The circular chromatic index of flower snarks

Mohammad Ghebleh* Daniel Král'†‡ Serguei Norine§
Robin Thomas¶

Abstract

We determine the circular chromatic index of flower snarks, by showing that $\chi'_c(F_3) = 7/2$, $\chi'_c(F_5) = 17/5$ and $\chi'_c(F_k) = 10/3$ for every odd integer $k \geq 7$, where F_k denotes the flower snark on $4k$ vertices.

1 Introduction

All graphs in this paper are finite and simple. A graph is *k-edge-colorable* if its edges can be colored using k colors in such a way that no two adjacent edges receive the same color. By a classical theorem of Vizing [11] every cubic graph is 4-edge-colorable, and hence cubic graphs fall into two categories: those that are 3-edge-colorable, and those that require four colors. Those of the latter kind that satisfy a mild connectivity requirement (cyclic 4-edge-connectivity) are called *snarks*. Snarks are of great interest [2, 4, 5, 12] because the non-existence of planar snarks is equivalent to

*Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, B.C., V5A 1S6, Canada. E-mail: mghebleh@sfu.ca.

†Institute for Mathematics, Technical University Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany. E-mail: kral@math.tu-berlin.de. The author is a postdoctoral fellow at TU Berlin within the framework of the European training network COMBSTRU.

‡Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: kral@kam.mff.cuni.cz.

§School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA. E-mail: snorine@math.gatech.edu. Partially supported by NSF Grant No. 0200595.

¶School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA. E-mail: thomas@math.gatech.edu. Partially supported by NSF Grants No. 0200595 and 0354742.

the Four Color Theorem [9], and it is known that a minimal counterexample to several important conjectures must be a snark, see e.g. [5, 13].

Are some snarks closer to being 3-edge-colorable than others? This question can be made precise using the concept of circular coloring, introduced by Vince [10] under the name of star coloring. For $r > 0$, an *r-circular edge-coloring* of a graph G is a mapping $c : E(G) \to [0, r)$ such that $1 \leq |c(e) - c(f)| \leq r - 1$ for every two adjacent edges e and f of G. The *circular chromatic index* of G is the infimum (in fact, the minimum) of all $r > 0$ such that G has an r-circular edge-coloring. The circular chromatic index of G is denoted by $\chi'_c(G)$. It is not hard to show that the *chromatic index* of G (the least k such that G is k-edge-colorable) is equal to $\lceil \chi'_c(G) \rceil$, and hence the circular chromatic number provides a finer measure of edge-colorability than the chromatic index. We refer the reader to the survey [14] for more details on circular colorings of graphs.

Zhu [14] asked whether there exists a snark with circular chromatic index four. Afshani et al. [1] answered the question in the negative by showing that the circular chromatic index of every bridgeless cubic graph is at most 11/3. The bound is tight as witnessed by the Petersen graph. However, the Petersen graph is the only known bridgeless cubic graph with circular chromatic index equal to 11/3, and so it seems natural to look for other examples among known families of snarks.

There is another result on circular edge colorings that motivated our work. Kochol [8] disproved the conjecture of Jaeger and Swart [3] that the girth of every snark is bounded by an absolute constant by constructing snarks of arbitrarily high girth. However, the conjecture holds in an approximate sense when relaxed to circular colorings: Kaiser et al. [6] proved that for every $\varepsilon > 0$ there exists an integer g such that every cubic bridgeless graph of girth at least g has circular chromatic index at most $3 + \varepsilon$. This result was extended in [7] to graphs with arbitrary maximum degree.

It is natural to ask whether perhaps the same conclusion (at least for cubic graphs) holds under the weaker assumption that the graph have *odd girth* at least g, i.e., that the graph have no odd cycle of length strictly less than g. We show that this is not the case by proving that the family of snarks known as flower snarks provide a counterexample. We were actually able to compute the circular chromatic index of flower snarks exactly. Let us recall that for an odd integer $k \geq 3$ the flower snark F_k, is the following graph [4]: the vertex set of F_k consists of $4(k)$ vertices $v_1, \ldots, v_k$ and $u_1^1, u_1^2, u_1^3, \ldots, u_k^1, u_k^2, u_k^3$. The graph is comprised of a cycle $u_1^1 \cdots u_k^1$ of length k and a cycle $u_1^2 \cdots u_k^2 u_1^3 \cdots u_k^3$ of length $2(k)$, and in addition, each vertex v_i is adjacent to u_i^1, u_i^2 and u_i^3.

