COLORED GRAPHS
WITHOUT COLORFUL CYCLES

RICHARD N. BALL, ALEŠ PULTR, AND PETR VOJTĚCHOVSKÝ

ABSTRACT. A colored graph is a complete graph in which a color has been assigned to each edge, and a colorful cycle is a cycle in which each edge has a different color. We first show that a colored graph lacks colorful cycles iff it is Gallai, i.e., lacks colorful triangles. We then show that, under the operation $m \circ n \equiv m + n - 2$, the lengths of omitted cycles in a colored graph form a monoid isomorphic to a submonoid of the natural numbers which contains all integers past some point. We conjecture that all such monoids are realized, and prove that several are.

We then characterize exactly Gallai graphs, i.e., graphs in which every triangle has edges of exactly two colors. We show that these are precisely the graphs which can be iteratively built up from three simple colored graphs, having 2, 4, and 5 elements, respectively. We then characterize, in two quite different ways, the monochromes, i.e., the connected components of maximal monochromatic subgraphs, of exact Gallai graphs. The first characterization is in terms of their reduced form, a notion which hinges on the important idea of a full homomorphism. And the second characterization is by means of a homomorphism duality

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Introduction

For the purposes of constructing coproducts of distributive lattices, the first two authors found certain edge-colorings of complete graphs to be useful. The specific colorings of use were those lacking colorful cycles of particular lengths. It turns out that such colorings exhibit a structure which may be of interest in its own right. We investigate that structure here.

The absence of short colorful cycles implies the absence of certain longer ones, and this fact leads to the concept of the spectrum, defined and analyzed in Section 2, culminating in Conjecture 2.6. Gallai colorings, i.e., colorings which lack colorful 3-cycles, constitute an extreme example of this phenomenon, for they have no colorful cycles at all (Proposition 2.2).

We therefore turn our attention to Gallai colorings. These colorings are known to have a simple and pleasing structure, which we review and elaborate in Section 3. We then impose the additional hypothesis of exactness, i.e., the hypothesis that every 3-cycle has edges of exactly two colors. The resulting structural description, given in Section 4, is especially sharp, and, in fact, the analysis can be considered to be complete. In Section 5 we recall the important notion of a full homomorphism, and some of its properties. Using this notion, together with the structural results from Section 4, we are able to characterize the monochromes, i.e., the components of the monochromatic subgraphs. We then elaborate this characterization in Section 6 by means of a homomorphism duality.

Gallai initiated the investigation of the colored graphs, which now bear his name, in his foundational paper [3]. Since then, these graphs have appeared in several different contexts, and for different reasons. We mention only two of the more recent investigations: Gyárfás and Simonyi showed the existence of monochromatic spanning brushes in [4]; Chung and Graham found the bound on the maximum number of vertices for a given number of colors in exact Gallai cliques in [2] (see Theorem 5.9). A good general background reference is the survey article [6].

1. Ground clearing

Graphs will be assumed to be finite, symmetric (undirected), and without loops. We denote a graph G by (V_G, E_G), where V_G and E_G designate the sets of vertices and edges of G, respectively. Symbols u, v, and w are reserved for vertices, with the edge connecting vertices u and v designated by uv. The symbol K is reserved for complete graphs.

An *edge coloring of a graph* G, or simply a *coloring of* G, is an assignment of an element of a finite set Γ of *colors* to each edge of G. We use lower-case Greek letters to designate the individual colors, upper-case Greek letters to designate sets of colors, $\overline{uv}$ to designate the color assigned to the edge uv, and $\overline{\bullet}$ to designate the coloring map itself. A *colored graph* is an object of the form $G = (V, E, \overline{\bullet})$, where (V, E) is a graph and $\overline{\bullet} : E \to \Gamma$ is a coloring.

In any graph, a *clique* is a complete subgraph induced by a nonempty subset of vertices. We denote cliques by symbols a, b, c, d, and for cliques a and b, we denote the set of edges joining their vertices by

$$ab \equiv \{uv \in E : u \in a,\ v \in b\}\,.$$

In most instances, our graphs will be complete, so that the cliques could be identified with the corresponding vertex subsets. Still, we prefer to speak of cliques to emphasize that we deal with edges rather than with vertices.

In a colored graph G, a clique is regarded as a colored graph under the restriction of the coloring of G. For cliques a and b, we designate by $\overline{ab}$ the set of colors of the edges of ab; note that $\overline{aa} = \emptyset$ if $|a| = 1$.

In any graph, an *n-cycle*, $n \geq 2$, is a sequence $(v_1, v_2, \ldots, v_n)$ of distinct vertices. Two cycles are regarded as identical if they can be made to coincide by a cyclical permutation $(v_i \longmapsto v_{j+i})$ of their elements, where all subscript arithmetic is performed $\bmod n$. 3-cycles are called *triangles*, 4-cycles are called *squares*, and so forth. The *edges of a cycle* $(v_1, v_2, \ldots, v_n)$ are those of the form $v_i v_{i+1}$.

In a colored graph, a cycle $(v_1, v_2, \ldots, v_n)$ is *colorful* if all its edges have different colors, i.e., if

$$\overline{v_i v_{i+1}} = \overline{v_j v_{j+1}} \Longleftrightarrow i = j \bmod n.$$

Note that a 2-cycle is never colorful. A *Gallai clique* is a clique which has no colorful triangles. An *exact Gallai clique* is a Gallai clique in which every triangle has edges of exactly two colors.

