Spring School on Combinatorics 2005

Eva Ondráčková, Tomáš Valla (eds.)

Spring school on Combinatorics is a traditional meeting organized for members of the Combinatorial Seminar at Charles University for nearly 30 years. By now it is well known internationally and it is regularly visited by our cooperating institutions in the DIMATIA and COMBSTRU networks. In the years 1999–2001, and again in 2004–2006, the school is supported by ERASMUS–SOCRATES Intensive Programme 503334–IC–1–2002–1–CZ–ERASMUS–IPUC–1 which includes participation of universities from Bonn, Berlin, Bordeaux, Barcelona, Pisa and recently Bergen.

The Spring Schools are organized by our undegraduate students and while the lectures are selected by senior people of KAM and ITI and other participating institutions, the lectures themselves are given by students (both graduate an undergraduate) only. This leads to unique atmosphere of the meeting which helps the students in further studies and their orientation.

This year the Spring School was organized in Borová Lada, a mountain village in Šumava with a great variety of possibilities for hiking a biking. Some of it is mirrored by photos in this volume.

We thank Eva Ondráčková and Tomáš Valla as the main organizers who also edited this volume. We also thank Martin Loebl, Pavel Valtr and other colleagues who took part both in the organization and in the Spring School itself.
We hope to meet all this year's participants in 2006 again!

Jan Kratochvíl, Jaroslav Nešetřil

List of all talks (in order of appearance)

Petr Škovroň

xofon@kam.mff.cuni.cz

Presented paper by A. V. Kostochka and V. Rödl

On graphs with small Ramsey number

(J. Graph Theory 37 (2001) 198–204)

Let $R(G)$ denote the minimum integer N such that for every bicoloring of the edges of K_N, at least one of the monochromatic subgraphs contains G as a subgraph. We show that for every positive integer d and each γ, $0 < \gamma < 1$, there exists $k = k(d, \gamma)$ such that for every bipartite graph $G = (W, U; E)$ with the maximum degree of vertices in W at most d and $|U| \leq |W|^\gamma$, we have $R(G) \leq k|W|$.

Josef Cibulka

josef.cibulka@seznam.cz

Presented paper by P.Balister, B.Bollobás, O.Riordan and R.H.Shelp

Graphs with large maximum degree containing no odd cycles of a given length

(http://www.sciencedirect.com/)

There are many results showing that every graph with enough edges, or satisfying some natural degree conditions, contains long cycles of certain lengths. Here we added the condition that the graph has at least one vertex of large degree.

Let us write $f(n, \Delta, C_{2k+1})$ for the maximal number of edges in a graph on n vertices with maximum degree Δ that contains no cycles of length 2k+1. For $\frac{n}{2} \leq \Delta \leq n - k - 1$ and n sufficiently large we show that $f(n, \Delta, C_{2k+1}) = \Delta(n - \Delta)$ and that the complete bipartite graph $K_{\Delta, n-\Delta}$ is the only one with that many edges. The upper bound holds for smaller Δ as well, but in that case the obvious bound $\frac{n\Delta}{2}$ is better.

The article also contains a proof that for every graph $|E(G)| \leq \frac{1}{2} \sum\limits_{v \in V(G)} l_G(v)$ where $l_G(v)$ means the length of a longest path starting in v. This is an improvement of an old result of Erdos and Gallai.

Christian Sloper
sloper@ii.uib.no

Fixed Parameter Complexity and Set Splitting

Basic Fixed Parameter Complexity Design

It is commonly believed that there is no polynomial time algorithm for NP-complete problems and that a combinatorial explosion in the running time is inevitable. Parameterized complexity is one of many recent inventions to combat this intractability of NP-complete problems. It is based on the idea that for many problems the 'natural' instances are not uniformly distributed among all possible instances. In many cases we can single out a parameter, that for most real-life applications is kept small. This can be the size of the output, the treewidth of the inputgraph or other information about the input or the output.

What we want to do in parameterized complexity is to design algorithms where the combinatorial explosion is limited to the parameter. That is, we wish to obtain an algorithm with a running time of the type $f(k)n^{O(1)}$ where $f(k)$ is an arbitrary (often exponential) function. The idea is then that if k is fixed, the running time will be polynomial. If k then increases much slower than n we will have an 'almost' polynomial algorithm for the problem and we might be able to solve instances of considerable size. If such an algorithm exists we say that the problem is Fixed Parameter Tractable or in the class FPT. However, not all problems fall into this class. While it has been known for some time that we can obtain a 2k kernel for Vertex Cover [1], for Dominating Set we now of nothing substantially better than an $n^{O(k)}$ algorithm trying every possible subset of size k. We substantiate the belief that there is no FPT-algorithm for Dominating Set and other problems by a completeness-hierarchy $FPT \subseteq W[1] \subseteq W[2] \subseteq \ldots \subseteq W[P]$ which is similar to $P \subseteq NP$. It is possible to show that Dominating Set is $W[2]$-complete and it is thus highly unlikely that it will have a FPT-algorithm.

We will now proceed to demonstrate a few positive techniques for designing algorithms in Fixed Parameter Complexity. Throughout this first part we will focus on the well known problem Vertex Cover.

Vertex Cover
Instance: Graph $G = (V, E)$
Parameter: A positive integer k
Question: $\exists S \subseteq V$, $|S| \leq k$ s.t. every edge $e \in E$ is incident to a vertex $s \subseteq S$?

The first method we will demonstrate is Bounded Search Tree. In bounded search tree we show how to limit the growth of the search tree. For Vertex Cover this is based on the simple observation that for every edge one of the endpoints must be selected. If we remove any of these vertices we are left with a smaller instance of vertex cover. So we have an obvious recursive algorithm that checks one edge, tries both possibilities to cover the edge and solves the two remaining subproblems. Since the number of vertices in a vertex cover is limited by k the depth of the recursion is also limited. To calculate the size of the recursion tree we get the following recursive equation $T(k) = 2 * T(k-1), T(0) = 1$. Which solves to $T(k) = 2^k$, and at each step algorithm does linear work, so we have our first algorithm for vertex cover with running time $O(2^k n)$.

An important method of obtaining FPT-algorithms is reduction to problem kernel. Here we attempt to reduce the graph in size such that the remainder (the kernel) has less than $g(k)$ vertices, where $g(k)$ is a function only dependent on k. Note that given such a kernel we can trivially construct a FPT-algorithm by testing every possible solution.

To reduce a graph G we commonly define a set of *reduction* rules. A reduction rule can be executed in polynomial time and identifies a subgraph of G that either is redundant, which can be safely deleted, or necessary, which must be in a solution. In either case the instance can be reduced in size. We will now give an easy example of how to obtain a $O(k^2)$ kernel for the Vertex Cover problem. We will base the argument on the observation that in any vertex cover C either $v \in C$ or $N(v) \in C$. Since we can select at most k vertices we must select every vertex of degree greater than k. From this we get a reduction rule stating.

Reduction rule. *$v \in V(G), deg(v) > k$ has a k-vertex cover $\iff G' = G[V - v]$ has a $k-1$ vertex cover.*

We can now iterate this reduction rule until we get a reduced instance G' with no vertex of size k or more. We can now argue that the size of G' is at most $O(k^2)$ as follows. Since maximum degree in in G' is k, any vertex in G can at most cover k edges. Thus G' cannot have more than k^2 edges, and the result follows.

Set Splitting

The second part of this abstract will discuss the problem Set Splitting.

k-SET SPLITTING

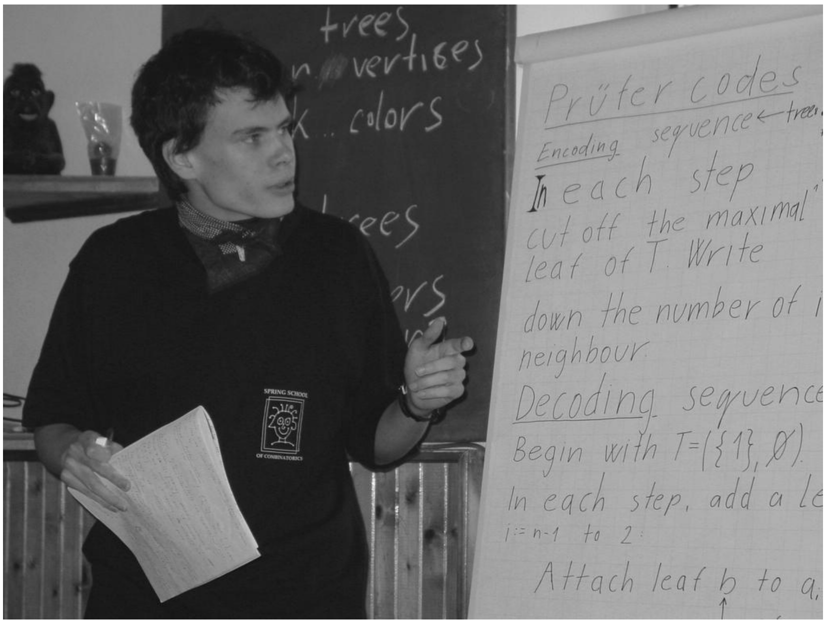
INSTANCE: A tuple $(X, \mathcal{F}, k)$ where $\mathcal{F}$ is a collection of subsets of a finite set X, and a positive integer k.

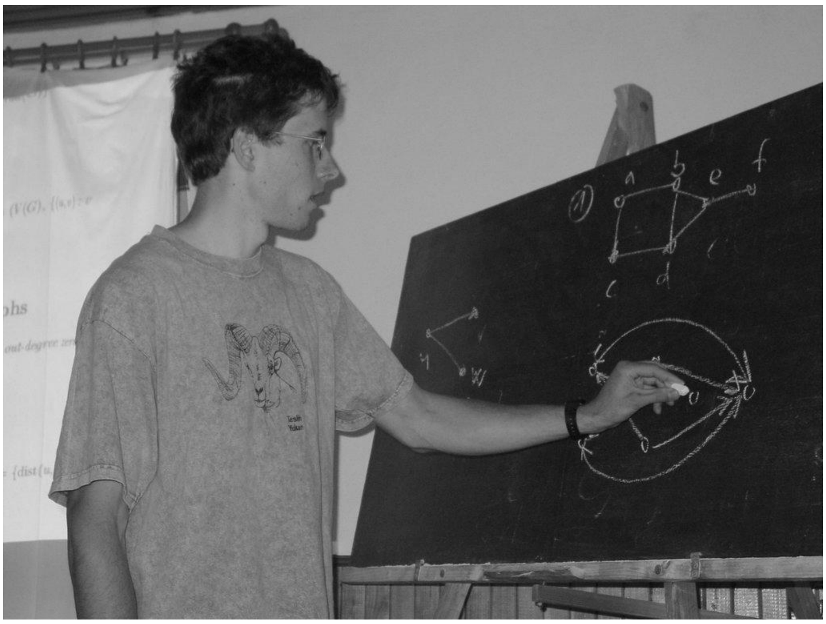
PARAMETER: k

QUESTION: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}, |\mathcal{F}'| \geq k$, and a partition of X into disjoint subsets X_0 and X_1 such that for every $S \in \mathcal{F}', S \cap X_0 \neq \emptyset$ and $S \cap X_1 \neq \emptyset$.

To improve the running time we show that any non-trivial solution of SET SPLITTING has a SET COVER of size at most k, we can then reduce the problem to 2^k instances of Max Sat with k clauses each.

Let a set cover be a subset $S \subseteq X$ such that for every set $P \in \mathcal{F}, P \cap S \neq \emptyset$.


trees
vertices
colors
Prüter codes
Encoding sequence ← trees
In each step
cut off the maximal
leaf of T. Write
down the number of
neighbour
Decoding sequence
Begin with T=({1},∅).
In each step, add a le
i:=n-1 to 2
Attach leaf b to a



We base this on a proof that an instance either has a set cover of size k or it has a k-Set Splitting.

Lemma 1. *Any instance $(X, \mathcal{F}, k)$ of Set Splitting that has a minimal set cover S, has a partitioning of X into disjoint subsets X_0 and X_1 such that at least $|S|$ sets are split.*

Theorem. *Set Splitting can be solved in time* $\mathcal{O}^*(2.6494^k)$

Proof. We obtain a minimal set cover S by greedily selecting vertices to cover all sets, by Lemma 1 we know that S has size less than k, otherwise we can immediately answer 'Yes'. Let $\mathcal{P} = \{P \mid P \in \mathcal{F}, P \nsubseteq S\}$. It is clear that $|\mathcal{P}| < k$, otherwise the partition $S, X \setminus S$ splits at least k sets. The remaining sets are only affected by how we partition S.