2 General bound

Let $\varepsilon > 0$ and set $r = 10/3 - \varepsilon$. We show that no flower snark has an r-circular edge-coloring. If $b - a < r + a - b$, then we say that b *follows* a; if $b - a > r + a - b$, then a *follows* b. Note that $\rho(x, y)$ is the distance between x and y on a circle of perimeter r.

The elements of $[0, r)$ are referred to as *colors*. For $0 \le a \le b \le r$, define $\rho(a, b) = \rho(b, a)$ to be $\min\{b - a, r + a - b\}$. Two colors x and y are *close* if $\rho(x, y) < 2/3$ and they are *far apart* if $\rho(x, y) > 2/3$. A sequence (c_0, c_1, c_2) of colors is of *type A* if c_0, c_1 and c_2 are pairwise far apart, and it is of *type B* if two of the colors are close and the remaining one is far apart from both of the other two. A sequence (c_0, c_1, c_2) of type A has *positive sign* if $0 \le c_i \le c_{i+1} \le c_{i+2} < r$ for some $i = 0, 1, 2$, where index arithmetic is taken modulo 3, and it has *negative sign* otherwise. In other words, the sequence (c_0, c_1, c_2) has positive sign if it can be obtained from the sorted sequence comprised of c_0, c_1 and c_2 by an even number of transpositions. We now define signs for sequences of type B. Let (c_0, c_1, c_2) be a sequence of colors of type B and let i, j, k be such that $\{i, j, k\} = \{0, 1, 2\}$ and c_i and c_j are close. The sequence (c_0, c_1, c_2) has *positive sign* if the color c_k follows both c_i and c_j, and we say that it has *negative sign* if c_i and c_j both follow the color c_k. Note that for sequences of type B the sign need not be defined and in case it is defined, it does not depend on the order of the elements in the sequence.

Similarly as in the original proof that flower snarks are not 3-edge-colored, a certain parity argument is also involved in our proof. The following lemma captures this:

Lemma 1. *Let c be an r-circular edge-coloring of a cubic graph G for $r = 10/3 - \varepsilon$ with $\varepsilon > 0$. Let v be a vertex of G, u_1, u_2 and u_3 be its neighbors, and e_i and f_i edges incident with u_i but not with v (for $i \in \{1, 2, 3\}$). If all the edges e_1, e_2, e_3, f_1, f_2 and f_3 are distinct, then the following holds: either $(c(e_1), c(e_2), c(e_3))$ and $(c(f_1), c(f_2), c(f_3))$ are both of type A and have the same sign or the two sequences are both of type B and have different signs (in particular, the signs of both of them are defined).*

Proof. For every color a let $I(a) = \{a + x : 1 \le x \le 4/3 - \epsilon\}$ and $J(a) = \{a - x : 1 \le x \le 4/3 - \epsilon\}$, where addition and subtraction is modulo r. Note that if the edges e, e' and e'' are distinct and share a vertex, then exactly one of $c(e')$ and $c(e'')$ belongs to $I(c(e))$ and the other belongs to $J(c(e))$.

