XI. Midsummer Combinatorial Workshop

Zdeněk Dvořák, ed.

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Contents

Preface

The Eleventh Prague Midsummer Combinatorial Workshop was held from July 26 to July 30, 2004 in our newly reconstructed building Malostranské náměstí 25. This of course contributed to the comfort of the participants as all the activities (including the lunches) could be taken on the same site. Besides, as it was expressed by several participants, the renovated faculty building surely belongs to the most beautiful math and CS departments in the world! The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with the DIMATIA centre. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Endre Szemeredi and Paul Seymour among the participants. The list of participants is included in this booklet.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students. Five undergraduate students from the USA and three undergraduate students from Charles University, together with their mentors Martin Mareš and Brenda Latka, took part in the workshop, within a joint DIMATIA-DIMACS program International REU (supported jointly by NSF and Czech ministry of education).

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest some of the atmosphere at the workshop from these proceedings, and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

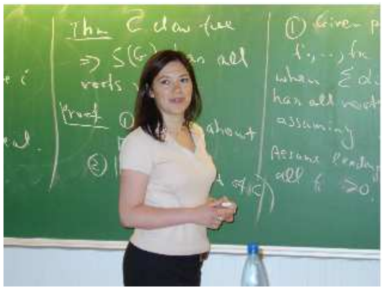
This volume was edited by Zdeněk Dvořák. Most of the contributions were supplied by the authors in electronic form. In a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

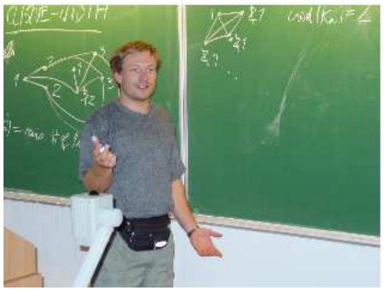
The Eleventh Midsummer Combinatorial Workshop was supported by Kontakt CS-US Grants and by our institute ITI (financed by the Ministry of Education of the Czech Republic as project LN00A056) and the publication of these series is supported by the newly approved ITI 1M0021620808. DIMATIA was the main organizer.

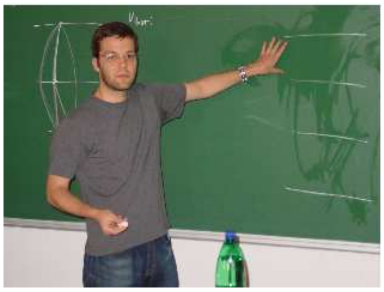
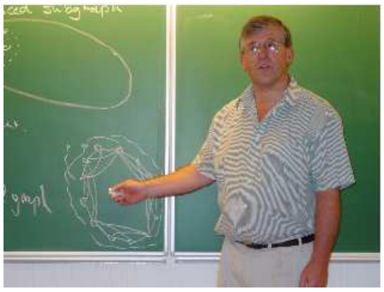
Based on our past experience and being encouraged by several participants, we hope to organize the Twelfth Prague Midsummer Combinatorial Workshop in the summer of 2005 (July 24-29). We hope to meet you there!

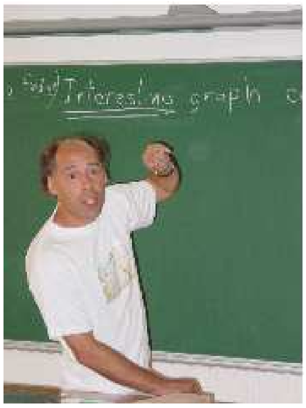
Jaroslav Nešetřil

Photos from the workshop







Interesting graph

Augmentations, arc reversals and deorientations in digraphs

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Abstract

We survey results and problems related to connectivity of digraphs.

1 Basics

Notation not defined here follows [1]. For a given digraph $D(V, A)$ and $X \subset V$ we denote by $d^-(X)$ the in-degree and by $d^+(X)$ the out-degree of X, that is, the number of arcs leaving, respectively entering X. The number of in-neighbours (out-neighbours) of X is denoted by $N^-(X)$ $(N^+(X))$.

A digraph $D = (V, A)$ is strong if and only if it contains a directed path from x to y for every choice of vertices $x, y \in V$. A digraph $D = (V, A)$ is ***k*-strong** if $D - X$ is strong for every subset X of at most $k - 1$ vertices. A digraph $D = (V, A)$ is ***k*-arc-strong** if $D - X$ is strong for every subset X of at most $k - 1$ arcs.

Proposition 1 *Let $D = (V, A)$ be a directed multigraph and let X, Y be subsets of V. Then the following holds:*

$$\begin{aligned} d^+(X) + d^+(Y) &= d^+(X \cup Y) + d^+(X \cap Y) + d(X, Y) \\ d^-(X) + d^-(Y) &= d^-(X \cup Y) + d^-(X \cap Y) + d(X, Y). \end{aligned} \tag{1}$$

2 Augmenting connectivity by adding new arcs

Definition 1 *Let $D = (V, A)$ be a directed multigraph. Then $\gamma_k(D)$ is the smallest integer γ such that*

$$\begin{aligned} \sum_{X_i \in \mathcal{F}} (k - d^-(X_i)) &\leq \gamma \text{ and} \\ \sum_{X_i \in \mathcal{F}} (k - d^+(X_i)) &\leq \gamma, \end{aligned}$$

for every subpartition $\mathcal{F} = \{X_1, \ldots, X_t\}$ of V.

Theorem 2 (Frank) *[7] Let $D = (V, A)$ be a digraph and k a natural number such that $\gamma_k(D) > 0$.*

(a) The minimum number $a'_k(D)$ of new arcs that must be added to D in order to give a k-arc-strong digraph $D' = (V, A \cup F)$ equals $\gamma_k(D)$.

(b) Using flows in networks one can find an optimal augmenting set of arcs in polynomial time.

Definition 2 *Let $D = (V, A)$ be a directed graph. Then $\gamma^*_k(D)$ is the smallest integer γ such that*

$$\begin{aligned} \sum_{X \in \mathcal{F}^-} (k - |N^-(X)|) &\leq \gamma \text{ and} \\ \sum_{X \in \mathcal{F}^+} (k - |N^+(X)|) &\leq \gamma, \end{aligned}$$

for every choice of subpartitions $\mathcal{F}^-, \mathcal{F}^+$ of V with the property that every $X \in \mathcal{F}^-$ satisfies $N^-[X] \neq V$ and every $X \in \mathcal{F}^+$ satisfies $N^+[X] \neq V$.

We call the number $\gamma^*_k(D)$ the **subpartition lower bound for the vertex-strong connectivity**.

Let $a_k(D)$ denote the minimum number of new arcs that must be added to a digraph $D = (V, A)$ in order to obtain a k-strong digraph. It is easy to see that $a_k(D)$ is well-defined provided that D has at least $k + 1$ vertices. We also call $a_k(D)$ the ***k*-strong augmentation number** of D.

Let X, Y be disjoint non-empty proper subsets of V. The ordered pair (X, Y) is a **one-way pair** in $D = (V, A)$ if D has no arc with tail in

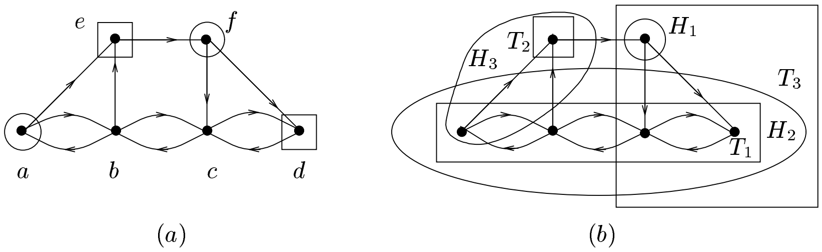

Figure 1: An example, due to Jordán, showing that the subpartition lower bound for vertex connectivity augmentation is not always attainable. The desired connectivity is $k = 2$ and the value $\gamma_2^*(D)$ is 2 and it is realized by the subpartitions $\{\{d\}, \{e\}\}, \{\{a\}, \{f\}\}$, respectively, (see (a)). Part (b) shows three pairwise independent one-way pairs $(T_1, H_1), (T_2, H_2), (T_3, H_3)$ (tails are indicated by boxes). This shows that $a_2(D) \geq 3$. In fact $a_2(D) = 3$, since adding the arcs af, ed, da will result in a 2-strong digraph.

X and head in Y. This definition is due to Frank and Jordán [9]. For such a pair (X, Y) we refer to X (Y) as the **tail** (**head**) of the pair. Let $h(X, Y) = |V - X - Y|$. The **deficiency** of a one-way pair (X, Y) with respect to k-strong connectivity is

$$\eta_k(X, Y) = \max\{0, k - h(X, Y)\}. \tag{2}$$

Lemma 3 (Frank and Jordán) *[9] A digraph $D = (V, A)$ is k-strong if and only if we have $h(X, Y) \geq k$ for every one-way pair (X, Y) in D.*

Two one-way pairs $(X, Y), (X', Y')$ are **independent** if either their heads or their tails are disjoint.

Note that one-way pairs that contribute to the sums in Definition 2 are always independent since either all heads or all tails are disjoint for those pairs.

Theorem 4 (Frank and Jordán) *[9] For every digraph D on at least $k+1$ vertices we have*

$$a_k(D) = \max_{\mathcal{F}}\{\sum_{(X,Y)\in\mathcal{F}} \eta_k(X, Y)\}, \tag{3}$$

where $\mathcal{F}$ is a family of independent one-way pairs in D.

Theorem 5 (Frank and Jordán) *[9] There exists a polynomial algorithm which given a digraph $D = (V, A)$ and a natural number k finds a minimum cardinality set F of new arcs to add to D so that the resulting graph is k-strong.* $\diamond$

The algorithm uses the ellipsoid algorithm and hence is not purely combinatorial.

Problem 1 *Find a purely combinatorial polynomial algorithm for determining $a_k(D)$.*

The existence of such an algorithm is open, except for the case when D is already $(k-1)$-strong. Here Frank has an algorithm.

3 Orienting edges and reversing arcs

Problem 2 *Given natural number k and a digraph $D = (V, A)$ on at least $k+1$ vertices, find a minimum set $F \subset A$ of arcs in D such that the digraph D' obtained from D by reversing every arc in F is k-strong.*

If such a subset exists, then we let $r_k(D) = |F|$, where F is a minimum cardinality subset of A, whose reversal makes the resulting digraph k-strong. Otherwise we let $r_k(D) = \infty$.

Since adding a new copy of an arc $xy \in A(D)$ have cannot affect the vertex connectivity of a digraph it is easy to see that if D is an oriented graph, then a good reversal exists if and only if there exist a k-strong orientation of the underlying graph $UG(D)$. Deciding the existence of such an orientation seems very hard and it is not known whether there is any function $f(k)$ will guarantee that every $f(k)$-connected graph has a k-strong orientation when $k > 2$. For $k = 2$ Jordán has proved recently (private communication) that $f(2) \leq 18$. He also proved that for eulerian graphs there is a polynomial algorithm for deciding the existence of a 2-strong orientation.

Conjecture 1 (Jackson and Thomassen) *[12] Every $2k$-connected graph has a k-strong orientation.*

Theorem 6 (Bang-Jensen and Jordan) *[2] If D is a semicomplete digraph on at least $3k-1$ vertices for some $k \geq 2$ then $a_k(D) = r_k(D)$.*

Conjecture 2 *[2] If D is a tournament on at least $2k+1$ vertices for some $k \geq 2$ then $a_k(D) = r_k(D)$.*

Conjecture 3 (Bang-Jensen, 1994) *For every tournament T with $n \geq 2k+1$ vertices we have $r_k(T) \leq \frac{k(k+1)}{2}$*

Using the following two easy observations it is easy to show that $r_k(T)$ is bounded by a function of k for every tournament T on at least $2k+1$ vertices.

(a) If D is a k-strong digraph and D' is obtained from D by adding a new vertex x and arcs from x to every vertex in a set X of k distinct vertices of D and arcs from every vertex of a set Y of k distinct vertices of D to x, then D' is also k-strong.

(b) If T is a tournament on at least $4k-1$ vertices, then T contains a vertex with in-degree and out-degree at least k.

Let $\mathcal{F}$ be a family of subsets of S and let $b : \mathcal{F} \to \mathcal{Z} \cup \{\infty\}$ be a function defined on $\mathcal{F}$. The function b is **fully submodular** on $\mathcal{F}$ if the inequality

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{4}$$

holds for every choice of members X, Y of $\mathcal{F}$. If (4) is only required to hold for intersecting (crossing) members of $\mathcal{F}$, then b is **intersecting (crossing)** submodular on $\mathcal{F}$.

A function $f : S \to \mathcal{R}$ is **modular** if $f(X)+f(Y) = f(X\cap Y)+f(X\cup Y)$ holds for every choice of $X, Y \subseteq S$.

