# Coloring squares of planar graphs with no short cycles

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**Abstract**

Wang and Lih conjectured that for every $g \geq 5$, there exists a number $M(g)$ such that the chromatic number of the square of every planar graph of girth at least $g$ and maximum degree $\Delta \geq M(g)$ is $\Delta + 1$. We disprove the conjecture for $g \in \{5, 6\}$ and prove the existence of the number $M(g)$ for $g \geq 7$. More generally, we show that every planar graph of girth at least 7 and maximum degree $\Delta \geq 190 + 2\lceil p/q \rceil$ has an $L(p, q)$-labeling of span at most $2p + q\Delta - 2$. For $q = 1$, the bound is tight for all pairs of $\Delta$ and $p$. We also show that the square of every planar graph of girth at least six and sufficiently large maximum degree $\Delta$ is $(\Delta + 2)$-colorable.

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# 1 Introduction

We study colorings of squares of planar graphs with no short cycles. The *square* $G^2$ of a graph $G$ is the graph with the same vertex set in which two vertices are joined by an edge if their distance in $G$ is at most two. The chromatic number of the square of a graph $G$ is between $\Delta+1$ and $\Delta^2+1$ where $\Delta$ is the maximum degree of $G$. However, it is not hard to infer from Brooks' theorem that there are only finitely many connected graphs for which the upper bound is attained. On the other hand, the chromatic number of the square of a planar graph is bounded by a function linear in the maximum degree (note that this does not follow directly from the 5-degeneracy of planar graphs [14]). In this paper, we show that if the girth (the length of the shortest cycle) of a planar graph is at least seven and its maximum degree is sufficiently large (at least 192), then the chromatic number of its square is the lowest possible, i.e., $\Delta+1$. This yields a proof of Conjecture 2 by Wang and Lih [31] for $g \geq 7$ (the conjecture is stated below).

Let us briefly survey the rich history of coloring of the squares of planar graphs. Wegner [32] proved that the squares of cubic planar graphs are 8-colorable. He conjectured that his bound can be improved:

**Conjecture 1 (Wegner 1977).** *Let $G$ be a planar graph with maximum degree $\Delta$. The chromatic number of $G^2$ is at most 7, if $\Delta = 3$, at most $\Delta+5$, if $4 \leq \Delta \leq 7$, and $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$, otherwise.*

If Conjecture 1 were true, the bounds would be the best possible. The reader is welcome to see Section 2.18 in [17] for more details. Though Conjecture 1 has been verified for several special classes of planar graphs, including outerplanar graphs [24], it remains open for all values of $\Delta$. However, there is a series of partial results. The following upper bounds on the chromatic number of the square of a planar graph with maximum degree $\Delta$ have been established: $8\Delta-22$ by Jonas [18], $3\Delta+5$ by Wong [33], $3\Delta+9$ for $\Delta \geq 8$ by Jendrol' and Skupien [15], $2\Delta+18$ for $\Delta \geq 12$ by Madaras and Marcionová [25], $2\Delta+25$ by van den Heuvel and McGuiness [14], $\lceil 9\Delta/5 \rceil+2$ for $\Delta \geq 749$ by Agnarsson and Halldórsson [2, 3], and $\lceil 9\Delta/5 \rceil+1$ for $\Delta \geq 47$ by Borodin, Broersma, Glebow and van den Heuvel [6]. The best known upper bounds are due to Molloy and Salavatipour [27, 28]: $\lceil 5\Delta/3 \rceil+78$ for all $\Delta$ and $\lceil 5\Delta/3 \rceil+25$ for $\Delta \geq 241$. Some of the above results were obtained by identifying so-called light structures in planar graphs—the reader is welcome to see the survey [16]. Coloring of higher powers of planar graphs was addressed by Agnarsson and Halldórsson [2, 3] who established an asymptotically tight upper bound on their chromatic numbers.

Besides ordinary colorings, we study $L(p,q)$-labelings of graphs. An $L(p,q)$-*labeling* of a graph $G$ is a labeling $c$ of the vertices by non-negative integers such that the colors (labels) assigned to neighboring vertices differ by at least $p$ and the colors of pairs of vertices at distance two differ by at least $q$. The least integer $K$ such that there exists a proper $L(p,q)$-labeling of $G$ by integers between 0 and $K$ is called the *span* and denoted by $\lambda_{p,q}(G)$. Clearly, if $p = q = 1$, an $L(p,q)$-labeling of $G$ is just a proper coloring of the square of $G$ with numbers between 0 and $K$ and $\chi(G^2) = \lambda_{1,1}(G) + 1$. Because of this close relation, we refer to the numbers used as vertex labels as to *colors*.

A special attention of researchers was devoted to $L(2,1)$-labelings, partly because of the conjecture of Griggs and Yeh [13] that $\lambda_{2,1}(G) \le \Delta^2$ for every graph $G$ with maximum degree $\Delta \ge 2$. This conjecture remains widely open, verified only for few classes of graphs including graphs of maximum degree two, chordal graphs [29] (see also [7, 23]) and Hamiltonian cubic graphs [19, 20]. For general graphs, the original bound $\lambda_{2,1}(G) \le \Delta^2 + 2\Delta$ from [13] was improved to $\lambda_{2,1}(G) \le \Delta^2 + \Delta$ in [8] and a recent more general result of Král' and Škrekovski [22] yields the present record $\lambda_{2,1}(G) \le \Delta^2 + \Delta - 1$. Optimal $L(p,q)$-labelings are also intensively studied for the class of planar graphs. The following bounds are known: $\lambda_{p,q}(G) \le (4q-2)\Delta + 10p - 38q - 23$ due to van den Heuvel et al. [14], $\lambda_{p,q}(G) \le (2q-1)\lceil 9\Delta/5 \rceil + 8p - 8q + 1$ if $\Delta \ge 47$ due to Borodin et al. [6], and $\lambda_{p,q}(G) \le q\lceil 5\Delta/3 \rceil + 18p + 77q - 18$ due to Molloy and Salavatipour [27]. The algorithmic aspects of $L(p,q)$-labelings also attracted a lot of attention of researchers [1, 5, 10, 11, 21, 26] because of potential applications in radio frequency assignment.