Let $c_i = c(vu_i)$, $a_i = c(e_i)$ and $b_i = c(f_i)$ for $i = 1, 2, 3$. Since the edges vu_1, vu_2 and vu_3 share a vertex, the sequence (c_1, c_2, c_3) is of type A. By symmetry, we may assume that it has positive sign, i.e., $c_1 \in I(c_3) \cap J(c_2)$, $c_2 \in I(c_1) \cap J(c_3)$, and $c_3 \in I(c_2) \cap J(c_1)$. Hence, any two colors in $I(c_3) \cup J(c_2)$ are close, and the same holds for $I(c_1) \cup J(c_3)$ and $I(c_2) \cup J(c_1)$. By reversing the roles of I and J, we may assume that $a_1 \in I(c_1)$ and $b_1 \in J(c_1)$. Assume first that a_1, a_2 and a_3 are pairwise far apart. Then, $a_3 \in I(c_3)$ (because $a_3 \notin J(c_3)$ since it is far apart from a_1), and similarly $a_2 \in I(c_2)$. Consequently, $b_3 \in J(c_3)$ and $b_2 \in J(c_2)$. The circular intervals $I(c_1)$, $I(c_2)$ and $I(c_3)$ are pairwise at distance at least $2/3$, and the same holds for $J(c_1)$, $J(c_2)$ and $J(c_3)$. It follows that (a_1, a_2, a_3) has positive sign, and that (b_1, b_2, b_3) is of type A and it also has positive sign.

The other case to consider is that two of a_1, a_2 and a_3 are close. By the symmetry, we may assume that a_2 and a_3 are close, i.e., $a_2 \in J(c_2)$ and $a_3 \in I(c_3)$. Hence, $b_2 \in I(c_2)$ and $b_3 \in J(c_3)$. We have that every member of $I(c_1)$ follows every member of $I(c_3) \cup J(c_2)$, because c_1 is at distance one from one end of the circular interval $I(c_1)$, and each of the circular intervals involved has length $1/3 - \epsilon$. Thus a_1 follows a_2 and a_3. Since $b_1, b_2 \in I(c_2) \cup J(c_1)$, we deduce that b_1 and b_2 are close, and it follows similarly as above that both b_1 and b_2 follow b_3, as desired. We conclude that both (a_1, a_2, a_3) and (b_1, b_2, b_3) are of type B and have different signs. □

We are now ready to deduce the lower bound on the circular chromatic indices of flower snarks:

Theorem 1. *For every $t \geq 1$, the circular chromatic index of the flower snark F_{2t+1} is at least $10/3$. Moreover, if $t \geq 3$, then $\chi'_c(F_{2t+1}) = 10/3$.*

Proof. Let $t \geq 1$ and suppose for a contradiction that c is an r-circular edge-coloring of F_{2t+1} with $r < 10/3$. We repeatedly apply Lemma 1 with $v = v_i$, $e_1 = u_i^1 u_{i-1}^1$, $e_2 = u_i^2 u_{i-1}^2$, $e_3 = u_i^3 u_{i-1}^3$, $f_1 = u_i^1 u_{i+1}^1$, $f_2 = u_i^2 u_{i+1}^2$ and $f_3 = u_i^3 u_{i+1}^3$ for $i = 1, \ldots, 2t+1$ (we set $u_0^1 = u_{2t+1}^1$, $u_{2t+2}^1 = u_1^1$, $u_0^2 = u_{2t+1}^3$, $u_{2t+2}^2 = u_1^3$, $u_0^3 = u_{2t+1}^2$ and $u_{2t+2}^3 = u_1^2$ where appropriate). In this way, we conclude that either the sequences $(c(u_{2t+1}^1 u_1^1), c(u_{2t+1}^3 u_1^2), c(u_{2t+1}^2 u_1^3))$ and $(c(u_{2t+1}^1 u_1^1), c(u_{2t+1}^2 u_1^3), c(u_{2t+1}^3 u_1^2))$ are both of type A and have the same sign (which is impossible since they differ by a single transposition) or they are both of type B and have different signs (which is again impossible by the definition of the sign in this case). This establishes that $\chi'_c(F_{2t+1}) \geq 10/3$.

