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Good weighted graphs

Posted by Petr Škovroň

Consider a finite set V. The pairs of elements of V have assigned real or infinite weights by a mapping $w : \binom{V}{2} \to R \cup \{\infty\}$. For simplicity of notation we put $w(a, b) = w(\{a, b\})$. Let us agree on putting $x < \infty$ for all $x \in R$.

Define w to be *good*, if

1. in any four-element subset, if there is a strictly maximal w, the second greatest value is not on the two opposite elements; formally, for every $a, b, c, d \in V$ and $T \in R$ with $w(a, b) \leq T$, $w(a, c) \leq T$, $w(a, d) \leq T$, $w(b, c) \leq T$, $w(b, d) \leq T$ and $w(c, d) > T$, we have $w(a, b) < T$; in yet other words, the ordering others $\leq w(a, b) < w(c, d)$ is prohibited,

2. for some $a, b \in V$ we have $w(a, b) = \infty$.

For a good w, define a *basis graph of* w $G = (V, E)$ by connecting all pairs of vertices of infinite w:

$$E = \{\{a, b\} : w(\{a, b\}) = \infty\}.$$

What graphs can we get? Is there a polynomial algorithm that for a given graph decides, whether it arises from some good w?

Obviously (by the second condition of the definition of good w), a graph with no edges is not a basis graph. A nontrivial example of a nonbasis graph consists of three nonadjacent edges on six vertices.

The problem is related to removing degeneracy in certain combinatorial models of linear programming.

A problem in extremal matroid theory

Posted by Anna de Mier

A flat F of a matroid is said to be *cyclic* if it is a union of circuits; or, equivalently, if F is not of the form $G \cup x$ for some flat G and some element $x \in F$. For instance, if M is the cycle matroid of some graph G, then a flat is cyclic if the corresponding set of edges is bridgeless.

We are interested in the following question:

Which is the maximum number of cyclic flats that a matroid on m elements can have?

This question was originally posed in the paper [1] devoted to several properties of cyclic flats and their lattice structure.

Current knowledge. A reasonable first try would be the direct sum of $m/2$ copies of $U_{1,2}$ (a parallel point). This gives $2^{m/2}$ cyclic flats for even m. A better bound, and the best one we know, is given by binary spikes. Recall that an *n-spike* is a rank-n matroid consisting of n 3-point lines $l_1, \ldots, l_n$ that have a common point a, called the tip, and such that every subset of $k < n$ such lines has rank $k+1$. In other words, if we label the elements in l_i as $\{a, x_i, y_i\}$ the only non-spanning circuits of an n-spike are the lines $\{a, x_i, y_i\}$, the 4-circuits $\{x_i, x_j, y_i, y_j\}$ for all $i \neq j$ and perhaps some circuit-hyperplanes of the form $\{z_1, z_2, \ldots, z_n\}$ where $z \in \{x, y\}$. The *binary n-spike* is the unique n-spike that is representable over GF(2) (for $n = 3$ it is the Fano plane). It has 2^{n-1} circuit-hyperplanes, since it can be shown that for any choice of $(z_1, \ldots, z_{n-1}) \in \{x_1, y_1\} \times \cdots \times \{x_{n-1}, y_{n-1}\}$ there is a unique $z_n \in \{x_n, y_n\}$ such that $\{z_1, \ldots, z_{n-1}, z_n\}$ is a circuit. The counting of cyclic flats for the binary n-spike goes as follows:

$$1+n+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n-2}+2^{n-1}+1 = 2^n-n+2^{n-1} = 3\cdot 2^{n-1}-n$$

This is for $m = 2n+1$. If $m = 2n$, the best example we know is obtained by deleting the tip from the binary n-spike. In this case the bound is $3 \cdot 2^{n-1} - 2n$. In terms of m, the maximum number of cyclic flats of an m-element matroid is at least

$$\begin{cases} 3 \cdot 2^{(m-3)/2} - (m-1)/2 & \text{if } m \text{ is odd} \\ 3 \cdot 2^{(m-2)/2} - m & \text{if } m \text{ is even} \end{cases}$$

(Note: for $m \leq 6$ these are not the optimal values.)

References

[1] J. Bonin and A. de Mier, The lattice of cyclic flats of a matroid, arXiv.org math.CO/0505689.