In this paper, we study colorings of the squares of planar graphs with no short cycles. There are several upper bounds on the chromatic number of the squares of such planar and non-planar graphs: if the girth of a (not necessarily planar) graph $G$ with maximum degree $\Delta$ is at least 7, then $\chi(G^2) \le O(\Delta^2/\log \Delta)$ [4]. Since the incidence graphs of finite projective planes have girth six and the chromatic number of their squares is $\Theta(\Delta^2)$, the assumption on the girth cannot be further decreased. The following bounds for planar graphs were proven by Wang and Lih [31]:

- $\lambda_{p,q}(G) \le (2q-1)\Delta + 4p + 4q - 4$ if $G$ is a planar graph of girth at least seven,
- $\lambda_{p,q}(G) \le (2q-1)\Delta + 6p + 12q - 9$ if $G$ is a planar graph of girth at

least six, and

- $\lambda_{p,q}(G) \le (2q-1)\Delta + 6p + 24q - 15$ if $G$ is a planar graph of girth at least five.

In addition, they conjectured the following:

**Conjecture 2 (Wang and Lih 2003).** *For any integer $g \ge 5$, there exists an integer $M(g)$ such that the chromatic number of the square of every planar graph $G$ of girth at least $g$ and maximum degree $\Delta \ge M(g)$ is $\Delta + 1$.*

In this paper, we prove the conjecture for $g \ge 7$ and show that it is not true for $g \in \{5, 6\}$. Our main result (Theorem 17) is the following: if $G$ is a planar graph of maximum degree $\Delta \ge 190 + 2p$, $p \ge 1$, and its girth is at least seven, then $\lambda_{p,1}(G) \le 2p + \Delta - 2$. This yields a proof of Conjecture 2 for $g \ge 7$. Our upper bound is tight for all pairs of $\Delta$ and $p$ with $\Delta \ge 190 + 2p$ as discussed at the end of the paper.

Wang and Lih [31] also conjectured that there exists a number $M(g)$ such that $\lambda_{2,1}(G) = \Delta + 1$ for every planar graph $G$ of girth at least $g$ and maximum degree $\Delta \ge M(g)$. However, this is not true even for trees (Proposition 24). On the other hand, Theorem 17 yields that the latter conjecture becomes true for $g \ge 7$ when $\Delta+1$ is replaced by $\Delta+2$. Moreover, an argument used in [27] applied to our results yields that $\lambda_{p,q}(G) \le 2p + q\Delta - 2$ for planar graphs with girth at least seven and sufficiently large maximal degree.

We also prove a weakened version of Conjecture 2 for girth $g = 6$ (Theorem 23): if $G$ is a planar graph of maximum degree $\Delta \ge 8821$ and its girth is at least six, then $\lambda_{1,1}(G) \le \Delta + 1$, i.e., $\chi(G^2) \le \Delta + 2$.

# 2 Preliminaries

In this section, we introduce notation used throughout the paper. All graphs considered in the paper are simple, i.e., without parallel edges and loops. A *$d$-vertex* is a vertex of degree exactly $d$. An *$(\le d)$-vertex* is a vertex of degree at most $d$. Similarly, an *$(\ge d)$-vertex* is a vertex of degree at least $d$. A *$k$-thread* is an induced path comprised of $k$ 2-vertices. The set of all the neighbors of a vertex $v$ is called the *neighborhood* of $v$ and the neighborhood enhanced by $v$ is called the *closed neighborhood* of $v$.

An *$\ell$-face* is a face of length $\ell$ (counting multiple incidences, i.e., bridges incident to the face are counted twice). If the boundary of a face $f$ forms a connected subgraph, then the subgraph formed by the boundary (implicitly equipped with the orientation determined by the embedding) is called the *facial walk*. A face $f$ is said to be *biconnected* if its boundary is formed by a single simple cycle. The neighbors of a vertex $v$ on the facial walk are called *$f$-neighbors* of $v$. Note that if $f$ is biconnected, then each vertex incident with $f$ has exactly two $f$-neighbors.

Let us consider a biconnected face $f$, and let $v_1, \ldots, v_k$ be $(\geq 3)$-vertices incident to $f$ listed in the order on the facial walk of $f$. The *type* of $f$ is a $k$-tuple $(\ell_1, \ldots, \ell_k)$ where $\ell_i$ is the length of the 2-thread between $v_i$ and $v_{i+1}$. In particular, if $v_i$ and $v_{i+1}$ are $f$-neighbors, then $\ell_i$ is zero. Two face types are considered to be the same if they can be types of the same face, i.e., they differ only by a cyclic rotation and/or a reflection.

If the face $f$ is biconnected and $v$ is a vertex incident to $f$, then the neighbors of $v$ that are not its neighbors on the facial walk are said to be *opposite* to the face $f$. Similarly, if both the faces $f_1$ and $f_2$ incident to an edge $uv$ are biconnected, then the faces incident to $v$ distinct from $f_1$ and $f_2$ are *opposite* to the vertex $u$ (with respect to the vertex $v$).

Some of our arguments used in Section 4 are based on elementary facts from the notion of list colorings (choosability). List colorings were introduced independently by Erdős, Rubin and Taylor [9] and Vizing [30]. A graph $G$ is said to be $\ell$-choosable if for any assignment of lists $L(v)$ of sizes $\ell$ to the vertices of $G$, there exists a proper coloring $c$ of $G$ such that $c(v) \in L(v)$ for every vertex $v$. The gap between the list chromatic number (the smallest $\ell$ for which the graph is $\ell$-choosable) and the usual chromatic number can be arbitrary large: for every integer $\ell$, there exists a bipartite graph that is not $\ell$-choosable. However, the only simple fact that we need in our consideration is the following: any cycle of even length is 2-choosable. The reader can figure out details of a simple proof of this statement him/herself or can consult [9].

# 3 Planar graphs of girth at least seven

In this section, we show that every planar graph of girth seven and sufficiently large maximum degree $\Delta$ is $(\Delta + 1)$-colorable and prove our general result on $L(p, q)$-labelings of such graphs. For an integer $D \geq 192$, a graph $G$ is *$D$-good* if its maximum degree is at most $D$ and it has an $L(p, 1)$-

labeling of span at most $D + 2p - 2$ for every $p \le (D - 190)/2$. A planar graph $G$ of girth at least 7 and maximum degree at most $D$ is said to be *$D$-minimal* if it is not $D$-good but every proper subgraph of $G$ is $D$-good. Clearly, if $G$ is $D$-minimal, then it is connected. Observe that every $\ell$-face of $G$ with $\ell \le 13$ is biconnected because of the girth assumption and that the facial walk of every $\ell$-face with $\ell \le 11$ induces a chordless cycle of $G$. A vertex of $G$ is said to be *small* if its degree is at most 95, and *big* otherwise.