2. The spectrum of a colored graph

In complete colored graphs, the absence of colorful cycles of a particular length implies the absence of certain longer colorful cycles. In particular, the absence of colorful triangles implies the absence of colorful cycles of any length. In this section we prove this fact (Proposition 2.2) and more, and make the general Conjecture 2.6. *The running assumption throughout this section is that we are dealing with a complete colored graph K.*

The *spectrum* of a coloring $\overline{\bullet}$ is the set of prohibited lengths of colorful cycles, designated

$$S(\overline{\bullet}) \equiv \{n \geq 2 : \text{there are no colorful } n\text{-cycles}\}.$$

Obviously, every spectrum contains 2, and contains all integers $n > |K|$. The set of all spectra will be denoted by $\mathcal{S}$.

On the set $\{2, 3, \dots\}$ define an operation $\circ$ by setting

$$m \circ n = m + n - 2.$$

The monoid so obtained is isomorphic to the additive monoid $\mathbb{N} = \{0, 1, \dots\}$ of natural numbers via $n \mapsto n - 2$; we denote it $\mathbb{N}(2)$.

Proposition 2.1. *Every spectrum $S \in \mathcal{S}$ is a submonoid of $\mathbb{N}(2)$ which is* eventually solid, *i.e., contains all integers $k \geq n$ for some n.*

Proof. Let C be an $(m + n - 2)$-cycle. There is a chord of C that makes C into an m-cycle conjoined with an n-cycle along the chord. Since $m \in S(\overline{\bullet})$, the color of the chord must match the color of some other edge from the m-cycle, and likewise that of some other edge of the n-cycle. This means that C is not colorful. □

Since 3 is the unique generator of $\mathbb{N}(2)$ corresponding to 1 in $\mathbb{N}$, we obtain the following insight.

Proposition 2.2. *If $3 \in S \in \mathcal{S}$ then S is the whole of $\mathbb{N}(2)$. In other words, if a colored graph contains no colorful triangles then it contains no colorful cycles at all.*

Proposition 2.3. *Assume that $S \in \mathcal{S}$ satisfies $4 \in S$. Then there is an odd integer $m \geq 3$ such that*

$$S = \{2, 4, 6, 8, \dots, m - 1, m, m + 1, m + 2, \dots\}.$$

Proof. The submonoid of $\mathbb{N}(2)$ generated by 4 consists of all positive even integers. Let m be the smallest positive odd integer in S. Then $m + 2 = 4 + m - 2 = 4 \circ m \in S$, $m + 4 = 4 \circ (m + 2) \in S$, and so forth. □

Corollary 2.4. *If a colored graph has no colorful squares and no colorful pentagons then it has no colorful n-cycles for any $n > 3$.*

The simplest question suggested by Proposition 2.3 is whether the integer m mentioned there can be any odd number, i.e., whether colored graphs without colorful squares can admit colorful m-gons for arbitrary odd integers m. The answer to this question is positive.

Proposition 2.5. *For every odd integer $m > 1$ there is a colored graph with m vertices having a colorful m-gon but no colorful squares.*

Proof. Let $m = 2k + 1$, and label the vertices v_i, $-k \le i \le k$. We employ a palette consisting of distinct colors α and β_i, $-k \le i \le k$, $i \ne 0$. For distinct indices i and j, set

$$\overline{v_i v_j} \equiv \begin{cases} \alpha & \text{if} \quad i \text{ and } j \text{ have the same parity,} \\ \beta_i & \text{if} \quad i \text{ and } j \text{ have different parity and } |i| > |j|\,. \end{cases}$$

The cycle $(v_{-k}, v_{-k+1}, \ldots, v_k)$ is colorful, with the color of the edges in order being

$$\beta_{-k}, \beta_{-k+1}, \ldots, \beta_{-1}, \beta_1, \ldots, \beta_{k-1}, \beta_k, \alpha.$$

It remains to show that there are no colorful squares. Let (v_i, v_j, v_k, v_l) be a square. Assume that the following happens at least twice around the square:

$(*)$ two consecutive vertices have the same parity.

Then at least two of the four edges are colored by α, and the square is not colorful. We can therefore assume that $(*)$ happens at most once. But $(*)$ cannot happen precisely once since odd and even indices must alternate but the square has four vertices, and so it never happens. Without loss of generality, let i have the maximum absolute value among the four indices. Since $(*)$ never happens, we conclude that $|j| < |i|$ and $|k| < |i|$. But then $\overline{v_i v_j} = \overline{v_i v_k} = \beta_i$. □

Figure 1 shows a colorful pentagon without colorful squares obtained by the construction of Proposition 2.5.

We have shown that if S is an eventually solid submonoid of $\mathbb{N}(2)$ containing 3 or 4 then $S \in \mathcal{S}$. We therefore conjecture that this simple pattern persists.

Conjecture 2.6. *Every eventually solid submonoid of $\mathbb{N}(2)$ appears as the spectrum of some colored graph.*

FIGURE 1. A colorful pentagon without colorful squares.

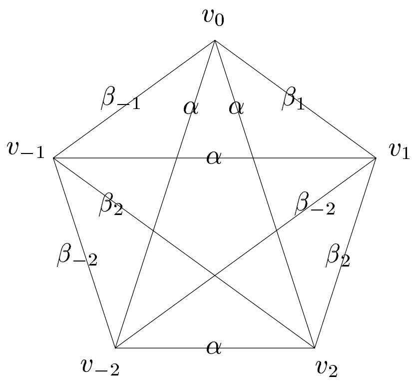
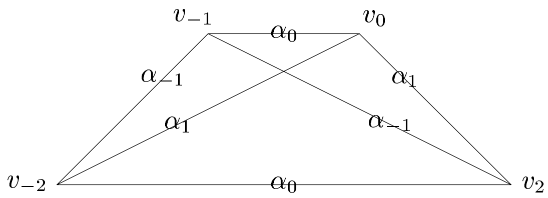


FIGURE 2. Construction of Proposition 2.7.