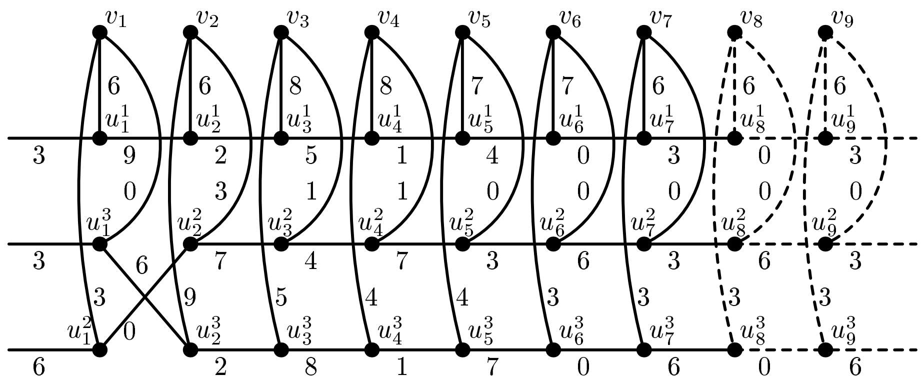

Figure 1: The construction of a 10/3-circular edge-coloring of large flower snarks. The colors of the edges are multiplied by three in the figure.

Figure 1 shows the construction of a 10/3-circular edge-coloring of F_{2t+1} for $t \geq 3$. The edges of the snark F_{2t+1} incident with the vertices $v_1, \ldots, v_7$, $u_1^1, \ldots, u_7^1$, $u_1^2, \ldots, u_7^2$ and $u_1^3, \ldots, u_7^3$ are always colored as in the figure. The remaining edges of F_{2t+1} are colored using the pattern shown on the dashed edges. Note that the colors in the figure are multiplied by three for better clarity. □

3 The flower snark F_3

An easy consequence of Lemma 1.3 of [14] is the following:

Proposition 1. *Let G be a graph. If $\chi'_c(G) = p/q$ (where p and q are relatively prime), then there exists a p/q-circular edge-coloring of G with colors $0/q, \ldots, (p-1)/q$ only and each of these colors is assigned to at least one edge of G.*

Since the edges colored with the colors $0/q, \ldots, (q-1)/q$ must form a matching in G, we obtain the following from Proposition 1:

Proposition 2. *Let G be a graph. If $\chi'_c(G) = p/q$ (where p and q are relatively prime), then q is at most the cardinality of a maximum matching of G.*

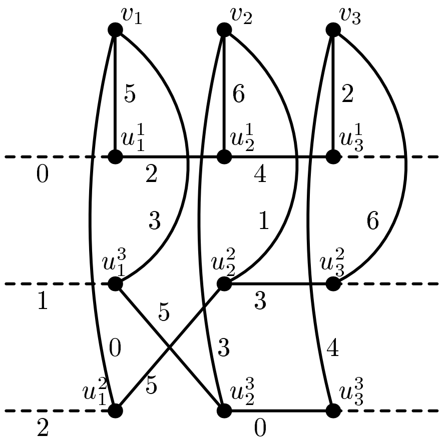

Figure 2: A 7/2-circular edge-coloring of F_3. The dashed edges wrap "around" the figure. The colors of all the edges are multiplied by two.

We are now ready to determine the circular chromatic index of F_3:

Theorem 2. *The circular chromatic index of F_3 is $7/2$.*

Proof. A 7/2-circular edge-coloring of F_3 can be found in Figure 2. By Theorem 1, the circular chromatic index of F_3 is at least 10/3. By Proposition 2, $\chi'_c(F_3) \in \{10/3, 17/5, 7/2\}$.

Assume that $\chi'_c(F_3) = 10/3$. By Proposition 1, there exists a 10/3-circular edge-coloring c of F_3 which is onto the set $\{0/3, \ldots, 9/3\}$. By the pigeon-hole principle, at least one of the colors is assigned to a single edge of F_3 (the size of F_3 is 18). Assume that $|c^{-1}(9/3)| = 1$. Observe that all the three sets $c^{-1}(\{0/3, 1/3, 2/3\})$, $c^{-1}(\{3/3, 4/3, 5/3\})$ and $c^{-1}(\{6/3, 7/3, 8/3\})$ of edges form matchings in F_3 and at least two of them are matchings of size six, i.e., perfect matchings. However, F_3 does not contain two disjoint perfect matchings since it is not 3-edge-colorable.

