prove a stronger result than Theorem 1 that is expressed in terms of 'chromatic span'. This concept is introduced in Section 2. The key lemma of the paper, which relates $\mathcal{F}$-free colourings of a cartesian product to so-called k-multiplicative Sidon sets, is proved in Section 3. In Section 4 we study k-multiplicative Sidon sets in their own right. The obtained bounds establish our main colouring results. The remaining sections contain numerous examples of the method, some of which generalise or improve upon existing results in the literature. In particular, we consider distance-2 colourings in Section 5, acyclic colourings in Section 6, and star colourings in Section 7.

2. Chromatic Span

Let c be a colouring of a graph G. The *span* of c is $\max\{|c(v) - c(w)| : vw \in E(G)\}$. (The number of colours is irrelevant.) The *chromatic span* of G, denoted by $\Lambda(G)$, is the minimum integer k for which there is a colouring of G with span k. Note that $\Lambda(G) \leq k$ if and only if there is a homomorphism from G into P_n^k for some n.

Let $[a,b] := [a,a+1,\dots,b]$ and $[b] := [1,b]$ for all integers $a \leq b$. We can assume that the range of a k-colouring is $[k]$. Thus $\Lambda(G) \leq \chi(G)-1$ for every graph G. Conversely, given a colouring c of G with span k, let $c'(v) := c(v) \bmod (k+1)$ for each vertex $v \in V(G)$. Then c' is a $(k+1)$-colouring of G. Thus

$$\Lambda(G) \;=\; \chi(G)-1 \;\;.$$

This suggests that chromatic span is pointless. Let the *$\mathcal{F}$-free chromatic span* of a graph G, denoted by $\Lambda(G,\mathcal{F})$, be the minimum integer k for which there is an $\mathcal{F}$-free colouring of G with span k.

Lemma 1. *Let $\mathcal{F}$ be a forbidden family. For every graph G,*

$$\Lambda(G,\mathcal{F})+1 \leq \chi(G,\mathcal{F}) \leq 2\cdot\Lambda(G,\mathcal{F})+1 \;\;.$$

Proof. Obviously $\Lambda(G,\mathcal{F}) \leq \chi(G,\mathcal{F})-1$. To prove that $\chi(G,\mathcal{F}) \leq 2\cdot\Lambda(G,\mathcal{F})+1$, let c be an $\mathcal{F}$-free colouring of G with span $k := \Lambda(G,\mathcal{F})$. For every vertex $v \in V(G)$, let $c'(v) := c(v) \bmod (2k+1)$. Clearly c' is a $(2k+1)$-colouring of G. For all $i \in [0,2k]$, let $V_i := \{v \in V(G) : c'(v) = i\}$, and for all $j \in \mathbb{Z}$, let $V_{i,j} := \{v \in V_i : c(v) = j(2k+1)+i\}$. Thus the V_i's are the colour classes of c' and the $V_{i,j}$'s are the colour classes of c. For $S,T \subseteq V(G)$ with $S \cap T = \emptyset$, let $G[S,T]$ be the subgraph of G with vertex set $S \cup T$ and edge set $\{vw \in E(G) : v \in S, w \in T\}$. Consider two edges $vw, xy \in E(G)$ with $v,x \in V_{i_1,j_1}$, $w \in V_{i_2,j_2}$, and $y \in V_{i_2,j_3}$. Since $|c(v)-c(w)| \leq k$ and $|c(x)-c(y)| \leq k$, we have $j_2 = j_3$. It follows that each bichromatic subgraph of c' is the union of disjoint bichromatic subgraphs of c. In particular, $G[V_{i_1},V_{i_2}] = \cup\{G[V_{i_1,j},V_{i_2,j}] : j \in \mathbb{Z}\}$ or $G[V_{i_1},V_{i_2}] = \cup\{G[V_{i_1,j},V_{i_2,j+1}] : j \in \mathbb{Z}\}$. Since each subgraph $G[V_{i_1,j},V_{i_2,j}]$ (or $G[V_{i_1,j},V_{i_2,j+1}]$ in the second case) is $\mathcal{F}$-free, c' is $\mathcal{F}$-free and $\chi(G,\mathcal{F}) \leq 2\cdot\Lambda(G,\mathcal{F})+1$. $\square$

Lemma 1 cannot be improved in general, since it is easily seen that $\Lambda(P_n^k,P_3) = k$ but $\chi(P_n^k,P_3) = 2k+1$. Thus chromatic span is of interest when considering $\mathcal{F}$-free colourings. We prove the following result, which with Lemma 1, implies Theorem 1.

Theorem 2. *Let $\mathcal{F}$ be a forbidden family. Let $G_1, G_2, \dots, G_d$ be graphs, each with $\mathcal{F}$-free chromatic span $\Lambda(G_i,\mathcal{F}) \leq k$ (which is implied if $\chi(G_i,\mathcal{F}) \leq$*

$k+1)$. *Then*

$$\Lambda(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, \mathcal{F}) \;\leq\; k(kd-k+1) \;\; , \;\; \textit{and}$$
$$\chi(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, \mathcal{F}) \;\leq\; 2k(kd-k+1)+1 \;\; .$$

Moreover, for all $\epsilon > 0$ *and for large* $d > d(k,\epsilon)$,

$$\Lambda(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, \mathcal{F}) \;\leq\; \frac{\boldsymbol{e}^{\gamma}}{1-\epsilon}\, dk \log k, \; \textit{and}$$
$$\chi(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, \mathcal{F}) \;\leq\; 1 + \frac{2\,\boldsymbol{e}^{\gamma}}{1-\epsilon}\, dk \log k \;\; .$$

3. The Key Lemma

Our results depend upon the following number-theoretic concept (where $\mathbb{N} := \{1,2,\dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$).

Definition 1. Let $k \in \mathbb{N}$. A set $A \subseteq \mathbb{N}$ is *k-multiplicative Sidon*[1] if for all $x, y \in A$ and for all $a, b \in [k]$, we have $ax = by$ implies $a = b$ and $x = y$. For brevity we write *k-multiplicative* rather than k-multiplicative Sidon.

Consider a cartesian product $\tilde{G} := G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d$ to have vertex set

$$V(\tilde{G}) = \{\tilde{v} : \tilde{v} = (v_1, v_2, \dots, v_d), v_i \in V(G_i), i \in [d]\} \;\; ,$$

where $\tilde{v}\tilde{w} \in E(\tilde{G})$ if and only if $v_i w_i \in E(G_i)$ for some i, and $v_j = w_j$ for all $j \neq i$; we say that the edge $\tilde{v}\tilde{w}$ is in *dimension i*.