Observe that if S was already partitioned into disjoint subsets X'_0, X'_1 every set in $\mathcal{P}$ has at least one member in X'_0 or in X'_1.

Assume we have a partitioning X'_0, X'_1 of S. For each set $R \in \mathcal{P}$, where R is not split by X'_0 and X'_1, create a clause C_R. If R contains an element in X'_0 add literals x_i for each element $x_i \in R - S$ to C_R, if R contains an element in X'_1 then add literals $\overline{x_i}$, for each element $x_i \in R - S$ to C_R.

Adding an element x to X'_0 now corresponds to setting variable x false, and vice versa. Observe that a set $R \in \mathcal{P}$ is split iff its clause C_R is satisfied. We can now employ Chen and Kanj's exact algorithm for Max Sat. There are 2^k different partitions of the set cover S, for each we construct an instance of Max Sat with at most k clauses. Thus we get a total running time of $\mathcal{O}^*(2^k * 1.3247^k) = \mathcal{O}^*(2.6494^k)$. $\diamondsuit$

References

[1] J. Chen, I. Kanj, and W. Jia. Vertex cover: Further Observations and Further Improvements. *Journal of Algorithms* Volume 41, 280–301 (2001).

Ondřej Suchý
ondra@s.cz

Presented paper by Peter J. Cameron and C. Y. Ku

Intersecting families of permutations

(http://dx.doi.org/10.1016/S0195-6698(03)00078-7)

Let S_n be the symmetric group on the set $X = \{1, 2, \ldots, n\}$. A subset S of S_n is intersecting if for any two permutations g and h in S, $g(x) = h(x)$ for some $x \in X$ (that is g and h agree on x). Deza and Frankl (J. Combin. Theory Ser. A 22 (1977) 352) proved that if $S \subseteq S_n$ is intersecting then $|S| \leq (n-1)!$. This bound is met by taking S to be a coset of a stabiliser of a point. We show that these are the only largest intersecting sets of permutations.

Definition. We say that $S \subseteq S_n$ is a coset of stabiliser of a point, iff there exist $x, y \in X$, such that $S = \{g | g \in S_n \wedge g(x) = y\}$.

Main theorem. *Let $n \geq 2$ and $S \subseteq S_n$ be an intersecting set of permutations such that $|S| = (n-1)!$. Then S is a coset of a stabiliser of one point.*

Alexandr Kazda

alexandr.kazda@seznam.cz

Presented paper by M. Cho, D. Kim, S. Seo, H. Shin

Colored Prüfer codes for k-edge colored trees

(The Electronic Journal of Combinatorics 11/2004, #N10)

The main goal of the article is to provide a bijective proof of the formula $k(n-2)\binom{n(k-1)}{n-2}$ for the number of k-edge colored trees on n vertices. These trees on n (labelled) vertices have their edges colored by k colors so that any two edges sharing a vertex have different colors. Denote by $C_{n,k}$ the set of all such trees (on the vertex set $\{1, 2, \ldots, n\}$).

The main idea of this proof is to count a different set $\widetilde{C}_{n,k}$ of colored trees using slightly modified Prüfer codes. Let r be a root edge of a tree T on n vertices. It is convinient to chose r as the first edge on the path from vertex 1 to vertex 2.* Call h the parent edge of e if it is $h \cap e \neq \emptyset$ and h has lower distance from r than e. The coloring f of T must satisfy the following conditions:

- $f(r) \in \{1, 2, \ldots, k\}$
- $f(e) \in \{1, 2, \ldots, k-1\}$ for $e \neq r$.
- If e, e' have a common parent edge then $f(e) \neq f(e')$.

These trees can be encoded (by an one-to-one mapping) to the set $P_{n,k}$ of arrays of the form:

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_{n-2} & 1 \\ c_1 & c_2 & \ldots & c_{n-2} & c_n \end{pmatrix}$$

where $a_i \in \{1, \ldots, n\}, c_i \in \{1, 2, \ldots, k-1\}$, the pairs (a_i, c_i) are distinct for $i \in \{1, 2, \ldots, n-2\}$ and $c_{n-1} \in \{1, 2, \ldots, k\}$. Obviously, there are exactly $k(n-2)!\binom{n(k-1)}{n-2}$ elements in $P_{n,k}$.

Encoding: Begin with the tree T and $i = 1$. In each step, cut off the leaf b in T with the largest number. Let a_i be the number of b's neighbor and c_i be

* The authors of the article have chosen r to be the edge connecting 1 with its minimal neighbor, which causes problems during encoding and decoding that were not adressed in the article.

the color of $\{a_i, b\}$. The last edge to be pruned is the root edge and thus the algorimthm produces an array from $P_{n,k}$.

Decoding: Start with $V = \{1\}, E = \emptyset$ and then for i from $n-2$ to 1 attach a leaf b to the vertex a_{i+1}. There are two possible choices of b: If it is $a_i \notin V$ then let $b = a_i$, else put $b = \min\{x \in \{1, 2, \ldots, n\} : x \notin V\}$. Color the edge $\{b, a_{i+1}\}$ by the color c_{i+1}. Finally, add the last remaining vertex b not in V to the vertex a_1. Color the new edge c_1.

A bijection between the set $\widetilde{C}_{n,k}$ and $C_{n,k}$ is then used to finish the proof. This bijection works as follows:

$$C_{n,k} \rightarrow \widetilde{C}_{n,k}$$

Let us have a tree T with the root edge r. Let f be a coloring of T by k colors. We will produce a new coloring g of T so that $(T, g) \in \widetilde{C}_{n,k}$ by "compressing" the colors of non-root edges. We put $g(r) = f(r)$ and then in each step define the color of the uncolored edge e with minimal distance from r. Let h be the parent edge of e. Then $g(e) = f(e) - 1$ if $f(e) > g(h)$ or $g(e) = f(e)$ if $f(e) \leq g(h)$.

$$\widetilde{C}_{n,k} \rightarrow C_{n,k}$$

The "decompression" works in a similar way: We set $f(r) = g(r)$ and then proceed with defining f for the yet uncolored edge e with minimal distance from e. Let again h be a parent of e. Then it is eighter $f(e) = g(e) + 1$ if $g(e) \geq f(h)$ or $f(e) = g(e)$ if $g(e) < f(h)$.

We have now obtained bijection between $P_{n,k}$ and $\widetilde{C}_{n,k}$ and a bijection between $\widetilde{C}_{n,k}$ and $C_{n,k}$. By composing them we get a bijection between $P_{n,k}$ and $C_{n,k}$, meaning that there are indeed $k(n-2)!\binom{n(k-1)}{n-2}$ k-edge colored trees on n vertices.

References

[1] J.H van Lint, R. M. Wilson. A Course in Combinatorics. Cambridge University Press, 1992.

[2] R.P. Stanley. Enumerative Combinatorics vol.2. Cambridge University Press, 1999.

Milan Tuharský

tuharsky@science.upjs.sk

Theory of light graphs

The participant has not submitted any abstract.

Ondřej Rucký
ondrej.rucky@seznam.cz

Excentric graphs

The talk is based mostly on the last article of the series about eccentric digraphs: *Joan Gimbert, Nacho Lpez, Mirka Miller, Joseph Ryan: Characterization of eccentric digraphs*, but there are also four preceding articles dealing with this topic: *Fred Buckley: The Eccentric Digraph of a Graph*, where the notion of eccentric digraph for graphs is introduced and eccentric digraphs of trees are studied; *James Boland, Mirka Miller: The Eccentric Digraph of a Digraph*, which just enhances the notion also for digraphs and states some basic problems; *James Boland, Fred Buckley, Mirka Miller: Eccentric Digraphs*, a slight extension of the previous article; and *Mirka Miller, Joan Gimbert, Frank Ruskey, Joseph Ryan: Iterations of eccentric digraphs*, where iterations of eccentric digraphs are considered.

Two main objectives of the talk are: to state and prove a complete characterization of eccentric digraphs and, second, to characterize (di)graphs whose eccentric digraphs are graphs (i.e. symmetric digraphs).

The eccentric digraphs describe naturally the relation of the greatest distance between every two vertices in a given (di)graph G: the *eccentricity* of a vertex u in G is the distance from u to the most remote vertex of it: $e(u) := \max\{dist(u,v) : v \in V(G)\}$; then, this most distant vertex is an *eccentric vertex* of u (there can be more than one of them, of course); *eccentric digraph* of a digraph G (denoted by $ED(G)$) describes this relation of being eccentric: $ED(G) := (V(G), \{(u,v) : v \text{ is eccentric vertex of } u \text{ in } G\})$; and finally a given digraph G is *eccentric* if $\exists$ digraph $H : ED(H) = G$.

Directly from its definition, eccentric digraphs must have some properties, for example, they cannot have a vertex with out-degree zero, hence cannot be acyclic. But it is proved much more: following two theorems give a complete characterization of eccentric digraphs (the second is restriction of the first for the undirected case):

Theorem 1. *A digraph G is eccentric if and only if $ED(\overline{G^-}) = G$.*

Theorem 2. *Let G be a graph of order $n > 1$. Then G is eccentric if and only if $\overline{G}$ is self-centered with radius 2 or $\overline{G}$ is the union of complete graphs.*

Where G^- (a *reduction* of G) is derived from the original by deleting all outgoing arcs from vertices with out-degree $n-1$ (where n is the order of G), *radius* is the minimum eccentricity in a (di)graph ($rad(G) := \min\{e(v) : v \in V(G)\}$) and a (di)graph is *self-centered* if the eccentricity of all its vertices is the same.

These theorems have two corollaries determining the eccentric character of some classes of graphs:

Corollary 3.

(i) Every non-connected graph with minimum degree > 0 is eccentric.

(ii) The eccentric graphs of radius 1 are the complete multipartite graphs with at least one partite set of cardinality 1.

(iii) Every connected graph with radius ≥ 3 or diameter ≥ 4 is eccentric.

Corollary 4. *A tree is eccentric if and only if its diameter is not equal to 3.*

Now the eccentric digraphs are characterized. But what condition must meet the original G to be its eccentric digraph symmetric (i.e. can be described as a graph)? This problem is only partially solved: the first proposition gives a characterization for graphs, the second for non-strongly connected digraphs.

Proposition 3. *Let G be a graph. Then the eccentric digraph $ED(G)$ is symmetric if and only if G is self-centered.*

Proposition 4. *Let G be a non-strongly connected digraph. Then $ED(G)$ is a symmetric digraph if and only if*

$$G = C_1 \cup \cdots \cup C_k \ \ (k \geq 2) \quad \text{or} \quad G = K_{n_0} \to (C_1 \cup \cdots \cup C_k) \ \ (k \geq 1),$$

where $C_1, \ldots, C_k$ are strongly connected digraphs.

The remaining unsolved case are the strongly connected but not symmetric digraphs. At least the following simple proposition holds:

Proposition 5. *Let G be a strongly connected digraph such that $ED(G)$ is symmetric. Then the following conditions hold:*

(i) $rad(G) > 1$, unless G is a complete digraph.

(ii) If $diam(G) = 2$ then G is a self-centered graph.

Of course many other interesting questions and problems can be studied for eccentric digraphs. Some of them are mentioned throughout the series, and other will certainly appear.

Robert Šámal

samal@matfyz.cz

Presented paper by Matt J. DeVos, Robert Šámal

Weak Pentagon Problem/Conjecture is true

Edges of a cubic graph of girth at least 17 can be 5-colored so that the complement of any color class is bipartite. Equivalent formulation is that any such graph admits a homomorphism to the Clebsch graph (Fig. 1).

This is an approach to Nešetřil's Pentagon Problem, it also provides a coloring version of results of Bondy and Locke and of Hopkins and Staton on the size of maximal bipartite subgraph.

Introduction and Statement of Results

We start the exposition by relating our result to a theorem about maxcut and to a conjecture about homomorphism to C_5.

Call a set $X \subseteq E(G)$ a *cut* if there is a set $U \subseteq V(G)$ such that X is the set of all edges between U and $V(G) \setminus U$ (we write $X = \delta(U)$). Let $\mathrm{MAXCUT}(G)$ be the maximum number of edges in a cut in G, we normalize and write

$$b(G) = \frac{\mathrm{MAXCUT}(G)}{|E(G)|}.$$

As determining $\mathrm{MAXCUT}(G)$ of a given graph G is known to be NP-complete, some bounds were seeked for. It is an easy exercise to show that $b(G) \geq 1/2$ for any graph G and $b(G) \geq 2/3$ when G is a cubic (i.e. 3-regular) graph. The former inequality is almost attained by a large complete graph, the latter is attained for $G = K_4$: any triangle contains at most two edges from any bipartite subgraph, and every edge of K_4 is in the same number of triangles. This shows triangle has a special role and rises a natural question to determine $b(G)$ for a cubic G that contains no triangle, or perhaps even no short circuits. In 1980's several authors independently considered this problem; the strongest results being

- $b(G) \geq 4/5$ for G with maximum degree 3 and no triangle (Bondy and Locke)
- $b(G) \geq 6/7 - o(1)$ for cubic G with girth (the length of the shortest circuit) tending to infinity (Zýka)

On the other hand, by considering random cubic graph we can prove that there are cubic graphs of arbitrarily high girth with $b(G) < 0.999$ (McKay).