We close this section with another construction supporting the conjecture:

Proposition 2.7. *For every $m > 2$ there is a colorful $2m$-gon without colorful $(2m-1)$-gons.*

Proof. Draw the complete graph K on $2m$ vertices in the usual way, as a convex $2m$-gon P on the perimeter and all remaining inner edges as straight line segments inside P. We say that two inner edges *cross* if they have a point in common that is not a vertex of K. Color P by $2m$ distinct colors. Pick four consecutive vertices on P, say v_{-2}, v_{-1}, v_1, v_2, and assume that $\overline{v_{-2}v_{-1}} = \alpha_{-1}$, $\overline{v_{-1}v_1} = \alpha_0$, $\overline{v_1v_2} = \alpha_1$. Color inner edges as follows: $\overline{v_{-2}v_1} = \alpha_1$, $\overline{v_{-1}v_2} = \alpha_{-1}$, all remaining inner edges are colored α_0. The clique $\{v_{-2}, v_{-1}, v_1, v_2\}$ is depicted in Figure 2.

Let H be a colorful $(2m-1)$-gon in the above coloring, and let n be the number of crossings among the edges of H. If $n = 0$, then H lies on P with the exception of one edge e that skips a vertex on P. If e skips v_{-1} then H has two edges colored α_1, namely $v_{-2}v_1$ and v_1v_2. If e skips v_1 then H has two edges colored α_{-1}, namely $v_{-2}v_{-1}$ and $v_{-1}v_2$. If e skips any other vertex, then H has two edges colored α_0, namely e and $v_{-1}v_1$.

We claim that $v_{-2}v_1$ and $v_{-1}v_2$ cannot both lie in H. Assume they do. If $v_{-1}v_1$ is also in H, then H has two edges colored α_0, since it must have another inner edge besides $v_{-2}v_1$, $v_{-1}v_2$. So $v_{-1}v_1$ is not in H. Since $\overline{v_{-2}v_{-1}} = \alpha_{-1}$, H must continue from v_{-1} via some inner edge colored α_0. Since $\overline{v_1v_2} = \alpha_1$, H must continue from v_1 via some edge colored α_0, a contradiction. We have proved the claim.

Assume $n = 1$. Since the crossing edges of H have distinct colors, say α and β, either the color α is α_1 or α_{-1}. There are therefore three scenarios: (i) $\alpha = \alpha_1$ and $\beta = \alpha_{-1}$. Then both $v_{-2}v_1$, $v_{-1}v_2$ are in H, contrary to the claim. (ii) $\alpha = \alpha_1$ and $\beta = \alpha_0$. Then $v_{-2}v_1$ is in H. But then H cannot continue from v_1, since all edges containing v_1 are colored α_1 or α_0. (iii) $\alpha = \alpha_{-1}$ and $\beta = \alpha_0$. Then we are in a situation dual to (ii). Assume $n \geq 2$. Then H has at least three inner edges, since two edges only cross once. Hence all three colors α_{-1}, α_0, α_1 must be assigned to inner edges of H, and we have once again violated the claim. □

3. Gallai cliques

The basic building blocks of Gallai cliques are the 2-cliques, i.e., cliques a such that $|\overline{aa}| \leq 2$, for a clique is Gallai iff it can be iteratively built up from 2-cliques. We flesh out this result in Theorem 3.7, in more detail than would be strictly necessary if that theorem were our only purpose. But the additional detail, and in particular the concept of factor clique, is necessary for the subsequent analysis of exact Gallai cliques in the following sections.

The fact that Gallai cliques can be iteratively built up from 2-cliques follows from Theorem 3.1. Following [4], we attribute this result to Gallai, for it is implicit in [3]. This theorem can also be found among the results of Cameron and Edmonds in [1], and a nice proof is in [4].

Let a be a clique in a colored graph, and let $\Delta \subseteq \Gamma$. A *Δ-relation on a* is an equivalence relation $R \subseteq a \times a$ such that for all $u, v \in a$,

$$(u, v) \notin R \Longrightarrow \overline{uv} \in \Delta.$$

A 2-*relation on* a is a Δ-relation on a for some $\Delta \subseteq \Gamma$ such that $|\Delta| \leq 2$. A Δ-relation is said to be *homogenous* if for all $u_i,\ v_i \in a$,

$$((u_1, u_2), (v_1, v_2) \in R \text{ and } (u_1, v_1) \notin R) \Longrightarrow \overline{u_1 v_1} = \overline{u_2 v_2}.$$

The descriptive adjectives of the relations apply to the partitions they induce, giving the terms Δ-*partition*, 2-*partition*, and *homogenous partition*.

To rephrase the definition, a Δ-partition of a is a pairwise disjoint family A of cliques whose union is a and which satisfies

$$\bigcup \{\overline{a_1 a_2} : a_i \in A,\ a_1 \neq a_2\} \subseteq \Delta,$$

and the relation is homogeneous if $|\overline{a_1 a_2}| = 1$ for all $a_i \in A$ such that $a_1 \neq a_2$.

Theorem 3.1. *A nonsingleton Gallai clique admits a nontrivial homogeneous* 2*-partition.*

It is already clear from Theorem 3.1 that Gallai cliques are iteratively built up from 2-cliques. What is necessary now is to identify, conceptually and notationally, the particular 2-cliques used in the formation of a given Gallai clique. Thus we are led to the notions of hereditary 2-clique and of tree 2-clique.

We inductively define a *hereditary* 2-*clique* as follows. A singleton clique is a hereditary 2-clique. If a clique admits a homogeneous 2-partition whose parts are hereditary 2-cliques then the clique itself is a hereditary 2-clique.