The proof presented in this section is based on the discharging method. We show that there is no $D$-minimal graph, i.e., all planar graphs of girth at least seven and maximum degree at most $D$ are $D$-good. In order to show this, we first describe configurations that cannot appear in a $D$-minimal graph (reducible configurations). In the proof, we consider a potential $D$-minimal graph and assign each vertex and each face a certain amount of charge. The amounts are assigned in such a way that their sum is negative. The charge is then redistributed among the vertices and faces according to the rules described in Subsection 3.3. It is shown that if the considered graph is $D$-minimal, then the final charge of every vertex and every face is non-negative after the redistribution. Since the sum of the initial charges is negative, we obtain a contradiction and conclude that there is no $D$-minimal graph.

## 3.1 Structure of $D$-minimal graphs

In this section, we identify configurations that cannot appear in $D$-minimal graphs. The following argument is often used in our considerations: we first assume that there exists a $D$-minimal graph $G$ that contains a certain configuration. We remove some vertices of $G$ and find a proper $L(p, 1)$-labeling of the new graph (the labeling exists because $G$ is $D$-minimal). We then recolor some of the vertices: at this stage, we state the properties that the new colors of the recolored vertices should have and say that the vertices are recolored with colors that have this property (and show that it is possible). If the original colors of such vertices have already the desired properties, then the vertices just keep their original colors. Finally, the labeling is extended to the removed vertices.

We have already seen that every $D$-minimal graph is connected. Similarly, it is not hard to see that the minimum degree of a $D$-minimal graph is at least two:

**Lemma 1.** *If $G$ is a $D$-minimal graph, then its minimum degree is at least* *two.*

*Proof.* Assume that $G$ contains a vertex $v$ of degree one (since $G$ is connected, it has no vertices of degree zero). Fix an integer $p \leq (D-190)/2$ such that $G$ has no proper $L(p,1)$-labeling of span $D+2p-2$. Let $v'$ be the neighbor of $v$ in $G$. Remove $v$ from $G$. Since $G$ is $D$-minimal, the obtained graph has a proper $L(p,1)$-labeling $c$ of span $D+2p-2$. We extend the labeling $c$ to $v$: the vertex $v$ cannot be assigned at most $2p-1$ colors whose difference from the color of $v'$ is less than $p$ and it cannot be assigned at most $D-1$ colors which are assigned to the other neighbors of $v'$. Therefore, there are at most $D+2p-2$ forbidden colors for $v$. In particular, there exists a color that can be assigned to $v$, and thus $c$ can be extended to $v$. This contradicts our assumption that $G$ is $D$-minimal. □

Next, we focus on 2-, 3- and 4-threads contained in $D$-minimal graphs:

**Lemma 2.** *If vertices $v$ and $w$ of a $D$-minimal graph $G$ are joined by a 2-thread, then at least one of the vertices $v$ and $w$ is big.*

*Proof.* Fix an integer $p \leq (D-190)/2$ such that $G$ has no proper $L(p,1)$-labeling with span $D+2p-2$. Let $v'w'$ be the 2-thread between $v$ and $w$ in $G$ (where $v'$ is the neighbor of $v$). Assume for the sake of contradiction that neither $v$ nor $w$ is big. Remove the vertices $v'$ and $w'$ from $G$. Since $G$ is $D$-minimal, there exists a proper $L(p,1)$-labeling $c$ of the obtained graph whose span does not exceed $D+2p-2$. We extend the labeling $c$ to the vertices $v'$ and $w'$.

Let $A_v$ be the set of the colors that differ by at least $p$ from the color of $v$ and are different from the colors of all the neighbors of $v$ and from the color of $w$. Similarly, let $A_w$ be the set of the colors that differ by at least $p$ from the color of $w$ and are different from the colors of all the neighbors of $w$ and from the color of $v$. Since $w$ is not a big vertex, the number of these colors is at least $(D+2p-1)-(2p-1)-94-1 \geq 2p$, since $D-95 \geq 2p$. Similarly, we have $|A_v| \geq 2p$.

Color now the vertices $v'$ and $w'$ by colors from $A_v$ and $A_w$ that differ by at least $p$ (observe that such colors always exist). The obtained labeling $c$ is a proper $L(p,1)$-labeling of $G$ with span at most $D+2p-2$. □

The following two statements readily follow:

**Lemma 3.** *No $D$-minimal graph $G$ contains a 4-thread.*

*Proof.* Assume that a $D$-minimal graph $G$ contains a 4-thread $vv'v''v'''$. By Lemma 2, $v$ or $v'''$ is big and $vv'v''v'''$ is not a 4-thread. □

**Lemma 4.** *If vertices $v$ and $w$ of a $D$-minimal graph $G$ are joined by a 3-thread, then both $v$ and $w$ are big.*

*Proof.* Let $v'v''v'''$ be the 3-thread joining $v$ and $w$. By Lemma 2, $v$ or $v'''$ is big. Since $v'''$ is a 2-vertex, $v$ is big. Similarly, we infer that $w$ is big. □

Next, we focus on cycles of lengths seven and eight contained in $D$-minimal graphs. Note that the boundary of every 7-face and 8-face is biconnected (because of the girth assumption), i.e., its boundary is a simple cycle of length seven or eight, and thus the following lemma can always be applied in such cases.

**Lemma 5.** *Let $v_1v_2v_3v_4v_5v_6v_7$ be a part of a 7-cycle or an 8-cycle contained in a $D$-minimal graph $G$. If $v_2$, $v_3$, $v_5$ and $v_6$ are 2-vertices, then $v_1$ or $v_7$ is a big vertex.*

*Proof.* Fix an integer $p \leq (D-190)/2$ such that $G$ has no proper $L(p,1)$-labeling with span $D+2p-2$. Note that the distance between the vertices $v_1$ and $v_7$ is at most two. Assume that neither $v_1$ nor $v_7$ is big. Remove the vertices $v_2$, $v_3$, $v_5$ and $v_6$ from $G$. Since $G$ is $D$-minimal, the new graph has an $L(p,1)$-labeling $c$ of span at most $2p+D-2$. Let $A$ be the set of colors $\gamma$ that differ from the color of $v_4$ by at least $p$ and such that no neighbor of $v_4$ is colored with $\gamma$. Since there are $2p+D-1$ colors available and the degree of $v_4$ in the new graph does not exceed $D-2$, we infer that $|A| \geq 2$.