A *cut complement* is simply a set $E(G) \setminus X$ for some cut X. The fact $b(G) \geq 4/5$ can be reformulated as

$$(\exists C \subseteq E(G)) \quad \frac{|C|}{|E(G)|} \leq \frac{1}{5} \text{ and } C \text{ is a cut complement.}$$

We prove a strengthening – a coloring version of this holds when G has high girth.

Theorem 1. *Let G be a graph with maximum degree 3 and girth at least 17. Then we can partition the edges of G into five cut complements. Moreover, there is a linear-time algorithm that computes this partition. (Unfortunately, the constant in this algorithm is too large to be practical.)*

We remark that the girth assumption is only forced by our proof, we have good reasons to believe the following:


Edge partitions of
Daniel Gonçalves

Conjecture 1. *Let G be a triangle-free graph with maximum degree 3. Then we can partition the edges of G into five cut complements.*

We conclude the introduction by relating our theorem to the Nešetřil's Pentagon Conjecture. Recall that a mapping $f : V(G) \to V(H)$ is a *homomorphism* if $f(u)f(v)$ is an edge of H for any edge uv of G.

Conjecture 2. *If G is a cubic graph of sufficiently high girth then there is a homomorphism from G to C_5.*

When we replace C_5 by C_3 we get an easy consequence of Brook's theorem. It is known that Conjecture 2 is false if we replace C_5 by C_{11} (Kostočka, Nešetřil, Smolíková) , by C_9 (Wormald and Wanless) and by C_7 (Hatami).

It is easy to prove directly that Conjecture 2, if true, implies Theorem 1. However, to explain this implication more deeply, we define a new class of graph mappings that provides the connection.

Call a mapping $g : E(G) \to E(H)$ *cut-continuous* if for every cut $X \subseteq E(H)$ its preimage $g^{-1}(X)$ is a cut in G. This concept is introduced in a paper by DeVos, Nešetřil and Raspaud (as a special case of tension-continuous mappings). In papers by Nešetřil and Šámal its properties are studied in more detail, in particular the relation between statements "there is a homomorphism from G to H" and "there is a cut-continuous mapping from G to H" is studied in greater detail. The first step in this project is the following easy lemma, reproved here for the reader's convenience. In the above-mentioned papers is shown that, suprisingly, the converse implication is often true.

Lemma 1. *Let $f : V(G) \to V(H)$ by a homomorphism. Then mapping $f^\sharp : E(G) \to E(H)$ defined by $f^\sharp(uv) = f(u)f(v)$ is cut-continuous.*

Proof. Let $X = \delta(U)$ be a cut in H. Then $(f^\sharp)^{-1}(X) = \delta(f^{-1}(U))$, hence it is a cut. $\diamondsuit$

Lemma 2. *For any graph G the following are equivalent.*

1. *There is a partition of $E(G)$ into five cut complements.*
2. *There is a cut-continuous mapping from G to C_5.*

Proof. Let $g : E(G) \to E(C_5)$ be a cut-continuous mapping. Any four edges of C_5 form a cut. Hence, their preimage is a cut in G. Consequently, if we color edge $e \in E(G)$ by $g(e)$, each of the color classes is a cut complement.

Conversely, every partitioning of $E(G)$ into five cut complements corresponds to a mapping $f : E(G) \to E(C_5)$ such that preimage of every set of four edges of C_5 is a cut. As 4-sets of edges of C_5 form a basis of the cut space of C_5, the mapping f is cut-continuous. $\diamondsuit$

A consequence of Lemma 1 and 2 is that trueness of Conjecture 2 impliess Theorem 1, as stated above. On the other hand we probably can't obtain Conjecture 2 using Theorem 1, as for example Figure 1 shows that Petersen graph has the desired partition (easily, it has no homomorphism to C_5).

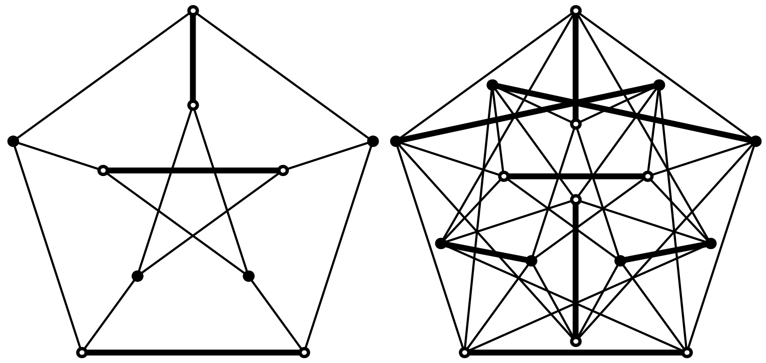


Figure 1.

Figure 1 shows Petersen and Clebsch graph with one cut complement emphasized, the respective bipartition of the vertex set is depicted, too. The other four cut complements are obtained by a rotation.

This Lemma shows that Reza in his PhD thesis (using results of Guenin) proves the same result from different assumption.

Theorem. *Any planar triangle-free graph admits a homomorphism to the Clebsch graph.*

Tomáš Ebenlendr

ebik@artax.karlin.mff.cuni.cz

Online machine covering

We study the problem of machine covering, also called bin covering. This is related to bin packing, which is generalized form of well known knapsack problem. All these problems are NP-complete. We consider the online aproximation of our problem.

Problem: Problem is to schedule jobs (put items into bins), such that all machines (bins) are covered (overfilled). In our model all machines have the covering level (bin size) equal. The jobs come one by one and online algorithm does not know the next job before it decides how to schedule the current one. It cannot change this decision later.

We want the algorithm to cover same number of machines as number of machines that can be covered in offline. This is impossible for any online algorithm, so we lower the covering level of machines (sizes of bins) for it. We ask how much must be the covering level lowered. The ratio between the original covering level and the covering level of algorithm we call covering ratio.

Upper bound: We constructed online algorithm that has the covering ratio of 11/6, and we analysed it by standard technique in bin packing, i.e. assigning weights to the jobs, and compare total weights of the machines of the optimal schedule and the schedule of algorithm. Previous algorithm was the trivial one with covering ratio of 2.

Lower bound: We also prooved that no online algorithm can have the covering ratio smaller than 43/24 by defining strategy for the enemy of the algorithm. This strategy leads to situation in which original schedule has covered one machine more than the algorithm.

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Our article is not published yet. Previous lower bound of 7/4 is in *Yossi Azar, Lea Epstein:* "On-line machine covering" in *Journal of Scheduling*, Vol. 1, pp 67–77, 1998.

Ondřej Plašil
oplasil@centrum.cz

Graph recurrence

The participant has not submitted any abstract.

Přemysl Holub
holubpre@kma.zcu.cz

Graph closures

The participant has not submitted any abstract.

Bernard Lidický
2berny@seznam.cz

Presented paper by Hal A. Kierstead

Asymetric graph coloring games

(http://www.interscience.wiley.com/)

We will introduce asymmetric version of graph coloring game and graph marking game played by Alice and Bob. Then we will give some bound for these games.

Graph coloring game is game played by Alice and Bob on a finite graph G using set of colors X. Alice is playing first. At the beginning all vertices of G are uncolored. In each turn player chooses any uncolored vertex v and colors it such that any colored neighbor of v has different color than v has. Alice wins if they finally color all vertices. Bob wins if at some time one of the players has no legal move. We may define *game chromatic number* of G as the least integer i such that Alice has a winning strategy when the game is played on G using i colors. Game chromatic number is denoted by $\chi_g(G)$.

Now we may define asymmetric version of the graph coloring game. The *(a,b)-coloring game* is played like coloring game but in each turn Alice marks a vertices and Bob marks b vertices. Simply we let players to color more vertices in one turn. You may note that (1,1)-coloring game is just an ordinary coloring game. The *(a,b)-game chromatic number* of G is the least integer i such that Alice has a winning strategy when (a,b)-coloring game is played on G using i colors. The (a,b)-game chromatic number is denoted by $\chi_g(G; a, b)$

Graph marking game is a simplified version of graph coloring game. It is useful for upper bound proofs. Graph marking game is played by Alice and Bob on a finite graph G. Alice is playing first. At the beginning of the game all vertices are unmarked. In each turn player chooses one unmarked vertex and marks it. The game end when all vertices are marked. Note that there is no winner or looser. We define $\mathrm{col}_g(G)$ as the least integer i such that Alice has a strategy when played on G such that in each turn every unmarked vertex has strictly less marked neighbors than i. It is easy to see that $\chi_g(G) \leq \mathrm{col}_g(G)$. Alice may use same strategy for coloring and marking game and color vertices by first-fit.

We also define *(a,b)-marking game.* Alice plays by marking a vertices and Bob plays by marking b vertices. We also will examine *(a,b)-game coloring number* denoted by $\mathrm{col}_g(G; a, b)$. It will be the least integer i such that after each marked vertex all unmarked vertices has strictly less than i marked neighbors. It is not sufficient to check marked neighbors after end of turn but we really need to check them when a vertex is marked. It then follows that $\chi_g(G; a, b) \leq \mathrm{col}_g(G; a, b)$.

Next we show the main theorem that gives upper and lower bounds for game coloring number and game chromatic number when played on class $\mathcal{F}$ of forests.

For a class of graphs $\mathcal{C}$ let

$$\chi_g(\mathcal{C}; a, b) = \max_{G \in \mathcal{C}} \chi_g(G; a, b) \text{ and } \mathrm{col}_g(\mathcal{C}; a, b) = \max_{G \in \mathcal{C}} \mathrm{col}_g(G; a, b)$$

Theorem. *Let a and b be positive integers.*

- *If $a < b$ then $\chi_g(\mathcal{F}; a, b) = \mathrm{col}_g(\mathcal{F}; a, b) = \infty$.*
- *If $b \leq a$ then $b + 2 \leq \chi_g(\mathcal{F}; a, b) \leq \mathrm{col}_g(\mathcal{F}; a, b) \leq b + 3$.*
- *If $b \leq a < \max\{2b, 3\}$ then $b + 3 \leq \chi_g(\mathcal{F}; a, b)$.*
- *If $4 \leq 2b \leq a < 3b$ then $\chi_g(\mathcal{F}; a, b) \leq b + 2 < b + 3 \leq \mathrm{col}_g(\mathcal{F}; a, b)$.*
- *If $3b \leq a$ then $\mathrm{col}_g(\mathcal{F}; a, b) \leq b + 2$.*

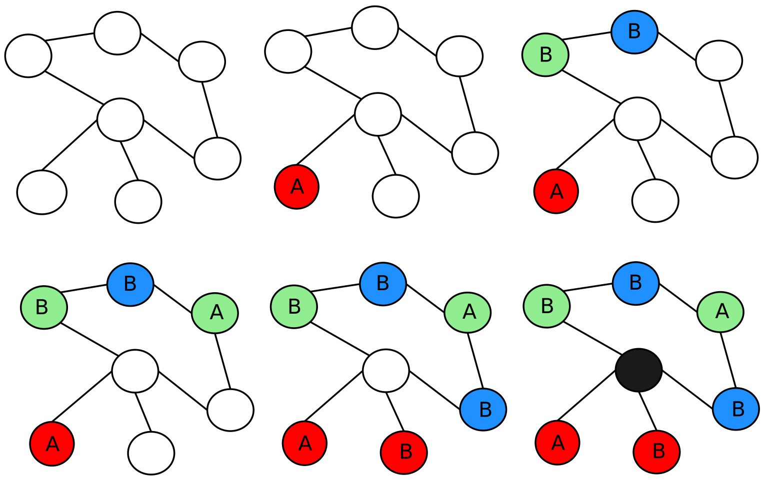
Proofs of upper bounds are based on Alice's strategy for graph coloring game on a tree graph. Proofs of lower bounds shows Bob's strategy. It works on trees with many vertices where non-leaf vertices has a large degree.

Since the main theorem isn't easy to read article provides a corollary of the main theorem.