Lemma 2. *Let $\mathcal{F}$ be a forbidden family. Let $G_1, G_2, \dots, G_d$ be graphs, each with $\mathcal{F}$-free chromatic span $\Lambda(G_i, \mathcal{F}) \leq k$ (which is implied if $\chi(G_i, \mathcal{F}) \leq k+1$). Let $S := \{s_1, s_2, \dots, s_d\}$ be a k-multiplicative set. Then*

$$\Lambda(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, \mathcal{F}) \;\leq\; k \cdot \max S \;\; .$$

Proof. Let $\tilde{G} := G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d$. For each $i \in [d]$, let c_i be an $\mathcal{F}$-free colouring of G_i with span k. For each vertex $\tilde{v} \in V(\tilde{G})$, let

$$c(\tilde{v}) \;:=\; \sum_{i \in [d]} s_i \cdot c_i(v_i) \;\; .$$

[1] Erdős [14, 15, 16] defined a set $A \subseteq \mathbb{N}$ to be *multiplicative Sidon* if $ab = cd$ implies $\{a,b\} = \{c,d\}$ for all $a,b,c,d \in A$; see [29, 31]. Additive Sidon sets have been more widely studied; see the classical papers [17, 32, 33] and the recent survey by O'Bryant [27].

For every edge $\tilde{v}\tilde{w} \in E(\tilde{G})$ in dimension i,

(1)
$$c(\tilde{w})-c(\tilde{v}) = \left(\sum_{j\in[d]} s_j \cdot c_j(w_j)\right) - \left(\sum_{j\in[d]} s_j \cdot c_j(v_j)\right) = s_i\big(c_i(w_i)-c_i(v_i)\big) .$$

Since $1 \leq |c_i(w_i) - c_i(v_i)| \leq k$ and $s_i \geq 1$, c is a colouring of $\tilde{G}$ with span $k \cdot \max S$.

Suppose, for the sake of contradiction, that c is not $\mathcal{F}$-free. That is, there is a bichromatic subgraph H of $\tilde{G}$ that is isomorphic to some graph in $\mathcal{F}$. First suppose that all the edges of H have the same dimension i. By Equation (1), and since H is connected, the edges $\{v_iw_i : \tilde{v}\tilde{w} \in E(H)\}$ induce a bichromatic subgraph of G_i that is isomorphic to a graph in $\mathcal{F}$, which is a contradiction. Thus not all the edges of H are in the same dimension. Since H is connected and has at least three vertices, H has two edges $\tilde{v}\tilde{x}$ and $\tilde{w}\tilde{x}$ with a common endpoint that are in distinct dimensions. Say $\tilde{v}\tilde{x}$ is in dimension i and $\tilde{w}\tilde{x}$ is in dimension $j \neq i$. Since H is bichromatic, $c(\tilde{v}) - c(\tilde{x}) = c(\tilde{w}) - c(\tilde{x})$. By Equation (1),

$$s_i\big(c_i(v_i) - c_i(x_i)\big) = s_j\big(c_j(w_j) - c_j(x_j)\big) .$$

Since c_i has span k, we have $1 \leq |c_i(v_i) - c_i(x_i)| \leq k$ and $1 \leq |c_j(w_j) - c_j(x_j)| \leq k$, which implies that S is not k-multiplicative. This contradiction proves that c is an $\mathcal{F}$-free colouring of $\tilde{G}$. $\square$

4. k-Multiplicative Sidon Sets

Motivated by Lemma 2, in this section we study k-multiplicative sets in their own right. We measure the 'size' of a k-multiplicative set by its density. The *density* of $A \subseteq \mathbb{N}$ is

$$\delta(A) := \lim_{n\to\infty} \frac{|A \cap [n]|}{n}$$

if the limit exists (otherwise the density is undefined). We say $A \subseteq N$ is *p-periodic* if $x \in A$ if and only if $x + p \in A$ for all $x \in \mathbb{N}$. Observe that if A is p-periodic then

(2)
$$\delta(A) = \frac{|A \cap [p]|}{p}.$$

The following theorem is our main result regarding k-multiplicative sets.

Theorem 3. *For all $k \in \mathbb{N}$, the maximum density of a k-multiplicative set is*

$$\Theta\left(\frac{1}{\log k}\right).$$

The lower and upper bounds in Theorem 3 are proved in Theorems 4 and 5, respectively. We start with a naive construction of a k-multiplicative set.

Lemma 3. *For all $k \in \mathbb{N}$, the set $R_k := \{x \in \mathbb{N} : x \equiv 1 \pmod{k}\}$ is k-multiplicative and has density $\delta(R_k) = 1/k$.*

Proof. Suppose that $ax = by$ for some $x, y \in R_k$ and $a, b \in [k]$. Then $x = pk + 1$ and $y = qk + 1$ for some $p, q \in \mathbb{N}$. Thus $(ap - bq)k = b - a$. Since $|b - a| \leq k - 1$, we have $a = b$ and $ap = bq$. Thus $p = q$ and $x = y$. That is, R_k is k-multiplicative. Since R_k is k-periodic, $\delta(R_k) = |R_k \cap [k]|/k = 1/k$ by Equation (2). □

Fix $k \in \mathbb{N}$. Let $\mathbb{P}_k := \{p_1, p_2, \ldots, p_\ell\}$ be the set of primes in $[k]$. Let

$$P_k := \prod_{i \in [\ell]} p_i \ .$$

Every $x \in \mathbb{N}$ can be uniquely represented as

$$x = \beta_*(x) \prod_{i \in [\ell]} p_i^{\beta_i(x)} \ ,$$

where $\beta_i(x) \in \mathbb{N}_0$ and $\beta_*(x)$ is not divisible by p_i for all $i \in [\ell]$. That is, $\gcd(\beta_*(x), P_k) = 1$. Let $\beta(x)$ be the vector $(\beta_1(x), \beta_2(x), \ldots, \beta_\ell(x))$. For all $x, y \in \mathbb{N}$,

$$\beta(x \cdot y) = \beta(x) + \beta(y) \text{ and } \beta_*(x \cdot y) = \beta_*(x) \cdot \beta_*(y) \ . \tag{3}$$

Lemma 4. *For all $k \in \mathbb{N}$, if $ax = by$ for some $a, b \in [k]$ and $x, y \in \mathbb{N}$, then $\beta_*(x) = \beta_*(y)$.*