A *tree* is a finite poset T in which every pair of unrelated elements has a common upper bound but no common lower bound. In such a poset we define

$$s \prec t \Longleftrightarrow (s < t \text{ and } \forall\, r\ \ (s \leq r \leq t \Longrightarrow r = s \text{ or } r = t)),$$

and we say that t *is the parent of* s, and that s *is a child of* t. We say that s *is an offspring of* t, and that t *is an ancestor of* s, if $s < t$. Elements s and t of T are said to be *siblings* if they are unrelated but share a parent. Note that every pair of unrelated elements are the offspring of siblings. A childless element is called a *leaf*, and the set of leaves is called the *yield* of the tree,

$$K(T) \equiv \{t : t \text{ is a leaf}\}.$$

The largest element of a tree is referred to as its *root*, and the *height* of a tree is the length of a longest path from a leaf to the root.

With a given tree T we associate two graphs. The *sibling graph* $S(T)$ has as vertices the elements of T and as edges all those of the form st, where

s and t are siblings. The *leaf graph* $K(T)$ is the complete graph on the yield of T. An *edge coloring of* $S(T)$, or simply a *coloring of* $S(T)$, is an assignment of a color, denoted $\widehat{st}$, to each edge st. We use $\widehat{\bullet}$ to denote the color map itself. If $\widehat{\bullet}$ has the additional property that for every $t \in T \setminus K(T)$

$$|\{\widehat{rs} : r \text{ and } s \text{ are distinct children of } t\}| \leq 2,$$

then we say that $\widehat{\bullet}$ is a 2-*coloring of* $S(T)$.

Proposition 3.2. *Let T be a tree. Any coloring $\widehat{\bullet}$ of $S(T)$ gives rise to a coloring $\overline{\bullet}$ of $K(T)$ by the rule*

$$\overline{st} \equiv \widehat{uv},$$

where u and v are the respective sibling ancestors of s and t. Such a coloring satisfies

$$\overline{st} = \overline{rt}$$

whenever s and r have a common ancestor unrelated to t, and any coloring of $K(T)$ with this property arises by this rule from a coloring of $S(T)$.

We refer to a clique a as a *tree clique* if there is some tree T and some coloring of $S(T)$ such that, when $K(T)$ is colored as in Proposition 3.2, a is isomorphic to $K(T)$. This means that there is a bijection from a onto the leaves of T which preserves the color of the edges. If the coloring of $S(T)$ is a 2-coloring, we refer to a as a *tree* 2-*clique.*

Proposition 3.3. *A tree* 2-*clique is Gallai.*

Proof. We induct on the height of the tree. Consider vertices u_i, $1 \leq i \leq 3$, in $a = K(T)$, where the edges of a derive their colors from a 2-coloring of $S(T)$ as in Proposition 3.2. Label the root of T as t_0, and its children t_1, $t_2, \ldots, t_n$. If all three vertices are offspring of a single t_i, the triangle they form lies in $V(\downarrow t_i)$, the tree 2-clique of the subtree rooted at t_i. Since this subtree has height less than that of T, the triangle is not colorful by the induction hypothesis. If two of the vertices, say u_1 and u_2, are offspring of one t_i, while the third vertex u_3 is the offspring of another t_j, $i \neq j$, then

$$\overline{u_1 u_3} = \widehat{t_i t_j} = \overline{u_2 u_3}.$$

If all three vertices are offspring of distinct children, say $u_i \leq t_{j_i}$ for distinct j_i, $1 \leq i \leq 3$, then because $S(T)$ carries a 2-coloring,

$$|\{\overline{u_i u_k} : 1 \leq i \neq k \leq 3\}| = \left|\left\{\widehat{t_{j_i} t_{j_k}} : 1 \leq i \neq k \leq 3\right\}\right| \leq 2.$$

Thus in any case the triangle formed by the u_i's is not colorful. $\square$

Proposition 3.4. *A clique a is a tree 2-clique iff it is a hereditary 2-clique.*

Proof. Given a hereditary 2-clique a, we build its tree inductively. If a is a singleton, its tree consists of a single root node. If a admits a homogeneous 2-partition into hereditary 2-cliques $a_1, a_2, \ldots, a_k$, then for each i there is, by the inductive hypothesis applied to a_i, a tree T_i and a 2-coloring of $S(T_i)$ such that a_i is isomorphic to $K(T_i)$. Denote the root of each T_i by t_i. Form the tree T for a by using a new root node t_0, by declaring the children of t_0 to be the t_i's, and by coloring the sibling edges of the root by the rule

$$\widehat{t_i t_j} \equiv \overline{a_i a_j},\ i \neq j.$$

The result is a 2-coloring of $S(T)$ which provides a natural isomorphism from a onto $K(T)$.

Now let a tree T be given, along with a 2-coloring of $S(T)$ and the corresponding coloring of $K(T)$ as in Proposition 3.2. We show by induction on the height of T that $K(T)$ is a hereditary 2-clique. If the height of T is 0 then T consists of the root alone, and $K(T)$ is a singleton and therefore a hereditary 2-clique. So suppose we have established the result for trees of height at most n, and consider a tree T of height $n+1$ with root t_0 and children of the root $t_1, t_2, \ldots, t_k$. Let T_i be $\downarrow t_i$, the subtree of T rooted at t_i. Then $a_i \equiv K(T_i)$ is a hereditary 2-clique by the inductive hypothesis, and the partition into the a_i's makes $a \equiv K(T)$ into a hereditary 2-clique as well. $\square$

Corollary 3.5. *A hereditary 2-clique is Gallai.*

The expression of a given hereditary 2-clique as a tree 2-clique is by no means unique. However, every such expression can be maximally refined, and this is the content of Proposition 3.6. This proposition will be required for the analysis in Section 4 of exact Gallai cliques.