Corollary. *Let a and b be positive integers.*

- *If* $b \leq a < 2b$ *or* $(a, b) = (2, 1)$ *then* $\chi_g(\mathcal{F}; a, b) = \mathrm{col}_g(\mathcal{F}; a, b) = b + 3$.
- *If* $2b \leq a < 3b$ *and* $b > 1$ *then* $\chi_g(\mathcal{F}; a, b) = b + 2$ *and* $\mathrm{col}_g(\mathcal{F}; a, b) = b + 3$.
- *If* $3b < a$ *then* $\chi_g(\mathcal{F}; a, b) = \mathrm{col}_g(\mathcal{F}; a, b) = b + 2$.

Example of (1,2)-coloring game with three colors:



Alice and Bob may play like drawn on the picture. Alice lost this game because she wan't able to color black vertex in last picture because it's neighbors have all three possible colors. But three colors are enough for Alice to win on this graph. She may color vertex with degree 4 in her first move an then all remaining vertices has degree strictly less then 3 so 3 tree colors will be enough for her. So $\chi_g(G; a, b) \leq 3$.

Marek Sulovský

skeeve@popelka.ms.mff.cuni.cz

Presented paper by Benjamin Doerr

Lattice approximation and linear discrepancy of totally unimodular matrices

The paper introduced to the lattice approximation problem, also known as a linear discrepancy problem and showed that this problem can be solved efficiently via linear programming. This method is optimal in the worst case. It seems to be the first time that linear programming is successfully used for a discrepancy problem.

In particular it derives an upper bound for the linear discrepancy of totally unimodular $m \times n$ matrix A:

$$lindisc(A) \leq \min\{1 - \frac{1}{n+1}, 1 - \frac{1}{m}\}$$

This bound is sharp. It proves Spencer's conjecture

$$lindisc(A) \leq (1 - \frac{1}{n+1}) herdisc(A)$$

for totally unimodular matrices.

Lattice approximation problem, linear discrepancy

Let $A \in \mathbf{R}^{m \times n}$ be any real matrix and $b = Ap, p \in \mathbf{R}^n$, a point of vector space generated by columns of A. The lattice approximation problem is to find a point $Az, z \in Z^n$, of the lattice $AZ^n = \{Az : z \in Z^n\}$ which is closest to b i.e. such that $\|Az - b\|_\infty$ is minimal. We also require $\|p - z\| \leq 1$ to hold, that is, z evolves from p by some rounding procedure.For a given A and p the approximation error of an optimal approximation is called *linear discrepancy of A with respect to p*. The worst case of inapproximability that can occur with the lattice generated by A is the linear discrepancy of A:

$$linidsc(A) = \max_{p \in [0,1]^n} lindisc(A, p) = \max_{p \in [0,1]^n} \min_{z \in \{0,1\}^n} \|A(p - z)\|_\infty$$

Theorem 1. *Let $A \in R^{m \times n}$ be a totally unimodular matrix and $p \in [0,1]^n$. Then there is a $z \in \{0,1\}^n$ such that*

$$\|A(p - z)\|_\infty \leq \min\{1 - \frac{1}{n+1}, 1 - \frac{1}{m}\}$$

This can be found efficiently by solving a linear optimization problem in R^n having $2(m+n)$ inequalities and up to m systems of linear equations of dimension smaller than $n \times n$.

Theorem 2. *Let $A \in R^{m \times n}$ be a totally unimodular matrix. Then the equality $lindisc(A) = 1 - \frac{1}{n+1}$ holds if and only if there is a collection of $n+1$ rows of A such that each n thereof are linarly independent. If $lindisc(A, p) = 1 - \frac{1}{n+1}$ for some $p \in [0,1]^n$, then $p_i \in \{\frac{1}{n+1}, \ldots, \frac{n}{n+1}\}$ for all $i \in [n]$.*

The second theorem shows that all the all the tight examples are similiar (somewhat generalized) to the one that lead Spencer to his conjecture.


Pilsner Urquell

Petr Škoda
petr.skoda@seznam.cz

Presented paper by Hamed Hatami, Xuding Zhu

The fractional chromatic number of graphs of maximum degree at most three

The paper shows the upperbound of fractional chromatic number of graphs with small degree. The fractional chromatic number could be defined as follows. Let *k-tuple n-coloring* of a graph $G = (V, E)$ be a mapping $f : V \to \binom{[n]}{k}$, where $\binom{[n]}{k}$ is the set of all k-subsets of set $1 \cdots n$, with an additional condition that for each two vertices $u, v \in V$ holds $f(u) \cap f(v) = \emptyset$. We simply color the vertices with sets of colors instead of only one color. Then the fractional chromatic number χ_f of G is the minimum of $\frac{k}{n}$ such that there exist a k-tuple n-coloring of G. We can easily see that the fractional chromatic number is greater or equal to classical chromatic number so it is more precise characterization of graph.

Main Theorem. *If G is triangle free with maximum degree at most three, then $\chi_f(G) \leq 3 - \frac{3}{64}$*

So it shows there is an upperbound and the trivial 3-coloring of the graph can be improved. The prove is based on constructing the $k+1$-tuple $3k$-coloring of the graph, where k is 63, but as authors say it can be slightly improved with some difficulties.

Conjecture. *Every triangle free graph with maximum degree at most three has fractional chromatic number at most $\frac{14}{5} = 3 - \frac{1}{5}$*

The authors thinks this Conjecture is true but the Main Theorem do not lead any closer to proving it.

Jiří Fink
fink@atrey.karlin.mff.cuni.cz

Presented paper by Ian Parberry

On the Computational Complexity of Optimal Sorting Network Verification

(http://hercule.csci.unt.edu/~ian/pubs/snverify.pdf)

A sorting network is a combinational circuit for sorting, constructed from comparison-swap units. The depth of such a circuit is a measure of its running time. It is reasonable to hypothesize that only the fastest (that is, the shallowest) networks are likely to be fabricated. It is shown that the problem of verifying that a given sorting network actually sorts is co-NP complete even for sorting networks of depth only $4 \log n + O(1)$ greater than optimal. This is shallower than previous depth bounds by a factor of two.

The Reduction

In order to show that NONSORT is NP-complete, it is sufficient to show that B3SAT $\propto$ NONSORT. Suppose we are given an instance of B3SAT, that is, a list of clauses $C = (C_1, \ldots, C_n)$ over a set of variables $V = \{v_1, \ldots, v_n\}$ such that every variable in V appears exactly three times in C. We will construct a comparator network with $5n$ inputs. An input $x = (x_1, \ldots, x_{5n})$ to the comparator network is said to correspond to assignment S for C iff for all $1 \leq i \leq n$, $x \in (0^5 \times 1^5)^n$, and $v_i \in S$ iff $x_{5n-4} = 1$. Our comparator network will sort only inputs that do not correspond to satisfying assignments for C, that is, inputs that do not correspond to any assignment, and inputs that correspond to nonsatisfying assignments. Therefore, it will be a sorting network iff C is not satisfiable.

Petr Kučera

kucerap@kti.ms.mff.cuni.cz

Augmentation problems

The participant has not submitted any abstract.

Eva Ondráčková

efa@matfyz.cz

Presented paper by Nati Linial, Avi Wigderson

Expander graphs I. – Introduction

(http://www.math.ias.edu/˜boaz/ExpanderCourse/)

Since the introduction of expander graphs in the 1970's they have turned out to be very significant. They have been used in solving problems in communication and the construction of error codes as well as a tool for proving results in number theory and computational complexity.

In the following, we define the notion of expansion and examine some linear algebraical properties of expanders.

Definition. The edge boundary of a set S, denoted ∂S, is $\partial S = E(S, \bar{S})$. This is the set of edges outgoing from S.

Definition. The expansion parameter of G, denoted by $h(G)$, is defined as follows:

$$h(G) = min_{\{S;|S| \leq \frac{n}{2}\}} \frac{|\partial S|}{|S|}.$$

We note that there are other notions of expansion that can be studied. The most popular is counting the number of neighbouring vertices of any small set S rather than the number of outgoing edges.

Definition. A family of expander graphs $\{G_i\}$, where $i \in \mathbf{N}$, is a collection of graphs with the following properties:

- The graph G_i is a d-regular graph of n_i vertices (d is the same constant for the whole family). $\{n_i\}$ is a monotone growing series that does not grow too fast (e.g. $n_{i+1} \leq n^2$).
- For all i, $h(G_i) \geq \varepsilon > 0$.

Now we define a similar (and somewhat simpler) graph property.

Definition. Let G be a two-sided graph with n vertices on each side. Let L be the vertices of the left side and R the vertices on the right. Assume that any vertex in L has d neighbours in R. We say that G is a (d, n)-hairy graph if it has the following two properties:

- For any $S \subseteq L$ such that $|S| \leq \frac{3n}{d}$: $\Gamma(S) \geq |S|\frac{d}{4}$
- For any $S \subseteq L$ such that $\frac{n}{3d} < |S| \leq \frac{n}{2}$: $\Gamma(S) \geq |S| + \frac{n}{3d}$

where $\Gamma(S)$ is the set of neighbours of S in G.

The existence of hairy graphs. *For each $d \geq 8$ and a sufficiently large n there exists a (d, n)-hairy graph.*

Moreover, we prove that for any $0 < \varepsilon < 1$ there exists an n_0 such that for all $n \geq n_0$ the following is true: the graph with both L and R containing n vertices and with every vertex in L having d random neighbours in R, is (d, n)-hairy with the probability $\frac{1}{\varepsilon}$.

The spectrum of an expander graph

The adjacency matrix of a graph G, denoted $A(G)$, is an $n \times n$ matrix that for each (u, v) contains the number of edges in G between the vertex u and the vertex v. Since the graph G is d-regular, the sum of each row and column is d. By definition the matrix $A(G)$ is symmetric, and therefore it has an orthonormal base $v_0, v_1, \ldots, v_{n-1}$, with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. The eigenvalues of $A(G)$ are called the spectrum of G.

The spectrum contains a lot of information regarding the graph. Here are some examples of observations that demonstrate this connection between the spectrum of a d-regular graph and its properties:

Spectral properties of a d-regular graph.

- $\lambda_0 = d$


METRIC SPACE (X, d)
OUR FOCUS
PRESERVE LENGTHS

- *the graph is connected iff* $\lambda_0 > \lambda_1$
- *the graph is bipartite iff* $\lambda_0 = -\lambda_{n-1}$.

In the rest, we will examine the connection between the expansion of a graph and its spectrum. In particular, the graph's second eigenvalue is related to the expansion parameter of the graph.

The spectral gap bounds.

$$\frac{d-\lambda_1}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_1)}$$

This theorem proves that $d - \lambda_1$, also known as the spectral gap, can give a good estimate on the expansion of a graph. Moreover, the graph is an expander ($h(G) > \varepsilon$) if and only if the spectral gap is bounded ($d - \lambda_1 > \varepsilon'$).

The expander mixing theorem. *Denote* $\lambda = \max(|\lambda_1|, |\lambda_{n-1}|)$. *Then for all* $S, T \subseteq V$*:*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda\sqrt{|S||T|}.$$

The lower bound for λ. *For every d-regular graph the following is true:*

$$\lambda \geq 2\sqrt{d-1} - o_n(1).$$

Vít Jelínek

jelinek@kam.mff.cuni.cz

Presented paper by V. Jelínek, J. Kynčl, R. Stolař, T. Valla

Monochromatic triangles in a two-colored plane

We investigate the following general problem, suggested by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus [1,2,3] in 1973: Let T be a given configuration of three points. Is it true that for every two-coloring χ of the plane we can find a monochromatic translated and rotated copy of T? It can be easily checked that if T is the vertex set of an equilateral triangle, and χ^* is a coloring of the plane consisting of alternating half-open parallel strips whose width is equal to the height of T, then χ^* does not contain a monochromatic copy of T. Erdős et al.[3] have conjectured that this example is essentially unique; more precisely, they made the following two conjectures:

Weaker conjecture. *If T is the vertex set of a non-equilateral triangle, then every coloring of the plane contains a monochromatic copy of T.*

Stronger conjecture. *If a coloring χ avoids a monochromatic copy of a triple of points T, then T is the vertex set of an equilateral triangle and χ is equal to the coloring χ^* defined above, up to a possible recoloring of the points on the boundary of the strips.*

In the past, it has been shown that the (weaker) conjecture holds for special classes of triangles, e.g. right triangles [4], but the general conjecture remains open.