Assume that $\chi'_c(F_3) = 17/5$. By Proposition 1, there exists a 17/5-circular edge-coloring c of F_3 which maps onto the set $\{0/5, \ldots, 16/5\}$. Observe that each of the colors is assigned to a single edge of F_3 except for one of the colors which is assigned to two edges of F_3. Assume that the exceptional color is 1/5. Both the sets $c^{-1}(\{0/5, 1/5, 2/5, 3/5, 4/5\})$ and $c^{-1}(\{1/5, 2/5, 3/5, 4/5, 5/5\})$ of edges of F_3 are perfect matchings, say

M_1 and M_2. By their choice, $|M_1 \cap M_2| = 5$. However, F_3 contains no two perfect matchings that differ at a single edge (in fact, no simple graph does). $\square$

4 The flower snark F_5

A construction of 17/5-circular edge coloring of F_5 depicted in Figure 3 shows that $\chi'_c(F_5) \leq 17/5$. We were not able to provide the matching lower bound without the assistance of a computer. By Theorem 1 and Proposition 2, it is enough to exclude the cases that $\chi'_c(F_5) = 10/3$ or $\chi'_c(F_5) = 27/8$. A brute force algorithm for finding such an edge-coloring will be too slow. Therefore, we designed a faster algorithm for verifying the existence of p/q-circular edge-coloring of F_{2t+1} based on the following idea: first, we construct an auxiliary graph of order p^3. The vertices of this graph are all the sequences of colors of length three. Two such sequences (a_1, a_2, a_3) and (a'_1, a'_2, a'_3) are joined by an edge if it is possible to extend the partial coloring $c(u^i_1 u^i_2) = a_i$ and $c(u^i_2 u^i_3) = a'_i$ to the three edges incident with the vertex v_2. It is not hard to observe that F_{2t+1} has a p/q-circular coloring if and only if the auxiliary graph contains a walk of length $2t+1$ from a vertex (a_1, a_2, a_3) to a vertex (a_1, a_3, a_2) for some choice of $a_1, a_2, a_3 \in \{0/q, \ldots, (p-1)/q\}$. Once the auxiliary graph is constructed (which may be done quite fast even if the brute force algorithm for determining the adjacency of its vertices is used), the existence of the walk can be decided in time linear in the size of the auxiliary graph. In this way, we verified that F_5 has neither 10/3-circular nor 27/8-circular edge-coloring. Based on the discussion of the previous two sections, we conclude:

Theorem 3. *The following holds for every $t \geq 1$:*

$$\chi'_c(F_{2t+1}) = \begin{cases} 7/2 & \text{if } t = 1, \\ 17/5 & \text{if } t = 2, \text{ and} \\ 10/3 & \text{otherwise.} \end{cases}$$

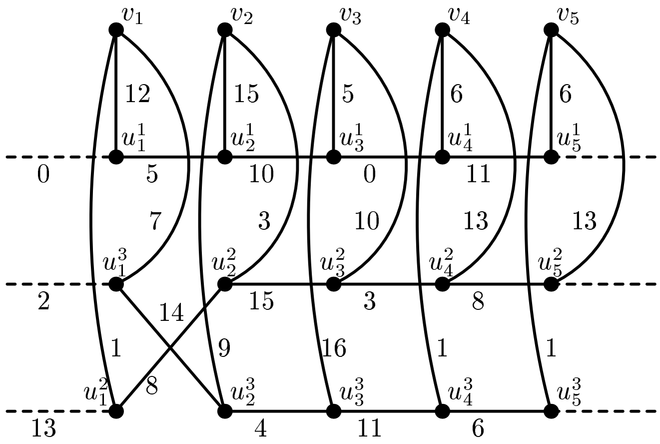

Figure 3: An 17/5-circular edge-coloring of F_5. The dashed edges wrap "around" the figure. The colors of all the edges are multiplied by five.

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