On the computational complexity of the $L_{(2,1)}$-labeling problem for regular graphs *

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Abstract. An $L_{(2,1)}$-labeling of a graph of span t is an assignment of integer labels from $\{0, 1, \dots, t\}$ to its vertices such that the labels of adjacent vertices differ by at least two, while vertices at distance two are assigned distinct labels.
We show that for all $k \geq 3$, the decision problem whether a k-regular graph admits an $L_{(2,1)}$-labeling of span $k+2$ is NP-complete. This answers an open problem of R. Laskar.

1 Introduction

Motivated by models of channel assignment in wireless communication [7, 6], generalized graph coloring and in particular the concept of $L_{(2,1)}$-labeling have drawn significant attention in the graph-theory community in the past decade [1].

Besides the practical aspects, also purely theoretical questions became very intersting. Among other we shall highlight a long-lasting conjecture of Griggs and Yeh that the span of any optimal $L_{(2,1)}$-labeling is upperbounded by $\Delta(G)^2$, where $\Delta(G)$ is the maximum degree of the given graph G [6]. So far this conjecture is still open, though it has been verified for various classes of graphs (e.g., for chordal graphs [11, 8] or for graphs of diameter at most two [6]).

We focus our attention on the computational complexity of the decision problem whether a given graph G allows an $L_{(2,1)}$-labeling of span at most λ.

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If λ is a part of the input, the problem becomes NP-complete by a reduction from the Hamiltonian path problem [6]. If λ is a fixed constant (i.e., the parameter of the problem), the computational complexity was settled in [3], by constructing a polynomial time algorithm for $\lambda \leq 3$ and by showing that the problem is NP-complete otherwise. The core argument of the NP-hardness proof is based on the fact that vertices of high degree may allow only extremal labels (i.e., 0 or λ) of the given spectrum.

In response to this fact, R. Laskar asked at the DIMACS/DIMATIA/-Rényi Workshop on Graph Colorings and their Generalizations (Rutgers University, 2003) what is the computational complexity of the $L_{(2,1)}$-labeling problem when restricted to regular graphs, hoping that the restriction might provide new ideas for a general proof of hardness results on distance constrained labelings. In this note we settle the computational complexity of the $L_{(2,1)}$-labeling problem on regular graphs in the following sense:

Theorem 1. *For every integer $k \geq 3$, it is NP-complete to decide whether a k-regular graph admits an $L_{(2,1)}$-labeling of span (at most) $\lambda = k + 2$.*

The result is the best possible in terms of the span, since no k-regular graph (for $k \geq 2$) allows an $L_{(2,1)}$-labeling of span $k+1$ (see e.g. a paper by Georges and Mauro [5] on labelings of regular graphs). Though our result is not totally unexpected, the reduction (namely the garbage collection) is surprisingly uneasy to design. It utilizes so called multicovers introduced in [10].

The paper is organized as follows: The next section provides necessary definitions and facts used latter. In Section 3 we prepare tools used in the construction and discuss their properties. The main result is then proven in Section 4.

2 Preliminaries

All graphs considered in this paper are finite and simple, i.e., with a finite vertex set and without loops or multiple edges. A graph G is denoted as a pair (V_G, E_G), where V_G stands for a finite set of vertices and E_G is a set of edges, i.e. unordered pairs of vertices. The distance $\mathrm{dist}_G(u, v)$ between two vertices u and v of a graph G is the length (the number of edges) of a shortest path connecting u and v. If two vertices belong to different components, we let their distance be unspecified.

The set of vertices adjacent to a vertex u is called *the neighborhood* of u and it is denoted by $N_G(u)$. The *degree* of a vertex u is the cardinality of its neighborhood, i.e., $\deg(u) = |N_G(u)|$. A graph is called *k-regular* if all its vertices are of degree k.

A vertex labeling by nonnegative integers $f : V_G \to \mathbb{Z}_0^+$ is called an *$L_{(2,1)}$-labeling* of G if $|f(u)-f(v)| \geq 2$ holds for any pair of adjacent vertices u and v, and the labels $f(u)$ and $f(v)$ are distinct whenever $\mathrm{dist}(u,v) = 2$.

The *span* of an $L_{(2,1)}$-labeling is the difference between the largest and the smallest label used. The parameter $\lambda_{(2,1)}(G)$ is the minimum possible span of an $L_{(2,1)}$-labeling of G. Such a labeling will be called *optimal*, and we may assume that it uses labels from the discrete interval $[0, \ldots, \lambda_{(2,1)}(G)]$.