Proof. By Equation (3), we have $\beta_*(a) \cdot \beta_*(x) = \beta_*(b) \cdot \beta_*(y)$. Since $a, b \leq k$, we have $\beta_*(a) = \beta_*(b) = 1$. Thus $\beta_*(x) = \beta_*(y)$. □

Theorem 4. *For all $k \in \mathbb{N}$, the set $S_k := \{s \in \mathbb{N} : \gcd(s, P_k) = 1\}$ is k-multiplicative and has density*

$$\delta(S_k) = \prod_{i \in [\ell]} \left(1 - \frac{1}{p_i}\right) \sim \frac{e^{-\gamma}}{\log k}.$$

Proof. Suppose that $ax = by$ for some $a, b \in [k]$ and $x, y \in S_k$. Thus $\beta_*(x) = \beta_*(y)$ by Lemma 4. Since $\gcd(x, P_k) = \gcd(y, P_k) = 1$, we have $\beta_i(x) = \beta_i(y) = 0$ for all $i \in [\ell]$. Hence $x = y$, which implies that $a = b$, and S_k is k-multiplicative.

It remains to compute the density of S_k. Let φ be Euler's totient function, $\varphi(x) := |\{y \in [x] : \gcd(x, y) = 1\}|$. If $q_1, q_2, \dots, q_r$ are the prime factors of x (with repetition), then

$$\varphi(x) = x \prod_{i \in [r]} \left(1 - \frac{1}{q_i}\right) .$$

Observe that S_k is P_k-periodic. By Equation (2),

$$\delta(S_k) = \frac{|S_k \cap [P_k]|}{P_k} = \frac{\varphi(P_k)}{P_k} = \prod_{i \in [\ell]} \left(1 - \frac{1}{p_i}\right) .$$

By Mertens' Theorem (see [20]), $\delta(S_k) \sim \boldsymbol{e}^{-\gamma}/\log k$; see Table 1. $\square$

The following corollary is a straightforward consequence of Theorem 4.

Corollary 1. *For all $k \in \mathbb{N}$, $\epsilon > 0$, and sufficiently large $n > n(k, \epsilon)$,*

$$\frac{(1-\epsilon)n}{\boldsymbol{e}^{\gamma} \log k} \leq |S_k \cap [n]| \leq \frac{(1+\epsilon)n}{\boldsymbol{e}^{\gamma} \log k} .$$

TABLE 1. The first 15 elements of the set S_k for each $k \leq 30$.

k	S_k	density
2	$\{1,3,5,7,9,11,13,15,17,19,21,23,25,27,29,\dots\}$	1/2
3, 4	$\{1,5,7,11,13,17,19,23,25,29,31,35,37,41,43,\dots\}$	1/3
5, 6	$\{1,7,11,13,17,19,23,29,31,37,41,43,47,49,53,\dots\}$	4/15
$7,\dots,10$	$\{1,11,13,17,19,23,29,31,37,41,43,47,53,59,61,\dots\}$	8/35
11, 12	$\{1,13,17,19,23,29,31,37,41,43,47,53,59,61,67,\dots\}$	16/77
$13,\dots,16$	$\{1,17,19,23,29,31,37,41,43,47,53,59,61,67,71,\dots\}$	192/1001
17, 18	$\{1,19,23,29,31,37,41,43,47,53,59,61,67,71,73,\dots\}$	3072/17017
$19,\dots,22$	$\{1,23,29,31,37,41,43,47,53,59,61,67,71,73,79,\dots\}$	55296/323323
$23,\dots,28$	$\{1,29,31,37,41,43,47,53,59,61,67,71,73,79,83,\dots\}$	110592/676039
29, 30	$\{1,31,37,41,43,47,53,59,61,67,71,73,79,83,89,\dots\}$	442368/2800733

We can now prove Theorem 2.

Proof of Theorem 2. Lemma 3 implies that $R := \{ik+1 : i \in [0, d-1]\}$ is k-multiplicative. Since $|R| = d$ and $\max R = dk - k + 1$, by using R as a k-multiplicative set in Lemma 2, we have $\Lambda(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, \mathcal{F}) \leq k(dk-k+1)$. This proves the first part of the theorem.

Let n be the minimum integer such that $|S_k \cap [n]| \geq d$. By Corollary 1, for $d > d(k, \epsilon)$,

$$\max\{S_k \cap [n]\} \leq n \leq \frac{\boldsymbol{e}^{\gamma}}{1-\epsilon} d \log k \ .$$

Using $S_k \cap [n]$ as a k-multiplicative set in Lemma 2, we have

$$\Lambda(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, \mathcal{F}) \leq \frac{\boldsymbol{e}^{\gamma}}{1-\epsilon} dk \log k \ .$$

The final claim in Theorem 2 follows from Lemma 1. $\square$

4.1. **Proof of Optimality.** We now prove that the lower bound in Theorem 4 is asymptotically optimal, which in turn completes the proof of Theorem 3.

Theorem 5. *For all $k \in \mathbb{N}$, $\epsilon > 0$, and sufficiently large $n > n(k, \epsilon)$, every k-multiplicative set $A \subseteq [n]$ satisfies*

$$|A| \leq \frac{(2+\epsilon)n}{\boldsymbol{e}^{\gamma} \log k} + \frac{2n}{\sqrt[4]{k}} = (2 + o(1))|S_k \cap [n]| \ .$$

To prove Theorem 5, we model k-multiplicative sets using graphs. Let $G_{n,k}$ be the graph with vertex set $V(G_{n,k}) := [n]$, where $xy \in E(G_{n,k})$ whenever $ax = by$ for some $a, b \in [k]$. Observe that a set $A \subseteq [n]$ is k-multiplicative if and only if A is an independent set of $G_{n,k}$. For each $s \in S_k \cap [n]$, let $G_{n,k,s}$ be the subgraph of $G_{n,k}$ induced by $X_{n,k,s} := \{x \in [n] : \beta_*(x) = s\}$.