Let $D = (V, A)$ be a directed multigraph let $\mathcal{F}$ be a family of subsets of V such that $\emptyset, V \in \mathcal{F}$ and let $b : \mathcal{F} \to \mathcal{Z} \cup \{\infty\}$ be fully submodular on $\mathcal{F}$. A function $x : A \to \mathcal{R}$ is a **submodular flow** with respect to $\mathcal{F}, b$ if it satisfies

$$x^-(U) - x^+(U) \leq b(U) \qquad \text{for all } U \in \mathcal{F}. \tag{5}$$

Theorem 7 (Edmonds and Giles) *[4] Let $D = (V, A)$ be a directed multigraph. Let $\mathcal{F}$ be a crossing family of subsets of V such that $\emptyset, V \in \mathcal{F}$, let $b : \mathcal{F} \to \mathcal{Z} \cup \{-\infty\}$ be crossing submodular on $\mathcal{F}$ with $b(\emptyset) = b(V) = 0$, and let $f \leq g$ be modular functions on A such that $f : A \to \mathcal{Z} \cup \{-\infty\}$ and $g : A \to \mathcal{Z} \cup \{\infty\}$. The linear system*

$$\{f \leq x \leq g \text{ and } x^-(U) - x^+(U) \leq b(U) \qquad \text{for all } U \in \mathcal{F}\} \tag{6}$$

is totally dual integral. That is if f, g, b are all integer valued, then the linear program $\min\{c^T x : x$ *satisfies* $(6)\}$ *has an integer optimum solution (provided it has a solution). Furthermore, if c is integer valued, then the dual linear program has an integer valued optimum solution (provided it has a solution).*

Theorem 8 (Frank) *[5] One can verify in polynomial time whether a given submodular flow problem has a feasible solution. If f, g, b are all integer valued and there exists a feasible submodular flow, then there exist a feasible integer valued submodular flow. Furthermore, if there is also a cost function on the arcs, then one can find a minimum cost feasible submodular flow in polynomial time.*

Let $G = (V, E)$ be an undirected graph. Let D be an arbitrary orientation of G. Clearly G has a k-arc-strong orientation if and only if it is possible to reorient some arcs of D so as to get a k-arc-strong directed multigraph. Suppose we interpret the function $x : A \rightarrow \{0, 1\}$ as follows: $x(a) = 1$ means that we reorient a in D and $x(a) = 0$ means that we leave the orientation of a as it is in D. Then G has a k-arc-strong orientation if and only if we can choose x so that the following holds:

$$d^-_D(U) + x^+(U) - x^-(U) \geq k \qquad \text{for all } \emptyset \neq U \subset V. \tag{7}$$

This is equivalent to

$$x^-(U) - x^+(U) \leq (d^-_D(U) - k) = b(U) \qquad \text{for all } U \subset V, U \neq \emptyset, V, \tag{8}$$

$$b(\emptyset) = b(V) = 0. \tag{9}$$

Observe that the function b is crossing submodular on $\mathcal{F} = 2^V$ (it is not fully submodular in general, since we have taken $b(\emptyset) = b(V) = 0$). Thus we have shown that G has a k-arc-strong orientation if and only if there exists a feasible integer valued submodular flow in D with respect to the functions $f \equiv 0, g \equiv 1$ and b.

Theorem 9 (Nash-Williams) *[11] A multigraph G has a k-arc-strong orientation if and only if G is $2k$-edge-connected.*

Proof: ([6], [10]) Suppose that G is $2k$-edge-connected, that is $d_G(X) \geq 2k$ for all proper non-empty subsets of V (by Menger's theorem). We claim that $x \equiv \frac{1}{2}$ is a feasible submodular flow. This follows from the following calculation:

$$\begin{aligned} d_D^-(U) + x^+(U) - x^-(U) &= d_D^-(U) + \frac{1}{2}d_D^+(U) - \frac{1}{2}d_D^-(U) \\ &= \frac{1}{2}d_D^-(U) + \frac{1}{2}d_D^+(U) \\ &\geq \frac{1}{2}(2k - d_D^+(U)) + \frac{1}{2}d_D^+(U) \\ &= k. \end{aligned} \tag{10}$$

Hence it follows from the integrality statement of Theorem 8 and the equivalence between (7) and (8) that there is a feasible integer valued submodular flow x in D with respect to f, g and b. As described above this implies that G has a k-arc-strong orientation where the values of x prescribe which arcs to reverse in order to obtain such an orientation from D. $\diamond$

Notice that by formulating the problem as a minimum cost submodular flow problem, we can also solve the weighted version where the two possible orientations of an edge may have different costs and the goal is to find the cheapest k-arc-strong orientation of the graph. This clearly includes the problem where we wish to find the minimum number of arcs to reverse in order to obtain a k-arc-strong directed multigraph, hence we have

Theorem 10 (Frank) *[5] Given a directed multigraph D, one can find in polynomial time the minimum number of arcs whose reversal in D results in a k-arc-strong directed multigraph.*

This includes the case when D has no such reversal which can be detected by checking whether the submodular flow problem above has a feasible solution.

Theorem 11 (Jackson, 1988) *[10] Every $2k$-arc-strong digraph contains a spanning k-arc-strong oriented graph.*

Conjecture 1 is a special case of the following conjecture.

Conjecture 4 (Jackson and Thomassen) *[12] Every $2k$-strong digraph contains a spanning k-strong oriented graph.*

This conjecture is open even for semicomplete digraphs (digraphs with no non-adjacent vertices). Here the best known bound is $3k-2$ see [1, Section 7.14].

Conjecture 5 (Frank) *[8] A graph $G=(V,E)$ has a k-strong orientation if and only if $G-X$ is $2(k-j)$-edge-connected for every set X of j vertices $(0 \leq j \leq k)$.*

A **mixed graph** is a graph $M=(V,A\cup E)$ which has both arcs (A) and undirected edges (E). A mixed graph M is k-arc-strong is the digraph we obtain from M by replacing every undirected edge by a directed 2-cycle is k-arc-strong.

By an **orientation** of a mixed graph M we mean any digraph that can be obtained by assigning an orientation to each edge of M. Using submodular flows one can decide in polynomial time whether a given mixed graph has a k-arc-strong orientation.

Problem 3 *Is there a polynomial algorithm for the following problem: Given a k-arc-strong mixed graph $M=(V,A\cup E)$ which does not have a k-arc-strong orientation. Find a maximum cardinality subset of E which can be oriented such that the resulting mixed graph is still k-arc strong.*

4 Deorienting arcs

By **deorienting** an arc ij in a digraph we mean the operation of adding the opposite arc ji to the digraph. We will only allow to deorient an arc which is not in a directed 2-cycle.

Let $\mathrm{Deor}_k(D)$ denote the minimum number of arcs one must deorient in D in order to obtain a k-arc-strong superdigraph of D. Clearly $\mathrm{Deor}_k(D) < \infty$ precisely when the underlying graph $UG(D)$ is k-edge-connected.

Since deorienting arcs is a special (restricted) form of augmentation and instead of reversing a set of arcs we may add the corresponding set of oppositely oriented arcs, we have that for every oriented graph D

$$r'_k(D) \geq \mathrm{Deor}_k(D) \geq a'_k(D)$$

As we have seen, the first number can be calculated in polynomial time using submodular flows and the third can be calculated in polynomial time using flows (via Franks algorithm).

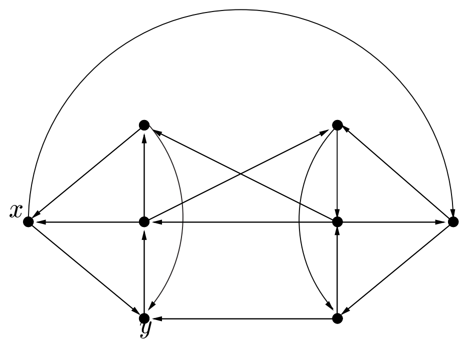

Figure 2: A digraph D with $r'_2(D) > \mathrm{Deor}_2(D) = 1$. All arcs into y are critical showing that $r'_2(D) > 1$.

Problem 4 *Is there a polynomial algorithm for calculating* $Deor_k(D)$*?*

In the special case of tournaments, the answer is yes.

Theorem 12 (Bang-Jensen and Yeo) *[3] Let T be a tournament on $n \geq 2k+1$ vertices. Then*

(a) $Deor_k(T) = r'_k(T)$

(b) One can calculate $Deor_k(T)$ and find an optimal set of arcs to deorient using flows in networks.

(c) $Deor_k(T) \leq \frac{k(k+1)}{2}$.

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The Roots of the Stable Set Polynomial of a Clawfree Graph Extended Abstract

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Joint work with Paul Seymour

A *stable set* in a graph is a set of pairwise non-adjacent vertices. The *stable set polynomial* of a graph G is the polynomial

$$S(G)(x) = \Sigma_{i\geq 0} a_i x^i$$

where a_i is the number of stable sets in G of size i.

Given a graph H, its *line graph* $L(H)$ is the graph whose vertex set is the set of edges of H, and two vertices are adjacent if they share an end in H. In [2] Heilmann and Lieb proved that if G is a line graph, then all the roots of $S(G)$ are real. This property does not hold for all graphs, since the stable set polynomial of a claw is

$$1 + 4x + 3x^2 + x^3$$

and not all its roots are real (a *claw* is the graph with vertex set $\{v_1, v_2, v_3, v_4\}$ and three edges v_1v_2, v_1v_3 and v_1v_4.)

A graph G is said to be *clawfree* if no induced subgraph of it is a claw. We answer a question of Hamidoune [1] that was later posed as a conjecture by Stanley [3].

Theorem 1 *If G is clawfree then all roots of $S(G)$ are real.*

[*]This research was conducted during the period the author served as a Clay Mathematics Institute Research Fellow.

Since all line graphs are clawfree, this extends the result of [2].

The proof of 1 consists of two parts. First we prove a lemma about polynomials, that allows us to deduce that non-negative linear combinations of certain polynomials have all roots real. Then we find a recursion formula for $S(G)$, describing $S(G)$ as a non-negative linear combination of polynomials, satisfying the hypotheses of the lemma. We then combine the two parts, applying the lemma to the recursive formula for $S(G)$, to conclude that all roots of $S(G)$ are real.

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Three Optimal Algorithms for Balls of Three Colors

Zdeněk Dvořák
Joint work with Vít Jelínek, Daniel Král', Jan Kynčl, and Michael Saks

We consider a game played by two players, Paul and Carol. Carol fixes a coloring of n balls with three colors. At each step, Paul chooses a pair of balls an asks Carol whether the balls have the same color. Carol truthfully answers yes or no. In the Plurality problem, Paul wants to find a ball with the most common color. In the Partition problem, Paul wants to partition the balls according to their colors. He wants to ask Carol the least number of questions to reach his goal. We describe optimal deterministic and probabilistic strategies for the Partition problem and an asymptotically optimal probabilistic strategy for the plurality problem.

Orders derived from locally constrained homomorphisms (extended abstract)

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Joint work with Daniël Paulusma[b] *and Jan Arne Telle*[c]

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1 Introduction

The well-known H-Coloring problem asks whether for a graph G (the instance) a graph homomorphism $f : V_G \to V_H$ exists, meaning that f is an edge preserving mapping. The H-Coloring problem is solvable in polynomial time if H is bipartite, and it is NP-complete otherwise [6].

In this paper we study *locally constrained* graph homomorphisms [9]. The adjective "locally constrained" expresses the condition that the mapping f restricted to the neighborhood of any vertex u must satisfy further properties. (See [15, 7] for a more general model of such conditions.) The mapping f may be required to be locally

[1] Supported by the Ministry of Education of the Czech Republic as project LN00A056.

- *bijective*, then f is called a local isomorphism or a *full covering projection* to H, and the corresponding decision problem whether such a mapping exists is called H-Cover [1, 14],
- *injective*, then it is called a local monomorphism or a *partial covering*, and the associated problem is H-PCover [8, 9],
- *surjective*, then we get a local epimorphism or a *locally surjective homomorphism* of H or a H-role assignment of G, and decision problem is H-Colordomination [15] also known as H-Role Assignment [12].

Full covers have important applications, for example in distributed computing [5], in recognizing graphs by networks of processors [2, 3], or in constructing highly transitive regular graphs [4]. Partial covers are used in distance constrained labelings of graphs [10] and also as indicators of the existence of homomorphisms of derivate graphs (line graphs) [17]. Locally surjective covers are of interest in social network theory where networks are modeled, in which individuals of the same social role relate to other individuals in the same way.

In [12] it is proven that for connected graphs H H-Colordomination is polynomially solvable if H contains at most two vertices, and NP-complete if H contains at least three vertices. This is in contrary to the related H-Cover and H-PCover problem for locally bijective and locally injective homomorphisms. For these problems many nontrivial graphs H are known, for which the associated problems are polynomially solvable, and where up today no computational complexity classification is known (or even conjectured).

We introduce (partial) orderings on a family of graphs, in which a graph H is smaller than a graph G, if a locally constrained homomorphism from G to H exists (Sect. 3.1). We show that a similar order can be derived also for matrices describing the degree structure of the graph (Sect. 3.3).

2 Preliminaries

A *graph homomorphism* from G to H is a vertex mapping $f : V_G \to V_H$ satisfying the property that whenever an edge (u, v) appears in E_G, then $(f(u), f(v))$ belongs to E_H as well, i.e., $f(N_G(u)) \subseteq N_H(f(u))$ for all $u \in V_G$.

Definition 1 *For graphs G and H we say that G* covers H fully*, denoted by $G \xrightarrow{B} H$, if a graph homomorphism $f : V_G \to V_H$ exists satisfying:*

$$\text{for all } u \in V_G : f(N_G(u)) = N_H(f(u)) \text{ and } |f(N_G(u))| = |N_G(u)|.$$

The function f is called a locally bijective homomorphism *or a* (full) covering projection *from G to H.*

We denote by $G \xrightarrow{I} H$ when G covers H partially*, i.e., if there exists a graph homomorphism $f : V_G \to V_H$ such that:*

$$\text{for all } u \in V_G : |f(N_G(u))| = |N_G(u)|.$$

The mapping f is then called a locally injective homomorphism *or a* partial covering projection *from G to H.*

Finally, G covers H surjectively*, denoted by $G \xrightarrow{S} H$, if G allows a homomorphism $f : V_G \to V_H$ satisfying:*

$$\text{for all } u \in V_G : f(N_G(u)) = N_H(f(u)).$$

In this case the function f is called a locally surjective homomorphism.