We extend the labeling $c$ to the removed vertices. Color the vertices $v_5$ and $v_3$ by distinct colors from $A$ in such a way that the colors of $v_5$ and $v_7$ are different, and the colors of $v_3$ and $v_1$ are also different. Since the colors of $v_7$ and $v_1$ are different (the distance of $v_7$ and $v_1$ in $G$ is at most two), this is always possible.

Color now the vertex $v_6$ by a color that differs by at least $p$ from the colors of $v_5$ and $v_7$ and that differ from the colors of $v_4$ and (at most 94) neighbors of $v_7$. Since there are at most $95+4p-2 \leq 2p+D-2$ forbidden colors for $v_6$, the vertex $v_6$ can be colored. Similarly, it is possible to color the vertex $v_2$. Since the obtained labeling is a proper $L(p,1)$-labeling with span at most $2p+D-2$, the graph $G$ is not $D$-minimal. □

The following result is an easy consequence of Lemma 5:

**Lemma 6.** *No $D$-minimal graph $G$ contains a pair of vertices joined by two 3-threads.*

*Proof.* Assume for the sake of contradiction that $G$ contains two vertices $v$ and $w$ joined by two 3-threads. The vertices $v$, $w$ and the two 3-threads joining them comprise an 8-cycle in $G$. By Lemma 5, at least one of the neighbors of $w$ in the 3-threads is big, but both the neighbors are 2-vertices. □

We now focus on 3-vertices in $D$-minimal graphs:

**Lemma 7.** *Let $v_1v_2v_3v_4$ be a path of a $D$-minimal graph $G$ where $v_2$ is a 3-vertex. If neither $v_1$ nor $v_4$ is big and $v_3$ is a 2-vertex, then the remaining neighbor $w$ of $v_2$ is big.*

*Proof.* Fix an integer $p \leq (D-190)/2$ such that $G$ has no $L(p,1)$-labeling of span $2p+D-2$. Assume that $w$ is not big. Remove the vertex $v_3$ from $G$. Since $G$ is $D$-minimal, there exists a proper $L(p,1)$-labeling of the obtained graph with span at most $2p+D-2$. We first change the color of $v_2$ and then we extend the labeling $c$ to the vertex $v_3$.

Recolor the vertex $v_2$ by a color that differs from the colors of $v_1$ and $w$ by at least $p$, and that is different from the colors of all the neighbors of $v_1$ and $w$ and from the color of $v_4$. Since neither $v_1$ nor $w$ is big, there are at most $2(2p-1)+2\cdot 94+1 \leq 2p+D-2$ forbidden colors for $v_2$. Hence, the vertex $v_2$ can be recolored.

Finally, color the vertex $v_3$ by a color that differs from the colors of $v_2$ and $v_4$ by at least $p$, and that is different from the colors of all the neighbors of $v_2$ and $v_4$. Since $v_2$ is a 3-vertex and $v_4$ is not big, there are at most $2(2p-1)+94+2 \leq 2p+D-2$ forbidden colors and $v_3$ can be colored. □

We finish this section by establishing a lemma on the structure of faces of type $(2,1,1)$:

**Lemma 8.** *The following configuration does not appear in a $D$-minimal graph $G$: a 7-face $f$ of type $(2,1,1)$ with one big and two 4-vertices such that both the 4-vertices of $f$ are adjacent only to small vertices.*

*Proof.* By Lemma 2, the big vertex incident to $f$ delimits the 2-thread. Let $v$ be the big vertex and $w$ the other vertex delimiting the 2-thread and let $v'v''$ be the 2-thread (the 2-vertex $v'$ is an $f$-neighbor of $v$). Let $w'$, $w''$ and

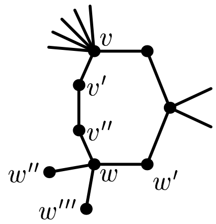


Figure 1: Notation used in the proof of Lemma 8.

$w'''$ be the neighbors of $w$ different from $v''$ (see Figure 1) and assume that $w'$ is an $f$-neighbor of $w$.

Fix an integer $p \leq (D-190)/2$ such that $G$ has no proper $L(p,1)$-labeling with span $2p+D-2$. Remove the vertices $v''$ and $w'$ from $G$. Since $G$ is $D$-minimal, there exists a proper $L(p,1)$-labeling $c$ of the new graph whose span is at most $2p+D-2$. Next, we change the color of $w$ and we extend the labeling $c$ to the vertices $v''$ and $w'$.

Recolor the vertex $w$ by a color that differs by at least $p$ from the colors $w''$ and $w'''$, and that is different from the colors of all the neighbors of $w''$ and $w'''$ and that is also different from the color of $v'$ and the other 4-vertex incident to $f$. Since none of the vertices $w''$ and $w'''$ is big, the number of colors forbidden for $w$ does not exceed $2(2p-1)+2\cdot 94+2 \leq D+2p-2$. Hence, the vertex $w$ can be recolored.

Next, color the vertex $w'$ by a color that differs from the colors of both the 4-vertices incident with $f$ by at least $p$ and that is also different from the colors of all the six neighbors of the 4-vertices. Since the number of such forbidden colors does not exceed $2(2p-1)+6 \leq D+2p-2$, the vertex $w'$ can be colored.