With an optimal labeling f we associate its *symmetric* labeling f', defined by $f'(u) = \lambda_{(2,1)}(G) - f(u)$. Clearly the symmetric labeling is also optimal.

$L_{(2,1)}$-labelings are closely related to graph covers: A *full covering projection* from a graph G to a graph H is a graph homomorphism $h : V_G \to V_H$ such that the neighborhood $N_G(u)$ of any vertex $u \in V_G$ is mapped bijectively on the neighborhood $N_H(h(u))$ of $h(u)$.

Similarly, if the mapping is locally injective, i.e., if $N_G(u)$ is mapped injectively into $N_H(h(u))$, we call h a *partial covering projection*. Obviously every full covering projection is also a partial covering projection.

The relationship between $L_{(2,1)}$-labelings and (partial) covering projections was discussed in [2]:

Proposition 1. *Every $L_{(2,1)}$-labeling of a graph G of span λ corresponds to a partial covering projection $G \to \overline{P_{\lambda+1}}$, and vice versa.*

In particular, $\overline{C_{\lambda+1}} \subset \overline{P_{\lambda+1}}$, hence every partial covering projection to $\overline{C_{\lambda+1}}$ is also an $L_{(2,1)}$-labeling of span at most λ.

Kratochvíl, Proskurowski and Telle [10] gave an explicit construction of a special multicover graph allowing many extensions to full covering projections. We will use it in our gadgets.

Proposition 2 ([10]). *For any regular graph F, there exists a graph H (called a* multicover *of F) with a distinguished vertex $u \in V_H$ such that any locally injective homomorphism $h' : N_H(u) \cup u \to F$ can be extended to a locally bijective homomorphism $h : H \to F$.*

3 Gadgets

3.1 Polarity gadget

Let $k \geq 3$ be a positive integer. Consider the graph F_p on $k+5$ vertices $v_1, \ldots, v_{k-1}, u_1, \ldots, u_4, x, y$, with edges defined as follows:

$$\begin{aligned} E(F_p) = {} & \{(v_i, v_j) \,|\, 1 \leq i, j \leq k-1, |i-j| \geq 2\} \\ & \cup \{(v_i, u_j) \,|\, 1 \leq i \leq k-1, j = 1,2,3\} \\ & \cup \{(v_i, u_4) \,|\, 2 \leq i \leq k-2, \} \\ & \cup \{(u_1, u_2), (u_3, u_4), (u_4, x), (u_4, y)\} \end{aligned}$$

See Fig. 1 for an example of such a graph. Observe also that each vertex except x and y is of degree k.

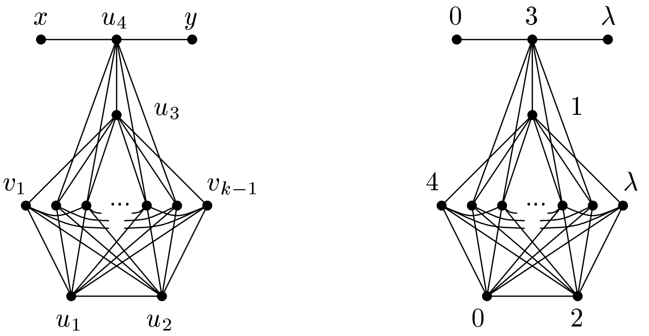

Fig. 1. The polarity gadget F_p and its $L_{(2,1)}$-labeling.

Lemma 1. *In the graph F_p, the pair of vertices x and y are labeled by* 0 *and λ (or vice versa) under any $L_{(2,1)}$-labeling f of span $\lambda = k+2$.*

Proof. The edge (u_1, u_2) participates in $k-1$ triangles. If both u_1 and u_2 were labeled by labels different from 0 and λ, then at most $\lambda - 4 < k-1$ labels would remain for $v_1, \ldots, v_{k-1}$, which is insufficient. So without loss of generality we may assume $f(u_1) = 0$, and then u_2 may get only two possible labels: 2 or λ. The latter would, however, exclude all possible choices for u_3.