Edge disjoint paths on the brick wall

Posted by Petr Kolman

In the maximum *edge disjoint paths* problem we are given a graph $G = (V, E)$ and a set $T = \{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}$ of k pairs of vertices. The objective is to connect a maximum number of pairs from T along edge disjoint paths.

The *brick wall* graph is depicted on Figure 1. The best known algorithm for the edge disjoint paths problem on the brick wall has approximation $O(\sqrt{|V|})$; the algorithm is simple: consider the pairs one by one and if there is a path between s_i and t_i of length at most $2\sqrt{|V|}$ that is edge disjoint with all paths established so far, use the path for the pair. Design an algorithm with a better approximation on the brick wall.

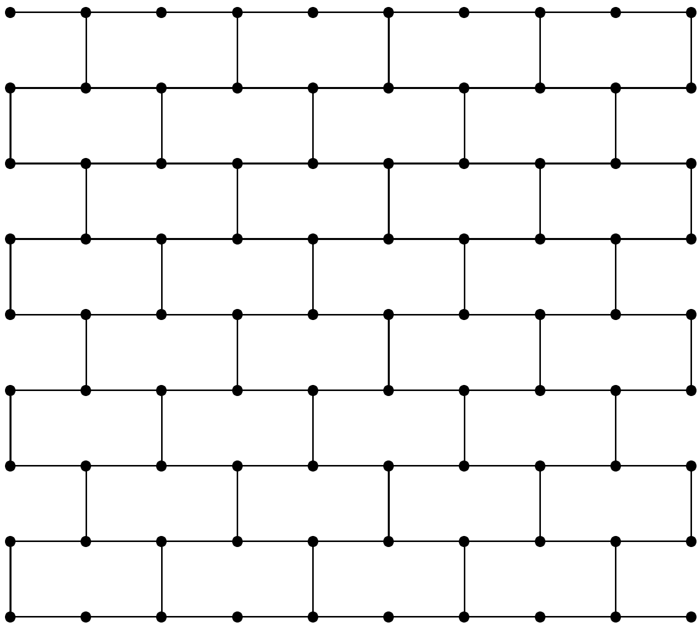

Figure 1: Brick wall 9×10.

Set Compression

Posted by Aleš Přívětivý

Let $\mathbf{F}$ be a prime field $\mathbf{Z}/q\mathbf{Z}$. Let us define *diameter* of a set $A \subseteq \mathbf{F}$ as the maximal distance of two elements of A, i.e.

$$\operatorname{diam}(A) = \max\{|a-b| : a, b \in A\},$$

where

$$|a-b| = \min(a-b, b-a).$$

For a fixed element j of field $\mathbf{F}$ we define a *j-shift* as a mapping which maps any subset $A \subseteq \mathbf{F}$ to the set containing all elements of A multiplied in the field $\mathbf{F}$ by j, i.e.

$$jA = \{ja : a \in A\}.$$

For a positive integer n, let $[n]$ denote the set $\{0, 1, \ldots, n-1\}$. For any $a \in \mathbf{Z}$ and $d_1, n_1 \in \mathbf{N}$ we define an *arithmetic progression* as a subset of Z given by the formula

$$AP(a, d_1, n_1) = \{a + id_1 : i \in [n_1]\}.$$

The definition of arithmetic progression can be extended to a sum of k arithmetic progressions, but in our problem we are interested only in sums of two arithmetic progression, defined for $a \in \mathbf{Z}$ and $d_1, d_2, n_1, n_2 \in \mathbf{N}$ as the set

$$AP_k(a, d_1, d_2, n_1, n_2) = \{a + i_1 d_1 + i_2 d_2 : i_1 \in [n_1], i_2 \in [n_2]\}.$$

Our goal is to find a set system $\mathcal{S} = \{A_1, A_2, \ldots, A_m\}$ on $\mathbf{F}$, satisfying

- every A in $\mathcal{S}$ is a subset of $\mathbf{F}$ and a sum of two arithmetical progressions with at least n elements,
- for every $j \in \mathbf{F}$ there exist a set A in $\mathcal{S}$ such that $\operatorname{diam}(jA) \leq \frac{1}{6}|\mathbf{F}|$,

with quantity $n/\sqrt{m}$ as large as possible.