Lemma 5. *The connected components of $G_{n,k}$ are $\{G_{n,k,s} : s \in S_k \cap [n]\}$.*

Proof. If $xy \in E(G_{n,k})$, then $\beta_*(x) = \beta_*(y)$ by Lemma 4, which implies that $x, y \in X_{n,k,s}$ for some $s \in S_k \cap [n]$. Thus distinct sets $X_{n,k,s}$ and $X_{n,k,t}$ are not joined by an edge of $G_{n,k}$. It remains to prove that each subgraph $G_{n,k,s}$ is connected. For each pair of vertices $x, y \in X_{n,k,s}$, let

$$f(x, y) := \sum_{i \in [\ell]} |\beta_i(x) - \beta_i(y)|.$$

We claim that x and y are connected by a path of $f(x,y)$ edges in $G_{n,k,s}$. The proof is by induction on $f(x,y)$. If $f(x,y) = 0$ then $x = y$ (since $\beta_*(x) = \beta_*(y) = s$) and we are done. Say $f(x,y) > 0$. Without loss of generality, $\beta_i(x) < \beta_i(y)$ for some i. Let $z := p_i x$. Then $z \in X_{n,k,s}$ and xz is an edge of $G_{n,k,s}$. Moreover, $\beta_i(z) = \beta_i(x) + 1$, which implies that $f(z,y) = f(x,y) - 1$. By induction, there is a path of $f(z,y)$ edges from z to y. Thus there is a path of $f(z,y) + 1 = f(x,y)$ edges from x to y. $\square$

Lemma 6. *Let $G_{n,k,s}$ be a connected component of $G_{n,k}$ with r vertices. Then the $\min\{k,r\}$ smallest elements of $X_{n,k,s}$ are $\{s, 2s, 3s, \dots, \min\{k,r\} \cdot s\}$, and they form a clique of $G_{n,k,s}$.*

Proof. Every element of $X_{n,k,s}$ is a multiple of s and is at least s. Now $is \in X_{n,k,s}$ for each $i \in [\min\{k,r\}]$. Thus the $\min\{k,r\}$ smallest elements of $X_{n,k,s}$ are $\{s, 2s, 3s, \dots, \min\{k,r\} \cdot s\}$, which clearly form a clique of $G_{n,k,s}$. $\square$

For all $x \in [n]$, let $N_k(x)$ be the closed neighbourhood of x in $G_{n,k}$. That is, $y \in N_k(x)$ if and only if $y \in [n]$ and $ay = bx$ for some $a, b \in [k]$.

Lemma 7. *Let $G_{n,k,s}$ be a connected component of $G_{n,k}$ with at least k vertices. Then $|N_k(x)| \geq \lfloor\sqrt{k}\rfloor$ for every $x \in X_{n,k,s}$.*

Proof. By Lemma 6, the k smallest elements of $X_{n,k,s}$ are $\{s, 2s, 3s, \dots, ks\}$, and they form a clique of $G_{n,k,s}$. In particular, $ks \leq n$.

Case (a). $x \leq \sqrt{k}s$: For each $a \in [\lfloor\sqrt{k}\rfloor]$, we have $ax \leq ks \leq n$. Thus $ax \in N_k(x)$ and $|N_k(x)| \geq \lfloor\sqrt{k}\rfloor$.

Case (b). $x > \sqrt{k}s$: First suppose that there is a prime p that divides x and $\sqrt{k} \leq p \leq k$. Then $\frac{ax}{p} \in [x]$ for each $a \in [p]$. Thus $\frac{ax}{p} \in N_k(x)$ and $|N_k(x)| \geq p \geq \sqrt{k}$. Now suppose that there is no prime divisor p of x with $\sqrt{k} \leq p \leq k$. Let $p_1 \leq p_2 \leq \dots \leq p_t$ be the prime factors of x with duplication. Since $x > \sqrt{k}$, for some $\ell \in [t]$, the integer $q := \prod_{i \in [\ell]} p_i$ divides x and $\sqrt{k} \leq q \leq k$. Thus $\frac{ax}{q} \in [x]$ for each $a \in [q]$. Thus $\frac{ax}{q} \in N_k(x)$ and $|N_k(x)| \geq q \geq \sqrt{k}$. $\square$

Proof of Theorem 5. Let $k' := \lfloor\sqrt{k}\rfloor$ and $k'' := \lfloor\sqrt{k'}\rfloor$. Note that $k'' \geq 1$ and $k'' > \frac{1}{2}\sqrt[4]{k}$. We proceed by studying the size of A within each connected component of the graph $G_{n,k'}$. That is, we consider A as the union of the disjoint sets $\{A \cap X_{n,k',s} : s \in S_{k'} \cap [n]\}$.

First consider $s \in S_{k'} \cap [n]$ for which $|X_{n,k',s}| \leq k'$. By Lemma 6, $X_{n,k',s}$ is a clique of $G_{n,k'}$. Since A is k-multiplicative, A is k'-multiplicative, and A

is an independent set of $G_{n,k'}$. Thus $|A \cap X_{n,k',s}| \leq 1$. The set $S_{k'} \cap [n]$ has exactly one element in $X_{n,k',s}$. Thus $|\cup\{A \cap X_{n,k',s} : s \in S_{k'} \cap [n], |X_{n,k',s}| \leq k'\}| \leq |S_{k'} \cap [n]|$. By Corollary 1,

(4)
$$\left|\bigcup\{A \cap X_{n,k',s} : s \in S_{k'} \cap [n], |X_{n,k',s}| \leq k'\}\right| \leq \frac{(1+\epsilon)n}{\boldsymbol{e}^{\gamma} \log k'} \leq \frac{(2+\epsilon)n}{\boldsymbol{e}^{\gamma} \log k}.$$

Now consider $s \in S_{k'} \cap [n]$ for which $|X_{n,k',s}| > k'$. We claim that $N_{k'}(x) \cap N_{k'}(y) = \emptyset$ for distinct $x, y \in A$. Suppose that $z \in N_{k'}(x) \cap N_{k'}(y)$ for some $x, y \in A$. Then $a_1 x = b_1 z$ and $a_2 y = b_2 z$ for some $a_1, a_2, b_1, b_2 \in [k']$. Thus $\frac{a_1 x}{b_1} = \frac{a_2 y}{b_2}$ and $(a_1 b_2)x = (a_2 b_1)y$. Since $a_1 b_2, a_2 b_1 \in [k]$ and A is k-multiplicative, $x = y$. This proves the claim. Now $N_{k'}(x) \subseteq X_{n,k',s}$ for each $x \in X_{n,k',s}$ by Lemma 5, and $|N_{k'}(x)| \geq k''$ by Lemma 7. Thus $|A \cap X_{n,k',s}| \cdot k'' \leq |X_{n,k',s}|$, and

$$\left|\bigcup\{A \cap X_{n,k',s} : s \in S_{k'} \cap [n], |X_{n,k',s}| > k'\}\right| \leq \frac{n}{k''} < \frac{2n}{\sqrt[4]{k}} . \tag{5}$$