When a clique a is expressed as a tree clique $K(T)$, for each $t \in T \setminus V(T)$ we refer to the clique of $S(T)$ of the form

$$\{s : s \prec t\}$$

as *the factor of a at t*. For $t_1 < t_2$ in $T \setminus V(T)$, we say that the factor at t_2 is *higher* than the factor at t_1.

A clique is said to be *irreducible* if it admits no nontrivial homogeneous partition. A clique is said to be a *hereditarily irreducible 2-clique* provided that it can be represented as a tree 2-clique with irreducible factors.

Proposition 3.6. *Every hereditary* 2*-clique is a hereditarily irreducible* 2*-clique.*

Proof. By a process of successive refinement, the cliques which arise in expressing a given hereditary 2-clique as a tree 2-clique can be rendered irreducible. Of course, the height of the tree typically increases. □

We summarize our results to this point.

Theorem 3.7. *The following are equivalent for a complete clique a in a colored graph.*

(1) *a is Gallai, i.e., a has no colorful triangles.*
(2) *a has no colorful cycles.*
(3) *a is a hereditary* 2*-clique.*
(4) *a is a hereditarily irreducible* 2*-clique.*
(5) *a is a tree* 2*-clique.*
(6) *For disjoint subcliques b and c of a,*

$$\left|\overline{bc} \smallsetminus \overline{bb}\right| \leq |c| \,.$$

(7) *For any subclique b of a,*

$$\left|\overline{bb}\right| \leq |b| - 1.$$

Proof. The equivalence of (1) and (2) is Proposition 2.2, that of (3) and (4) is Proposition 3.6, that of (3) and (5) is Proposition 3.4, the implication from (3) to (1) is Corollary 3.5, and the implication from (1) to (3) yields to a simple induction based on Theorem 3.1. (6) implies (1) by taking $|b| = 2$ and $|c| = 1$, and (1) implies (6) by a simple induction on $|c|$. Finally, (7) implies (1) by taking $|b| = 3$, and (1) implies (7) by a simple induction on $|b|$ based on (6). □

4. Exact Gallai cliques

Now we turn our attention to exact Gallai cliques, i.e., complete cliques in which every triangle has edges of exactly two colors. Their analysis requires consideration of the monochromatic subgraphs of a colored graph $G = (V, E, \overline{\bullet})$. More explicitly, for each color α we have the (uncolored) graph

$$G(\alpha) \equiv (V, \{e \in E : \overline{e} = \alpha\}) \,.$$

A subgraph of G is called *monochromatic* if it is a subgraph of $G(\alpha)$ for some α. A *monochrome* of G is a component of one of the $H(\alpha)$'s, i.e., a maximal connected monochromatic subgraph of G, considered as an uncolored graph.

Although the monochromes in Gallai cliques can be as complicated as one wishes (Proposition 4.1), the monochromes in exact Gallai cliques are fairly simple (Proposition 5.6), and the monochromes of the irreducible factors of exact Gallai cliques are simple indeed (Proposition 4.4).

A subgraph is said to *span* a graph if every vertex of the graph is a vertex of the subgraph. The nontrivial part of the next result is due to Gallai [3].

Proposition 4.1. *A Gallai clique has a spanning monochrome, and every connected graph is a spanning monochrome in a Gallai clique.*

Proof. Given a nonsingleton Gallai clique a, use Theorem 3.7 to represent it as a tree 2-clique $K(T)$. The factor b at the root is a 2-clique which may be assumed to have at least two nodes. By a remark of Erdös, one of the monochromes $b(\alpha)$ spans b. When we recall that the edges of a inherit their colors from the edges of b as in Proposition 3.2, it then follows that $a(\alpha)$ spans a. Finally, a connected graph can be made into a monochrome by coloring its edges with one color, then adding the missing edges and giving them a second color. This makes a 2-clique, which is Gallai. □

Corollary 4.2. *A clique is Gallai iff every subclique has a spanning monochrome.*

Proof. A triangle is a subclique. □

We will need to refer to several specific uncolored graphs.

Notation 4.3. *The* k-path *is*

$$P_k \equiv (\{v_i : 0 \le i \le k\}, \{v_i v_{i+1} : 0 \le i < k\}),$$

and the k-cycle *is*

$$C_k \equiv (\{v_i : 0 \le i \le k-1\}, \{v_i v_{i+1} : 0 \le i < k-1\} \cup \{v_{k-1} v_0\}).$$

We introduce one more special graph which will play a role in Section 6.

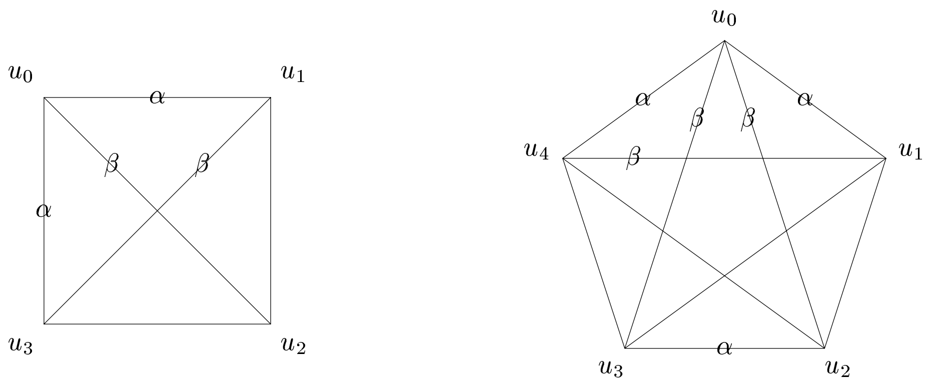
$$A \equiv (\{v_i : 1 \le i \le 5\}, \{v_0 v_1, v_1 v_2, v_1 v_4, v_2 v_3, v_3 v_4, v_4 v_5\}).$$

We say that a clique a in a colored graph is *simple* if it is complete, and if either

- $|a| = 2$, or
- $|a| = 4$, and a has two monochromes isomorphic to P_3, or
- $|a| = 5$, and a has two monochromes isomorphic to C_5.