Now up to a symmetry of labelings we have $f(u_1) = 0$, $f(u_2) = 2$ and for the vertices $v_1, \ldots, v_{k-1}$ remain the labels $4, 5, \ldots, \lambda$. Since F_p restricted onto these vertices is the complement of a path on $k-1$ vertices, only two possible labelings exist: either $f(v_i) = i + 3$ or $f(v_i) = \lambda + 1 - i$. In both cases (they are equivalent under an automorphism of F_p) only one possible label remains for the vertex u_3, namely $f(u_3) = 1$. We deduce by similar arguments that $f(u_4) = 3$.

Finally, since u_4 is adjacent to vertices labeled by $1, 5, \ldots, \lambda - 1$, its remaining neighbors x and y must be labeled either one by 0 and the other one by λ as claimed.

3.2 Swallowing gadget

In our construction we involve multicovers allowing two different $L_{(2,1)}$-labelings as follows. Let H, u be a multicover of the k-regular graph $\overline{C_{k+3}}$. For the swallowing gadget we take two copies H_1, H_2 of the graph H (with the notation that the copy of vertex v in H_i is denoted by v_i, $i = 1, 2$), insert two new vertices x, y and modify the edge set as follows:

$$E(F_s) = (E(H_1) \cup E(H_2) \setminus \{(u_1, v_1), (u_2, v_2)\}) \cup \{(x, v_1), (y, v_2), (u_1, u_2)\}$$

where v is an arbitrary neighbor of u in H. Observe that again all vertices except x and y are of degree k. The construction of the swallowing gadget is illustrated in Fig. 2.

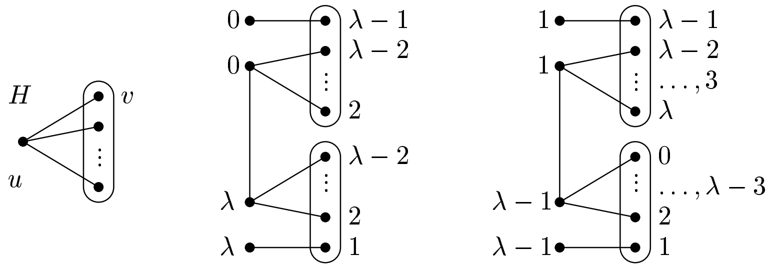

Fig. 2. The swallowing gadget F_s and the two of its $L_{(2,1)}$-labelings.

Lemma 2. *The graph F_s allows two $L_{(2,1)}$-labelings f and f', such that $f(v_1) = f'(v_1) = \lambda - 1$, $f(v_2) = f'(v_2) = 1$, while $f(x) = 0$, $f'(x) = 1$ and $f(y) = \lambda$, $f'(y) = \lambda - 1$.*

Proof. We label the vertices of $\overline{C_{k+2}}$ by integers $[0, \lambda]$, in a usual sequential way. For the construction of f we choose the covering projection from H_1 to $[0, \lambda]$ where u is mapped on 0 and v on $\lambda - 1$. The remaining neighbors of u are mapped onto $2, 3, \ldots, \lambda - 1$. On H_2 we use the symmetric labeling and get a valid $L_{(2,1)}$-labeling of F_s, since the "central" vertices u_1 and u_2 got labels 0 and λ which are sufficiently separated for the desired $L_{(2,1)}$-labeling f.

Similarly f' can be obtained in a similar way from an $L_{(2,1)}$-labeling of H where u is mapped on 1, the vertex v on $\lambda - 1$ and the remaining neighbors of u are mapped on the set $3, 4, \ldots, \lambda - 2, \lambda$.

Both labelings are schematically depicted in Fig. 2.

3.3 Coupling gadget

Let a be an integer in the range $[1, \lambda - 1]$. Set $T = \{1, 2, \ldots, a - 1, a + 1, \ldots, k\}$. We construct the following graph (called the *coupling gadget*) F_c^a on $k^2 + 2k + 1$ vertices $\{v_i^t : i = 0, 1, \ldots, \lambda,\ t \in T\} \cup \{u_1, u_2, x, y\}$ by setting the edges as follows:

$$\begin{aligned} E(F_c^a) = &\{(v_i^t, v_j^t) \mid 0 \leq i, j \leq \lambda, |i - j| \geq 2, t \in T\} \\ &\setminus \{(v_t^t, v_\lambda^t), (v_{\lambda-t}^t, v_0^t) \mid t \in T\} \\ &\cup \{(u_1, v_t^t), (u_2, v_{\lambda-t}^t) \mid t \in T\} \\ &\cup \{(u_1, x), (u_2, y)\} \end{aligned}$$

An example of the coupling gadget for a particular choice $\lambda = 5, a = 2$ is depicted in Fig 3. Observe that similarly as above all vertices except x and y are of degree k.