In our research, we focused on testing whether the two conjectures hold for special classes of plane colorings. We obtained the following results:

First result. *If χ is a two-coloring of the plane such that the points colored with one of the colors form a closed set, then χ contains a monochromatic copy of every point triple T.*

Second result. *The weaker conjecture holds for colorings χ that partition the plane into a tiling of monochromatic plygonal regions (i.e. regions whose boundary is a union of segments). The regions may be unbounded and their boundary may consist of infinitely many segments, but we assume that every bounded subset of the plane is intersected by only finitely many boundary segments. We make no assumptions about the coloring of the boundary.*

Third result: Among the *polygonal colorings* described in the previous paragraph, there are some counterexamples to the stronger conjecture of Erdős et al. We are able to give a complete characterization of these "polygonal" counterexamples, and we prove that every polygonal coloring χ can be either turned into such counterexample by recoloring some of the points on the boundary of the regions, or χ contains a monochromatic copy of any triple T in such a way that the vertices of the copy avoid the boundary.

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Vojtěch Kulvait
kulvait@atlas.cz

Presented paper by Nati Linial and Avi Wigderson

Expander graphs II. – Applications

Metric space with arbitrary metrics can be embedded into Euclidian space with common metrics. Generally there is some distortion of the distances. The expanders with common graph metrics are metric spaces hardest to embed into Euclidian space.

Definition. We call the tuple (X, d) a *metric space* if X is set and d is a metric function such that

- $d : X \times X \to \mathbf{R}^+$
- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) = d(x, z) + d(z, y)$

Definition. Metric space L2 $(\mathbf{R}^n, ||.||)$ is an Euclidian metric space with common metrics.

Given the metric spaces (X, d) and $(\mathbf{R}^n, ||.||)$ and a transformation $f : X \to \mathbf{R}^n$ we define:

$$\begin{aligned}
\text{expansion}(f) &= \max_{x_1, x_2 \in X} \frac{||f(x_1) - f(x_2)||}{d(x_1, x_2)} \\
\text{contraction}(f) &= \max_{x_1, x_2 \in X} \frac{d(x_1, x_2)}{||f(x_1) - f(x_2)||} \\
\text{distortion}(f) &= \text{expansion}(f) \cdot \text{contraction}(f)
\end{aligned}$$

Finding the Minimal Distortion

Given a metric space (X, d) we denote its minimal distortion by $C_2(X, d)$.

Theorem 1 (Bourgain 1985). *Any n-point metric space (X, d) can be embedded into $O(\log n)$ dimensional Euclidean space with $O(\log n)$ distortion.*

Theorem 2 (Linial, London, and Rabinovich 1995). *There is a polynomial time algorithm which computes $C_2(X, d)$.*

Theorem 3. *There exists explicit formula for finding $C_2(X, d)$.*

Embedding expander graphs in L2

Let $G = (V, E)$ be a k-regular graph, $|V| = n$. From theorem 2 this graph can be embedded with distortion $O(\log n)$ in L2. Indeed, take the expander and put it as a simplex in $\mathbf{R}^n$. Since every two nodes of the simplex have distance 1 we get that $\text{ex}(G) = 1$, and $\text{co}(G) = \text{dis}(G) = \text{diam}(G)$. Since G is an expander




then $\mathrm{diam}(G) = O(\log n)$. As upcoming Theorem 3 says we can not embed G with lower distortion than $\log(n)$.

Lemma. *Let $H = (V, E)$ be a graph with the same vertex set as G. Two vertices are adjacent in H if their distance in G is at least* $\log k(n) - 2$. *Then there exists matching in H on $n/2$ edges.*

Theorem. *Let $G = (V, E)$ be a k-regular graph, $|V| = n$. Then $C_2(G) = O(\log n)$.*

Ondřej Zajíček

santiago@mail.cz

Presented paper by Omer Reingold, Salil Vadhan, Avi Wigderson

Expander graphs III. – The zig-zag construction

The Construction of Constant Degree Expanders With The Zig-Zag Product

The seminar is based on definition of zig-zag graph product. Using zig-zag graph product on small and large graph, the resulting graph is larger than the large one, but its degree only depends on the small one. Expanding property of product result is inherited from both. The heart of article (and seminar) is the proof of this property of zig-zag product. Iteration of this product (with combination of common graph operations like graph squaring and tensor graph poduct) on fixed graph gives family of constant degree expanders.

Definitions

(N, D, λ)-graph is a D-regular graph on N vertices, whose normalized adjacency matrix has second eigenvalue (in absolute value) at most λ.

Rot_G - Rotation map of graph, function from $N \times D$ to $N \times D$, which describes a graph G. Having fixed numbering of vertices and neighours of every vertex, $Rot_G(a, b) = (c, d)$ means that b-th neighbour of vertex a is vertex c and d-th neigbour of vertex c is vertex a.

i'th power of graph G is graph on same set of vertices which has edge for every connected edge sequence of length i.

Given D_1-regular graph G_1 on $[N_1]$ and D_2-regular graph G_2 on $[N_1]$, then $G_1 \otimes G_2$ (tensor product of graphs G_1 and G_2) is $(D_1 \cdot D_2)$-regular graph on $[N_1 \cdot N_2]$, whose rotation map is as follows:

To compute $Rot_{G_1 \otimes G_2}((v, w), (i, j))$:

1. Let $(\tilde{v}, \tilde{i}) = Rot_{G_1}(v, i)$
2. Let $(\tilde{w}, \tilde{j}) = Rot_{G_2}(w, j)$

3. Output $((\tilde{v}, \tilde{w}), (\tilde{i}, \tilde{j}))$

Straightforward idea – for first dimension use first graph and for second dimension use second graph.

Given D_1-regular graph G_1 on $[N_1]$ and D_2-regular graph G_2 on $[D_1]$, then $G_1 ⓩ G_2$ (zig-zag product of graphs G_1 and G_2) is D_2^2-regular graph on $[N_1 \cdot D_1]$, whose rotation map is as follows:

To compute $Rot_{G_1 ⓩ G_2}((v, w), (i, j))$:

1. Let $(k, l) = Rot_{G_2}(w, i)$
2. let $(m, n) = Rot_{G_1}(v, k)$
3. Let $(o, p) = Rot_{G_2}(n, j)$
4. Output $((m, o), (p, l))$

Intuition is that every vertex of G_1 is expanded to cloud – copy of G_2. (i, j)-th edge from vertex v makes a step in a cloud (by i), then jump across clouds (position in cloud is used to select edge in G_1) and finally step in a cloud (by j).

Theorem 1. *If G_1 is an (N_1, D_1, λ_1)-graph, then G_1^2 is an $(N_1, D_1^2, \lambda_1^2)$-graph.*

Theorem 2. *If G_1 is an (N_1, D_1, λ_1)-graph and G_2 is a (N_2, D_2, λ_2)-graph, then $G_1 \otimes G_2$ is a $(N_1 \cdot N_2, D_1 \cdot D_2, max(\lambda_1, \lambda_2))$-graph.*

Main theorem 1. *If G_1 is an (N_1, D_1, λ_1)-graph and G_2 is a (D_1, D_2, λ_2)-graph, then $G_1 ⓩ G_2$ is a $(N_1 \cdot D_1, D_2^2, f(\lambda_1, \lambda_2))$-graph, where $f(\lambda_1, \lambda_2) \leq \lambda_1 + \lambda_2 + \lambda_2^2$ and $f(\lambda_1, \lambda_2)) \leq 1$ when $\lambda_1 \leq 1$ and $\lambda_2 \leq 1$. $Rot_{G_1 ⓩ G_2}$ can be computed in time $poly(logN, logD_1, logD_2)$ with one oracle query to Rot_{G_1} and two oracle queries Rot_{G_2}.*

Main theorem 2 – Family of expanders. *Let H be an (D^8, D, λ)-graph for some D and λ. Define:*

1. $G_1 = H^2$
2. $G_2 = H \otimes H$
3. $G_n = \left(G_{\lceil (t-1)/2 \rceil} \otimes G_{\lfloor (t-1)/2 \rfloor}\right)^2 ⓩ H$

Then every G_n is an (D^{8t}, D^2, λ_t)-graph, where $\lambda_t = \lambda + O(\lambda^2)$. Moreover, Rot_{G_t} can be computed in time $poly(t, logD)$, with $poly(t)$ oracle queries to Rot_H.

Marek Krčál
marek.krcal@seznam.cz

Hall's Theorem for Hypergraphs

(J Graph Theory 35: 83–88, 2000)

We prove a hypergraph version of Hall's theorem. The proof is topological.

Systems of disjoint representatives

Hypergraph is set of subset (called edges) on some ground set (set of vertices). Let's consider some hypergraph G. *Matching (in G)* is set of disjoint edges of G.

Let $H \subseteq G$ is matching. Then *width(H)* is $\min\{|M|; M \subseteq G$ is matching and every edge from H is intersected by some edge from M$\}$.

Let $\mathcal{H} = \{H_1, H_2, \dots, H_n\}, H_1, H_2, \dots, H_n \subseteq G$. Then *System of disjoint representatives of* $\mathcal{H}$ is a mapping $f : \mathcal{H} \longrightarrow \bigcup \mathcal{H}$ such that for every $i \neq j \in \mathbf{N}$ there is $f(H_i) \in H_i$ and $f(H_i) \cap f(H_j) = \emptyset$.

Theorem (if version). *Let G be a hypergraph and $\mathcal{H} = \{H_1, \dots, H_n\}$ such that $H_1, \dots, H_n \subseteq G$. Let for every $\mathcal{B} \subseteq \mathcal{H}$ exists a matching $M_\mathcal{B} \subseteq \cup\mathcal{B}$ such that width$(M_\mathcal{B}) \geq |B|$. Then there exists a system of disjoint representatives of $\mathcal{H}$.*

Theorem (if and only if version). *$\forall G$ hypergraph $\forall \mathcal{H} = \{H_1, H_2, \dots, H_n\}$, $H_1, H_2, \dots, H_n \subseteq G$: ($\forall \mathcal{B} \subseteq \mathcal{H}$: $\exists$ matching $M_\mathcal{B} \subseteq \cup\mathcal{B}$: ($\forall \mathcal{D} \subseteq \mathcal{C} \subseteq \mathcal{H}$: for all $|\mathcal{C}| - 1$ edges from $M_\mathcal{D}$ there exists an edge from $M_\mathcal{C}$ which is disjoint with all of them.)) $\Leftrightarrow \exists$ System of disjoint representatives.*

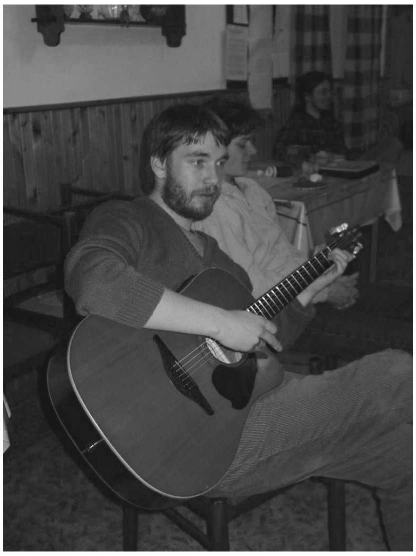
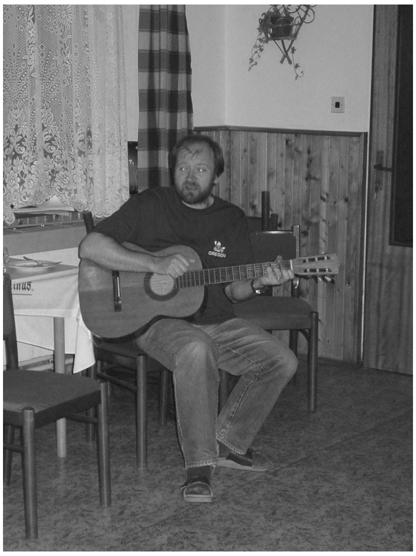
Jan Hladký
hladk@seznam.cz

Presented paper by R. Aharoni, E. Berger, R. Ziv

A Version of König's Theorem

(Combinatorica, Volume 22 (p. 335–343))

Definition. *Definitions:* Let H and F be two hypergraphs on the same vertex set. A subset C of F is said to be H-covering if every edge in H meets some edge from C. The F-width $w(H, F)$ of H is the minimal size of an H-covering set of edges from F. The F-matching width $mw(H, F)$ of H is the maximum, over all matchings M in H, of $w(M, H)$. We write $w(H)$ for $w(H, H)$, $mw(H)$ for $mw(H, H)$.


The deficiency $def(A)$ of a family A of hypergraphs is the minimal natural number d such that $mw(\bigcup B) \geq |B| - d$ for every subfamily B of A.