Note that whenever G covers H fully, it also covers H both partially and surjectively. Hence, any result valid for locally injective or for locally surjective homomorphisms is also valid for full covering projections.

We provide here also an alternative definition of these three kinds of mappings via subgraphs induced by preimages of edges. Surprisingly, upto our knowledge, this useful definition didn't yet appear in literature.

Observation 1 *Let $f : G \to H$ be a graph homomorphism. For every edge (x, y) of H, the subgraph of G induced by $f^{-1}(x) \cup f^{-1}(y)$ is*

- *a* perfect matching *if and only if f is locally bijective,*
- *a* matching *iff f is locally injective,*
- *a* bipartite graph without isolated vertices *iff f is locally surjective.*

For a finite connected graph F we define the *universal cover* T_F as the only (possibly infinite) tree which allows a locally bijective homomorphism $T_F \xrightarrow{B} F$. The vertices of T_F can be represented as walks in F starting in a fixed vertex u that do not traverse the same edge in two consecutive steps.

Edges in T_F connect those walks that differ in the presence of the last edge. The mapping $f_0 : T_f \xrightarrow{B} F$ sending a walk in $V(T_F)$ to its last vertex is a locally bijective homomorphism.

It was shown in [11] that if a connected graph G partially covers a connected graph H, then a locally injective homomorphism $g' : T_G \xrightarrow{I} T_H$ can be derived from any homomorphism $g : G \xrightarrow{I} H$. Moreover, the same construction provides a locally surjective homomorphism $g' : T_G \xrightarrow{S} T_H$ from a homomorphism $g : G \xrightarrow{S} H$.

In the sequel we consider all isomorphism classes of connected simple graphs. We assume that each of these classes is represented by one of its elements, and these representatives form the set $\mathcal{C}$, called the *set of connected (unlabeled) graphs.*

3 Homomorphism-induced orders

It is well known that graph homomorphisms define a quasiorder on the class of all graphs, which can be factorized into a partial order (for an overview of these results see e.g. recent monograph [13]).

We show that a similar interesting structure exists on the class of connected graphs $\mathcal{C}$ for locally constrained homomorphisms. For this purposes we will view $\xrightarrow{B}$, $\xrightarrow{I}$ and $\xrightarrow{S}$ as binary relations on $\mathcal{C}$, denoted by $(\mathcal{C}, \xrightarrow{*})$ if necessary, where $*$ will indicate the appropriate local constraint.

3.1 Partial orders on graphs

We first show that $(\mathcal{C}, \xrightarrow{*})$ is a partial order for any of the three local constraints $* = B, I, S$:

Observe first that for any $G \in \mathcal{C}$ the identity mapping $i : G \to G$ clarifies that all three relations $\xrightarrow{*}$ are *reflexive.*

Also for any kind of the local constraint, the composition of two graph homomorphisms is a graph homomorphism of the same sort (composition of two local bijections is a local bijection, etc.) Hence each $\xrightarrow{*}$ is also *transitive.*

Proposition 2 ([11]) *If G allows a locally injective and also a locally surjective homomorphism to the same connected graph H, then both these homomorphisms are locally bijective.*

Finally, for $G, H \in \mathcal{C}$ suppose that $f : G \xrightarrow{*} H$, $g : H \xrightarrow{*} G$, where f, g and consequently also $f \circ g : G \xrightarrow{*} G$ are all of the same local constraint.

But the identity mapping assure also $G \xrightarrow{B} G$, hence $f \circ g$, and consequently f and g are graph isomorphisms. As $G \simeq H$ we get that all three relations are *antisymmetric*.

We note here that the antisymmetry of $\xrightarrow{I}$ was proved by an iterative proof in [17] in a different setting.

Proposition 3 *All three relations* $(\mathcal{C}, \xrightarrow{B}), (\mathcal{C}, \xrightarrow{I})$ *and* $(\mathcal{C}, \xrightarrow{S})$ *are partial orders.*

In view of Proposition 2 we can conclude:

$$(\mathcal{C}, \xrightarrow{B}) = (\mathcal{C}, \xrightarrow{I}) \cap (\mathcal{C}, \xrightarrow{S})$$

3.2 Degree partitions and their matrices

During studying locally bijective homomorphisms, it was shown that the notion of degree partition and the associated degree refinement matrices provide a necessary condition for the existence of such a mapping. Moreover, these matrices are computable in polynomial time, so this condition can be tested in polynomial time as well. (This among others imply that the H-PCover problem is NP-complete whenever the corresponding H-Cover problem is NP-complete [9].)

These notions are based on the property of locally bijective homomorphisms (originally graph isomorphisms) that these mappings maintain vertex degree and this property propagates also on the neighborhood of any vertex.

Definition 2 *The* degree partition *of a graph G is a partition of the vertex set V_G into the disjoint sets $\mathcal{B} = \{B_1, \dots, B_k\}$, called* blocks *such that whenever two vertices u and v belong to the same block B_i, then for any $j \in \{1, \dots, k\}$ it holds $|N(u) \cap B_j| = |N(v) \cap B_j| = m_{i,j}$.*

The matrix $M^{\mathcal{B}}$ such that $(M^{\mathcal{B}})_{i,j} = m_{i,j}$ is called degree partition matrix *for $\mathcal{B}$.*

Observe that the incidence matrices appear as a special kind of degree partition matrices. Moreover, for any graph, the degree partition with the minimum number of classes is unique upto a permutation of its classes. Such a partition can be computed (in $O(n^3)$ time) by the following iterative procedure:

Algorithm for minimal degree partition
Input: A graph G
Output: The minimal degree partition $\mathcal{B}$

0. Set $\mathcal{B}^0 = \{B_1^0\} = \{V_G\}$, $t = 1$.
1. For each vertex u compute the degree vector
 $\overrightarrow{d(u)} := \Big(|N(u) \cap B_1^t|, |N(u) \cap B_2^t|, \dots\Big)$.
2. Set $t := t + 1$ and define the new partition $\mathcal{B}^t$ of V_G such that
   - $u, v \in B_i^t$ if and only if $\overrightarrow{d(u)} = \overrightarrow{d(v)}$,
   - $u \in B_i^t$, $v \in B_{i'}^t$ $i \leq i'$ if and only if $\overrightarrow{d(u)} \leq_{\text{Lex}} \overrightarrow{d(v)}$, where $\leq_{\text{Lex}}$ is the lexicographic order on integer sequences.
3. If $\mathcal{B}^t = \mathcal{B}^{t-1}$ then set $\mathcal{B} = \mathcal{B}^t$ and stop, otherwise continue by the step 1.

As a side-effect the procedure provides also an unique canonical ordering of the blocks and in the rest we assume that the blocks of any minimal partition $\mathcal{B}$ are ordered in this manner The unique matrix $\text{drm}(G) = M^{\mathcal{B}}$ is then called the *degree refinement matrix* of G.

Clearly each degree partition matrix is symmetric in the sense that symmetric entries are either both zero or both non-zero. On the other hand for no such a matrix is a degree refinement matrix. E.g. the degree partition matrix $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$ with $k \geq 1$, describes bipartite k-regular graphs, but the degree refinement matrix of this class is the 1×1 matrix (k).

The test whether a given matrix M is a degree refinement matrix, i.e. whether there exists a graph G and its minimal degree partition $\mathcal{B}$ s.t. $M = M^{\mathcal{B}} = \text{drm}(G)$, can be resolved in polynomial time by an algorithm similar for computing the minimal degree partition. Assume that the given matrix M is of order k. We will introduce a row partition $\mathcal{R} = \{R_1, R_2, \dots\}$ of the set of row indices $\{1, 2, \dots, k\}$, which will imitate the degree partition $\mathcal{B}$ of the vertex set V_G of a possible graph G with that degree refinement matrix.

Algorithm constructing degree refinement matrix

Input: A degree partition matrix M

Output: A degree refinement matrix M', which encode the same class of graphs as M.

0. Set $\mathcal{R}^0 = \{R_1^0\} = \{1,\dots,k\}$, $t = 1$.

1. For each row $r = 1,\dots,k$ compute the row-degree vector $\overrightarrow{d(r)} := \left(\sum_{i\in R_1^t} m_{r,i}, \sum_{i\in R_2^t} m_{r,i}, \dots\right)$.

2. Set $t := t+1$ and define the new partition $\mathcal{R}^t$ of $\{1,\dots,k\}$ such that

   – $r, s \in B_i^t$ if and only if $\overrightarrow{d(r)} = \overrightarrow{d(s)}$,

   – $r \in B_i^t$, $s \in B_{i'}^t$ $i \leq i'$ if and only if $\overrightarrow{d(r)} \leq_{\mathrm{Lex}} \overrightarrow{d(s)}$,

3. If $\mathcal{R}^t = \mathcal{R}^{t-1}$ then set $M' = \begin{pmatrix} \overrightarrow{d(r)} : r \in R_1^t \\ \overrightarrow{d(r)} : r \in R_1^t \\ \vdots \end{pmatrix}$ and stop,

   otherwise continue by the step 1.

Clearly, for any graph G allowing a (possibly non-minimal) degree partition $\mathcal{B}$ with matrix M, and any vertex u belonging to a class B_r in any time interval t the degree vector of and the associated row-degree vectors are equal: $\overrightarrow{d(u)} = \overrightarrow{d(r)}$.

Let us note, that for a given matrix M it would not be worth to try to construct implicitly a graph G, such that $\mathrm{drm}(G) = M$, since even for $M = (k)$, such smallest G is the complete graph on $k+1$ vertices, and its size is exponential in the size of M if k is encoded in binary. On the other hand (i.e. if the exponential space and time is not an issue), given an positive integer-valued matrix M with all zeros placed symmetrically around the diagonal, a graph G and and its degree partition $\mathcal{B}$ with associated matrix M can be constructed on at most $n = \mathrm{lcm}(\{m_{i,j} : m_{i,j} \neq 0, i \neq j\} \cup \{m_{i,i}+1\})$ vertices. (We leave this simple assertion as an exercise.)

3.3 Derived orders on matrices

Let us recall here a folklore fact that (e.g. proved explicitly in a wider context in [16]) a locally bijective homomorphism from a graph G to a graph

H may exists only if G and H have the same degree refinement matrix:

Theorem 4 ([16]) *Two finite connected graphs G and H have a common degree refinement matrix if and only if they share a common finite cover as well as iff their universal covers are isomorphic.*

In view of this theorem we can define the universal cover T_M associated with a degree refinement matrix M as the universal cover $T_G = T_M$ of any graph G s.t. $\mathrm{drm}(G) = M$.

Observe that the symmetric and transitive closure of the partial order $\xrightarrow{B}$ is an equivalence relation whose classes can be naturally represented by degree refinement matrices. Speaking in the negative manner, whenever two graphs have different representatives (i.e. matrices), no locally bijective homomorphism can exist.

The above observation lead Jan Arne Telle to ask whether the other two kinds of locally constrained homomorphisms would be conditioned by the existence of a well defined relation (e.g. a nontrivial partial order) on the degree refinement matrices. Here we prove, that such a relation exists and moreover, that it is a partial order.

Definition 3 *Denote by $\mathcal{M}$ the set of all degree refinement matrices of connected graphs and define three relations $\xrightarrow{B}$, $\xrightarrow{I}$, and $\xrightarrow{S}$ resp., on $\mathcal{M}$, where for two matrices $M, N \in \mathcal{M}$ holds $M \xrightarrow{*} N$ if there exist finite graphs G and H such that $M = \mathrm{drm}(G)$ and $N = \mathrm{drm}(H)$, where $G \xrightarrow{*} H$ for the appropriate local constraint.*

As stated above $(\mathcal{M}, \xrightarrow{B})$ is a trivial order where no distinct element are comparable. For the other two relations, the reflexivity of the relation follows straightforwardly from existence the identity mapping on any underlying graph (where at least one must exist to assert the membership of the matrix in $\mathcal{M}$).

The other two properties would require more effort: For proving antisymmetry we involve again the notion of universal cover. Assume (according to the definition) that $M \xrightarrow{I} N$ and $N \xrightarrow{I} M$, where the locally injective homomorphism $f : G_1 \xrightarrow{I} H_1$ justifies the first comparison and $g : G_2 \xrightarrow{I} H_2$ the second. But $f' : T_M \xrightarrow{I} T_N$ is a locally injective homomorphism between the associated universal covers as well as the derived mapping $g' : T_N \xrightarrow{I} T_M$. As in the previous section we get that $f' \circ g'$ is an automorphism of T_M, then the universal covers T_M and T_N are isomorphic and hence $M = N$ due to Theorem 4.

The antisymmetry of $\xrightarrow{S}$ can be proven according to exactly same arguments: By the composition of the derived homomorphisms we get that the universal covers T_N and T_M are isomorphic, and hence both correspond to the same matrix.