Currently best known lower bound is from [Heb04], giving lower bound $n/\sqrt{m} = \Omega\left(|\mathbf{F}|^{1/3}\right)$. The upper bound is due to random coloring from combinatorial discrepancy $n/\sqrt{m} = O\left(|\mathbf{F}|^{1/2}\right)$.

References

[BS95] J. Beck and V. Sós. *Discrepancy Theory.* In Handbook of Combinatorics, pages 1405-1446. North-Holland, Amsterdam, 1995.

[Heb04] N. Hebbinghaus. Discrepancy of Sums of Arithmetic Progressions. *In Electronic Notes in Discrete Mathematics 17C*, pages 185-189, 2004.

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Coloring of arithmetic progression

Posted by Jaroslav Nešetřil

This problem is originally from Jacob Fox from MIT.

Suppose we divide a set$\{1, 2, \ldots, 2N\}$ into two similar parts A and B. The question is whether there exists an arithmetic progression of length k such that 'the coloring' of this progression will be abababababab or bababababa or aaaaaabbbbbb or bbbbaaaa.

The degree of cube graph

Posted by Jaroslav Nešetřil

This problem is originally from S. Lev.

Suppose we have a graph of cube Q_n. It has 2^n vertices and independence number α equal to the half of that. We take a subgraph with $\alpha + 1$ vertices. This is no longer independent set. Let d be the minimum degree. We ask for the value of d. The hypothesis is $\sqrt{n}$.

Plurality of symmetric difference

Posted by Jaroslav Nešetřil

Suppose that X is large set and $\mathcal{M}$ is system of subsets of X such that each subset of X is a symmetric difference of two sets from $\mathcal{M}$. It is possible to find out any $Y \subset X$ such that Y can be defined as such symmetric difference in at least k different ways. In more formal way we can ask ourself whether for every k there exists X such that for every M with the above mentioned properties exists Y.

The core of M-graph

Posted by Robert Šámal

Let M be a finite abelian group—even the case $M = \mathbb{Z}_2$ is not known and interesting. We will say that a graph G is an *M-graph* if it is a Cayley graph on some power of M; explicitly, if there is an integer k and $S \subseteq M^k \setminus \{0\}$ satisfying $S = -S$, such that $V(G) = M^k$ and $E(G) = \{uv, v - u \in S\}$. ($\mathbb{Z}_2$-graphs are also called cube-like graphs; they have been introduced by Lovász as an example of graphs, for which every eigenvalue is an integer.)

The *core* of a graph G is a graph $G' \subseteq G$ such that G maps homomorphically to G' and G' is minimal with this property.

Question 1 *Is the core of each M-graph an M-graph?*

Let G be a directed graph, M an abelian group and $B \subseteq M$ such that $B = -B$. A mapping $\varphi : E(G) \to M$ is a *flow* if it 'sums to zero around each vertex'. If a flow φ uses only values from B, we call it an *(M, B)-flow.* Let $\mathrm{Cay}(M, B)$ be the Cayley graph with vertices M and B as the connection set.

Conjecture 1 (DeVos) *Let M, M' be abelian groups, let $B \subseteq M$, $B' \subseteq M'$ satisfy $B = -B$, $B' = -B'$. Let there be a homomorphism from $\mathrm{Cay}(M, B)$ to $\mathrm{Cay}(M', B')$. Then every graph with (M, B)-flow admits an (M', B')-flow.*

Outdegree of vertex-disjoint subgraphs

Posted by Robert Šámal

Conjecture 1 (Thomassé) *Let D be a digraph with minimum outdegree d and with directed girth at least g. Then D has a (directed) path of length $(g-1)d$.*

(Even the case $g = 3$ is open.) If true, this conjecture may be hard to prove since it implies the Caccetta-Häggkvist conjecture. (Let D be a digraph of order n in which every vertex has outdegree at least d. Then D contains a directed cycle of length g such that $(g-1)d < n$.)

Question 2 (Alon) *Does every directed graph, where each vertex has out-degree at least 10^{1000} contain two vertex-disjoint subgraphs, in which each vertex has outdegree at least 2?*

Computational complexity of CSP for a description of reducts

Posted by Manuel Bodirský

What is the computational complexity of the following problem.

INPUT: A 4-uniform hypergraph with vertex set V and edge set E.