We used the version of Hall's theorem by Aharoni and Haxell (the one presented at SSC05 by Marek Krčál) and extended it into following:

Theorem 1. *Every family A of hypergraphs has a subfamily D of size at most $def(A)$, such that $A \setminus D$ has a choice function of disjoint edges.*

Theorem 2. *Let $H_1 \subseteq H_2$ be two families of subtrees of a tree. Then $mw(H_1, H2) = w(H_1, H_2)$.*

Theorem 2 follows immediately from Lemma 1:

Lemma 1. *Let $H_1 \subseteq H_2$ be families of subtrees of a given tree, let $c, d \in H_1$ be any two intersecting edges. Then $w(H_1 \setminus \{c\}, H2) = w(H_1, H_2)$ or $w(H_1 \setminus \{d\}, H2) = w(H_1, H_2)$.*

Lemma 1 is proved by contradiction: we consider the minimal H_1-covering C, and suppose that $w(H_1 \setminus \{c\}, H2) = w(H_1, H_2) - 1$ and $w(H_1 \setminus \{d\}, H2) = w(H_1, H_2) - 1$, and obtain contradiction with the minimality of C.

Definition. A hypergraph H is a *point-tree hypergraph* if, for some set X and a tree T whose vertex set is disjoint from X, each edge $e \in H$ is of the form $x \cup V(t)$, where $x = x(e) \cup X$ and $t = t(e)$ is a subtree of T . For such a hypergraph we denote by $\sigma(H)$ the number $w(H, F)$, where $F = x(e) : e \in H \cup V(t(e)) : e \in H$. The *matching number* $\nu(H)$ of a hypergraph H is the maximal size of a matching in H.

The main result is Theorem 3 which follows quite straightforward from Theorem 1 and Theorem 2. This extends König's theorem:

Theorem 3. *Let H be a point-tree hypergraph. Then $\sigma(H) \leq \nu(H)$*

Using Theorem 3 a conjuncture of B. Reed for chordal graphs is proved.

Jakub Černý

kuba@kam.mff.cuni.cz

Beautiful proofs of geometric theorems

In the talk we presented several geometric theorems which seems to be hard and have complicated proofs, but there were discoverd very simple proofs using beautiful tricks.

The first is so called *Slope problem.* We have n points in the plane, not all lying on the line. How many slopes can be determined by the lines passing through each two points? Many of these lines can be parallel so they determine the same slope. There are constructions of n points in the plane determining at least $n-1$ slopes for n odd and at least n slopes for n even. We show that it cannot be less. In the proof we reduce the slope problem to another combinatorial problem – to so called *allowable sequences.* In the second step of the proof we easily show, that the sequence has the lenght at least n.

The second theorem is about *Geometric graphs.* Geometric Graph is a graph drawn in the plane that the vertices are points in the plane and edges are straight line segments. There are many problems on geometric graphs. One of the big class of problems are extremal questions (Turan type questions). How many edges can have a geometric graph on n vertices without containing a forbidden subconfiguration? I.e. geometric graphs with no two crossing edges are plane graphs. They have at most $3n-6$ edges. What about similar question. How many edges can a geometric graph with no two disjoint edges have? By a nice dischargeing technique it can be shown, that at most n. Moreover there are geometric graphs on n vertices with n edges. So this result is tight. A little more general question is the following. How many edges can a geometric graph on n vertices with no $k+1$ pairwise disjoint edges have? It can have at most $k^4 n$ edges. This bound is not the best possible, but it has a very nice proof using Dilworth theorem. We introduce an order relation on disjoint edges, apply Dilworth theorem to partition the edge set to at most k antichains and for each antichain we apply the previos result for two disjoint edges, because every antichain doesn't contain disjoint edges.

The two disjoint edges are parallel iff their convex hull is quadrilateral. The third theorem says that a geometric graph on n vertices with no two parallel edges can have at most $2n-2$ edges. Its proof uses Davenport-Schinzel sequences of order two.

The first proof can be found in [1], the second in [2] and the third one in [3].

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Tomáš Gavenčiak

gavento@atrey.karlin.mff.cuni.cz

Presented paper by Jiří Matoušek

Sixteen miniatures

The participant has not submitted any abstract.

Martin Pergel
perm@kam.mff.cuni.cz

Acyclic choosability of graphs with small maximum degree

The participant has not submitted any abstract.

Rudolf Stolař
riv@email.cz

Fractal sequences and restricted nim

The participant has not submitted any abstract.

Tomáš Vyskočil
tiger@matfyz.cz

Edge partition of planar graphs into two outerplanar graphs

The participant has not submitted any abstract.

Tomáš Valla
valla@kam.mff.cuni.cz

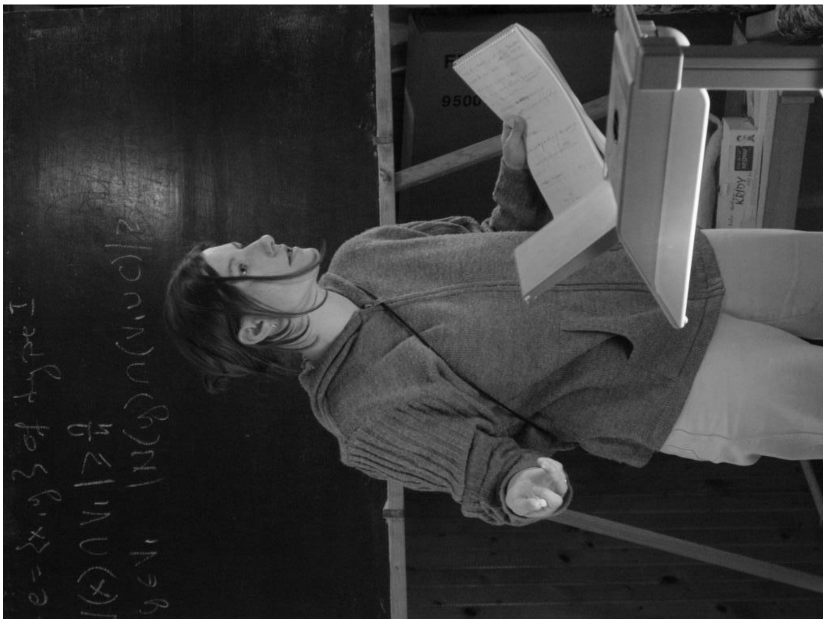
Presented paper by Vašek Chvátal

Expander application: Optimal sorting networks

Introduction

First we will describe the concept of sorting networks. A *sorting network* consist of n wires which hold n input numbers to be sorted. Then it contains elementary devices called *comparators* attached to exactly two wires which rearranges the numbers on the wires such that the lesser is placed on the left wire and the greater on the right wire. Some comparators can be placed on the same level, some not. The computation of a sorting network is parallel, the input bubbles


Proofs from THE BOOK



down through the comparators and the time complexity is measured by this number of parallel steps, thus in the number of levels, also called the *depth.*

Every correct sorting network has to have at least $\Omega(\log n)$ levels. To see this consider arbitrary sorting network with depth d. Choose one wire and note that on every level the number of possible input wires doubles. Therefore, to cover all n input wires one needs $d = \log n$ levels. First we presented the simple *bitonic sorter* scheme that yields the time $\Theta(\log^2 n)$.

In the main part of the talk we showed the basic components of the $\Theta(\log n)$ sorting network. The first $O(\log n)$ sorting network was built in 1983 by Ajtai, Komlós and Szemerédi [1] but was terribly complicated. In 1990 Paterson [2] described a much simpler scheme which was later more simplified and some details were filled in by Chvátal.

The Network

A *perfect halver* is a device, which reads a wires and outputs the smaller $a/2$ number at the first $a/2$ output wires and the bigger $a/2$ number at the second $a/2$ output wires. Given a perfect halver construction we could design a sorting network consisting of a balanced tree of $\log n$ depth and at each node having a perfect halver as a subroutine. Unfortunaly, one can show that also this module needs $\Omega(\log a)$ levels, thus resulting into an $\Omega(\log^2 n)$ network.

Instead, we will construct an *imperfect halver.* Like the output wires of a perfect halver, the output wires of the weaker module are split into blocks B_L, B_R such that $|B_L| = |B_R| = a/2$. Unlike a perfect halver, the weaker module may misdirect a small fraction of the smaller $a/2$ input keys into B_R and it may misdirect a small fraction of the larger $a/2$ input keys to B_L. A partial compensation for this defect is an explicitly designated set F of output wires (typically about 5% of their total) such that for every input of a distinct keys, most of the smallest keys end up in $F \cap B_L$ and most of the largest keys end up in $F \cap B_R$.

At each time t, each node that holds any wires at all uses them as input wires of an imperfect module; between times t and $t+1$, it sends all the wires of the output block F to its parent, it sends all the wires of the output block $B_L \setminus F$ to its left child and it sends all the wires of the ouput block $B_R \setminus F$ to its right child. The n wires are distributed throughout the tree in such a way that the actual number of wires held in a node x depends only on t and on the depth i of x; we let $a(i,t)$ denote this number. To relocate wires between times t and $t+1$, each node on level i sends $\pi(i,t)$ wires to its parent and it sends $\chi(i,t)$ wires to each of its two children.

Of course one has to carefully set all these numbers. We omit the details. Then, we have to prove that the network really sorts and that it sorts fast. Let $\alpha(t)$ and $\omega(t)$ denote the top and the bottom level, respectively, that contain nonempty nodes at time t. It could be proven that $\alpha(t)$ first oscillate between 0 and 1 and then begins a periodic zig-zag descent with period four and the average speed of one level per two iterations. Similarly, $\omega(t)$ descends steadily in a periodic zig-zag movement with period 3 and the average speed of one level per three iterations. We omit all technical proofs and details, as well as the extremely complicated analysis of the body, therefore having

Theorem. *The designed network is correct and works in $O(\log n)$ time.*

The Modules

By *bipartite (n, d, μ)-expander,* we shall mean a bipartite graph G such that

(i) G has n vertices in each part,

(ii) the edge-set is the union of d matchings,

(iii) every nonempty set S of vertices in one part of G has

$$|N_G(S)| > \min\{\mu|S|, n - |S|\}.$$

Theorem. *If μ and d are positive integers such that*

$$(\mu+1)e^{\mu+2}\left(\frac{\mu}{\mu+1}\right)^d < \frac{1}{3},$$

then, for every positive integer n, there is a bipartite (n, d, μ)-expander.

The proof is probabilistic. We describe an algorithm that is able to generate all graphs consisting of all unions of d matchings, which are not expanders. We count the number $\mathcal{B}$ of these graphs and we prove that the probability there exist an expander is strictly less than 1, thus actually proving the existence of arbitrary large expander.

As a first step, we construct a simpler device than it is actually used. By and *strong $(2n, \varepsilon)$-halver,* we shall mean a comparator network on $2n$ wires with the output wires collected in equally sized block B_L, B_R so that, for every $k = 1, 2, \ldots, n$,

(i) the network places at most εk of its k smallest input keys into output block B_R and

(ii) the network places at most εk of its k largest input keys into output block B_L.

Theorem. *For every positive ε there is a positive integer d such that, for every positive integer n, there is a strong $(2n, \varepsilon)$-halver of depth d.*

The construction is quite straighforward from the expander graph G. The $2n$ wires are identified with the $2n$ vertices of G. The t-th layer in the series decomposition of $E(G)$ into d layers is defined by the t-th matching M_t of $E(G)$: its comparators are precisely the edges of M_t. It is quite easy to show that this method yield a valid $(2n, \varepsilon)$-halver.

Finally, we recursively construct the really used device, called $(a, f, \varepsilon_B, \varepsilon_F)$-*separator* (the constants here control the quality of the module). We omit the construction details.

Theorem. *For every choice of positive $\varepsilon_B, \varepsilon_F$ and δ, there is a positive integer d such that, for every choice of positive even integers a and f such that $\delta a \leq f \leq a$, there is and $(a, f, \varepsilon_B, \varepsilon_F)$-separator of depth d.*

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Diana Piguet

diana@kam.mff.cuni.cz

The Loebl conjecture

Loebl conjectured that, if a graph of order n has at least half of its vertices of degree at least $\frac{n}{2}$, then it contains, as a subgraph, any tree with at most $\frac{n}{2}$ edges.

My talk was divided into two parts: the first part was about a result by myself and Maya Stein, the second part was about a result by Ajtai, Komlós and Szemerédi.