For the transitivity property of $\xrightarrow{I}$ assume that $M \xrightarrow{I} N$ and $N \xrightarrow{I} Q$ and let locally injective homomorphisms $f : G_1 \xrightarrow{I} H_1$ witnesses the first and $g : G_2 \xrightarrow{I} H_2$ the second relation, where all four graphs are finite a connected. If $H_1 = G_2$ then the composition $f \circ g$ would justify $M \xrightarrow{I} Q$. If this is not the case, we can only relay on the fact that H_1 and G_2 share the same degree refinement matrix. Now the transitivity property of $\xrightarrow{I}$ follows from the construction established in the next lemma.

Lemma 5 *Let for finite connected graphs G_1, G_2, H_1, H_2 hold that $G_1 \xrightarrow{I} H_1$ and $G_2 \xrightarrow{I} H_2$ where H_1 and G_2 share the same degree refinement matrix. Then there exists a finite connected graph F such that $F \xrightarrow{I} H_2$ and $F \xrightarrow{B} G_1$.*

The same assertion can be proved also for the order $\xrightarrow{S}$ with exactly same arguments, the only difference is that the preimage in G_1 of any edge $(x, y) \in E_{H_1}$ is a spanning bipartite graph, and the same holds also for the preimage of any corresponding $(x_i, y_j) \in E_{F'}$ in F.

Hence we can conclude:

Theorem 6 *All the three relations $(\mathcal{M}, \xrightarrow{B})$, $(\mathcal{M}, \xrightarrow{I})$ and $(\mathcal{M}, \xrightarrow{S})$ are partial orders and moreover*

$$(\mathcal{M}, \xrightarrow{B}) = (\mathcal{M}, \xrightarrow{I}) \cap (\mathcal{M}, \xrightarrow{S}) = (\mathcal{M}, \{(M, M) : M \in \mathcal{M}\}).$$

Corollary 7 *For an arbitrary kind of local constraint $* = B, I, S$ the order $(\mathcal{M}, \xrightarrow{*})$ arises as a factor of the order $(\mathcal{C}, \xrightarrow{*})$, when we unify the graphs that have the same degree refinement matrices.*

Finally let us note that any partial covering projection $G \xrightarrow{I} H$ can be extended to a full covering projection $G' \xrightarrow{B} H$ where $G \subseteq G'$. This yields an alternative definition of the order $(\mathcal{M}, \xrightarrow{I})$:

Observation 8 *For any pair of matrices $M, N \in \mathcal{M}$ it holds $M \xrightarrow{I} N$ if and only if there exists graphs G and H with degree refinement matrices M and N, resp. such that G is a subgraph of H.*

Similarly, for universal covers hold that

Theorem 9 *For any degree refinement matrices $M, N \in \mathcal{M}$ it holds that $M \xrightarrow{I} N$ if and only if $T_M \subseteq T_N$, and similarly $M \xrightarrow{S} N$ iff $T_N \subseteq T_M$.*

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Uniform Abelian Group Labels on Hamiltonian Cycles

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We consider labeling edges of graphs with elements from abelian groups. Particular attention is given to graphs where the labels on any two hamiltonian cycles sum to the same value. We find several characterizations for such labelings for cubes, complete graphs and complete bipartite graphs. This extends work done in a variety of places, where the group under consideration is the integers with standard addition.

Let G be a graph of order n. By *group* we shall mean an abelian group. We shall let $(A, +)$ represent an arbitrary group. Given $x \in A$ let $2x$ represent $x + x$. We shall refer to $2x$ as *twice* the value of x. We shall say y is *even* if it is twice the value of some group element. A *vertex labeling* of G is a mapping from $V(G)$ into A. An edge labeling is defined similarly. When clarity allows, we will refer to an edge labeling simply as a labeling. An edge labeling γ is *induced* by a vertex labeling α if $\gamma(uv) = \alpha(u) + \alpha(v)$, for each edge uv. We will say a labeling is *induced* if it is induced by some vertex labeling. Given a labeled graph G, the *weight* of G, denoted $\omega(G)$, is the sum of the edge labels. This term will often be applied to edges in a subgraph. Given a labeled graph G, if we speak of a subgraph H we will mean that H is labeled with the labeling on $E(G)$ restricted to $E(H)$. We wish to study labelings which have the property that given any pair of hamiltonian cycles, the cycles have the same weight.

Let us say a labeling is *hamilton-stable* if every pair of hamiltonian cycles has the same weight. Suppose the vertices of a circuit appear sequentially as v_1, v_2, $\ldots$, v_k. We shall represent the circuit as $(v_1, v_2, \ldots, v_k)$. At times it will be important to similarly represent a circuit by sequentially listing its edges e.g. $(e_1, e_2, \ldots, e_k)$. In a graph labeled by γ, an even circuit $(e_1, e_2, \ldots, e_k)$ is *balanced* if $\gamma(e_1) - \gamma(e_2) + \ldots - \gamma(e_k) = 0$. A labeling γ is *C_4-balanced* if each 4-cycle is balanced. Let us say an odd cycle $(e_1, e_2, \ldots, e_k)$ is *even* if $\gamma(e_1) - \gamma(e_2) + \gamma(e_3) - \gamma(e_4) + \ldots + \gamma(e_k)$ is even. If the labels on any pair of k-factors have the same weight, we will say the labeling is k-stable. Thus, if a labeling is 2-stable then it is hamilton-stable. This leads us to define the following problem.

HAMILTON STABLE
Instance: A labeled graph G.
Question: Is G not hamilton-stable?

Theorem 1 *The Hamilton Stable problem remains NP-complete when restricted to integer labeled graphs that are planar, cubic, 3-connected and have face girth at least five.*

Observation 2 *Given r and s where $1 \leq s \leq r$, a labeling of an r-regular graph is s-stable if and only if it is $(r - s)$-stable.*

Let Q_n denote the n-dimensional cube.

Theorem 3 *A labeling of Q_n is induced if and only if it is C_4-balanced.*

A proof is by induction on the dimension.

A 3-cube may be hamilton-stable yet fail to be C_4-balanced. Our next result shows that for larger dimensions, this is not the case.

Theorem 4 *For $n \geq 4$, a labeling of Q_n is C_4-balanced if and only if it is hamilton-stable.*

Thus, while being hamilton-stable is a global property, it is equivalent to a local property.

Theorem 5 *A labeling of Q_3 with elements from the real numbers is hamilton-stable if and only if it is C_4-balanced.*

Theorem 6 *If a labeling of Q_n is 1-stable then it is hamilton-stable.*

Theorem 7 *For $n \geq 4$ and k fixed between 1 and $n - 1$, a labeling of Q_n is hamilton-stable if and only if it is k-stable.*

Similar results can be stated for complete and complete bipartite graphs. However, the proofs are somewhat different.

The circular chromatic index of graphs of high girth

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Xuding Zhu (National Sun yat-sen University, Kaohsiung)

Colorings of graphs form a prominent topic in graph theory. Several relaxations of usual colorings have been introduced and intensively studied. In this talk, we focused on circular colorings of line graphs. A proper circular k-edge-coloring, for a real $k \geq 1$, is a coloring by real numbers from the interval $\langle 0, k)$ such that the difference modulo k of the colors γ_1 and γ_2 assigned to incident edges is at least one, i.e., $1 \leq |\gamma_1 - \gamma_2| \leq k - 1$.

A classical theorem of Vizing states that the edges of every graph G with maximum degree D can be colored by at most $D + 1$ colors so that no two incident edges have the same color, i.e., the chromatic index of G is at most $D + 1$. We show that for every $\varepsilon > 0$, there exists g such that the circular chromatic index of a graph G with maximum degree D whose girth is at least g does not exceed $D + \varepsilon$. Note that the circular chromatic index must be at least D because the line graph of such graph G contains a clique of order D. A key ingredient of our proof is recent results on systems of independent representatives and hypergraph matchability of Aharoni, Haxell, Meshulam and others.

Our research has been motivated by a conjecture of Jaeger and Swart 1979 that high girth cubic graphs have chromatic index three, which was disproved by Kochol in 1996. Our results imply that the conjecture is true when relaxed to circular colorings: the circular chromatic index of high girth cubic graphs is close to three.

At the end of the talk, we posed several open problems and questions including the following ones:

- Can the assumption that the girth of a graph G is high be replaced by a weaker assumption that its odd girth is high?

- What is the least g such that circular chromatic index of every cubic bridgeless graph of girth at least g is at most $5/2$?

For the latter problem, the best upper bound on g which we know is 14.

Reptile Simplices Could Help the Net Police But Probably They Do Not Exist

Jiří Matoušek

A simplex S is called an *m-reptile* if it can be tiled without overlaps by simplices $S_1, S_2, \ldots, S_m$ that are all congruent and similar to S. The only m-reptile d-simplices that seem to be known for $d \geq 3$ have $m = k^d$, $k \geq 2$. We prove, using eigenvalues, that there are no 2-reptile simplices of dimensions $d \geq 3$. This investigation has been motivated by a probabilistic packet marking problem in theoretical computer science, introduced by Adler in 2002.

Comments on the Erdős-Turán conjecture

Jaroslav Nešetřil
Joint work with Oriol Serra

Abstract

An old conjecture of Erdős and Turán states that the representation function of an additive basis can not be bounded. We survey some of the main results related to this conjecture.

Let $(G, *)$ be a semigroup and $X \subset G$. A subset $A \subset X$ is a *basis* of X if $X \subset A * A$. When X is an infinite set, a subset A is an *asymptotic basis* if $X \setminus (A * A)$ is finite.

For $g \in G$ we denote by $r_A(g)$ the number of pairs $(a, a') \in A \times A$ such that $g = a * a'$. We call r_A the *representation function* of A in $(G, *)$.

Given a positive integer k, we say that X has the *k-Erdős-Turán property* $ET(k)$ if, for any base A of X, there is an element $x \in X$ with $r_A(x) \geq k$. We say that X has the Erdős-Turán property ET if it has $ET(k)$ for every $k \in \mathbb{N}$.

A famous conjecture of Erdős and Turán [5] formulated in 1941 states that the set of positive integers with addition has the ET property.

Here we survey some of the main results connected to this conjecture.

Even if the conjecture is originally formulated for sets which are additive basis, it can be easily shown that sets with large density do have unbounded representation functions, regardless of them being a basis.

Erdős [1] proved in 1964 that $(\mathbb{N}, \cdot)$ does have the ET property. Nešetřil and Rödl [8] gave a simple proof of this result by using Ramsey Theorem. Puš [10] extended this result by showing that a semigroup with an infinite set of primes and a finite number of units has the ET property.

A broader view of the problem can be taken by considering more general binary operations. We can for example show that direct sums of lattices do have the ET property. In particular, the family of finite sets of a countable set with union or intersection do have the ET property. The proof goes along the lines of Neštřil-Rödl method.

As another application of this method, the conjecture was proved to be true for the class of so-called d-bounded additive basis of $\mathbb{N}$. We discuss this application to the original problem and some possible limitations of the method for this case.

On the negative side, it is known that the group of integers with addition has basis with unique representation (up to commutativity), a result which can be extended to any abelian free group. Ruzsa [11] shows that, for any prime p, there are bases of $\mathbb{Z}_p \times \mathbb{Z}_p$ whose representation function is bounded by 18. This result has been recently extended by Haddad and Helou [6] by showing that the class of cyclic groups does not have the ET-property. These authors also show that vector spaces over finite fields of odd characteristic or algebraic closed fields do not have the ET–property.

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Perfect Matching Preservers

(Extended Abstract)

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Joint work with Martin Loebl

A subset $M \subseteq E(G)$ of the edge set of a graph G is called *a matching* if no two edges in M have a vertex in common. A matching M is called *a perfect matching* if each vertex of G is incident with one edge in M. A graph G is *k-extendable*, where k is an integer, if every matching in G of size at most k can be extended to a perfect matching in G.

In this note we characterize the bipartite graphs which have the same perfect matchings. We will achieve it by a full description of matching preservers defined as follows: A bijection $\psi : E(G) \to E(G')$ is called *a matching preserving* mapping, or *a matching preserver*, if $M \subseteq E(G)$ forms a perfect matching in G if and only if $\psi(M)$ does in G'. The question of the existence of matching preservers arise from the theory of permutation linear preservers and our result answers the conjecture of Brualdi (R. Brualdi: Linear Preservers and Diagonal Hypergraphs. Linear Alg. Applics., to appear) on the structure of permanent preserving permutation linear operators on coordinate subspaces.

If G is bipartite then we say that a twist is *a bitwist* if the corresponding vertices u, v belong to different parts of G. It is easy to observe that bitwists do not preserve only circuits but also perfect matchings. In the language of bipartite graphs, the conjecture of Brualdi can be stated as follows.

Conjecture 1 *Let G and G' be two bipartite 1-extendable graphs and let $\psi : E(G) \to E(G')$ be a matching preserver. Then there is a sequence of bitwists of G resulting in a graph isomorphic to G' and ψ is induced by this isomorphism.*

We show that this is not true: one more operation is needed to describe matching preservers.

Let G be a 2-connected bipartite graph and G_1, G_2, G_3 be subgraphs of G and let a_i, b_i be vertices of G_i both from the same part of G_i, $i = 1, 2, 3$.

Further let G be obtained from a disjoint union of G_1, G_2, G_3 by identifying pairs b_1, a_2; b_2, a_3 and b_3, a_1. Then we can define *bitransposition* as follows: Let G' be a graph obtained from a disjoint union of G_1, G_2, G_3 by identifying pairs b_1, a_3; b_2, a_1 and b_3, a_2. The graph G' is said to be obtained from G by bitransposition of G_1 and G_2. The operation of bitransposition also preserves both circuits and perfect matchings.