Corollary 1 and Equations (4) and (5) imply that

$$|A| \leq \frac{(2+\epsilon)n}{\boldsymbol{e}^{\gamma} \log k} + \frac{2n}{\sqrt[4]{k}} \leq \frac{(2+o(1))n}{\boldsymbol{e}^{\gamma} \log k} = (2+o(1))|S_k \cap [n]| .$$

□

4.2. **An Improved Construction.** While $S_k \cap [n]$ is a k-multiplicative set whose cardinality is within a constant factor of optimal, larger k-multiplicative sets in $[n]$ can be constructed. Recall that $\mathbb{P}_k = \{p_1, p_2, \dots, p_\ell\}$ is the set of primes in $[k]$. Let $\alpha_i := \lfloor \log_{p_i} k \rfloor + 1$ for each $p_i \in \mathbb{P}_k$. Define

$$T_k := \{x \in \mathbb{N} : \beta_i(x) \equiv 0 \pmod{\alpha_i}, i \in [\ell]\} .$$

Lemma 8. *For each $k \in \mathbb{N}$, the set T_k is k-multiplicative.*

Proof. Suppose that $ax = by$ for some $a, b \in [k]$ and $x, y \in T_k$. By Equation (3),

$$\beta_i(a) + \beta_i(x) = \beta_i(b) + \beta_i(y)$$

for all $i \in [\ell]$. Now $\beta_i(x) \equiv \beta_i(y) \equiv 0 \pmod{\alpha_i}$ since $x, y \in T_k$. Thus $\beta_i(a) \equiv \beta_i(b) \pmod{\alpha_i}$. Now $p_i^{\beta_i(a)} \leq a \leq k$. Thus $\beta_i(a) \leq \lfloor \log_{p_i} k \rfloor = \alpha_i - 1$. Similarly $\beta_i(b) \leq \alpha_i - 1$. Hence $\beta_i(a) = \beta_i(b)$ for all $i \in [\ell]$. Thus $a = b$ and $x = y$. Therefore T_k is k-multiplicative. □

We now set out to determine the density of T_k. Observe that $S_k = \{x \in \mathbb{N} : \beta_i(x) = 0, i \in [\ell]\} \subset T_k$. Thus (if it exists) the density of T_k is at least that of S_k.

Consider $A, B \subseteq \mathbb{N}$ with $A \cap B = \emptyset$. If $\delta(A)$ and $\delta(B)$ exist, then $\delta(A \cup B) = \delta(A) + \delta(B)$. The following lemma extends this idea to an infinite union, where

$$\overline{\delta}(A) := \sup_{n \to \infty} \frac{|A \cap [n]|}{n} .$$

Lemma 9. *Let $A_1, A_2, \ldots \subseteq \mathbb{N}$ such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$. Suppose that for each $i \in \mathbb{N}$, $\delta(A_i)$ exists and $\overline{\delta}(A_i) \leq c \cdot \delta(A_i)$ for some constant $c \geq 1$. Let $A := \bigcup_i A_i$. Then $\delta(A) = \sum_i \delta(A_i)$.*

Proof. Let $\delta := \sum_i \delta(A_i)$. Let $\epsilon > 0$ be an arbitrary positive number. Let r_ϵ be the least integer such that

$$\sum_{i > r_\epsilon} \delta(A_i) < \frac{\epsilon}{c} .$$

Let n_ϵ be the minimum integer such that for all $n > n_\epsilon$ and for all $i \in [r_\epsilon]$,

$$\left| \frac{|A_i \cap [n]|}{n} - \delta(A_i) \right| < \frac{\epsilon}{r_\epsilon} .$$

Let $n > n_\epsilon$, $X := A \cap [n]$, $X_i := X \cap A_i$ and $X^* := \cup\{X_i : i > r_\epsilon\}$. We have $|X_i| < c \cdot \delta(A_i) n$. Thus

$$|X^*| < cn \sum_{i > r_\epsilon} \delta(A_i) < \epsilon n .$$

Therefore

$$\begin{aligned}
\left| \frac{|X|}{n} - \delta \right| &= \left| \left(\sum_{i \in [r_\epsilon]} \frac{|X_i|}{n} - \delta(A_i) \right) + \frac{|X^*|}{n} - \sum_{i > r_\epsilon} \delta(A_i) \right| \\
&< \sum_{i \in [r_\epsilon]} \left| \frac{|X_i|}{n} - \delta(A_i) \right| + \frac{|X^*|}{n} + \sum_{i > r_\epsilon} \delta(A_i) \\
&< r_\epsilon \frac{\epsilon}{r_\epsilon} + \frac{\epsilon n}{n} + \frac{\epsilon}{c} < \epsilon \left(2 + \frac{1}{c} \right) < 3\epsilon .
\end{aligned}$$

This proves that $\delta(A) = \delta$. $\square$

Theorem 6. *The set T_k is k-multiplicative with density*

$$\delta(T_k) = \delta(S_k) \prod_{i \in [\ell]} \left(1 + \frac{1}{p_i^{\alpha_i} - 1} \right) = \prod_{i \in [\ell]} \left(1 - \frac{1}{p_i} \right) \left(1 + \frac{1}{p_i^{\alpha_i} - 1} \right) .$$

Proof. For all $A \subseteq \mathbb{N}$ and $t \in \mathbb{N}$, let $t \cdot A := \{ta : a \in A\}$. If $\delta(A)$ exists then

$$\delta(t \cdot A) \;=\; \frac{\delta(A)}{t} \quad . \tag{6}$$

Now, for all $v \in \mathbb{N}_0^\ell$, let

$$S_k^v \;:=\; \left(\prod_{i\in[\ell]} p_i^{v_i\alpha_i}\right) \cdot S_k \quad .$$

Note that $S_k^v \cap S_k^w \;=\; \emptyset$ for distinct $v, w \in \mathbb{N}_0^\ell$. For all $v \in \mathbb{N}_0^\ell$ we have $\frac{\overline{\delta}(S_k^v)}{\underline{\delta}(S_k^v)} = \frac{\overline{\delta}(S_k)}{\underline{\delta}(S_k)}$. Now $T_k = \bigcup\{S_k^v : v \in \mathbb{N}_0^\ell\}$. By Lemma 9,

$$\delta(T_k) \;=\; \sum_{v\in\mathbb{N}_0^\ell} \delta(S_k^v) \quad .$$

By Equation (6) with $A = S_k$ and $t = \prod_i p_i^{v_i\alpha_i}$,

$$\delta(T_k) \;=\; \sum_{v\in\mathbb{N}_0^\ell} \delta(S_k) / \prod_{i\in[\ell]} p_i^{v_i\alpha_i} \quad .$$