Theorem (D. P. and Maya Stein). *The Loebl conjecture is true for trees with diameter at most 5, where we understand the diameter of a graph as the maximum distance between two of its vertices.*

The main idea of the proof is the following: we subdivide the set of vertices of that graph G into two sets V_1 and V_2, where V_1 is the set of vertices of degree at least $\frac{n}{2}$ and V_2 is its complement, i.e. $V_2 := V \setminus V_1$. We set also the set $B \subseteq V_1$ as the set of vertices in V_1 that have at least $\frac{n}{4}$ neighbours in V_1 and $C \subseteq V_2$ as the set of vertices in V_2 that have at least $\frac{n}{4}$ neighbours in V_1. Then we show that if there exists an edge $\{x, y\}$ that either

– $|N(x) \cap V_1| \geq \frac{n}{4}$, $y \in V_1$ and $|N(y) \cap (V_1 \cup C)| \geq \frac{n}{4}$ or – $x \in B$, $y \in V_1$ and $|N(y) \cap V_1| \geq \frac{n}{8}$,

then we can embed in the graph any tree T with $diam(T) \leq 5$. The second part of the proof is to show that the graph G contains an edge with one of the mentioned properties. A more detailed proof of the result should soon appear in ITI and KAM-series.

The second part of the talk was on a paper from Ajtai, Komlós and Szemerédi: *On a conjecture of Loebl* (Preprint). The result of this paper is the following theorem:

Theorem (Ajtai, Komlós, Szemerédi). *For any $\pi > 0$, there exists an n_0 such that for any $n \geq n_0$, the following holds: If G is a graph of order n such that at least $(1+\pi)\frac{n}{2}$ of its vertices have degree at least $(1+\pi)\frac{n}{2}$, then G contains as a subgraph any tree with at most $\frac{n}{2}$ edges.*

The main tool in the proof is Szemerédi's regularity lemma.

Theorem (Szemerédi). *For any $\varepsilon, \alpha > 0$ and $m \in \mathbf{N}$, there exists an $n_0, M \in \mathbf{N}$ such that for any graph $G = (V, E)$ on $n \geq n_0$ vertices, there exists a partition $V = V_0 \cup V_1 \cup \ldots \cup V_N$ of its vertices with $m \leq N \leq M$ such that 1) $|V_0| \leq \varepsilon n$, 2) $|V_1| = |V_2| = \ldots = |V_N|$, 3) all but at most εN^2 pairs (V_i, V_j), $i, j \neq 0$ are regular. We say that (V_i, V_j) is regular if for any $U_i \subseteq V_i$, $U_j \subseteq V_j$ such that $|U_i|, |U_j| \geq \alpha |V_i|$, we have that*

$$\left| \frac{e(U_i, U_j)}{|U_i||U_j|} - \frac{e(V_i, V_j)}{|V_i||V_j|} \right| < \varepsilon,$$

where $e(A, B)$ is the number of edges between A and B.

After stating this theorem, I showed some typical use of it: to erase "ugly" edges, i.e. edges that: –are contained inside a cluster (partition classes are called clusters), –have non-empty intersection with V_0, –are contained in non-regular pairs, –are contained in regular pairs with low density ($< p$).

There are so few such edges (choosing ε, m, p in a proper way) that the resulting graph G_p has similar property as the graph G: at least $(1+\frac{\pi}{2})\frac{n}{2}$ vertices have degree at least $(1+\frac{\pi}{2})\frac{n}{2}$. Also I showed that we can estimate the degree of most vertices of a cluster in G_p by knowing the average degree of this cluster.

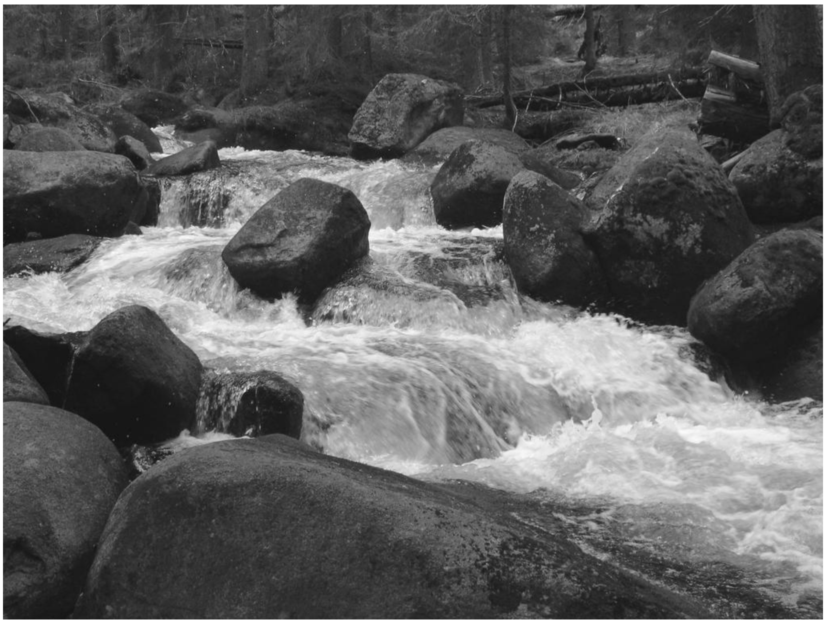
I finished my talk by showing the basic ideas of the rest of the proof, i.e. that we partition the tree T into T_A and T_B in a particular way, that we find two neighbouring clusters A and B in G_p with high average degree such that a matching M covers almost all of their neighbours, that we partition M into M_A and M_B depending on $|T_A|$ and $|T_B|$ and that, at the end, we embed the tree T such that T_A is embedded into $A \cup M_A$ and T_B is embedded into $B \cup M_B$.

Julia Böttcher

boettche@informatik.hu-berlin.de

3-coloring sparse 3-colorable graphs in polynomial expected time

The participant has not submitted any abstract.



Jan Kára

kara@kam.mff.cuni.cz

Presented paper by Jan Kára, Jan Kratochvíl, David R. Wood

On the complexity of the balanced vertex ordering problem

Introduction

We study the complexity of finding a 'balanced' ordering of the vertices of the graph that is used by a several graph drawing algorithms as a starting point. Here balanced means that neighbours of each vertex v are as evenly distributed to the left and right of v as possible. The problem of determining such an ordering was recently studied by Biedl [1]. We solve a number of open problems from [1] and study a few other related problems.

Let $G = (V, E)$ be a multigraph without loops. An *ordering* of G is a bijection $\sigma : V \to \{1, \ldots, |V|\}$. For $u, v \in V$ with $\sigma(u) < \sigma(v)$, we say that u is *to the left* of v and that v is *to the right* of u. The *imbalance* of $v \in V$ in σ, denoted by $B_\sigma(v)$, is

$$\big||\{e \in E : e = \{u, v\}, \sigma(u) < \sigma(v)\}| - |\{e \in E : e = \{u, v\}, \sigma(u) > \sigma(v)\}|\big|.$$

When the ordering σ is clear from the context we simply write $B(v)$ instead of $B_\sigma(v)$. The *imbalance* of ordering σ, denoted by $B_\sigma(G)$, is $\sum_{v \in V} B_\sigma(v)$. The minimum value of $B_\sigma(G)$, taken over all orderings σ of G, is denoted by $M(G)$. An ordering with imbalance $M(G)$ is called *minimum*. Clearly the following two facts hold for any ordering:

E very vertex of odd degree has imbalance at least one.

T he two vertices at the beginning and at the end of any ordering have imbalance equal to their degrees.

These two facts imply the following lower bound on the imbalance of an ordering. Let $\mathrm{odd}(A)$ denote the number of odd degree vertices among the vertices of $A \subseteq V$. Let $(d_1, \ldots, d_n)$ be the sequence of vertex degrees of G, where $d_i \le d_{i+1}$ for all $1 \le i \le n - 1$. Then

$$B_\sigma(G) \ge \mathrm{odd}(V) - (d_1 \bmod 2) - (d_2 \bmod 2) + d_1 + d_2.$$

An ordering σ is *perfect* if the above inequality holds with equality. *Perfect ordering* is the decision problem whether a given multigraph G has a perfect ordering. This problem is clearly in NP.

Results

Whether the balanced ordering problem is efficiently solvable for planar graphs with maximum degree four is of particular interest since a number of algorithms for producing orthogonal drawings of planar graphs with maximum degree four start with a balanced ordering of the vertices. We answer this question in the negative:

Theorem. *The perfect ordering problem is NP-complete for planar graphs with maximum degree four.*

As the problem we reduce from we use the planar *2-in-4sat.* The NP-completeness of this problem is also shown in our paper. Next we study the case of regular graphs and prove:

Theorem. *The perfect ordering problem is NP-complete for 5-regular multigraphs.*

Using a few lemmas we also show that:

Corollary. *It is NP-hard to find a minimum ordering for 5-regular simple graphs.*

In the end we describe algorithms solving at least some special cases in a polynomial time. The algorithms are base on the following lemma:

Lemma. *There is an $O(n + m)$ time algorithm to test whether a multigraph G with n vertices and m edges has an ordering in which a given list of vertices imbalanced $= (v_1, \ldots, v_k)$ are the only imbalanced vertices, and $\sigma(v_i) < \sigma(v_{i+1})$ for all $1 \leq i \leq k - 1$.*

The following theorem is a consequence of the previous lemma:

Theorem. *There is an algorithm that, given an n-vertex m-edge multigraph G, computes a minimum ordering of G with at most k imbalanced vertices (or answers that there is no such ordering) in time $O(n^k \cdot (m + n))$.*

Corollary. *There is a polynomial time algorithm to determine whether a given multigraph G has an ordering with imbalance less than a fixed constant c.*

Corollary. *The perfect ordering problem is solvable in time $O(n^2(n + m))$ for any n-vertex m-edge multigraph with all vertices of even degree.*

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Jan Foniok
foniok@kam.mff.cuni.cz

The homomorphism order of graphs

By a 'graph', we mean a simple undirected graph without loops.

Let $G = (V, E)$, $H = (W, F)$ be two graphs. A mapping $f : V \to W$ is a *homomorphism* from G to H if for every edge $\{x, y\} \in E$ the image $\{f(x), f(y)\} \in F$, i.e. if the mapping preserves edges. In such a case we write $f : G \to H$.

It is obvious that the identity mapping on the vertex set is a homomorphism from an arbitrary graph to itself, and that the composition of two homomorphisms is a homomorphism as well.

Therefore the binary relation $\to$ on the class of all graphs, defined by $G \to H$ iff there exists a homomorphism from G to H, is a reflexive and transitive relation, a *quasiorder*. If $G \to H$, we say that *G is homomorphic to H*.

If $G \to H$ and $H \to G$, we say that G and H are *homomorphically equivalent* and write $G \leftrightarrow H$. Clearly, this relation is an equivalence relation on the class of graphs. The quasiorder $\to$ induces a partial order on the set of $\leftrightarrow$-equivalence classes; this partial order is called the *homomorphism order* and denoted by $\mathcal{C}$.

When talking about this order, we (somewhat carelessly) speak about the order of graphs, while we in fact mean the order of equivalence classes. We also use the notation $G \leq H$ instead of $G \to H$ and we write $G < H$ if $G \to H$ and $H \not\to G$. If both $G \not\to H$ and $H \not\to G$, we write $G \parallel H$.

Besides the quasiorder of graphs and the partial order of equivalence classes, there is yet another point of view for $\mathcal{C}$. In each equivalence class there exists a unique (up to isomorphism) distinguished graph called a core.

We say that a graph G is a *core* if it is not homomorphic to any proper subgraph of itself.

It can be shown that two cores are isomorphic if and only if they are homomorphically equivalent. Moreover, every graph is homomorphically equivalent to a core (consider the smallest of images of all homomorphisms from G to G). Consequently, every graph G is homomorphically equivalent to a unique core; it is called *the core* of G.

The homomorphism order is a lattice: every 2-element subset $\{G, H\}$ has its supremum $G \cup H$ and its infimum $G \times H$. The first is the disjoint union of the two graphs and the latter the categorical product of graphs; for $G = (V, E)$ and $H = (W, F)$ we have $V(G \times H) = V \times W$ and $E(G \times H) = \big\{\{(u, v), (x, y)\} : \{u, x\} \in E,\ \{v, y\} \in F\big\}$.

Now, let's show the existence of an infinite antichain in $\mathcal{C}$. In fact, it is a consequence of the following much stronger statement.

Theorem 1. *Whenever G_1, G_2, ..., G_m are non-bipartite graphs that are mutually incomparable, i.e. for all $1 \leq k < l \leq m$ we have $G_k \parallel G_l$, then there exists a graph G such that $G \parallel G_k$ for all $1 \leq k \leq m$.*

For the proof, we make use of the following famous theorem by Erdős (1959).

Theorem 2 (Erdős Magic). *For arbitrary positive integers g and k, there exists a graph G of girth at least g and chromatic number at least k.*

Note that $\chi(G) \leq k$ if and only if $G \to K_k$. Therefore, if $\chi(H) < \chi(G)$, then $G \not\to H$. Also note that $C_{2k+1} \to C_{2l+1}$ if and only if $k \geq l$. Consequently, we have that $G \to H$ implies that the odd girth of G is at least as large as the odd girth of H.