Theorem 1 *Let G and G' be two bipartite 1-extendable graphs and let $\psi : E(G) \to E(G')$ be a matching preserver. Then there is a sequence of bitwists and bitranspositions of G resulting in a graph isomorphic to G' and ψ is induced by this isomorphism.*

Prisms, why, how and what

Moshe Rosenfeld[1]

Abstract

A brief history of prisms over graphs and the Prague summer workshops.

1 Before the summer workshops...

The *prism* over a graph G is the Cartesian product $G \square K_2$ of G with the complete graph K_2. In plain words, we take two disjoint copies of a graph G and add a perfect matching that matches a vertex in G with its clone in the other copy. Initially, my study of prisms was motivated by D. Barnette's conjecture that all simple 4-polytopes are Hamiltonian (these are certain 4-regular, 4-connected graphs). Prisms over simple 3-polytopes (planar, 3-connected cubic graphs) are examples of simple 4-polytopes. In 1973 [19] we observed that if G is the graph of a simple 3-polytope and the 4 color conjecture is true then the prism over G is Hamiltonian. After the establishment of the 4-color theorem, we carried this study further. In 1978 while visiting Simon Fraser University, B. Alspach and I [1] observed that not only the prisms over certain cubic graphs are Hamiltonian but actually they can be decomposed into two disjoint Hamiltonian cycles. We conjectured that the prisms over all cubic, 3-connected graphs can be decomposed into two disjoint Hamiltonian cycles. This conjecture is still open.

2 Prague summer workshops

I presented the decomposition conjecture in the Prague 2001 summer workshop. Immediately following the workshop, together with R. Čada, T. Kaiser and Z. Ryjáček [6] we proved that prisms over 3-connected, cubic, bipartite planar graphs have a Hamiltonian decomposition.

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In the 2002 summer workshop, just before the floods, we noted a close connection between prisms and Hamiltonian cycles and paths in graphs.

The hunt for Hamiltonian cycles in graphs precedes Graph Theory by hundreds of years. The famous knight's tour problem on a chess board, a search for a Hamiltonian path or cycle in a graph of order 64 can be traced back to 840 AD. Its first mathematical formulation, or more precisely, the first mathematical paper on the subject was the paper by L. Euler [12] in 1759. Officially, the notion of Hamiltonian cycle was introduced almost a 100 years later by Hamilton. Thus the mathematical gestation period (the time elapsed from its first mathematical appearance to the formal introduction of the concept) of Hamiltonian cycles is about 100 years. George Jelliss' web site http://www.ktn.freeuk.com/ is full of gems on the history and current affairs of the knight's tour.

Nowadays we are flooded by theorems, conjectures, refuted conjectures, web sites all devoted to finding Hamiltonian cycles in graphs. The search for Hamiltonian cycles can be broken roughly into three categories:

- Specific graphs, such as the knight's tour or I. Havel's "mid cube" graph, Kneser graphs and many more.
- If G has property P then G is Hamiltonian.
- Find long cycles in graphs.

The long search for Hamiltonian cycles has left us with many theorems, intricate constructions, conjectures, some refuted and many still open (not surprisingly the difficult ones). The vast list of theorems, open problems, conjectures and counter examples can be easily found in the numerous survey papers on Hamiltonian cycles such as Ron Gould's recent survey [14], Adrian Bondy's [4] a quarter century earlier and many more.

As the remaining problems on Hamiltonian cycles are tough, the *tough* search for alternative problems. Various approaches have been adopted by researchers to modify the Hamilton cycle (or path) and search for closely related structures in graphs. Researchers are also trying to develop measures of "*closeness*" to Hamiltonicity. An excellent natural example of such a measure was proposed by B. Jackson and N. Wormald [16].

A Hamilton cycle is a spanning closed walk that visits each vertex exactly once and a Hamiltonian path is a spanning tree of maximum degree 2. It is then very natural to explore the following modifications:

Instead of searching for a Hamilton cycle in a graph, search for a spanning, closed walk in which every vertex is visited at most twice (or k times).

Instead of searching for a Hamilton path, look for a spanning tree of maximum degree 3 (or more). We call these spanning structures *k-walks* and *k-trees* respectively.

It is not hard to show that any graph with a k-tree has a k-walk, and that the existence of a k-walk guarantees the existence of a $(k+1)$-tree (see [16]). This creates the following hierarchy among families of graphs:

1-walk (Hamilton cycle) $\Rightarrow$ 2-tree (Hamilton path) $\Rightarrow$ 2-walk $\Rightarrow$ 3-tree $\Rightarrow \ldots$

This hierarchy provides a natural setting for defining *closeness* to being Hamiltonian. For more information on k-walks, k-trees and related topics refer, e.g., to [10].

In 2002 we observed that *having a Hamiltonian prism* is *'sandwiched'* between the existence of a 2-tree and the existence of a 2-walk. That is:

$$\text{2-tree} \Rightarrow \textbf{Hamiltonian prism} \Rightarrow \text{2-walk} \tag{1}$$

and all of the implications are sharp. This means that *having a Hamiltonian prism* is "closer" to Hamiltonian than having a closed 2-walk. This observation naturally led us to revisit and consider many theorems and conjectures on Hamiltonian cycles and paths. As am example, consider the following sequence of proofs: D. Barnette [2] proved in 1966 that planar 3-connected graphs have a spanning 3-tree. Gao and Richter [13] improved Barnette's result. They proved that 3-connected planar graphs have 2-walks, or are even "closer" to Hamiltonian graphs. Are 3-connected planar graphs even closer to being Hamiltonian?

Conjecture 1 *The prisms over* 3*-connected planar graphs are Hamiltonian.*

Immediately following the workshop, we expanded our team and managed to show that certain families of graphs, such as Kleetopes, generalized Halin graphs, 2-connected line-graphs and others have a Hamiltonian prism [17]. Kleetopes are planar triangulations, but not all planar triangulations are Kleetopes. Probably the next step should be to prove that all 3-connected planar triangulations have Hamiltonian prisms. We are now considering more related questions. Nash-Williams [18] conjectured that 4-regular, 4-connected graphs are Hamiltonian. This was disproved by Meredith [5]. Since these graphs are Eulerian they have a closed 2-walk. Are these graphs closer than having a 2-walk to being Hamiltonian?

Conjecture 2 *The prisms over* 4*-connected* 4*-regular graphs are Hamiltonian.*

In 1973, Chvátal [8] stated the beautiful conjecture that every 2-tough graph is Hamiltonian. 27 Years later, in 2000, the conjecture was disproved by Bauer, Broersma and Veldman [3] by constructing non-Hamiltonian $9/4 - \varepsilon$-tough graphs (for small ε). A weaker form of the conjecture, that there is a fixed k such that k-toughness implies Hamiltonicity, is still open. The construction from [3] was modified by Ellingham and Zha [11] who constructed $(17/24 - \varepsilon)$-tough graphs with no 2-walk. An upper bound for toughness that guarantees the existence of a 2-walk was also obtained in [11]: every 4-tough graph has a 2-walk. These are the best bounds available, but a conjecture from [16] states that the truth is much closer to the lower bound: namely that 1-tough graphs have a 2-walk. This would improve a result of Win [21] that all 1-tough graphs have a spanning 3-trees. Clearly, if G is t-tough then $G \square K_2$ is 2t-tough. Thus if G has a Hamiltonian prism it must be $\frac{1}{2}$-tough. We were able to construct $(9/8 - \varepsilon)$-tough graphs whose prisms are not Hamiltonian. These led us to the following conjecture:

Conjecture 3 *If G is 2-tough then the prism over G is Hamiltonian.*

The notorious "middle two levels" problem (as Tom Trotter refers to it in his web site): "Let n = 2k+1 be an odd integer and consider the subset lattice consisting of all subsets of $\{1, \ldots, 2k+1\}$. The middle two levels consisting of the sets of size k and $k+1$ form a bipartite graph. A k subset is connected by an edge to all $k+1$ subsets that contain it. It is conjectured that this graph is Hamiltonian. Known to be true for n at most 15. The origins of this problem are not completely clear, but it was first posed to me by Ivan Havel in the summer of 1982. Havel in fact showed me a reprint in which the problem was posed. [15]"

Papers considering this problem proved existence of long cycles in the middle level graph e.g. [20] or constructed explicit complete factorization hoping that two disjoint 1-factors will produce the elusive Hamiltonian cycle. Days before the 2004 summer workshop, using the modular explicit factorization [9], we were able to show that the prism over the middle two levels graph is Hamiltonian. Other related graphs, such as the "odd graph", the graph whose vertices are the k subsets of $\{1, \ldots, 2k+1\}$ where two vertices (k-subsets) are connected by an edge if the two subsets are disjoint (also known as the Kneser Graph $K(2k+1, k)$) so far resisted our attempts to prove that the prisms over these graphs are Hamiltonian. We conjecture:

Conjecture 4 *The prisms over the Kneser Graphs $K(2k+1, k)$ are Hamiltonian.*

More generally, the Kneser Graphs $K(n, k)$ are known to be Hamiltonian for $n \geq 3k$ [7]. We conjecture:

Conjecture 5 *The prisms over the Kneser Graphs $K(n, k)$ are Hamiltonian.*

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On the Regularity Method for Hypergraphs

Mathias Schacht
joint work with Brendan Nagle, Vojtěch Rödl, and Jozef Skokan

The Regularity Lemma of Szemerédi [22], proved to be a powerful tool in Combinatorics. This lemma states that all sufficiently large graphs can be approximated, in some sense, by random graphs. Since "random-like" graphs are often easier to handle than arbitrary graphs, the Regularity Lemma is especially useful in situations when the problem in question is easier to prove for random graphs.

Let $G = (V, E)$ be a graph and A, $B \subseteq V$ be a pair of disjoint sets of vertices of G. Denote by $e(A, B)$ the number of edges of G between A and B. The *density* of the pair (A, B) is defined by $d(A, B) = e(A, B)/(|A||B|)$. The pair is called ε*-regular* if for any $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, we have $|d(A, B) - d(A', B')| < \varepsilon$.

Theorem 1 (Szemerédi's Regularity Lemma) *For every $\varepsilon > 0$ there exist a T_0 such that the vertex set $V(G)$ of any graph G can be partitioned into $t \leq T_0$ classes $V(G) = V_1 \cup \cdots \cup V_t$, so that all but εt^2 pairs (V_i, V_j) are ε-regular.*

Many applications of the Regularity Lemma are based on its accompanying Counting Lemma (see, e.g., [11, 12] for a survey).

Theorem 2 (Counting Lemma) *If G is an ℓ-partite graph with $V(G) = V_1 \cup \cdots \cup V_\ell$ and $|V_i| = n$ for all $i \in [\ell]$, and all pairs (V_i, V_j) are ε-regular of density d for $1 \leq i < j \leq \ell$, then G contains $(1 \pm f_\ell(\varepsilon))d^{\binom{\ell}{2}} \times n^\ell$ cliques K_ℓ of order ℓ, where $f_\ell(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

We discuss a generalization of Szemerédi's Regularity Lemma from graphs to k-uniform hypergraphs, which allows us to prove an accompanying Counting Lemma. Unlike for graphs, there are several "natural ways" to define "regularity" for k-uniform hypergraphs. Consequently, various forms of a

Regularity Lemma for hypergraphs have been already considered in [1, 2, 4, 6, 15]. None of these Regularity Lemmas seemed to admit a companion counting result (i.e., a corresponding generalization of Theorem 2). The first attempt of developing a Hypergraph Regularity Lemma with a corresponding Counting Lemma was undertaken by Frankl and Rödl in [5] for 3-uniform hypergraphs. Recently, Rödl and Skokan [19] established a generalization of this Regularity Lemma to k-uniform hypergraphs for any $k \geq 3$.

Analogously to the feature that Szemerédi's Regularity Lemma decomposes a given graph into an ε-regular partition, this Hypergraph Regularity Lemma decomposes the edge set of a given k-uniform hypergraph into constantly many "blocks", almost all of which are, in a specific sense, "quasi-random". The concept of hypergraph regularity which plays the analogous role of the ε-regular pair is, unfortunately, considerably more technical than its graph counterpart, and we cannot give the precise definitions here.

Just as Theorem 2, the Counting Lemma, is an important companion statement to Szemerédi's Regularity Lemma, most applications of the Hypergraph Regularity Lemma from [19] require a similar companion lemma - the "general Counting Lemma". Analogously to Theorem 2, the general Counting Lemma estimates the number of copies of the clique $K_\ell^{(k)}$ (i.e., the complete k-uniform hypergraph on ℓ vertices) contained in an appropriate collection of "dense and regular blocks" within a regular partition provided by the Hypergraph Regularity Lemma. Such a Counting Lemma was established for special cases ($k = 3$, $\ell > 3$ and $k = 4$, $\ell = 5$) in [5, 13, 18]. Recently, in [14] we succeeded to prove the general Counting Lemma for any $\ell > k \geq 2$, reducing it to an earlier result from [10]. This Counting Lemma together with the Hypergraph Regularity Lemma of [19] can be viewed as a generalization of the Regularity Method from graphs to uniform hypergraphs. A similar extension was independently obtained by Gowers [9].