Thus

$$\delta(T_k) \;=\; \delta(S_k) \sum_{v\in\mathbb{N}_0^\ell} \prod_{i\in[\ell]} p_i^{-v_i\alpha_i} \;=\; \delta(S_k) \prod_{i\in[\ell]} \frac{p_i^{\alpha_i}}{p_i^{\alpha_i}-1} \;=\; \delta(S_k) \prod_{i\in[\ell]} \left(1 + \frac{1}{p_i^{\alpha_i}-1}\right) \quad .$$

The result follows by substituting the expression for $\delta(S_k)$ from Theorem 4; see Table 2. $\square$

TABLE 2. The first 15 elements of the set T_k for each $k \le 15$.

k	T_k	density
2	$\{1,3,4,5,7,9,11,12,13,15,16,17,19,20,21,\dots\}$	$2/3$
3	$\{1,4,5,7,9,11,13,16,17,19,20,23,25,28,29,\dots\}$	$1/2$
4	$\{1,5,7,8,9,11,13,17,19,23,25,29,31,35,37,\dots\}$	$3/7$
$5,6$	$\{1,7,8,9,11,13,17,19,23,25,29,31,37,41,43,\dots\}$	$5/14$
7	$\{1,8,9,11,13,17,19,23,25,29,31,37,41,43,47,\dots\}$	$5/16$
8	$\{1,9,11,13,16,17,19,23,25,29,31,37,41,43,47,\dots\}$	$7/24$
$9,10$	$\{1,11,13,16,17,19,23,25,27,29,31,37,41,43,47,\dots\}$	$7/26$
$11,12$	$\{1,13,16,17,19,23,25,27,29,31,37,41,43,47,49,\dots\}$	$77/312$
$13,14,15$	$\{1,16,17,19,23,25,27,29,31,37,41,43,47,49,53,\dots\}$	$11/48$

We now show that $\delta(T_k)$ approaches $\delta(S_k)$ for large k.

Proposition 1. *For all $k \in \mathbb{N}$,*

$$\delta(S_k) \;<\; \delta(T_k) \;=\; c_k \cdot \delta(S_k) \;\;,$$

for some constant $c_k \to 1$ for large k.

Proof. By the Prime Number Theorem, $\ell \leq \mathcal{O}(k/\log k)$. Thus

$$\begin{aligned} c_k \;&=\; \prod_i \left(1 + \frac{1}{p_i^{\alpha_i} - 1}\right) \;<\; \prod_i \left(1 + \frac{1}{k-1}\right) \;\leq\; \left(1 + \frac{1}{k-1}\right)^{\mathcal{O}(k/\log k)} \\ &\leq\; \exp(\mathcal{O}(1/\log k)) \;\to\; 1 \;\;. \end{aligned}$$

□

The case $k = 2$ was previously studied by Tamura [34] and Allouche *et al.* [6]. Observe that $T_2 = \{2^{2i}(2j+1) : i, j \in \mathbb{N}_0\}$. Theorem 6 with $k = 2$ was proved by Allouche *et al.* [6], who also proved that T_2 has the maximum density out of all 2-multiplicative sets. Interesting relationships with the Thue-Morse sequence were also discovered.

Proposition 2 ([6])**.** *The set T_2 is 2-multiplicative and has density $2/3$. For all $d \in \mathbb{N}$, the d-th smallest element of T_2 is at most $\frac{3}{2}d + \mathcal{O}(\log d)$.*

Theorem 7. *Let $\mathcal{F}$ be a forbidden family. Let $G_1, G_2, \ldots, G_d$ be graphs, each with $\Lambda(G_i, \mathcal{F}) \leq 2$ or $\chi(G_i, \mathcal{F}) \leq 3$. Let t be the d-th smallest element of T_2. Then*

$$\begin{aligned} \Lambda(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, \mathcal{F}) \;&\leq\; 2t \;\leq\; 3d + \mathcal{O}(\log d) \;\;, \textit{ and} \\ \chi(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, \mathcal{F}) \;&\leq\; 4t + 1 \;\leq\; 6d + \mathcal{O}(\log d) \;\;. \end{aligned}$$

Proof. By Lemma 1, $\chi(G_i, \mathcal{F}) \leq 3$ implies $\Lambda(G_i, \mathcal{F}) \leq 2$. The result follows by applying Lemma 2 with the d smallest elements of the 2-multiplicative set T_2 from Proposition 2. □

5. P_3-free Colourings

Recall that a colouring is P_3-free if vertices at distance at most two receive distinct colours. Let $\Delta(G)$ be the maximum degree of the graph G. Since a vertex and its neighbours receive distinct colours in a P_3-free colouring,

$$\chi(G, P_3) \;\geq\; \Delta(G) + 1 \;\;. \tag{7}$$

Let $Q_d := K_2 \,\square\, K_2 \,\square\, \cdots \,\square\, K_2$ be the d-dimensional hypercube. P_3-free colourings of Q_d (and more generally, colourings of powers of Q_d) have been extensively studied [21, 26, 28, 36]. Wan [36] proved that

$$d + 1 \leq \chi(Q_d, P_3) \;\leq\; 2^{\lceil \log_2(d+1) \rceil} \;\;.$$

While our methods are not powerful enough to obtain the above upper bound, for grid graphs we have the following result that was first proved by Fertin *et al.* [18].

Example 1 ([18]). *Every d-dimensional grid graph $G := P_{n_1} \square P_{n_2} \square \cdots \square P_{n_d}$ satisfies $\chi(G, P_3) \leq 2d + 1$, with equality if every $n_i \geq 3$.*

Proof. The lower bound follows from Equation (7) since $\Delta(G) = 2d$ if every $n_i \geq 3$. Colour the i-th vertex in P_n by i. We obtain a P_3-free colouring of P_n with span 1. Thus $\Lambda(P_n, P_3) = 1$, and the upper bound follows from Theorem 2 with $k = 1$. □

Example 1 highlights the utility of chromatic span. A weaker bound on $\chi(G, P_3)$ is obtained if the P_3-free chromatic number, $\chi(P_n, P_3) = 3$, is used rather than the the P_3-free chromatic span, $\Lambda(P_n, P_3) = 1$.