Thus, to prove Theorem 1 it suffices to let Erdős Magic produce a graph G having both chromatic number and (odd) girth greater than all of the graphs G_k.

Another interesting property of the homomorphism order is its universality.

Theorem 3. *Every countable partially ordered set is isomorphic to a subposet of* $\mathcal{C}$.

The proof of this theorem is complicated, but it is easy to show universality for finite posets. It is well-known that every finite poset is isomorphic to a suborder of $(2^X, \subseteq)$ for a finite set X. We may assume that $X = \{1, 2, \ldots, m\}$. Let G_1, G_2, ..., G_m be mutually incomparable graphs such that the odd girth of G_k is $2k+1$ and the chromatic number of G_k is k. We construct a suborder of $\mathcal{C}$ isomorphic to $(2^X, \subseteq)$ by assigning to each subset M of X the graph

$$G_M = \bigcup_{k \in M} G_k.$$

One can easily check that $M \subseteq N$ if and only if $G_M \to G_N$.

Finally, we characterise all gaps in $\mathcal{C}$. A *gap* is a pair of graphs (F, H) such that $F < H$ and such that there is no graph G with $F < G < H$.

Theorem 4. *The only gap in the homomorphism order is* (K_1, K_2).

It is enough to prove that whenever F, H are cores, $F < H$ and H is not bipartite, there exists a graph G such that $F < G < H$.

Let J be an (Erdős Magic) graph with odd girth greater than the odd girth of H and $\chi(J) > |V(F)|^{|V(H)|}$. Let $G = F \cup (J \times H)$.

Obviously, $F \to G$ and $G \to H$ (recall that $J \times H$ is the infimum of the two graphs).

Assume $H \to G$. This implies the existence of a connected component C of H such that $C \to J \times H$ (otherwise H would be homomorphic to F, contrary to our assumptions). Therefore $C \to J$, because $J \times H \to J$ (the product is the infimum). This is a contradiction, since the odd girth of J is bigger than the odd girth of C.

Now suppose that $G \to F$. Let $f : J \times H \to F$. For every vertex u of J define the mapping $f_u : V(H) \to V(F)$ by $f_u(x) = f(u, x)$. The number of all mappings from $V(H)$ to $V(F)$ is $|V(F)|^{|V(H)|} < \chi(J)$, so there exist two adjacent vertices u, v of J such that $f_u = f_v$. We claim that this mapping $f_u = f_v$ is a homomorphism from H to F, contradicting that $F < H$. Indeed, if $\{x, y\}$ is an edge of H,

$$\{f_u(x), f_u(y)\} = \{f_u(x), f_v(y)\} = \{f(u, x), f(v, y)\} \in E(F),$$

because $\{(u,x),(v,y)\}$ is an edge of $J \times H$ and f is a homomorphism.

That finishes the proof of Theorem 4.

Aleš Přívětivý

privetivy@kam.mff.cuni.cz

Presented paper by B. Barak, R. Impagliazzo, A. Wigderson

Extracting randomness using few independent sources

In this work we give the first deterministic extractors from a constant number of weak sources whose entropy rate is less than $1/2$. Specifically, for every $\delta > 0$ we give an explicit construction for extracting randomness from a constant (depending polynomially on $1/\delta$) number of distributions over $\{0,1\}^n$, each having min-entropy δn. These extractors output n bits, which are 2^{-n} close to uniform. This construction uses several results from additive number theory, and in particular a recent one by Bourgain, Katz and Tao and of Konyagin.

We also consider the related problem of constructing randomness dispersers. For any constant output length m, our dispersers use a constant number of identical distributions, each with min-entropy $\Omega(\log n)$ and outputs every possible m-bit string with positive probability. The main tool we use is a variant of the "stepping-up lemma" used in establishing lower bound on the Ramsey number for hypergraphs (Erdős and Hajnal).

Definition. U_n denotes the uniform distribution over the set $\{0,1\}^n$ and $\mathrm{dist}(\mathcal{X},\mathcal{Y})$ denotes the *statistical distance* of two distributions $\mathcal{X}$ and $\mathcal{Y}$. That is,

$$\mathrm{dist}(\mathcal{X},\mathcal{Y}) = \frac{1}{2} \sum_{i \in \mathrm{Supp}(\mathcal{X}) \cup \mathrm{Supp}(\mathcal{Y})} \Big| Pr[\mathcal{X}=i] - Pr[\mathcal{X}=i] \Big|.$$

Definition. The *min-entropy* of a random variable $\mathcal{X}$, denoted by $H^\infty(\mathcal{X})$ is defined as

$$H^\infty(\mathcal{X}) = \min_{i \in \mathrm{Supp}(\mathcal{X})} -\log Pr[\mathcal{X}=i].$$

Theorem (Existence of a multiple sample extractor). *For every constant $\delta > 0$ there exists a constant $l = (1/\delta)^{O(1)}$ and a polynomial time computable function* $\mathrm{Ext} : \{0,1\}^{n \cdot l} \to \{0,1\}^n$ *such that for every independent random variables $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_l$ over $\{0,1\}^n$ satisfying $H^\infty(\mathcal{X}_i) > \delta n$ for every $i = 1, \ldots, l$, it holds that*

$$\mathrm{dist}(\mathrm{Ext}(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_l), U_n) < 2^{-\Omega(n)}.$$


H-colouring
STOP!


VEGETARIÁNI

Martin Cetkovský
martin@alikuvkoutek.cz

Presented paper by Reinhard Diestel, Christof Rempel

Dense minors in graphs of large girth

(Combinatorica 25 (1) (2005) pages 111–116)

Theorem (Thomassen 1983). *For any integer k, every graph G of girth $g(c) > 4k-3$ and $\delta(G) \geq 3$ has a minor H with $\delta(H) > k$.*

Our Main aim is to reduce the upper bound for the required girth to the correct order of magnitude:

Theorem. *For any integer k, every graph G of girth $g(c) > 6\log k + 3$ and $\delta(G) \geq 3$ has a minor H with $\delta(H) > k$.*

Corollary. *There exists a constant $c \in \Re$ such that every graph G of girth $g(c) \geq 6\log r + 3\log\log r + c$ and $\delta(G) \leq 3$ has a K_r minor.*

Lemma (Mader). *In a graph of girth $g(G) > 2d+1$ and $\delta(G) \leq 3$ the d-ball around a vertex x is a tree T_x sending at least $|T_x| - 2$ edges to the rest of G.*

Lemma. *Let T be a tree with root r in which no vertex has exactly one successor, and let $d \in N$. Then $\sum_{i \geq d} 2^{d-i} \left|L_T^i\right| \geq \left|V_{T,r}^d\right|$.*

Breef overview of the proof: Put $\lfloor \log k \rfloor =: d$ and let X be a maximal set of vertices such that $d(x,y) > 2d$ for all distinct $x, y \in X$. Starts from $T_x^0 := \{x\}$ for all $x \in X$ and using induction we construct trees T_x which partition the entire vertex set of G. So T_x contain all the vertices of G at distance at most d from x, T_x are induced subgraphs in G and $d(w,y) \leq d(v,x) + 1$ whenever $vw \in E(G)$ with $v \in T_x$ and $w \in T_y$.

Let us use Lemma to estimate the number of edges leaving a tree T_x. Let T_x' denote the subgraph of G induced by T_x and all its neighbours in G which is again a tree.

Now we show using lemmas and simple observations that if T_x sends edges to at least 2^{d+1} other trees T_y, contracting all the trees T_x with $x \in X$ we then obtain a minor of G of minimum degree at least $2^{d+1} > k$, as desired.

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List of All Participants

ROBERT BABILON
Charles University, Prague
Czech Republic
babilon@kam.mff.cuni.cz

JULIA BÖTTCHER
Humboldt-Universität, Berlin
Germany
boettche@informatik.hu-berlin.de

MARTIN CETKOVSKÝ
Charles University, Prague
Czech Republic
martin@alikuvkoutek.cz

JOSEF CIBULKA
Charles University, Prague
Czech Republic
josef.cibulka@seznam.cz

JAKUB ČERNÝ
Charles University, Prague
Czech Republic
kuba@kam.mff.cuni.cz

VERONIKA DOUCHOVÁ
Charles University, Prague
Czech Republic
veronika@kam.mff.cuni.cz

TOMÁŠ EBENLENDR
Charles University, Prague
Czech Republic
ebik@artax.karlin.mff.cuni.cz

JIŘÍ FINK
Charles University, Prague
Czech Republic
fink@atrey.karlin.mff.cuni.cz

JAN FONIOK
Charles University, Prague
Czech Republic
foniok@kam.mff.cuni.cz

Tomáš Gavenčiak
Charles University, Prague
Czech Republic
gavento@atrey.karlin.mff.cuni.cz

Jan Hladký
Charles University, Prague
Czech Republic
hladk@seznam.cz

Petr Hliněný
Technical University of Ostrava
Czech Republic
petr.hlineny@vsb.cz

Přemysl Holub
University of West Bohemia, Pilsen
Czech Republic
holubpre@kma.zcu.cz

Vít Jelínek
Charles University, Prague
Czech Republic
jelinek@kam.mff.cuni.cz

Tomáš Kaiser
University of West Bohemia, Pilsen
Czech Republic
kaisert@kma.zcu.cz

Jan Kára
Charles University, Prague
Czech Republic
kara@kam.mff.cuni.cz

Alexandr Kazda
Charles University, Prague
Czech Republic
alexandr.kazda@seznam.cz

Jan Kratochvíl
Charles University, Prague
Czech Republic
honza@kam.mff.cuni.cz

Dan Král'
Charles University, Prague
Czech Republic
kral@kam.ms.mff.cuni.cz

Marek Krčál
Charles University, Prague
Czech Republic
marek.krcal@seznam.cz

Petr Kučera
Charles University, Prague
Czech Republic
kucerap@kti.ms.mff.cuni.cz

Vojtěch Kulvait
Charles University, Prague
Czech Republic
kulvait@atlas.cz

Petr Lopez Kučera
Czech University of Agriculture, Prague
Czech Republic
kucera@pef.czu.cz

Bernard Lidický
Charles University, Prague
Czech Republic
2berny@seznam.cz

Martin Loebl
Charles University, Prague
Czech Republic
loebl@kam.mff.cuni.cz

Martin Mareš
Charles University, Prague
Czech Republic
mj@ucw.cz

Jaroslav Nešetřil
Charles University, Prague
Czech Republic
nesetril@kam.mff.cuni.cz

Eva Ondráčková
Charles University, Prague
Czech Republic
efa@matfyz.cz

Ondřej Pangrác
Charles University, Prague
Czech Republic
pangrac@kam.mff.cuni.cz

Martin Pergel
Charles University, Prague
Czech Republic
perm@kam.mff.cuni.cz

Diana Piguet
Charles University, Prague
Czech Republic
diana@kam.mff.cuni.cz

Ondřej Plašil
Charles University, Prague
Czech Republic
oplasil@centrum.cz

Aleš Přívětivý
Charles University, Prague
Czech Republic
privetivy@kam.mff.cuni.cz

Ondřej Rucký
University of West Bohemia, Pilsen
Czech Republic
ondrej.rucky@seznam.cz

Christian Sloper
University of Bergen
Norway
sloper@ii.uib.no

Rudolf Stolař
Charles University, Prague
Czech Republic
riv@email.cz

Ondřej Suchý
Charles University, Prague
Czech Republic
ondra@s.cz

Marek Sulovský
Charles University, Prague
Czech Republic
skeeve@popelka.ms.mff.cuni.cz

Robert Šámal
Charles University, Prague
Czech Republic
samal@matfyz.cz

Petr Škoda
Charles University, Prague
Czech Republic
petr.skoda@seznam.cz

Petr Škovroň
Charles University, Prague
Czech Republic
xofon@kam.mff.cuni.cz

Riste Škrekovski
University of Lyublyana
Slovenia
skrekovski@yahoo.com

Milan Tuharský
Pavol Jozef Šafárik University in Košice
Slovak Republic
tuharsky@science.upjs.sk

Tomáš Valla
Charles University, Prague
Czech Republic
valla@kam.mff.cuni.cz

Pavel Valtr
Charles University, Prague
Czech Republic
valtr@kam.mff.cuni.cz

Tomáš Vyskočil
Charles University, Prague
Czech Republic
tiger@matfyz.cz

Ondřej Zajíček
Charles University, Prague
Czech Republic
santiago@mail.cz

More information about the Spring School can be found at
`http://kam.mff.cuni.cz/~spring/`