For the sake of concise exposition in what follows, we shorten the phrase "the triangle with vertices u_0, u_1 and u_2" to "the triangle $u_0 u_1 u_2$."

FIGURE 3. Proving Proposition 4.4.



Proposition 4.4. *A clique is simple iff it is a nonsingleton irreducible Gallai 2-clique.*

Proof. Let a be a nonsingleton irreducible Gallai 2-clique. a cannot have six or more elements, for the most basic form of Ramsey's Theorem ([8]) asserts that a 2-clique with six vertices has a monochromatic triangle. a cannot have three elements, for identification of the two vertices connected by the edge with minority color constitutes a nontrivial homogenous partition.

Let $a = \{u_0, u_1, u_2, u_3\}$. Without loss of generality $\overline{u_0u_1} = \overline{u_0u_3} = \alpha$. If $\overline{u_0u_2} = \alpha$ then $\{\{u_0\}, \{u_1, u_2, u_3\}\}$ is a nontrivial homogeneous partition, hence $\overline{u_0u_2} = \beta \neq \alpha$. The triangle $u_0u_1u_3$ cannot be monochromatic, hence $\overline{u_1u_3} = \beta$. We are now in the situation depicted in Figure 3, and it is easy to see that a is simple.

Let $a = \{u_0, u_1, u_2, u_3, u_4\}$. Without loss of generality $\overline{u_0u_1} = \overline{u_0u_4} = \alpha$, and $\overline{u_0u_3} = \overline{u_1u_4} = \beta$. If $\overline{u_0u_2} = \alpha$ then $\overline{u_1u_2} = \beta$, but in that case any color assigned to u_2u_4 would result in a colorful triangle. Thus $\overline{u_0u_2} = \beta$, $\overline{u_2u_3} = \alpha$, and we are in the situation depicted in Figure 3. It is then easy to see that a is simple. □

Theorem 4.5. *A clique is exactly Gallai iff it is a hereditarily irreducible 2-clique with simple factors, such that higher factors use different colors than lower factors.*

Proof. Let a by a hereditarily irreducible 2-clique with simple factors. As long as higher factors use different colors than lower factors, the argument given in Proposition 3.3 can be readily modified to show that every triangle has edges of exactly two colors. Now consider an exact Gallai clique in a colored graph. Apply Theorem 3.7 to express it as a tree 2-clique with irreducible factors. Then these factors are exact by Proposition 4.4, and clearly higher factors use differed colors than lower factors, since otherwise a monochromatic triangle exists. $\square$

5. Full homomorphisms

A *full homomorphism* ([5]) is a map $f : G \to H$ between (uncolored) graphs such that for all $v_i \in V_G$,

$$v_1 v_2 \in E_G \Longleftrightarrow f(v_1) f(v_2) \in E_H.$$

Note that the identity map is a full homomorphism, and that the composition of full homomorphisms is itself a full homomorphism. Thus, graphs with full homomorphisms constitute a category. For our purposes we need only a few simple properties of these maps, given in the following lemmas. In these lemmas and in what follows, we reserve the term embedding for the identity map on an induced subgraph.

Lemma 5.1. *An embedding is a full homomorphism, and every full homomorphism factors into a full surjection followed by an embedding. That is, each full homomorphism $f : G \to H$ factors as $f = jf'$,*

$$G \xrightarrow{f'} f(G) \xrightarrow{j=\subseteq} H,$$

where f' is the map $v \longmapsto f(v)$ onto the induced subgraph with vertex set $f(G)$, and j is the embedding of this subgraph into H.

A graph $G = (V, E)$ is said to be *reduced* if for all $v_i \in V$,

$$v_1 E = v_2 E \Longrightarrow v_1 = v_2.$$

Lemma 5.2. *A full homomorphism out of a reduced graph is injective, hence an embedding.*

Proof. Let $f : G \to H$ be a full homomorphism, let G be reduced, and let $v_i \in V$ satisfy $f(v_1) = f(v_2)$. Since for any $v \in V$ we have

$$v_1 v \in E_G \Longleftrightarrow f(v_1) f(v) \in E_H \Longleftrightarrow f(v_2) f(v) \Longleftrightarrow v_2 v \in E_G,$$

it is clear that $v_1 E_G = v_2 E_G$. Because G is reduced, $v_1 = v_2$. $\square$

For our purposes, a graph can be replaced by a specific reduced version as follows. For a given graph $G = (V_G, E_G)$, fixed for the next several lemmas, define $\widehat{G} \equiv (V_{\widehat{G}}, E_{\widehat{G}})$ by setting

$$V_{\widehat{G}} \equiv \{vE : v \in V_G\}, \; E_{\widehat{G}} \equiv \{(v_1 E)(v_2 E) : v_1 v_2 \in E_G\}.$$

We first show that this definition makes sense.

Lemma 5.3. *If $u_1 E = u_2 E$ and $v_1 E = v_2 E$ then*

$$u_1 E v_1 \iff u_2 E v_2.$$

Proof. Since

$$u_1 E v_1 \iff v_1 \in u_1 E = u_2 E \iff u_2 \in v_1 E = v_2 E \iff u_2 E v_2,$$

the result is clear. □

We define the canonical map $r_G : G \to \widehat{G}$ by the rule $v \longmapsto vE$.

Lemma 5.4. *$\widehat{G}$ is reduced and r_G is a full surjection. And any function $h_G : \widehat{G} \to G$ which satisfies $h_G(vE) \in vE$ for all $v \in V_G$ constitutes a full homomorphism such that $r_G h_G$ is the identity map on $\widehat{G}$.*

The significance of $\widehat{G}$ is that it is the smallest full quotient of G.