Example 2. *Let G be the d-dimensional graph $G := P^2_{n_1} \square P^2_{n_2} \square \cdots \square P^2_{n_d}$. Let t be the d-th smallest element of T_2. Then*

$$\chi(G, P_3) \leq 4t + 1 \leq 6d + \mathcal{O}(\log d) \ ,$$

and if each $n_i \geq 5$ then $\chi(G, P_3) \geq 4d + 1$.

Proof. Equation (7) implies the lower bound since $\Delta(G) = 4d$ if each $n_i \geq 5$. Obviously $\Lambda(P^2_n, P_3) \leq 2$. Thus the upper bound follows from Theorem 7; see Table 3. □

TABLE 3. Upper bound on $\chi(G, P_3)$ for $G := P^2_{n_1} \square P^2_{n_2} \square \cdots \square P^2_{n_d}$ or $G := C_{n_1} \square C_{n_2} \square \cdots \square C_{n_d}$.

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$\chi(G, P_3) \leq$	5	13	17	21	29	37	45	49	53	61	65	69	77	81	85	...

Example 3. *Let G be the graph $P^k_{n_1} \square P^k_{n_2} \square \cdots \square P^k_{n_d}$. If there exists $n_i, n_j \geq k$ then $\chi(G, P_3) \geq k^2$, and if every $n_i \geq 2k + 1$ then $\chi(G, P_3) \geq 2dk + 1$. For an upper bound, we have*

$$\chi(G, P_3) \leq 2k(kd - k + 1) + 1 \ .$$

Moreover, for all $\epsilon > 0$ and for large $d > d(k, \epsilon)$,

$$\chi(G, P_3) \leq 1 + \frac{2\,e^{\gamma}}{1 - \epsilon}\, dk \log k \ .$$

Proof. If $n_i, n_j \geq k$ then G^2 contains a k^2-vertex clique, and $\chi(G, P_3) = \chi(G^2) \geq k^2$. The second lower bound follows from Equation (7) since $\Delta(G) = 2dk$ if every $n_i \geq 2k+1$. Obviously $\Lambda(P_n^k, P_3) \leq k$. Thus the upper bounds follow from Theorem 2. $\square$

Example 4. *The d-dimensional Hamming graph $G := K_n \,\square\, K_n \,\square\, \cdots \,\square\, K_n$ satisfies*

$$\max\{n^2, d(n-1)+1\} \;\leq\; \chi(G, P_3) \;\leq\; 2n(nd-n+1)+1 \ .$$

Moreover, for all $\epsilon > 0$ and for large $d > d(n, \epsilon)$,

$$\chi(G, P_3) \;\leq\; 1 + \frac{2\,\boldsymbol{e}^{\gamma}}{1-\epsilon}\, d \cdot n \log n \ .$$

Proof. Since G^2 contains an n^2-vertex clique, $\chi(G, P_3) = \chi(G^2) \geq n^2$. The second lower bound follows from Equation (7) since $\Delta(G) = d(n-1)$. Obviously $\Lambda(K_n, P_3) = n$. Thus the upper bounds follow from Theorem 2. $\square$

Example 5. *The d-dimensional toroidal grid $G := C_{n_1} \,\square\, C_{n_2} \,\square\, \cdots \,\square\, C_{n_d}$ satisfies*

$$2d+1 \leq \chi(G, P_3) \;\leq\; 4t+1 \;\leq\; 6d + \mathcal{O}(\log d) \ ,$$

where t is the d-th smallest element of T_2.

Proof. The lower bound follows from Equation (7) since G is $2d$-regular. Say $C_n = (v_1, v_2, \ldots, v_n)$. By considering the vertex ordering

$$(v_1, v_n; v_2, v_{n-1}; \ldots; v_i, v_{n-i+1}; \ldots; v_{\lfloor n/2 \rfloor}, v_{\lceil n/2 \rceil}) \ ,$$

of C_n, we see that $C_n \subset P_n^2$. Thus the upper bound follows from Example 2. $\square$

Fertin *et al.* [19] studied P_4-free colourings of toroidal grids, and proved that the minimum number of colours is at most $2d^2 + d + 1$, and at most $2d+1$ in the case that $2d+1$ divides each n_i. Thus Example 5 gives a linear upper bound on the P_3-free chromatic number of toroidal grids, where even for the weaker notion of P_4-free colourings, only a quadratic upper bound was previously known.

Example 6. *Let G be the graph $C_{n_1}^k \,\square\, C_{n_2}^k \,\square\, \cdots \,\square\, C_{n_d}^k$. If there exists $n_i, n_j \geq k$ then $\chi(G, P_3) \geq k^2$, and if every $n_i \geq 2k+1$ then $\chi(G, P_3) \geq 2dk+1$. For an upper bound, we have*

$$\chi(G, P_3) \leq 4k(2kd - 2k + 1) + 1 \ .$$

Moreover, for all $\epsilon > 0$ and for large $d > d(k,\epsilon)$,

$$\chi(G,P_3) \;\leq\; 1 + \frac{4\,\boldsymbol{e}^{\gamma}}{1-\epsilon}\, d \cdot k \log(2k) \;\; .$$

Proof. The lower bounds are the same as in Example 3. As proved in Example 5, $C_n \subset P_n^2$. Thus $C_n^k \subset P_n^{2k}$, and the upper bound follows from Example 3. $\square$

For $p \in \mathbb{N}$, an $\mathrm{L}(p,1)$-*labelling* of a graph G is a P_3-free colouring of G with the additional property that the colours given to adjacent vertices differ by at least p. Such colourings arise in frequency assignment problems. It is easily seen that Lemmata 1 and 2 can be generalised for $\mathrm{L}(p,1)$-labellings. Thus the above examples for P_3-free colourings can be generalised to this setting—we omit the details.