Finally, we color the vertex $v''$ by a color that differs from the colors of $v'$ and $w$ by at least $p$ and that is different from the colors of the vertices $v$, $w'$, $w''$ and $w'''$. Since there are at most $4p+2 \leq D+2p-2$ forbidden colors, the labeling $c$ can be also extended to the vertex $v''$. □

## 3.2 Initial charge

We now describe the amounts of initial charge of vertices. The initial charge of a $d$-vertex $v$ is set to

$$\mathrm{ch}(v) = d - 3,$$

and the initial charge of an $\ell$-face $f$ to

$$\mathrm{ch}(f) = \ell/2 - 3.$$

It is easy to verify that the sum of initial charges is negative:

**Proposition 9.** *If $G$ is a connected planar graph, then the sum of all initial charges of the vertices and faces of $G$ is $-6$.*

*Proof.* Since $G$ is connected, Euler's formula yields that $n + f = m + 2$ where $n$ is the number of the vertices of $G$, $m$ is the number of its edges and $f$ is the number of its faces. The sum of initial charges of the vertices of $G$ is equal to

$$\sum_{v \in V(G)} (d(v) - 3) = 2m - 3n.$$

The sum of initial charges of the faces of $G$ is equal to

$$\sum_{f \in F(G)} \left( \frac{\ell(f)}{2} - 3 \right) = m - 3f.$$

Therefore, the sum of initial charges of all the vertices and faces is $3m - 3n - 3f = -6$. $\square$

Note that the amounts of initial charge were chosen such that each face of size at least 6 (consequently, each face of a $D$-minimal graph) has non-negative charge, the charge of 6-faces is zero and only 2-vertices have negative charge of $-1$ unit.

## 3.3 Discharging rules

Next, the charge is redistributed among the vertices and faces of a (potential) $D$-minimal graph by the following rules:

**R1** Each face $f$ sends charge of $1/2$ to every incident 2-vertex.

**R2** Each 4-vertex sends charge of $1/4$ to every incident face.

**R3** Each small ($\geq 5$)-vertex sends charge of $5/16$ to every incident face.

**R4** Each big vertex adjacent to a 3-vertex $w$ sends charge of $5/16$ to the opposite face through $w$.

**R5** Each big vertex adjacent to a 4-vertex $w$ sends charge of $1/16$ to each of the two opposite faces through $w$.

**R6** If $v$ is a big vertex incident to a face $f$ and $v_1$ and $v_2$ are its $f$-neighbors, then $v$ sends the following charge to $f$:

| | |
|---|---|
| $1/2$ | if $k = 0$, |
| $3/4$ | if $k = 1$, |
| $15/16$ | if $k = 2$ and the type of $f$ is not $(3, 2)$, and |
| $1$ | if the type of $f$ is $(3, 2)$, |

where $k$ is the number of 2-vertices in set $\{v_1, v_2\}$.

If there are multiple incidences, the charge is sent according to the appropriate rule(s) several times, e.g., if a 2-vertex $v$ is incident to a bridge, then it is incident to a single face $f$ and $f$ sends charge of $1/2$ to $v$ twice by Rule R1.

### 3.4 Final charge of vertices

In this subsection, we analyze the final amounts of charge of vertices.

**Lemma 10.** *If a graph $G$ is $D$-minimal, then the final charge of every ($\leq 4$)-vertex is zero.*

*Proof.* The initial charge of a 2-vertex $v$ is $-1$ and it receives charge of $1/2$ from each of the two incident faces by Rule R1. Therefore, its final charge is zero. Since a 3-vertex does not receive or send out any charge, its final charge is zero. Similarly, a 4-vertex sends charge of $1/4$ to each of the four incident faces by Rule R2. Since its initial charge is 1, its final charge is also zero. $\square$

**Lemma 11.** *If a graph $G$ is $D$-minimal, then the final charge of every small ($\geq 5$)-vertex is non-negative.*

*Proof.* Consider a small vertex $v$ of degree $d \geq 5$. The vertex $v$ sends charge of $5/16$ to each of the $d$ incident faces by Rule R3. Hence, it sends out charge of at most $5d/16$. Since the initial charge of $v$ is $d - 3 \geq 5d/16$, the final charge of $v$ is non-negative. $\square$

The analysis of final charge of big vertices needs finer arguments:

**Lemma 12.** *If a graph $G$ is $D$-minimal, then the final charge of every big vertex is non-negative.*

*Proof.* Let $v$ be a big vertex of degree $d$. Let $v_1, \ldots, v_d$ be the neighbors of $v$ in a cyclic order around the vertex $v$ and let $f_1, \ldots, f_d$ be the faces incident to $v$ in the order such that the $f_i$-neighbors of $v$ are the vertices $v_i$ and $v_{i+1}$. Note that some of the faces $f_i$ can coincide. Let $\varphi(v_i)$ be the amount of charge sent from $v$ through a vertex $v_i$. Similarly, $\varphi(f_i)$ is the amount of charge sent to $f_i$. Note that this is a slight abuse of our notation since the faces $f_i$ are not necessarily mutually distinct—in such case, $\varphi(f_i)$ is the amount of charge sent from $v$ because of this particular incidence to $f_i$.

We show that the following holds for every $i = 1, \ldots, d$ (indices are modulo $d$):

$$\frac{\varphi(v_i)}{2} + \varphi(f_i) + \varphi(v_{i+1}) + \varphi(f_{i+1}) + \frac{\varphi(v_{i+2})}{2} \leq \frac{31}{16}. \qquad (1)$$

Summing (1) over all $i = 1, \ldots, d$ yields the following:

$$\sum_{i=1}^{d} (2\varphi(v_i) + 2\varphi(f_i)) \leq \left(2 - \frac{1}{16}\right) d. \qquad (2)$$

Recall now that the initial charge of $v$ is $d - 3$. Because $v$ is big, its degree $d$ is at least 96. Since the charge sent out by $v$ is at most $d - d/32$ by (2), the final charge of $v$ is non-negative. Therefore, in order to establish the statement of the lemma, it is enough to show that the inequality (1) holds.

Let us fix an integer $i$ between 1 and $d$. We distinguish several cases according to which of the vertices $v_i$, $v_{i+1}$ and $v_{i+2}$ are of degree 2:

**None of the vertices $v_i$, $v_{i+1}$ and $v_{i+2}$ is a 2-vertex.** In this case, the vertex $v$ sends through each of the vertices $v_i$, $v_{i+1}$ and $v_{i+2}$ charge at most $5/16$ by Rules R4 and R5, i.e., $\varphi(v_i), \varphi(v_{i+1}), \varphi(v_{i+2}) \leq 5/16$. By Rule R6, both the faces $f_i$ and $f_{i+1}$ receive charge of $1/2$ from $v$, i.e., $\varphi(f_i), \varphi(f_{i+1}) \leq 1/2$. Hence, the sum (1) of charges is at most $13/8 < 31/16$.