Lemma 5.5. *r_G is the smallest full surjection out of G. That is, if $f : G \to H$ is a full surjection then there is a unique full surjection $g : H \to \widehat{G}$ such that $gf = r_G$.*

Proof. If $f(v_1) = f(v_2)$ then we claim that $v_1 E_G = v_2 E_G$. For if $v \in v_1 E_G$ then $f(v_1) E_H f(v)$, hence $f(v_2) E_H f(v)$ and $v_2 E_G v$, and conversely. Thus we can define g by setting $g(f(v)) \equiv vE$. It is routine to verify that g has the properties claimed for it. □

It follows from Lemma 5.5 that G is reduced iff r_G is an isomorphism. We refer to $\widehat{G}$ as the *reduced form of G*, and we refer to the isomorphism type of $\widehat{G}$ as the *type of G*. Note that if G is connected then so is its type.

Exact Gallai monochromes are characterized by their types.

Proposition 5.6. *Monochromes of exact Gallai cliques are of type P_1, P_3, or C_5, and every graph of one of these types appears as a (spanning) monochrome in an exact Gallai clique.*

Proof. According to Theorem 4.5, we may think of an exact Gallai clique as a tree 2-clique $K(T)$ with simple factors. Let $G = (V, E)$ be a monochrome in $K(T)$, i.e., a component of $K(T)(\alpha)$ for some color α. Now every edge of

E inherits its color from that of an edge connecting a sibling pair in $S(T)$ as in Proposition 3.2, and all these sibling pairs have a common parent t because higher factors use different colors than lower factors. Let $b = \{t' \in T : t' \prec t\}$ be the factor at t, and let H be $b(\alpha)$. Let $f : G \to H$ be the map which takes each $v \in V$ to its unique ancestor in b. Then f is clearly a full homomorphism, and since b is simple, H is isomorphic to P_1, P_3, or C_5. □

An induced subgraph of a reduced graph need not be reduced. The reduced induced subgraphs of C_5 are P_1, P_3, and C_5 (the remaining P_2 reduces to P_1), the very graphs used to define simple cliques. This observation permits a second characterization of exact Gallai types in Corollary 5.8, a result which uses the following trivial lemma.

Lemma 5.7. *There exists a full homomorphism from G into H iff the type of G is embedded in the type of H.*

Proof. In light of the $r_X : X \to \widehat{X}$ and $h_X : \widehat{X} \to X$ from Lemma 5.4, there is an $f : G \to H$ iff there is a $\widehat{f} : \widehat{G} \to \widehat{H}$. By Lemma 5.2 the latter is an embedding. □

Corollary 5.8. *A connected graph is an exact Gallai monochrome iff it can be mapped into C_5 by a full homomorphism.*

C_5 reappears in a pivotal role in Section 6.

Most questions about exact Gallai cliques can now be answered by straightforward calculations. For example, we offer a concise proof of Theorem 1 of [2].

Theorem 5.9. *The largest number of vertices of an exact Gallai clique colored by k colors is $5^{\frac{k}{2}}$ if k is even and $2 \cdot 5^{\frac{k-1}{2}}$ if k is odd.*

Proof. Theorem 4.5 permits us to view an exact Gallai clique as a tree 2-clique with factors isomorphic to P_1, P_3, or C_5, in which higher factors use different colors than lower factors. Since each factor contributes exactly two colors, the objective of building an exact Gallai clique with the fewest colors and most vertices is obviously achieved by using factor cliques isomorphic to C_5. The result follows. □

6. Homomorphism dualities

In this section, all graphs are assumed to be connected. Let $\mathcal{M}$ be a class of graph homomorphisms. We write

$$G \to_{\mathcal{M}} H$$

to mean that there is an $f : G \to H$ in $\mathcal{M}$. Otherwise we write

$$G \nrightarrow_{\mathcal{M}} H.$$

Two sets $\mathcal{A}$ and $\mathcal{B}$ of graphs are said to be in a *homomorphism duality* ([7]) if

$$\forall A \in \mathcal{A}\ (A \nrightarrow_{\mathcal{M}} G) \Longleftrightarrow \exists B \in \mathcal{B}\ (G \to_{\mathcal{M}} B)\,.$$

In this section we take $\mathcal{M}$ to be the class of full homomorphisms.

Theorem 6.1. *We have the homomorphism duality*

$$\{C_3, P_4, A\} \nrightarrow_{\mathcal{M}} G \quad \textit{iff} \quad G \to_{\mathcal{M}} C_5,$$

and the graphs G characterized by this condition are precisely the monochromes in exact Gallai cliques.

Proof. The condition displayed on the right characterizes the monochromes in exact Gallai cliques by Lemma 5.7 combined with Proposition 5.6. The same lemma also shows that the condition displayed on the right implies the one on the left. Thus we have only to show that for any connected graph G,

$$\{C_3, P_4, A\} \nrightarrow_{\mathcal{M}} G \;\Rightarrow\; G \to_{\mathcal{M}} C_5.$$

Suppose $G = (V, E)$ contains a copy of C_5, designated as in 4.3. First observe that for every $v \in V$ there is an index i for which $vv_i \in E$. Indeed, if this were not the case then there would exist vertices u and w and index j such that $uw, wv_j \in E$ but $uv_j \notin E$. (Consider the last three vertices on a shortest path from v to C_5.) In order to prevent $\{u, w, v_j, v_{j+1}, v_{j+2}\}$ and $\{u, w, v_j, v_{j-1}, v_{j-2}\}$ from being copies of P_4, we would have to have uv_{j+2}, $uv_{j-2} \in E$, but then we would have a triangle. Furthermore, v cannot be connected with only one $v_j \in C_5$, or else there would be a P_4-path $\{w, v_j, v_{j+1}, v_{j+2}, v_{j+3}\}$. To avoid triangles, v cannot be connected with two neighboring points v_j, v_{j+1} of C_5. Therefore,

- for every $v \in V$ there is exactly one i, $0 \le i \le 4$, such that vv_{i-1}, $vv_{i+1} \in E$; set $f(v) = i$.