These generalizations can be applied to several extremal hypergraph problems. In particular, answering a question of Erdős, Frankl, and Rödl [3], the following theorem was proved in [17]

Theorem 3 *Suppose an n-vertex k-uniform hypergraph $\mathcal{H}$ contains only $o(n^\ell)$ copies of $K_\ell^{(k)}$. Then one can delete $o(n^k)$ edges of $\mathcal{H}$ to make it $K_\ell^{(k)}$-free.*

It is known that this theorem can be used to give an alternative proof the well-known Density Theorem of Szemerédi [21] regarding the upper density of sets containing no arithmetic progression of fixed length (see [5, 17]). Moreover, it can also be used to derive combinatorial proofs to some of the density theorems of Furstenberg and Katznelson [7, 8] (see [9, 16, 20]).

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Open problems of separoids

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A *separoid* is a set S endowed with a symmetric relation $\dagger \subset \binom{2^S}{2}$ defined on its pairs of disjoint subsets which is closed as a filter in the natural partial order induced by the inclusion. That is, if $A, B \subseteq S$ then

- $A \dagger B \implies A \cap B = \emptyset$,

•• $A \dagger B \;\&\; B \subset B' \;(\subseteq S \setminus A) \implies A \dagger B'$.

The pair $A \dagger B$ is a *Radon partition*, each part (A and B) is a *component* and the union $A \cup B$ is the *support* of the partition. The separoid is *acyclic* if $A \dagger B \implies |A||B| > 0$.

Theorem 1 (Arocha et al. [1] and Strausz [2, 9]) *Every finite separoid can be represented by a family of convex sets in some Euclidean space; that is, for every separoid S there exists a family of convex sets $\mathcal{F} = \{C_i \subseteq \mathbb{E}^d : i \in S\}$ such that, for $A, B \subseteq S$*
(∗)
$$A \dagger B \iff \langle C_a \in \mathcal{F} : a \in A\rangle \cap \langle C_b \in \mathcal{F} : b \in B\rangle \neq \emptyset \quad (\textit{and} \quad A \cap B = \emptyset),$$
where $\langle\cdot\rangle$ denotes the convex hull. Furthermore, if the separoid is acyclic, then such a representation can be done in the $(|S|-1)$-dimensional Euclidean space.

For the acyclic case, let S be identified with the set $\{1, \dots, n\}$. For each element $i \in S$ and each pair $A \dagger B$ such that $i \in A$, let $\rho^i_{A\dagger B}$ be the point of $\mathbb{R}^n$ defined by

$$\rho^i_{A\dagger B} = e_i + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B} e_b - \frac{1}{|A|}\sum_{a\in A} e_a\right],$$

where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$. Observe that all such points are in the affine hyperplane spanned by the basis. Now, each element $i \in S$ is represented by the convex hull of all such points:

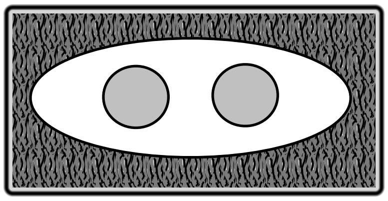
$$i \mapsto C_i = \left\langle \rho^i_{A\dagger B} : i \in A \text{ and } A \dagger B \right\rangle.$$

From here, it is easy to prove $(*)$. •

Question 1 *¿Is there a geometric representation theorem for infinite separoids? Guess: YES.*

The minimum dimension where the separoid S can be represented is the *geometric dimension* of S and it is denoted by $\mathrm{gd}(S)$. The (*combinatorial*) *dimension* $\mathrm{d}(S)$ is the minimum d such that every subset with $d+2$ elements is the support of a Radon partition. The separoid S is in *general position* if no $\mathrm{d}(S)+1$ elements are the support of a Radon partition. The separoid S is a *point separoid* if it can be represented by a family of points $P \subset \mathbb{E}^d$ in some Euclidean space (cf. [3]).

Theorem 2 (Bracho & Strausz [4]) *Let S be a separoid in general position. S is a point separoid if and only if* $\mathrm{d}(S) = \mathrm{gd}(S)$.



Marcos: *The smallest non-point separoid S such that* $\mathrm{d}(S) = \mathrm{gd}(S)$.

Question 2 *¿Is there a polynomial-verifiable property of separoids Ψ which makes the following statement true: S is a point separoid if and only if $\Psi(S)$ and $\mathrm{d}(S) = \mathrm{gd}(S)$? Guess: YES.*

Given two separoids S and T, a mapping $h\colon S \to T$ is a *homomorphism* if the image of minimal Radon partitions are minimal Radon partitions; that is, for $A, B \subseteq S$,

$$A \dagger B \text{ minimal} \implies h(A) \dagger h(B) \text{ minimal.}$$

Theorem 3 (Nešetřil & Strausz [5]) *The category of separoids endowed with homomorphisms is a universal category. That is, every category can be represented by an induced subcategory of separoids' homomorphisms.*

Question 3 *¿Is the subcategory of point separoids universal? Guess: NO.*

The *homomorphisms order* of separoids is defined with the relation

$$S \preceq T \iff \exists h\colon S \longrightarrow T,$$

and identifying those separoids S and S' which $S \preceq S'$ and $S' \preceq S$. A partially ordered set $(X, \leq)$ is *fractal* if for each interval $[x, y] := \{z \in X : x \leq z \leq y\}$ there exists a monotone and injective function $\iota\colon X \hookrightarrow [x, y]$.

Theorem 4 (Nešetřil [6]) *The homomorphisms order of graphs is fractal.*

Question 4 *¿Is the homomorphisms order of separoids fractal? Guess: YES.*

A pair of disjoint subsets $A \cap B = \emptyset$ which are not a Radon partition, are said to be *separated* and denoted $A \mid B$. A mapping $\mu\colon S \to T$ is a *morphism* if the preimage of separations are separations; that is, for $C, D \subseteq T$,

$$C \mid D \implies \mu^{-1}(C) \mid \mu^{-1}(D).$$

A morphism is a *monomorphism* if it is injective.

Theorem 5 (Strausz [8]) *Let S be a d-dimensional separoid of order $|S| = (k-1)(d+1)+1$. Suppose that in addition there exists a monomorphism $\mu\colon S \longrightarrow P$ into a d-dimensional point separoid P in general position. Then there exists a k-colouring $\varsigma\colon S \to \{1, \ldots, k\}$ such that each pair of chromatic classes are the components of a Radon partition; that is,*

$$1 \leq i < j \leq k \implies \varsigma^{-1}(i) \dagger \varsigma^{-1}(j).$$

Question 5 *¿How far the additional hypothesis can be weakened —the hypothesis "$\exists\mu\ldots$"— while maintaining the conclusion of the previous Theorem? ¿Is there a purely combinatorial Tverberg-type theorem for separoids? Guess: Not much and...I do not know.*

A morphism is an *epimorphism* if it is surjective. An epimorphism $\varsigma\colon S \to T$ is a *chromomorphism* if the preimage of minimal Radon partitions are Radon partitions; that is, for $C, D \subseteq T$,

$$C \dagger D \text{ minimal} \implies \varsigma^{-1}(C) \dagger \varsigma^{-1}(D).$$

The *complete* separoid K_k is the separoid of k elements such that $i \dagger j$, for all $i, j \in K_k$. The (k, d)-*Tverberg number* $\vartheta(k, d)$ is the minimum n such that all d-dimensional separoids of order at least n maps onto K_k with a chromomorphism; that is, $\vartheta(k,d)$ is minimal with the property

$$|S| \geq \vartheta(k, \mathrm{d}(S)) \implies \exists\varsigma : S \longrightarrow K_k.$$

Theorem 6 (Montellano et al. [7]) *For all $d > 0$ and $k > 2$ it follows that*

$$(k-1)(d+1)+1 < \vartheta(k,d) < \binom{k}{2}(d+1)+1.$$

Furthermore, for each $d > 0$ there exists a constant C_d such that for all $k \geq d+2$,

$$\vartheta(k,d) \leq C_d k \log k.$$

Question 6 *¿Is it true that $\vartheta(k,d) \in O(kd)$? ¿Can $\vartheta(k,d)$ be precisely determined? Guess: YES* and... *I do not know, I am working on that.*

Metaproblem. *Choose your favourite problem from graph theory and/or discrete geometry and generalise it to include all separoids. Now, solve it!*

For instance:
¿Does the *perfect separoids* conjecture hold?
¿Does *Erdős-Szekeres* theorem hold for separoids?
¿Do *visibility separoids* have chromatic number bounded by a function of their clique number?

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Weak Pentagon Problem
or
On a Certain 5-Edge-Coloring

Robert Šámal
Joint work with Matt J. DeVos *†

Abstract

We prove that edges of a cubic graph of girth at least 16 can be 5-colored so that the complement of any color class is bipartite (see Fig. 1). Equivalent formulation is that any such graph admits a homomorphism to the Clebsch graph (see Fig. 2). This is an approach to the Pentagon Conjecture; it also provides a coloring version of results of [1] and [5] on the size of maximal bipartite subgraph.

For a graph G let $\mathrm{MAXCUT}(G)$ be the maximum number of edges in a bipartite subgraph of G, we normalize and write

$$b(G) = \frac{\mathrm{MAXCUT}(G)}{|E(G)|} .$$

It is an easy exercise to show that $b(G) \geq 1/2$ for any graph G and $b(G) \geq 2/3$ when G is cubic, that is 3-regular graph. The former inequality is almost attained by a large complete graph, the latter is attained for $G = K_4$: any triangle contains at most two edges from any bipartite subgraph, and every edge of K_4 is in the same number of triangles. This shows triangle has a special role and rises a natural question to determine $b(G)$ for a (cubic) G that contains no triangle, or perhaps even no short circuits. In 1980's several authors independently considered this problem ([5], [1], [11], ...); the strongest results being

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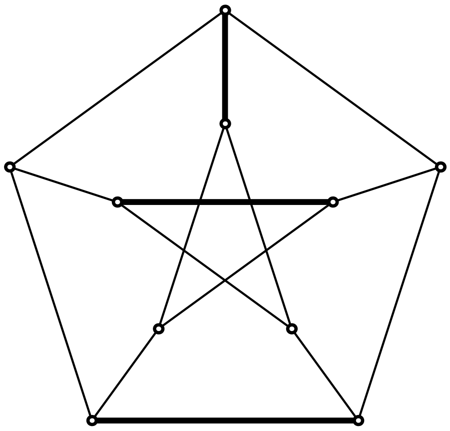

Figure 1: Petersen graph with edges partitioned into five odd-circuit blockers.

- $b(G) \geq 4/5$ for G with maximum degree 3 and no triangle ([1])
- $b(G) \geq 6/7 - o(1)$ for cubic G with girth (the length of the shortest circuit) tending to infinity [11]

On the other hand, by considering random cubic graph we can prove that there are cubic graphs of arbitrarily high girth with $b(G) < 0.999$ ([7]).

Call a set $F \subseteq E(G)$ an odd-circuit blocker (OCB) if it contains an edge from every odd circuit of G, that is if $G \setminus F$ is bipartite. The fact $b(G) \geq 4/5$ can be reformulated as

$$(\exists F \subseteq E(G)) \frac{|F|}{|E(G)|} \leq \frac{1}{5} \text{ and } F \text{ is an OCB.}$$

We prove a coloring version of this holds when G has high girth.

Theorem 1 *Let G be a graph with maximum degree 3 and girth at least 16. Then we can partition the edges of G into five odd-circuit blockers. Moreover, there is a linear-time algorithm that computes this partition.*

We remark the girth assumption is only forced by the proof, for all what we know it may be enough to suppose G is triangle-free.

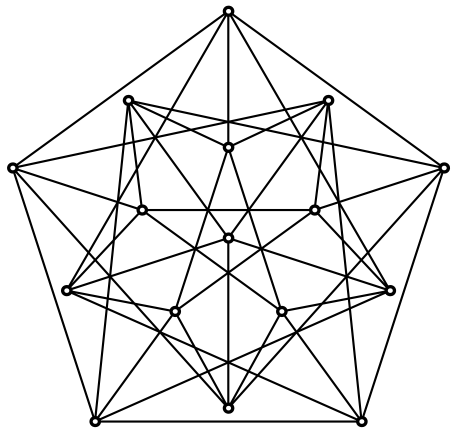

Figure 2: Clebsch graph

We conclude the Introduction by the relation of our theorem to the Nešetřil's Pentagon Conjecture ([6]). Recall that a mapping $f : V(G) \to V(H)$ is a *homomorphism* if $f(u)f(v)$ is an edge of H for any edge uv of G.

Conjecture 1 *If G is a cubic graph of sufficiently high girth then there is a homomorphism from G to C_5.*

When we replace C_5 by C_3 we get an easy consequence of Brook's theorem. It is known that the Conjecture is false if we replace C_5 by C_11 ([6]), by C_9 ([10]) and (proved recently) by C_7 ([4]). If the conjecture is true, our theorem follows immediately (possibly with a different girth assumption): If f is a homomorphism from G to C_5, color the edge $e \in E(G)$ by $f(e)$. As no odd circuit can be homomorphically mapped to a proper subgraph of C_5 we see that each color class is an OCB. On the other hand we probably can't obtain the conjecture using Theorem 1, as for example Figure 1 shows that Petersen graph has desired partition but it has no homomorphism to C_5.

Call a mapping $g : E(G) \to E(H)$ *cut-continuous* if for every cut $X \subseteq E(H)$ the set $g^{-1}(X)$ is a cut in G. This concept is introduced in [2]. In [9] the relation between statements "there is a homomorphism from G to H" and "there is a cut-continuous mapping from G to H" is studied in greater detail. Here we just show how Theorem 1 can be reformulated using this notion.