Lemma 3. *The graph F_c^a allows an $L_{(2,1)}$-labeling f of span λ such that $f(x) = a$, $f(y) = \lambda - a$, $f(u_1) = \lambda$ and $f(u_2) = 0$.*

Proof. Set $f(v_i^t) = i$ for $i = 0, 1, \ldots, \lambda,\ t \in T$. (See Fig. 3 for an example.) An easy check shows that it is a valid $L_{(2,1)}$-labeling.

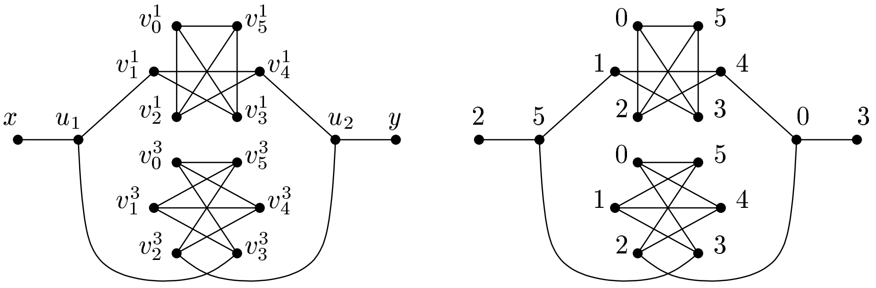

Fig. 3. The coupling gadget F_c^2 for $\lambda = 5$ and its $L_{(2,1)}$-labeling.

4 Main result

Theorem 2. *For every integer $k \geq 3$, it is* NP*-complete to decide whether a k-regular graph admits an $L_{(2,1)}$-labeling of span (at most) $\lambda = k + 2$.*

Proof. The problem is clearly in NP. Moreover, no k-regular graph admits an $L_{(2,1)}$-labeling of span less than $k + 2$ (as long as $k \geq 1$), so we can restrict our attention only to labelings of span exactly $k + 2$.

We prove the theorem by a reduction from the NOT-ALL-EQUAL 3-SATISFIABILITY problem. The input of this problem is a Boolean formula Φ in conjunctive normal form with exactly three literals in each clause and the question is whether it is NAE-satisfiable, i.e, if a truth assignment exists such that each clause contains at least positively and at least one negatively valued literal. Determining whether Φ is NAE-satisfiable has been shown NP-complete by Schaefer [12] (see also [4, Problem LO3]).

Without loss of generality we may assume that with each clause C the formula Φ contains also its complementary clause C' consisting of the complements of all literals in C. In particular, each variable has then the same number of positive and negative occurrences.

From such a formula Φ we construct a k-regular graph G such that G allows an $L_{(2,1)}$-labeling f of span $k + 2$ if and only if Φ is NAE-satisfiable. The graph G is constructed by local replacements of variables and clauses by variable and clause gadgets described below. (Consult Fig. 4 for details of the construction.)

Variable gadgets: Assume first that $k \neq 4$. For each variable with t positive and t negative occurrences, we insert in G $2t$ copies of the polarity gadget F_p, arranged in a circular manner, i.e., the vertex y_i of the i-th gadget will be identified with the vertex x_{i+1} of the consequent gadget. (The last and the first gadgets are joined accordingly as well.)

The vertices x_i with odd indices will represent positive occurrences of the associated variable, while even indices will be used as negated occurrences of the variable. For $k \geq 5$, we conclude the construction of each variable gadget by inserting $t(k+1)$ new vertices $v_i^s,\ i = 1, \dots, k+1,\ s = 1, \dots, t$, and t triples of coupling gadgets F_c^1, F_c^3 and F_c^3 linked by the following edges:

- (v_i^s, v_j^s) if $|i - j| \geq 2$, i.e., each $(k+1)$-tuple with the same upper index induces the complement of a path on $k+1$ vertices
- $(v_i^s, x_{2s-1}), (v_i^s, x_{2s})$ if $i \neq 1, 3, \lambda - 3, \lambda - 1$

Moreover, for each $s = 1, \dots, t$ the vertices v_1^s and $v_{\lambda-1}^s$ are identified with the x, y vertices of its uniquely associated coupling gadget F_c^1, and similarly v_3^s and $v_{\lambda-1}^s$ are merged with the x, y of a pair of F_c^3s.