We need to demonstrate that the map $v \longmapsto v_{f(v)}$ is a full homomorphism. If $uv \in E$ then we must have $f(u) = f(v) \pm 1$, since otherwise

$$\left\{v_{f(u)-1}, v_{f(u)+1}\right\} \cap \left\{v_{f(v)-1}, v_{f(v)+1}\right\} \neq \emptyset,$$

resulting in a triangle. Finally, if $f(u) = i$ and $f(v) = i + 1$ then $uv \in E$ lest $\{u, v_{i-1}, v_{i-2}, v_{i+2}, v\}$ be a copy of P_4.

Suppose (V, E) does not contain a copy of C_5. Then the longest induced path is a copy of P_k, $k = 1$, 2, or 3. Choose such a path in G, call it P_k, and designate its vertices as in 4.3. Since $P_k \to_{\mathcal{M}} C_5$, it suffices to construct a strong homomorphism $f : G \to P_k$.

If $k = 1$ then G, by connectedness, is P_1 itself and the statement is obvious. So suppose $k = 2$, so that P_k is $\{v_0, v_1, v_2\}$. Then for every $v \in V$ we have either vv_1 or vv_0 in E, and in the latter case we also have vv_2 in E, since otherwise there would be a P_3-path. Set

$$f(v) = \begin{cases} v_1 & \text{if} \quad vv_0, vv_2 \in E \\ v_0 & \text{if} \quad v_1 \in E \end{cases} .$$

(Note that the range of f is actually a P_1-path. This is not surprising, for the reduced form of a P_2-path is a P_1-path, so that by Lemma 5.7, G admits a strong homomorphism into a P_2 path iff it admits a strong homomorphism into a P_1-path.) Now if $uv \in E$ we could not have $f(u) = f(v) = v_i$, for there would be the triangle uvv_{1-i}. And if $f(u)f(v)$ is an edge, say $f(u) = v_1$ and $f(v) = v_0$, then, in order to prevent $\{u, v_0, v_1, v\}$ from being a P_3-path, we have to have $uv \in E$.

It remains only to handle the case in which $k = 3$. We claim that each $v \in V$ has to be immediately connected with some $v_i \in P_3$. For otherwise consider the last three points, call them u, w and v_i, on a shortest path connecting v to P_3. Note that, since u is not connected to P_3, avoiding a P_4-path requires a second edge (other than wv_i) joining w to P_3. If i is 0 then the possibilities for the second edge are wv_1, wv_2 and wv_3, but these choices lead to a copy of C_3, a copy of A, or a copy of P_4, respectively. If i is 1 then the possibilities are wv_0, wv_2 and wv_3, but these choices lead to a copy of C_3, a copy of C_3, or a copy of A, respectively. Symmetrical arguments rule out the possibility that i could be 2 or 3, and the claim is proven.

Set

$$f(v) \equiv \begin{cases} v_0 & \text{if} \quad vv_1 \in E \text{ and } vv_3 \notin E \\ v_1 & \text{if} \quad vv_0 \in E \text{ and } vv_2 \in E \\ v_2 & \text{if} \quad vv_1 \in E \text{ and } vv_3 \in E \\ v_3 & \text{if} \quad vv_2 \in E \text{ and } vv_0 \notin E \end{cases} .$$

The definition is correct, for if $vv_0 \in E$ then $vv_2 \in E$ to prevent $\{v, v_0, v_1, v_2, v_3\}$ from being either P_4 or C_5, and similarly, if $vv_3 \in E$ then also $vv_1 \in E$. And the value of the function at any argument is unique, since otherwise we would have a copy of C_3. For the same reason, if $f(u) = f(v)$ then $uv \notin E$.

Now we must show that if $f(u)$ and $f(v)$ are connected by an edge then so are u and v. If $f(u) = v_0$ and $f(v) = v_1$ then $uv \in E$, since otherwise we would have an A-subgraph $\{u, v_1, v_0, v_2, v_3\}$; likewise $f(u) = v_2$ and

$f(v) = v_3$ imply $uv \in E$. If $f(u) = v_1$ and $f(v) = v_2$ then $uv \in E$, since otherwise we would have a P_4-path $\{u, v_0, v_1, v, v_3\}$.

At last, we must show that if $f(u)$ and $f(v)$ are not connected by an edge then neither are u and v. If $f(u) = v_0$ and $f(v) = v_2$ then $uv \notin E$ because of the triangle uvv_1, and similarly $uv \notin E$ if $f(u) = v_1$ and $f(v) = v_3$. Finally, if $f(u) = v_0$ and $f(v) = v_3$ then $uv \notin E$ since otherwise we would have an A-subgraph $\{v_0, v_1, u, v, v_2, v_3\}$. □

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(Ball and Vojtěchovský) Department of Mathematics, University of Denver, Denver, Colorado 80208, U.S.A.

E-mail address, Ball: `rick.ball@nsm.du.edu`

URL: `www.math.du.edu/~rball`

(Pultr) Department of Applied Mathematics and ITI, MFF, Charles University, CZ 11800 Praha 1, Malostranské nám. 25

E-mail address, Pultr: `pultr@kam.ms.mff.cuni.cz`

E-mail address, Vojtěchovský: `petr@math.du.edu`

URL: `www.math.du.edu/~petr`