**The vertex $v_{i+1}$ is not a 2-vertex and one of $v_i$ and $v_{i+2}$ is a 2-vertex.** By symmetry, we can assume that $v_i$ is a 2-vertex and $v_{i+2}$ is a

($\geq 3$)-vertex. Since $v_i$ is a 2-vertex, $v$ sends no charge through it, i.e., $\varphi(v_i) = 0$. By Rule R6, $\varphi(f_i) = 3/4$ and $\varphi(f_{i+1}) = 1/2$. By Rules R4 and R5, the amounts of charge sent from $v$ through $v_{i+1}$ and $v_{i+2}$ do not exceed 5/16, i.e., $\varphi(v_{i+1}), \varphi(v_{i+2}) \leq 5/16$. Therefore, the sum (1) is bounded by $3/4 + 1/2 + 3/2 \cdot 5/16 < 31/16$.

**The vertex $v_{i+1}$ is not a 2-vertex and both $v_i$ and $v_{i+2}$ are 2-vertices.** The vertex $v$ sends charge of 3/4 to both the faces $f_i$ and $f_{i+1}$ by Rule R6, i.e., $\varphi(f_i) = \varphi(f_{i+1}) = 3/4$. No charge is sent through the vertices $v_i$ and $v_{i+2}$, i.e., $\varphi(v_i) = \varphi(v_{i+2}) = 0$. The amount of charge sent through $v_{i+1}$ is at most 5/16 (charge can be sent through it only by Rule R4 or Rule R5), i.e., $\varphi(v_{i+1}) \leq 5/16$. We conclude that the sum (1) is at most $2 \cdot 3/4 + 5/16 < 31/16$.

**The vertex $v_{i+1}$ is a 2-vertex and neither $v_i$ nor $v_{i+2}$ is a 2-vertex.** The vertex $v$ sends charge of 3/4 to both the faces $f_i$ and $f_{i+1}$ by Rule R6, i.e., $\varphi(f_i) = \varphi(f_{i+1}) = 3/4$. The amount of charge sent through each of $v_i$ or $v_{i+2}$ is at most 5/16 (charge can be sent through it only by Rule R4 or Rule R5), i.e., $\varphi(v_i), \varphi(v_{i+2}) \leq 5/16$. Since no charge is sent through $v_{i+1}$, i.e., $\varphi(v_{i+1}) = 0$, the sum (1) is at most $2 \cdot 3/4 + 5/16 < 31/16$.

**The vertex $v_{i+1}$ is a 2-vertex and one of $v_i$ and $v_{i+2}$ is a 2-vertex.** By symmetry, we can assume that $v_i$ is a 2-vertex and $v_{i+2}$ is a ($\geq 3$)-vertex. Since $v_i$ and $v_{i+1}$ are 2-vertices, $v$ sends no charge through $v_i$ or $v_{i+1}$, i.e., $\varphi(v_i) = \varphi(v_{i+1}) = 0$. By Rule R6, the face $f_i$ receives charge of at most 1 and the face $f_{i+1}$ charge of at most 3/4, i.e., $\varphi(f_i) \leq 1$ and $\varphi(f_{i+1}) \leq 3/4$. Finally, the charge sent from $v$ through $v_{i+2}$ is at most 5/16, i.e., $\varphi(v_{i+2}) \leq 5/16$. We infer that the sum (1) is bounded by $\leq 1 + 3/4 + 5/32 < 31/16$.

**All the vertices $v_i$, $v_{i+1}$ and $v_{i+2}$ are 2-vertices.** There is no charge sent from $v$ through any of the vertices $v_i$, $v_{i+1}$ and $v_{i+2}$, i.e., $\varphi(v_i) = \varphi(v_{i+1}) = \varphi(v_{i+2}) = 0$. If at least one of the faces $f_i$ and $f_{i+1}$ is not a $(3,2)$-face, then the total amount of charge sent to both of them by Rule R6 is at most $15/16 + 1 = 31/16$ as desired. In the rest, we consider the case when both the faces $f_i$ and $f_{i+1}$ are $(3,2)$-faces. Let $v'$ be the other big vertex incident to $f_i$ and $f_{i+1}$. The vertex $v_{i+1}$ lies in a 2-thread or a 3-thread shared by the faces $f_i$ and $f_{i+1}$. If the faces $f_i$ and $f_{i+1}$ share a 2-thread, then the vertices $v$ and

$v'$ are joined by two 3-threads—this is impossible by Lemma 6. On the other hand, if they share a 3-thread, then the vertices $v$ and $v'$ together with the two 2-threads form a 6-cycle contradicting the girth assumption.

$\square$

### 3.5 Final charge of faces

In this subsection, we analyze the final amounts of charge of faces. First, we start with faces that are not biconnected.

**Lemma 13.** *Let $f$ be a face of a $D$-minimal graph $G$. If $f$ is not biconnected, then its final charge is non-negative.*

*Proof.* Let $P$ be the facial walk of $f$. Since $f$ is not biconnected, $P$ consists of two or more blocks. In particular, it contains at least one cut-vertex. In addition observe that the end-blocks of $P$ are cycles of length at least seven. Let $C_1$ and $C_2$ be two different end-blocks of $P$ and $w_1$ and $w_2$ be their cut-vertices (note that $w_1$ may be equal to $w_2$), respectively.

Let $k$ be the number of incidences of $f$ with ($\geq 3$)-vertices, counting multiplicities. If $w_1 \neq w_2$, then each of $w_1$ and $w_2$ contributes by at least two to $k$, thus $w_1$ and $w_2$ together contribute by at least 4 to $k$. Otherwise, the vertex $w_1 = w_2$ contributes by at least two to $k$.

Since the length of $C_1$ is at least seven, it has at least one ($\geq 3$)-vertex different from $w_1$ by Lemma 3. If $C_1$ contains exactly one such ($\geq 3$)-vertex, then it has a 3-thread (it cannot have a 4-thread by Lemma 3), and the vertex $w_1$ is big by Lemma 4. Similar statements hold for $C_2$. Therefore $k \geq 4$, and if $w_1 \neq w_2$ or $w_1 = w_2$ is small, then $k \geq 6$.

If $f$ is an $\ell$-face, its initial charge is $\ell/2-3$. The face $f$ sends out charge of $(\ell-k)/2$ by Rule R1. If $k \geq 6$, then this is at most $\ell/2-3$ and thus the final charge of the face is non-negative.