QUESTION: Is there a graph G on the vertex set V such that every edge $\{v_1, v_2, v_3, v_4\}$ from E of the input induces one of the following graphs in G

1. two isolated points and an edge,
2. one isolated point and a 3-vertex path P_3,
3. the 4-vertex path P_4,
4. an edge attached to the triangle K_3,

5. K_4 without an edge.

BACKGROUND: The relation E is a relation that appears in the study of the reducts of the the countably infinite random graph by Simon Thomas. It shows, that the permutation group F consisting of the automorphisms, anti-automorphisms, and "switching-permutations" of the random graph is not 4-transitive. The above computational problem is a classical constraint satisfaction problem (CSP) for a description of the reduct of the random graph that has F as its automorphism group.

The centre of points in plane

Posted by Jiří Fiala

We have n points in a plane and we want to find such a point k (a centre) such that the size of all angles determined by half lines lead from such centre k to particular points will be as close as possible to value $\frac{2\pi}{n}$.

There are few results already known. If we assume that all angles are same we can solve the problem in linear time. However in this case such a centre does not need to exists.

The numbers of graphs containing no k-*crossing* and no k-*nesting*

Posted by Vít Jelínek

Assume not oriented simple graphs on vertex set $\{1, 2, \ldots, n\}$. Next we assume two disjoint edges $\{i, j\}$ and $\{k, l\}$ with $i < j$, $k < l$ and $i < k$. We say that these two edges *cross* themselves if $i < k < j < l$ and we say that these two edges are *nested* if $i < k < l < j$. Then k-*crossing* is k-tuple of edges from which each two cross themselves. Analogously k-*nesting* is k-tuple of edges from which each two are nested.

The problem is to prove or disprove that for every n and k is the number of graphs containing no k-*crossing* the same as the number of graphs containing no k-*nesting*.

There are some already known results.

1. There exist similar relations for specific classes of graphs. For example for perfect matching (i.e. only graphs with all degrees equal to one) this proposition holds. This proposition also holds for graphs of which each vertex is connected with at most on larger vertex and at most one small vertex (i.e. only disjoint union of monotone paths). Both proofs are difficult.

2. It is true if $n = 2k + 1$ (and for $n \leq 2k$ it is trivial).

3. Atilla Por has tested this problem by computer and he has not found any counterexample.

Complexity of no-rainbow coloring problem

Posted by Jan Kára

Let $H = (V, E)$ be a 3-uniform hypergraph. We say that a coloring c of vertices of H is *no-rainbow* if for each hyperedge $e \in E$ there are two vertices $u, v \in e$ with $c(u) = c(v)$. What is the complexity of decision of the following question: does H have a no-rainbow coloring using 3 colors (every color must be used at least once)? Can you solve the problem for k colors?

Complexity of recognizing intersections graphs of congruent triangles

Posted by Jan Kratochvíl

The following problem reflects a question of Aleš Pultr asked after Jan Kratochvíl's talk at Workshop on Graph Classes, Width Parameters and Optimization held in Prague on October 17-19, 2005. The answer is still not clear to me and could result in an interesting problem.

What is the computational complexity of recognizing intersections graphs of congruent triangles, whose bases are placed in two parallel lines? (If the bases lie in one line, then we get exactly interval graphs.)

On graphs with large independent sets everywhere

Posted by Jiří Matoušek

It is true that for every $\alpha > 0$ there exists $\beta > 0$ such that if G is a graph in which every subset $S \subseteq V(G)$ contains an independent set of size at least $\alpha|S|$ and $\omega : V(G) \to [0, \infty)$ is a weight function on vertex set, then every subset $S \subseteq V(G)$ contains an independent subset of weight at least $\beta\omega(S)$? Here $\omega(S) = \sum_{v \in S} \omega(S)$.

Comments: The first property, that every subset S of vertices contain an independent subset of size at least αS, is sometimes called a *Pisier-type* property (for independent sets). So here we ask whether a Pisier-type property for independent sets implies a *weighted* Pisier-type property for independent sets. The first thing to realize is perhaps that there are graphs with a Pisier-type property for independent sets, for suitable fixed $\alpha > 0$, and with an arbitrarily large chromatic number. (If no such graph existed, a positive answer to the question will be easy.).