When $k = 4$, we join polarity gadgets in a similar way: Use t copies of the polarity gadget, the x-vertices represent positive occurrences and the y-vertices negations. Now with a help of the $t(k+1)$ new vertices $v_1^1, \dots, v_{k+1}^t$, we define the remaining edges as:

- (v_i^s, v_j^s) if $|i - j| \geq 2$, i.e., each $(k+1)$-tuple with the same upper index induces the complement of a path on $k+1$ vertices
- $(v_i^s, y_s), (v_i^s, x_{s+1})$ for $i = 2, 4$

As above, two coupling gadgets F_c^1, F_c^3 are joined to vertices v_1^s and v_5^s (gadget F_c^1) and to v_3^s (gadget F_c^3); both connections terminate in v_3^s). See Fig 4 (right) for a detail of this construction.

Observe that at this moment vertices of variable gadgets are of degree $k - 1$ (the x_i's) or of degree k (all others).

Clause gadgets: Each clause gadget consists of $k+3$ vertices $z_1, z_2, z_3, w_1, \dots, w_k$, and of the following edges:

- (w_i, w_j) if $|i - j| \geq 2$, inducing the complement of a path on k vertices
- (z_i, w_j) if $i = 1, 2, 3$ and $2 \leq j \leq k - 2$

Clause gadgets of complementary clauses C and C' are joined by use of

- three swallowing gadgets F_s where for each $i = 1, 2, 3$, the vertices x, y are identified with z_i and z'_i. (Both z vertices must represent the same variable — one a positive occurrence, the other one a negated occurrence),
- two coupling gadgets F_c^k where both x's are merged with w_k and both y's with w'_k,
- a coupling gadget F_c^1 between w_1 and w'_1,
- the edge (w_1, w'_1).

Completing the construction: Finally, all variable and clause gadgets are composed together as follows: The x-vertices of variable gadgets are linked in one-by-one manner to the z-vertices of clause gadgets such that edges between gadgets represent the variable-clause incidence relation in Φ between the associated variables and clauses. As was already noted above, vertices x_i with odd i indicate positive occurrences of the associated variable, while those with an even i represent negations. (Formally, if a variable v occurs positively in a clause c, we pick a unique x_i, i odd, of the gadget representing v and a unique z_j of the clause gadget representing c and insert into G the edge (x_i, z_j). Similarly for a negated variable we choose x_i with an even index i.) This concludes the construction of the graph G, see Fig. 4 for an illustration.

Clearly G is k-regular. It remains to be shown that G admits an $L_{(2,1)}$-labeling of span $k+2$ if and only if Φ is NAE-satisfiable. In particular, we prove that the NAE-satisfying Boolean assignments ϕ are in one-to-one correspondence with valid labelings f of G via the equivalence:

(*) $\phi(v) = \text{TRUE} \Leftrightarrow f(x_1) = 0$ for x_1 of the variable gadget representing v.

Assume first that Φ is NAE-satisfied by an assignment ϕ. Then the partial of x_1's can be extended to all variable gadgets such that each x_i is inside the gadget incident with vertices $2, 3, \ldots, \lambda - 2$. (Just set $f(v_i^s) = i$ and extend it to the polarity and coupling gadgets.) Consider a clause C and its gadget. Without loss of generality we may assume that C contains one positively and two negatively valued literals, i.e., (up to an permutation of indices) z_1 is adjacent to a variable vertex labeled by λ, and z_2, z_3 to vertices labeled by 0. We extend f onto the clause gadget by letting $f(z_1) = 0$, $f(z_2) = \lambda - 1$, $f(z_3) = \lambda$ and $f(w_i) = i$. For the complementary clause C', we label its gadget symmetrically and extend f to the remaining swallowing and coupling gadgets to get a valid labeling of the entire graph G.