If $k < 6$, then $w_1 = w_2$ is a big vertex (this follows from our previous discussion) and it has two incidences with $f$. Therefore $f$ receives charge of at least one unit from $w_1$ by Rule R6 and its final charge is $\ell/2-3-(\ell-k)/2+1 \geq 0$. $\square$

Next, we analyze biconnected faces starting with 7-faces:

**Lemma 14.** *The final charge of each 7-face $f$ of a $D$-minimal graph $G$ is non-negative.*

*Proof.* The initial charge of the face $f$ is $1/2$. By Lemma 3, $f$ does not contain a 4-thread, and thus the face $f$ is incident to at least two $(\geq 3)$-vertices. We distinguish five cases according to the number of $(\geq 3)$-vertices incident to $f$:

**The face $f$ is incident to two $(\geq 3)$-vertices.** In this case, the type of $f$ is $(3, 2)$. By Lemma 4, both the $(\geq 3)$-vertices are big and each of them sends charge of 1 unit to $f$ by Rule R6. Since $f$ sends out charge of $5/2$ to the five incident 2-vertices, its final charge is zero.

**The face $f$ is incident to three $(\geq 3)$-vertices.** Since $f$ sends out charge of two units to the incident 2-vertices, it is enough to show that it receives charge of at least $3/2$ from the incident $(\geq 3)$-vertices. Since $G$ does not contain a 4-thread by Lemma 3, the type of $f$ is $(3, 1, 0)$, $(2, 2, 0)$ or $(2, 1, 1)$.

If $f$ is incident to two big vertices, then each of them sends charge of at least $3/4$ to $f$ by Rule R6, and the final charge of $f$ is non-negative. In the rest, we assume that $f$ is incident to at most one big vertex. Consequently, the type of $f$ is $(2, 2, 0)$ or $(2, 1, 1)$ by Lemma 4 and $f$ is incident to exactly one big vertex by Lemma 2.

Assume that the type of $f$ is $(2, 2, 0)$. By our assumption, $f$ is incident to a single big vertex and, by Lemma 2, this vertex delimits both the 2-threads of $f$. However, Lemma 5 yields that one of the other two $(\geq 3)$-vertices is also big (contrary to our assumption).

The final case to consider is that the type of $f$ is $(2, 1, 1)$. Let $v$ be the big vertex incident to $f$. By Lemma 2, $v$ delimits the 2-thread. Since both $f$-neighbors of $v$ are 2-vertices, $v$ sends charge of $15/16$ to $f$. Let $v'$ be any of the other two $(\geq 3)$-vertices incident to $f$. If $v'$ is a 3-vertex, its neighbor opposite to $f$ is big by Lemma 7 and it sends (through $v'$) charge of $5/16$ to $f$ by Rule R4. If $v'$ is a 4-vertex, it sends charge of $1/4$ to $f$, and if $v'$ has a big neighbor opposite to $f$, then the big neighbor sends $f$ additional charge of $1/16$ by Rule R5. Finally, if $v'$ is a small $(\geq 5)$-vertex, it sends charge of $5/16$ to $f$ by Rule R3. We conclude that if $f$ receives total charge of less than $3/2$, then both the $(\geq 3)$-vertices incident to $f$ are 4-vertices with no big neighbors. However, this is impossible by Lemma 8.

**The face $f$ is incident to four $(\geq 3)$-vertices.** Since $f$ is incident to three 2-vertices, it sends out charge of $3/2$. We show that, on the

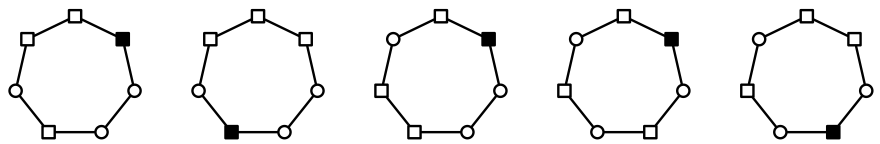

Figure 2: All configurations (up to symmetry) of a 7-face of types $(2,1,0,0)$, $(2,0,1,0)$ and $(1,1,1,0)$ when the face is incident to a single big vertex. The big vertices are represented by full squares, the small $(\geq 3)$-vertices by empty squares and the 2-vertices by circles. Note that a 2-thread must be bounded by at least one big vertex by Lemma 2.

other hand, it receives charge of at least one unit from the incident $(\geq 3)$-vertices. This will imply that the final charge of $f$ is non-negative. If $f$ is incident to two big vertices, then it receives charge of at least $1/2$ from each of them, i.e., charge of at least one unit in total. Hence, we can assume in the rest that $f$ is incident to at most one big vertex. In particular, by Lemma 4, $f$ has no 3-thread. Therefore, the type of $f$ is one of the following: $(2,1,0,0)$, $(2,0,1,0)$ or $(1,1,1,0)$.

Assume first that $f$ is incident to no big vertex. By Lemma 2, the type of $f$ is $(1,1,1,0)$. Let $v$ be any of the four $(\geq 3)$-vertices incident to $f$. Note that $v$ has an $f$-neighbor that is a 2-vertex. If $v$ is a $(\geq 4)$-vertex, then $f$ receives charge of at least $1/4$ units from $v$ by Rules R2 and R3. If $v$ is a 3-vertex, then its neighbor opposite to $f$ is big by Lemma 7 and it sends charge of $5/16$ through $v$ to $f$ by Rule R4. Since the choice of $v$ was arbitrary, the amount of charge sent from (or through) each incident $(\geq 3)$-vertex is at least $1/4$ and $f$ receives charge of at least 1 unit in total.

We now consider the case that exactly one vertex incident to $f$ is big. We say that a vertex $x$ incident to $f$ has Property $S$ if the following conditions are satisfied:

1. $x$ is small,
2. both $f$-neighbors of $x$ are small, and
3. one of the $f$-neighbors of $x$ is a 2-vertex with no big $f$-neighbor.

It is routine to check that the following claim holds (consult Figure 2): unless the type of $f$ is $(2,1,0,0)$ and the big vertex delimits both the

2-thread and the 1-thread of $f$, the face $f$ is incident to two different $(\geq 3)$-vertices $w_1$ and $w_2$ that have Property $S$.