Geometric blocking sets

Posted by Jiří Matoušek

Let P be an n-point set in the plane in general position (no three points on a common line). Let $b(P)$ denote the smallest size of a point set B such that $B \cap P = \emptyset$ and for every two points $p, q \in P$ the segment pq contains a point of B. Let $b(n)$ be the maximum of $b(P)$ over all n-point sets P in general position. The problem is to obtain any nontrivial asymptotic estimates of $b(n)$. This problem is inspired by Ramsey-type question of Kára et al.

Proper edge colorings of K_n with colorful K_4's

Posted by Jiří Matoušek

This is combinatorial abstraction of previous problem (which may or may not be useful). What is the smallest number $f(n)$ of colors such that there exists a coloring of the edge set of complete graph K_n by $f(n)$ colors, such that, first, no two edges sharing a vertex have the same color, and second, at least 5 colors are used on the edges of every subgraph of K_n induced by 4 vertices.

It may be that the answer is known. A simple upper bound is $O(n \log(n))$. If we require only the second condition, there is a coloring by $O(n)$ colors.

Mapping of plannar graph with girth 9

Posted by Nigussie Yared

Let C_8 be the the circular graph on 8 vertices. Is it true that every planar graph with girth 9 maps to C_8? How about girth 7?

k-EPT graphs

Posted by Martin Pergel

Let k-EPT graphs be intersection graphs of paths in a tree with intersection defined only for graphs with at least k common edges.

It is known that characterization of k-EPT graph is NP-complete. There are also some results about coloring, but there is still one open question. It is about the inclusion of k-EPT and $(k+1)$-EPT. It is known [Golumbic, Lipshteyn, Stern] that there exist $(k+1)$-EPT graphs which are not k-EPT. It is also known that k-EPT graph is also k'-EPT graph for $k' \geq k^2 - 2k + 2$. The question is to find the better estimate of k' or to find graphs which are k-EPT and which are not k''-EPT for $k < k'' < k'$.

Optically all k-EPT is also $2k$-EPT (subdivision of each edge), but how the situation change for $2k+1$-EPT? The question [Jamison, Mulder] is if the $k' = k + 1$ is enough.

Coloring of points in plane

Posted by Moshe Rosenfeld

Is it possible to color the points of the plane in a finite number of colors so that two points whose Euclidean distance is an odd integer get distinct colors?

Note: If you take 4 points in the plane then at least 1 of the 6 distances determined by them will not be an odd integer.

Hamiltonity of prism

Posted by Moshe Rosenfeld

Is the prism over a 4-connected, 4-regular graph hamiltonian?

Tightness of edge-critical k-forest

Posted by Ricardo Strausz

The question is short. Are edge-critical k-forest tight? For better insight into that problem there are some previous results.

A k-graph is a uniform hypergraph of rank k; i.e, a set of vertices V and a family E of subsets of V, called the edges, all of order k ($E \subseteq \binom{V}{k}$). For $k = 2$ we have a simple graph. Let $H = (V, E)$ be a k-graph. The cardinal $n = |V|$ is the order and the cardinal $m = |E|$ is the size.

H is said to be *tight* if for every k-coloring of its vertices (not necessarily proper, what ever that means), there exists an edge which uses the k colors. Clearly a 2-graph —or a simple graph, if you will— is tight if and only if it is connected.

Theorem 1 (Arocha, Bracho and Neumann-Lara) *The minimum size of a tight 3-graph of order n is n(n-2)/3. (This can be generalised for all k.)*

H is said to be a *k-forest* if for every edge, there exists a k-coloring (again, not necessarily proper) in which ONLY such an edge uses the k colors. Again, a 2-graph is a 2-forest if and only if it is a forest (it is acyclic).

A k-forest which is tight is called a *k-tree.*

Observation 1 *If H is an edge-critical (minimal) tight k-graph, then it is a k-tree.*

Theorem 2 (Lovasz) *The maximum size of a* 3*-forest of order* n *is* $\frac{n(n-1)}{2}$*.* *(Again, this can be generalised to all* k*.)*

Theorem 3 (Strausz) *If* H *is a* k*-forest of order* n *and size* $\binom{(n-1)}{(k-1)}$ *then it is tight (and therefore a* k*-tree).*

Are all edge-critical (maximal) k-forests tight (and therefore k-trees)?

Observe that there are maximal k-forest of size strictly less that the maximum proved by Lovasz. Even for $k = 3$, as far as I know, the question remains open for $\frac{n(n-2)}{3} < m < \frac{n(n-1)}{2}$.