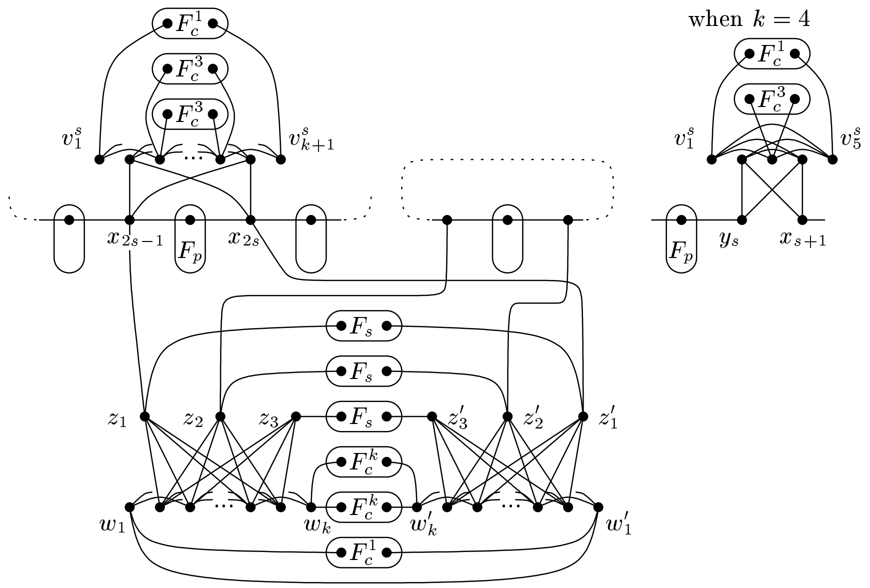

Fig. 4. Construction of G, variable gadgets in the upper part, two complementary clause gadgets at the bottom. The different construction of the variable gadget for $k = 4$ shown on the right side.

In the opposite direction, it is easy to observe that in a valid $L_{(2,1)}$-labeling f of G of span $k+2$ the following arguments hold:

- Up to symmetry the polarity gadgets allow only one possible labeling, where in each variable gadget $f(x_i) \neq f(x_{i+1})$, $f(x_i) \in \{0, \lambda\}$.
- For $k \geq 5$, the $k-3$ common neighbors $\{v_2^s, v_4^s, v_5^s, \ldots, v_{\lambda-4}^s, v_{\lambda-2}^s\}$ of x_{2s-1} and x_{2s} must be labeled by the set $\{2, 4, 5, \ldots, \lambda-4, \lambda-5\}$ regardless the labeling of x_{2s-1} and x_{2s}. (Note that each x_i gets neighbors labeled 3 and $\lambda - 3$ inside the polarity gadgets.)
  Similarly for $k = 4$, it holds that $f(y_s) \neq f(x_{s+1})$ and $f(z_2^s), f(z_4^s) \in \{2, 4\}$.
- If x_i is labeled 0, then its neighbor z_j is labeled either λ or $\lambda - 1$ and symmetrically if $f(x_1) = \lambda$ then $f(z_j) \in \{0, 1\}$.
- In each clause gadget the labels of z_1, z_2 and z_3 must be distinct since they share a common neighbor (e.g., the vertex w_1). Then from both sets $\{0, 1\}$ and $\{\lambda - 1, \lambda\}$ at least one label is used on $\{z_1, z_2, z_3\}$.

Then ϕ defined by (*) is a NAE-satisfying assignment for Φ, i.e., each clause contains some positively as well as also some negatively valued literals. This concludes the proof of NP-hardness of the problem. Membership in NPis obvious.

5 Conclusion

We have shown NP-hardness of determining the minimum span of $L(2,1)$-labelings of regular graphs by proving that for every $k \geq 3$, the decision problem whether $\lambda_{(2,1)}(G) \leq k+2$ is NP-complete for k-regular graphs G. Note that the bound $k+2$ is the minimum possible, no k-regular graph allows an $L(2,1)$-labeling of span less than $k+2$.

We conjecture that for every $k \geq 3$, there exists a constant c_k (depending on k) such that the decision problem $\lambda_{(2,1)}(G) \leq \lambda$ restricted to k-regular graphs is NP-complete for every fixed $\lambda \in \{k+2, k+3, \ldots, c_k\}$ and polynomially solvable for all other values of λ. The latter is certainly true for small λ (i.e., $\lambda \leq k+1$). The upper bound is more interesting. In particular, if our conjecture is true, it still remains a question how far is the c_k from $\lambda_k = \max\{c : \exists k\text{-regular } G \text{ s.t. } \lambda_{(2,1)}(G) > c\}$. We conjecture that $c_k \neq \lambda_k$, i.e., that in the upper part of the spectrum there will be space for nontrivial polynomial time algorithms. Note finally that $\lambda_k \leq k^2 + k - 2$ follows from [9], and that $\lambda_k \leq k^2 - 1$ if the conjecture of Griggs and Yeh is true.

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