Lemma 2 *For any graph G the following are equivalent.*

1. *There is a partition of $E(G)$ into five odd-circuit blockers.*

2. *There is a cut-continuous mapping from G to C_5.*

3. *There is a homomorphism from G to the Clebsch graph (depicted in Figure 2).*

This Lemma shows that Reza ([8] using results of Guenin [3]) proves the same result from different assumption.

Theorem 3 *Any planar triangle-free graph admits a homomorphism to the Clebsch graph.*

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A new multiplicative graph

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Keywords: Categorical product, Multiplicative graphs, Hedetniemi's conjecture.
AMS 1991 Subject Classification: 05C15.

Abstract

A graph K is called *multiplicative* if whenever a categorical product of two graphs admits a homomorphism to K, then one of the factors also admits a homomorphism to K. We use right adjoints in (the skeleton of) the category of graphs to provide a new example of a multiplicative graph

1 Introduction: Multiplicativity

The *categorical product* of two graphs G and H is the graph $G \times H$ with vertex-set $V(G \times H) = V(G) \times V(H)$ and edge-set

$$E(G \times H) = \{[(u,v),(w,x)] : [u,w] \in E(G) \text{ and } [v,x] \in E(H)\}.$$

A graph K is called *multiplicative* if whenever a product $G \times H$ admits a homomorphism (that is, an edge-preserving map) to K, one of the factors G, H also admits a homomorphism to K. This concept was introduced in [2] and [3] in connection with Hedetniemi's conjecture which states that the chromatic number of a categorical product of graphs is equal to the minimum of the chromatic numbers of the factors. In terms of multiplicativity, this conjecture is equivalent to the statement that every complete graph is multiplicative.

The fact that K_2 is multiplicative is an easy consequence of König's characterisation of bipartite graphs as the graphs that do not contain odd cycles. However, as this characterisation amounts to a polynomial characterisation

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of bipartite graphs, no obvious generalisation is available for larger complete graphs. A considerable breakthrough was obtained by El-Zahar and Sauer [1] who used "topological" properties of the 3-colourings of cycles to prove that K_3 is multiplicative. Their result was generalised by Haggvist et al [2] who proved that all odd cycles are multiplicative. No other graph has been shown to be multiplicative since.

In this note we show that the family of known multiplicative graphs can be enriched by using the right adjoints in the category of graphs, which have been described by Pultr [6]. The idea of the method is apparently due to Roman Bacik and was communicated to the author by Xuding Zhu. We will limit ourselves to one example here, and discuss the generalisations and limitations of the method afterwards.

2 A new multiplicative graph

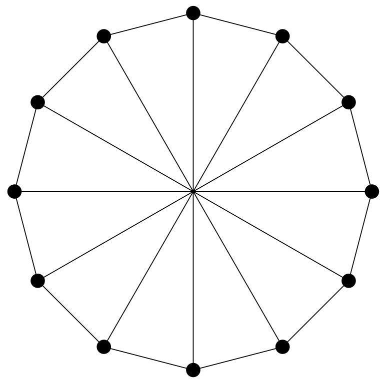

Figure 1: M

The reader can easily verify that the graph M above can be described as follows: Its vertices are the couples (A, B) of nonempty subsets of $\{0, 1, 2\}$ such that $A \cap B \neq \emptyset$, and its edges are the pairs $\{(A, B), (A', B')\}$ such that $A \cap A' = \emptyset$, $B \cap B' = \emptyset$, $A \cap B' \neq \emptyset$ and $B \cap A' \neq \emptyset$.

Next we define a construction Φ_c making a graph $\Phi_c(G)$ from an input graph G. The vertices of $\Phi_c(G)$ are the homomorphisms from K_2 to G, (where $V(K_2) = \{0, 1\}$), and the edges of $\Phi_c(G)$ are the pairs $\{f, g\}$ such that there exists a homomorphism h from the 6-cycle C_6 to G such that $f(0) = h(0), f(1) = h(1), g(0) = h(3)$ and $g(1) = h(4)$ (where $V(C_6) = \{0, 1, \ldots, 6\}$). Thus, the vertices of $\Phi_c(G)$ correspond to the arcs of G, and the edges of $\Phi_c(G)$ join opposite arcs in a closed walk of length 6 in G. The functorial definition of Φ_c has the following consequence (whose proof is a straightforward application of the definition:

Lemma 1 $\Phi_c(G \times H) = \Phi_c(G) \times \Phi_c(H)$

We also have the following:

Lemma 2 *$\Phi_c(G)$ is 3-colourable if and only if G admits a homomorphism to M.*

Sketch of proof. A finite verification shows that $\Phi_c(M)$ is 3-colourable. Hence if there is a homomorphism from G to M, then there is a homomorphism from $\Phi(G)$ to $\Phi(M)$, from which we can pull back a 3-colouring of G. Conversely, if $\psi : \Phi(G) \mapsto K_3$ is a homomorphism (where $V(K_3) = \{0, 1, 2\}$), then we can define a homomorphism $\hat{\psi} : G \mapsto M$ by putting $\hat{\psi}(u) = (A_u, B_u)$, where $A_u = \{\psi(f) : f(1) = u\}$ and $B_u = \{\psi(f) : f(0) = u\}$.

■

Corollary 3 *M is multiplicative.*

Proof. Let G, H be graphs such that there is a homomorphism from $G \times H$ to M. Then $\Phi_c(G \times H)$ is 3-colourable by the second lemma whence $\Phi_c(G) \times \Phi_c(H)$ is 3-colourable by the first lemma. Since K_3 is multiplicative by [1], $\Phi_c(G)$ or $\Phi_c(H)$ is 3-colourable, whence G or H admits a homomorphism to M by the first lemma.

■

3 Concluding comments

The method employed above can be generalized as follows: If K and M are graphs and Φ is a construction making a graph $\Phi(G)$ out of the graph G, with the following properties:

(i) $\Phi(G \times H) \simeq \Phi(G) \times \Phi(H)$,

(ii) $\Phi(G)$ admits a homomorphism to M if and only if G admits a homomorphism to K.

Then if M is multiplicative, so is K. In [7] we showed that this method is sufficient to prove the multiplicativity of the circulants used in the definition of the circular chromatic number up to (but not including) K_4. There seems to be a "topological gap" preventing the method from being useful to prove the multiplicativity of other complete graphs, so it may be worth the while to investigate semi-lattice homomorphisms from the category of graphs to other semi-lattices.

On the other hand, the arc graph construction of Poljak and Rödl is also a right adjoint. In [5] and [4], it has been used to show that if the function f is bounded above, then the bound is at most 9, where

$$f(n) = \min\{\chi(G \times H) : \chi(G) \geq \chi(H) \geq n\}.$$

Other right adjoints could perhaps be used to sharpen this bound.

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Colouring Visibility Graphs*

David R. Wood[‡§]
Joint work with Jan Kára[§] *and Attila Pór*[§]

Let $P \subseteq \mathbb{R}^2$ be a set of points in the plane. Let $\overline{vw}$ denote the closed line-segment between points $v \in \mathbb{R}^2$ and $w \in \mathbb{R}^2$. Two distinct points $v, w \in P$ are *visible* with respect to P if $P \cap \overline{vw} = \{v, w\}$. The *visibility graph* $\mathcal{V}(P)$ of P has vertex set P, with an edge between every pair of visible points in P. This paper studies the following open problem.

Open Problem 1 *Are visibility graphs χ-bounded? That is, is there a function f such that $\chi(\mathcal{V}(P)) \leq f(\omega(\mathcal{V}(P)))$ for every finite point set P?*

As an example, consider the integer lattice $P = \{(x, y) : x, y \in \mathbb{Z}\}$, as illustrated in Figure 1. Let $f((x, y)) = (x \bmod 2, y \bmod 2)$ for all $(x, y) \in P$. For any two monochromatic points (x_1, y_1) and (x_2, y_2) in P, both $|x_1 - x_2|$ and $|y_1 - y_2|$ are even. Thus the midpoint of the segment $\overline{(x_1, y_1)(x_2, y_2)}$ is in P, and (x_1, y_1) and (x_2, y_2) are not visible. Hence f is a 4-colouring of $\mathcal{V}(P)$. There is no 3-colouring since $\{(0, 0), (1, 0), (1, 1), (0, 1)\}$ is a 4-clique. Therefore $\chi(\mathcal{V}(P)) = 4$.

The bipartite visibility graphs are easily characterised as follows.

Theorem 1 *Let P be a finite point set. Then the following are equivalent:*
(a) $\chi(\mathcal{V}(P)) \leq 2$,
(b) *all the points in P are collinear,*
(c) $\mathcal{V}(P)$ *has no K_3 subgraph.*

Eppstein [2] characterised the planar visibility graphs as follows, as illustrated in Figure 2.

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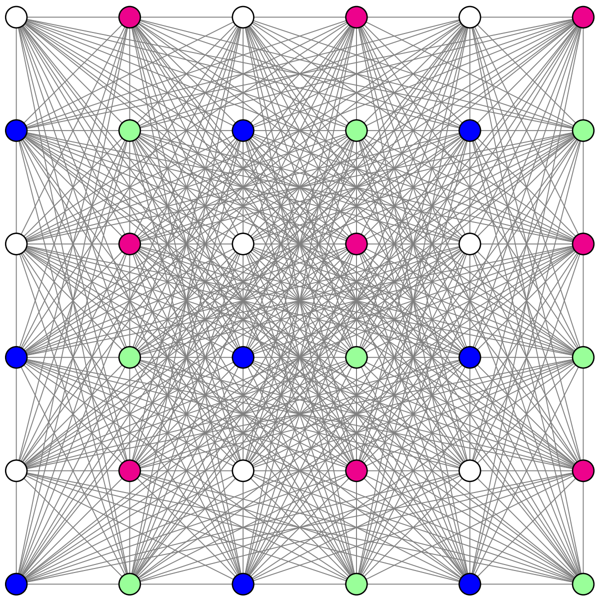

Figure 1: 4-colouring of the visibility graph of the integer lattice.

Lemma 2 ([2]) *Let P be a point set. Then $\mathcal{V}(P)$ is planar if and only if at least one of the following conditions hold:*
(a) *all the points in P are collinear,*
(b) *all the points in P, except for one, are collinear,*
(c) *all the points in P are collinear, except for two non-visible points,*
(d) *all the points in P are collinear, except for two points $v, w \in P$, such that the line-segment $\overline{vw}$ does not intersect the line-segment that contains $P \setminus \{v, w\}$,*
(e) *$\mathcal{V}(P)$ is the drawing of the octahedron shown in Figure 2(e).*

Lemma 2 and a result of Develin *et al.* [1] are the key ingredients to the proof of the following characterisation of 3-colourable visibility graphs.

Theorem 3 *Let P be a finite point set. Then the following are equivalent:*
(*i*) $\chi(\mathcal{V}(P)) \leq 3$,
(*ii*) *P satisfies conditions* (a), (b), (c) *or* (e) *in Lemma 2,*
(*iii*) *$\mathcal{V}(P)$ has no K_4 subgraph.*

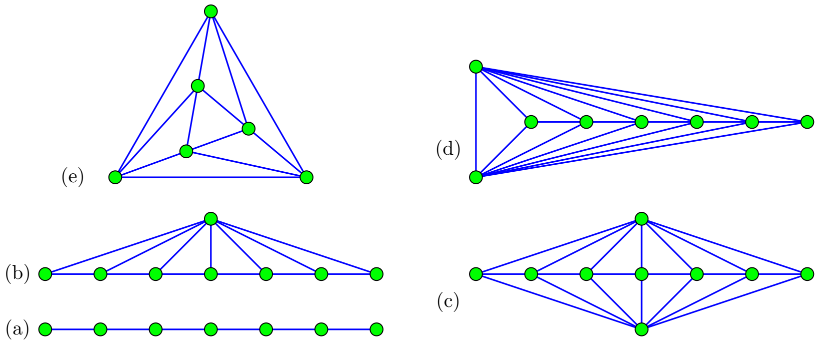

Figure 2: The planar visibility graphs.

Our main result is the following super-polynomial lower bound on the chromatic number of a visibility graph.

Theorem 4 *There are constants $c_1, c_2, c_3, c_4 > 0$ and an infinite sequence of visibility graphs $G_0, G_1, G_2, \ldots$, such that $\omega(G_i) \to \infty$ and*

$$\chi(G_i) \;\geq\; (c_1 \log \omega(G_i))^{c_2 \log \omega(G_i)} \;=\; (c_3\, \omega(G_i))^{c_4 + \log\log \omega(G_i)} \;.$$

The next interesting case in Open Problem 1 is $\omega(\mathcal{V}(P)) = 4$. There is a visibility graph $\mathcal{V}(P)$ with $\omega(\mathcal{V}(P)) = 4$ and $\chi(\mathcal{V}(P)) = 5$. It is an open problem whether every visibility graph with $\omega(\mathcal{V}(P)) \leq 4$ has $\chi(\mathcal{V}(P)) \leq 5$.

Acknowledgements

Thanks to Ricardo Strausz for stimulating discussions.

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A simple proof of the circular chromatic number of series parallel graphs

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Abstract

This paper gives a simple proof of the result of Z. Pan and X. Zhu [2], that gives an upper bound for $\chi_c(G)$ of a series-parallel graph G.