6. Acyclic Colourings

Recall that a colouring with no bichromatic cycle is *acyclic.* The following elementary lower bound is well known, where half the average degree of a graph G is denoted by

$$\overline{d}(G) \;:=\; \frac{|E(G)|}{|V(G)|} \;\; .$$

Lemma 10. *Every graph G (with at least one edge) has acyclic chromatic number $\chi(G,\mathcal{C}) > \overline{d}(G)+1$.*

Proof. Say G has an acyclic k-colouring. Let n_i be the number of vertices in the i-th colour class. Let $m_{i,j}$ be the number of edges between the i-th and j-th colour classes. Thus $m_{i,j} \leq n_i + n_j - 1$. Hence

$$|E(G)| \;=\; \sum_{1\leq i<j\leq k} m_{i,j} \;\leq\; \sum_{1\leq i<j\leq k} n_i + n_j - 1 \;\;=\; \sum_{1\leq i\leq k} (k-1)n_i \;-\binom{k}{2} \;=\; (k-1)|V(G)| - \binom{k}{2} \;\; .$$

Now $k \geq 2$ since G has at least one edge. Thus $k \geq (|E(G)|+1)/|V(G)|+1 > \overline{d}(G)+1$. $\square$

It is easily seen that a cartesian product satisfies

$$\overline{d}(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d) \;=\; \sum_{i\in[d]} \overline{d}(G_i) \;\; . \tag{8}$$

The following theorem, which was proved for paths by Fertin *et al.* [18], gives a special case when a $(k+1)$-colouring can be obtained from a colouring with span k, rather than the $(2k+1)$-colouring guaranteed by Lemma 1.

Proposition 3. *For all trees $T_1, T_2, \ldots, T_d$, the acyclic chromatic number*

$$\chi(T_1 \,\square\, T_2 \,\square\, \cdots \,\square\, T_d, \mathcal{C}) \;\leq\; d+1 \;\;,$$

with equality if every $|V(T_i)| \geq d$.

Proof. Let $\tilde{G} := T_1 \,\square\, T_2 \,\square\, \cdots \,\square\, T_d$. First we prove the lower bound. By Lemma 10 and Equation (8), and since $|V(T_i)| \geq d$,

$$\chi(\tilde{G}, \mathcal{C}) \;>\; \overline{d}(\tilde{G}) + 1 \;=\; 1 + \sum_{i \in [d]} \frac{|V(T_i)| - 1}{|V(T_i)|} \;=\; d + 1 - \sum_{i \in [d]} \frac{1}{|V(T_i)|} \;\geq\; d \;\;.$$

Hence $\chi(\tilde{G}, \mathcal{C}) \geq d + 1$.

Now we prove the upper bound. Root each tree T_i at some vertex r_i. For each vertex $v \in V(T_i)$, let $c_i(v)$ be the distance between r_i and v in T_i. Then c_i is a colouring of T_i with span one. For each vertex $\tilde{v} \in V(\tilde{G})$, let

$$c(\tilde{v}) \;:=\; \sum_{i \in [d]} i \cdot c_i(v_i) \;\;.$$

For each edge $\tilde{v}\tilde{w} \in E(\tilde{G})$ in dimension i,

(9)

$$c(\tilde{w}) - c(\tilde{v}) \;=\; \left(\sum_{j=1}^{d} j \cdot c_j(w_j) \right) - \left(\sum_{j=1}^{d} j \cdot c_j(v_j) \right) \;=\; i\big(c_i(w_i) - c_i(v_i)\big) \;=\; \pm i \;\;.$$

Thus c is a colouring of $\tilde{G}$ with span d. Let $c'(\tilde{v}) := c(\tilde{v}) \bmod (d+1)$. Obviously c' is a $(d+1)$-colouring of $\tilde{G}$. We claim that c' is acyclic.

Consider each edge of T_i to be oriented away from the root r_i. Orient each edge $\tilde{v}\tilde{w} \in E(\tilde{G})$ in dimension i according to the orientation of $v_i w_i$. That is, orient $\tilde{v}$ to $\tilde{w}$ so that $c_i(w_i) - c_i(v_i) = 1$. Clearly the orientation of $\tilde{G}$ is acyclic.

Suppose that on the contrary there is a vertex $\tilde{v} \in V(\tilde{G})$ that has two incoming edges $\tilde{u}\tilde{v}$ and $\tilde{w}\tilde{v}$ for which $c'(\tilde{u}) = c'(\tilde{w})$. Thus $c(\tilde{u}) \equiv c(\tilde{w}) \pmod{(d+1)}$ and

$$c(\tilde{u}) - c(\tilde{v}) \;\equiv\; c(\tilde{w}) - c(\tilde{v}) \pmod{(d+1)} \;\;.$$

Let i and j be the dimensions of $\tilde{u}\tilde{v}$ and $\tilde{w}\tilde{v}$, respectively. By Equation (9),

$$i(c_i(u_i) - c_i(v_i)) \;\equiv\; j(c_j(w_j) - c_j(v_j)) \pmod{(d+1)} \;\;.$$

By the orientation of edges, $c_i(u_i) - c_i(v_i) = 1$ and $c_j(w_j) - c_j(v_j) = 1$. Thus $i \equiv j \pmod{(d+1)}$, which implies that $i = j$. Hence $\tilde{u} = \tilde{w}$ since v_i has only one incoming edge in T_i (from its parent). Thus every vertex of $\tilde{G}$ has at most one incoming edge in each bichromatic subgraph H (with respect to the colouring c'). Hence H has an acyclic orientation with at most one incoming edge at each vertex. Therefore H is a forest, and c' is the desired acyclic colouring of $\tilde{G}$. □

7. P_4-free Colourings

Recall that a colouring with no bichromatic P_4 is a *star colouring*.

Example 7. *For trees $T_1, T_2, \ldots, T_d$, the star chromatic number*

$$\chi(T_1 \,\square\, T_2 \,\square\, \cdots \,\square\, T_d, P_4) \leq 2d + 1 \ .$$

Proof. Root each tree T_i at some vertex r_i. For each vertex $v \in V(T_i)$, let $c_i(v)$ be the distance between r_i and v in T_i. (This is the same colouring used in Proposition 3.) Obviously c_i is a P_4-free colouring of T_i with span one. The result follows from Theorem 2 with $k = 1$. Also note that the same lower bound from Proposition 3 applies for the star chromatic number. □

Example 8. *Let $\mathcal{G}$ be a minor-closed graph family that is not the class of all graphs. Then there is a constant $c = c(\mathcal{G})$ such that for all graphs $G_1, G_2, \ldots, G_d \in \mathcal{G}$,*

$$\chi(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, P_4) \ \leq \ cd \ .$$

Proof. Nešetřil and Ossona de Mendez [24] proved that there is a constant c_1 (bounded by a small quadratic function of the maximum chromatic number of a graph in $\mathcal{G}$) such that every graph $G \in \mathcal{G}$ has star-chromatic number $\chi(G, P_4) \leq c_1$. By Theorem 1, there is constant c_2 (bounded by a small quadratic function of c_1) such that $\chi(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d, P_4) \leq c_2 d$. □

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