1 Introduction

For two integers $k \geq d \geq 1$, a (k,d)-*coloring* of a graph G is a coloring c of the vertices of G with colors $0, 1, 2, \ldots, k-1$ such that $d \leq |c(x) - c(y)| \leq k - d$, whenever $\{x, y\}$ is an edge of G. The infimum of the ratio k/d for which there exists a (k,d)-coloring is denoted by $\chi_c(G)$ and we say $\chi_c(G)$ is the *circular chromatic number* of G.

Finding a tight upper bound for $\chi_c(G)$, in a class of K_n-minor-free graphs, $(\mathcal{G}/K_n)$, is a difficult problem even for small n. The case $n \geq 5$ is unsolved. For planar graphs, to date the best known upper bound is given by Zhu, [5], but we are far from the conjectured upper bound. Pan and Zhu have settled this problem for $\mathcal{G}/K_4$ (the class of *series-parallel-graphs*) in [2] and proved in [3] that their bound is indeed the best possible by constructing a

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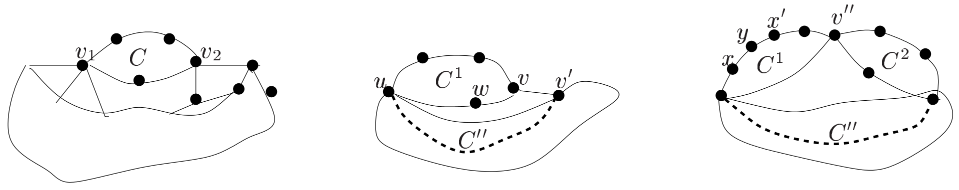

Figure 1: Example of an eye C (left), a 3-lens v (middle) and a 4-lens v''(right). The labels C'', w, v', x, x', y are used in proof of Lemma 3.

graph G, with $\chi_c(G)$ arbitrarily close to the given bounds. The following is the main result of [2], for which we give a new proof. We rewrite their theorem in this way for convenience in our proof.

Theorem 1 (Pan, Zhu) *Suppose* $r \in \{-1, 1, 3\}$ *and* $G \in \mathcal{G}/K_4$ *has odd-girth* g. *If* $g \geq 6k + r$, *then* $\chi_c(G) \leq 2g'/(g' - 1)$, *where* $g' = 4k + (r + |r|)/2$.

The results by Pan and Zhu are based on the so-called *labelling method* and a clever construction. Although the labelling method has been quite useful in several proofs, it leads to long case analysis and calculations, and so resulted omitting of some technical details. In this note, we give a short structural proof of Theorem 1.

We assume graphs are finite and simple. Let G, G' be graphs. A *homomorphism* from G to G' is a mapping $f{:}V(G) \to V(G')$ which preserves adjacency, (i.e., $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(G')$). The notation $G \to G'$ means there is a homomorphism from G to G'. Note that "$\to$" is a reflexive and transitive relation.
Let G be a graph. If $C \subseteq G$, is an induced cycle in G then (C, v_1, v_2) is *an eye* of G, if C has exactly two vertices v_1 and v_2 of degree at least 3. Let (C^1, v, u) be an eye of G. Then v is a *3-lens* if degree$(v) = 3$ in G. If degree$(v) = 4$ and $v \in V(C^1) \cap V(C^2)$, for some eye $(C^2, v, u'), u \neq u'$, then we call v a *4-lens* of G. (See Figure 1). We say v is a *lens* when its degree is irrelevant. A simple graph G is *hom-simple* if every eye of G is an odd cycle.

Lemma 2 *Every 2-connected hom-simple graph* $G \in \mathcal{G}/K_4$ *which is not a cycle contains a lens.*

Proof: Let $G \in \mathcal{G}/K_4$ be 2-connected. Let $\bar{G} \in \mathcal{G}/K_4$ be the graph we obtain from G by replacing every path of degree 2 vertices by an edge, and by identifying resulting parallel edges. Since G is hom-simple and not a cycle, we have $|V(\bar{G})| > 2$, and that $\bar{G}$ is simple and 2-connected. Note that $\bar{G} \in \mathcal{G}/K_4$ and so a degree 2 vertex $\bar{v}$ in $\bar{G}$ exists and $\bar{v}$ is a lens in G, because G is hom-simple. $\square$

The following is a key lemma which we use in the next section.

Lemma 3 (Folding lemma for $\mathcal{G}/K_4$) *Let $G \in \mathcal{G}/K_4$ be a 2-connected hom-simple graph of odd-girth$(G) = g > 3$. If G has a 3-lens v or if G has eyes (C^1, v, u) and (C^2, v, u'), such that v is a 4-lens and $|C^1| > g$ or $|C^2| > g$, then there exists a graph $G' \in \mathcal{G}/K_4$ of same odd-girth, and a 4-lens $z \in V(G')$, with eyes $(C_1', u', z), (C_2', z, w')$, such that $G \to G'$, $|V(G')| < |V(G)|$ and $|C_1'| = |C_2'| = g$.*

Proof: Let G be contrary to the lemma, where $|V(G)|$ is minimal. Let v be a lens of an eye (C^1, v, u) of G. If v is a 3-lens as in Figure 1 middle, let H be obtained by identifying v' with a neighbor, w of $v, w \neq u$. If v is a 4-lens as in Figure 1, right, and $|C^1| > g > 3$, let H be obtained by identifying vertices x and x' of C^1, that have a common degree 2 neighbor y and identifying y with a neighbor of x or x'. We see that H remains 2-connected and hom-simple. Moreover, H has odd girth g, for otherwise some cycle C'', where $E(C'') \cap E(C^1) \neq \emptyset$ has length g (See Figure 1). But this implies the edges of $E(C^1) \backslash E(C'')$ can be identified with $E(C'')$, contrary to G being minimal. Hence, in both cases we have $G \to H$ and $|H| < |G|$. The induction assumption and transitivity of "$\to$" obtains a contradiction and the result follows. $\square$

2 Proof of Theorem 1

Let $G \in \mathcal{G}/K_4$ of odd girth g be a counterexample to Theorem 1, with $|V(G)|$ as small as possible. For integers $q \geq p \geq 1$, let G_q^p denote the graph with vertex set $\{0, 1, \ldots, q-1\}$ in which i is adjacent to j when $p \leq |i - j| \leq q - p$. To prove $\chi_c(G) \leq p/q$, it suffices to show $G \to G_q^p$. We prove that if $r \neq -1$ then $G \to G_{4k+r}^{2k+(r-1)/2} = C_{4k+r}$ and if $r = -1$, $G \to G_{8k}^{4k-1} = V_{8k}$, which contradicts the choice of G.

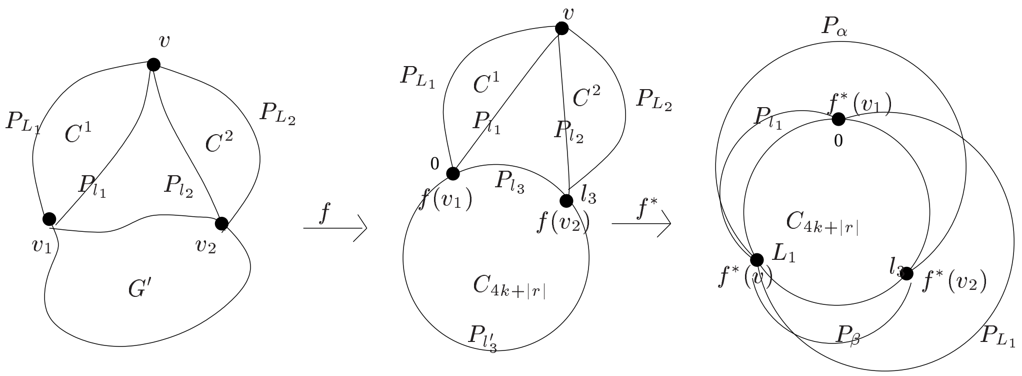

Figure 2: Left: G with its 4-lens v and eyes C^1 and C^2. The label P_n denotes a path of length $n \geq 1$. In the right, $\{P_\alpha, P_\beta\} = \{P_{L_2}, P_{l_2}\}$ such that $L_1 - l_3 = \beta$ modulo 2.

It is easy to see that G must be 2-connected, because G_q^p is vertex-transitive and so inductively a homomorphism $f_i : H_i \to G_q^p$ for each 2-connected component $H_i, i = 1, 2, \dots, m \geq 2$, can be extended to $f : G \to G_q^p$, a contradiction. By Lemma 3, G has odd-girth $g = 6k + r$, $r \in \{-1, 1, 3\}$, and a 4-lens v, such that $|C^1| = |C^2| = g$ (i.e. in Figure 2, $L_i + l_i = 6k + r$, $i = 1, 2$).

Let G' be obtained by deleting C^1 and C^2. Then by induction, $f : V(G') \to G_q^p$ exists. Note that if $G_q^p = V_{8k}$, then $f(v_1)$ and $f(v_2)$ can be found on some $C_{4k+1} \subseteq V_{8k}$ (See Figure 3, left). We may assume $f(v_1) = 0$ and $f(v_2) = l_3$, $l_3 + l_3' = 4k + |r|$, $l_3 < l_3'$ and that $L_1 \geq L_2 > l_2$. Then $l_2 \geq l_1$. We may also assume that $L_1 < 4k + (r + |r|)/2$, for otherwise $G \backslash P_{L_1} \to C_{4k+|r|}$ can be extended to $G \to C_{4k+|r|}$, and we are done. Then, we have $l_1 > 2k$, if $r \neq -1$ and $l_1 \geq 2k$, if $r = -1$. In addition, we can assume $l_3 \geq 2$, for if $0 \leq l_3 \leq 1$, we clearly have $G \to C_{4k+|r|}$. It follows, $l_1 + l_2 > l_3'$.

Let $\{\alpha, \beta\} = \{L_2, l_2\}$ such that $L_1 \cong \beta + l_3$ modulo 2. Then $\beta > L_1 - l_3$, for otherwise we have $L_1 \geq \beta + l_3$ and $\alpha \geq l_1 + l_3$. Hence, in the middle graph in Figure 2, we may identify P_{L_1} with $P_\beta \cup P_{l_3}$ and P_α with $P_{l_1} \cup P_{l_3}$. Since $l_1 + \beta \geq l_1 +, \dots, _2 > l_3'$, we get $G \to C_{4k+|r|}$. We now extend f by f^* as follows: If $r = -1$, $L_1 = L_2 = 3k$, and $l_3 = 2k$, map G to V_{8k} by letting $f^*(v) = 5k$. (See Figure 3, right). Otherwise let $f^*(v) = L_1$,

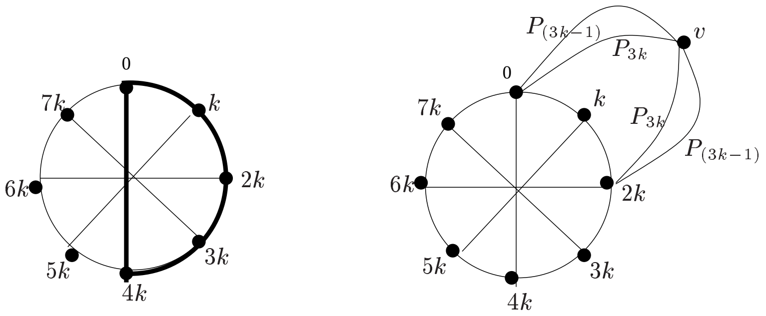

Figure 3: Left: A C_{4k+1} subgraph in V_{8k} is highlighted. Right: Case $r = -1$, $L_1 = L_2 = 3k$. By symmetry, $f^*(v) = 5k$ is a unique solution for $G \to V_{8k}$.

(Figure 2, right) and we verify f^* is a map to $C_{4k+|r|}$ by showing each path: $P_{L_1}, P_{l_1}, P_\beta$ and P_α depicted in Figure 2, can be identified with their corresponding subpaths of $C_{4k+|r|}$. For P_{L_1}, we have $L_1 < 4k + |r|$, because $L_1 < 4k + (r + |r|)/2 \leq 4k + |r|$. Note also that $L_1 > l_3$ since $l_3 < l_3'$ and $l_3 + l_3' \leq L_1 + l_1$. For P_{l_1}, we have $l_1 \geq 4k + |r| - L_1$, because $L_1 + l_1 = 6k + r$. For P_β, we have $\beta > L_1 - l_3$, as shown above. For P_α we show, $\alpha \geq l_3 + (4k + |r| - L_1)$. Substituting $\alpha + \beta = 6k + r$ and rearranging we shall verify:

$$L_1 - \beta \geq (l_3 - 2k) + (|r| - r) \qquad (*)$$

Note that $L_1 - \beta \geq 0$. If $r \neq -1$, then $|r| - r = 0$, and so if $2k \geq l_3$, we are done. Otherwise, $l_3 = 2k + 1$, then $L_1 \not\equiv \beta$ modulo 2, (i.e. $L_1 > \beta$), which implies (*) holds. Next, let $r = -1$, (i.e. $|r| - r = 2$ and $l_3 \leq 2k$). By assumption if $l_3 = 2k$, then $L_1 \neq \beta$ and so $L_1 - \beta = 2t, t \geq 1$ and if $l_3 < 2k$, we see once more (*) holds.
Note that the case $l_3 \geq 2k + 1$ is symmetric, since $l_3' = 4k + 1 - l_3 \leq 2k$. This concludes the proof that no counterexample exists to Theorem 1.

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