Under the assumption that the type of $f$ is not $(2,1,0,0)$, we show that the face $f$ receives charge of at least $1/4$ from (or through) each of $w_1$ and $w_2$: if $w_i$ is a $(\geq 4)$-vertex, then $f$ receives charge of at least $1/4$ from it. Otherwise, $w_i$ is a 3-vertex and, by Lemma 7, its neighbor opposite to $f$ is big. Consequently, it sends through $w_i$ charge of $5/16$ to $f$. Since $f$ receives in addition the charge of at least $1/2$ from the big vertex, its final charge is non-negative as desired.

It remains to consider the case when the type of $f$ is $(2,1,0,0)$ and the big vertex delimits both the 2-thread and the 1-thread of $f$. In this case, $f$ receives charge of $15/16$ from the incident big vertex by Rule R6. Moreover, there exists a vertex $w$ that has Property $S$ (consult Figure 2). Similarly as in the previous paragraph, the charge sent from $w$ to $f$ is at least $1/4$. Altogether, $f$ receives charge of at least 1 and the final charge is thus non-negative.

**The face $f$ is incident to five $(\geq 3)$-vertices.** The face $f$ sends out charge of 1 unit to the two incident 2-vertices. Thus it is enough to show that the face $f$ receives charge of at least $1/2$ from incident vertices. If $f$ is incident to a big vertex, then $f$ receives charge of at least $1/2$ from it by Rule R6. We assume in the rest that $f$ is incident only to small vertices. In particular, $f$ has no 2-thread (by Lemma 2).

Let $v$ be a 2-vertex incident to $f$ and let $v^-$ and $v^+$ be the two $f$-neighbors of $v$. Note that both $v^-$ and $v^+$ are $(\geq 3)$-vertices. If $v^-$ is a $(\geq 4)$-vertex, it sends charge of at least $1/4$ to $f$. If $v^-$ is a 3-vertex, then its neighbor opposite to $f$ is big by Lemma 7, and it sends charge of $5/16$ through $v^-$ to $f$. Similarly, $f$ receives charge of at least $1/4$ from (or through) $v^+$. Hence, the total charge received by $f$ from the vertices $v^-$ and $v^+$ is at least $1/2$ and the final charge of $f$ is non-negative.

**The face $f$ is incident to six or seven $(\geq 3)$-vertices.** Since the face $f$ is incident to at most one 2-vertex, it sends out charge of at most $1/2$ and its final charge is non-negative.

□

Next, we analyze the final charge of 8-faces.

**Lemma 15.** *The final charge of each biconnected* 8*-face* $f$ *of a* $D$*-minimal graph* $G$ *is non-negative.*

*Proof.* First note that the initial charge of the face $f$ is one. By Lemma 3, the face $f$ does not contain a 4-thread. Therefore, the face $f$ is incident to at least two ($\geq 3$)-vertices. We distinguish five cases based on the number of ($\geq 3$)-vertices incident to the face $f$:

**The face $f$ is incident to two ($\geq 3$)-vertices.** Since $f$ does not contain a 4-thread, the type of $f$ is $(3,3)$. However, this is impossible by Lemma 6.

**The face $f$ is incident to three ($\geq 3$)-vertices.** Since $f$ sends out charge of $5/2$ to the incident 2-vertices, it is enough to show that it receives charge of at least $3/2$ from the incident ($\geq 3$)-vertices. Since $f$ does not contain a 4-thread, the type of $f$ is $(3,2,0)$, $(3,1,1)$ or $(2,2,1)$.

If the type of $f$ is $(3,2,0)$ or $(3,1,1)$, then the 3-thread is delimited by two big vertices (by Lemma 4) and $f$ receives from each of them charge of at least $3/4$ by Rule R6. Hence, the final charge of $f$ is non-negative.

Assume that the type of $f$ is $(2,2,1)$. It is enough to show that $f$ is incident to at least two big vertices because each of them would send charge of $3/4$ to $f$ by Rule R6. If this is not the case, then $f$ is incident to exactly one big vertex that is common to the two 2-threads by Lemma 2. However, by Lemma 5, at least one of the other two ($\geq 3$)-vertices is also big. We conclude that $f$ is incident to at least two big vertices.

**The face $f$ is incident to four ($\geq 3$)-vertices.** Since $f$ is incident to four 2-vertices, $f$ sends out charge of two units. We claim that it also receives charge of at least one unit from the incident vertices. This will imply that the final charge of $f$ is non-negative. If $f$ is incident to two big vertices, then it receives charge of at least $1/2$ from each of them and the claim holds. We assume in the rest that $f$ is incident to at most one big vertex. In particular, by Lemma 4, $f$ has not a 3-thread.

Assume that $f$ contains a 2-thread. Let $v$ and $v'$ be the vertices delimiting the 2-thread. By Lemma 2, $v$ or $v'$ is big, say $v$. Since $v$ is incident to a 2-vertex, it sends charge of at least $3/4$ to $f$ by Rule R6. If $v'$ is a $(\geq 4)$-vertex, then $f$ receives charge of at least $1/4$ from $v'$ and the final charge of $f$ is non-negative. Otherwise, $v'$ is a 3-vertex incident to a 2-thread and its $f$-neighbor not contained in the 2-thread is a small vertex. By Lemma 7, the neighbor of $v'$ opposite to $f$ is a big vertex. Hence, the face $f$ receives charge of $5/16$ from the big neighbor of $v'$ and thus its final charge is non-negative.

In the rest, we assume that $f$ has neither a 3-thread nor a 2-thread. Consequently, the type of $f$ must be $(1,1,1,1)$. Let $v_1$, $v_2$, $v_3$ and $v_4$ be the $(\geq 3)$-vertices incident to $f$ in the order as they appear on the facial walk of $f$. If $f$ is incident to two or more big vertices, it receives charge of $15/16$ from each of them and its final charge is positive. Assume that $f$ is incident to a single big vertex, say $v_1$. Note that $f$ receives charge of $15/16$ from $v_1$ by Rule R6. If $v_3$ is a $(\geq 4)$-vertex, it sends charge of $1/4$ to $f$ and the final charge of $f$ is non-negative. If $v_3$ is a 3-vertex, then its neighbor opposite to $f$ is big (by Lemma 7), $v_3$ sends charge of $5/16$ to $f$, and thus the final charge of $f$ is non-negative.

It remains to consider the case when the type of $f$ is $(1,1,1,1)$ and $f$ is incident to no big vertex. Let us consider a vertex $v_1$. If $v_1$ is $(\geq 4)$-vertex, it sends charge of at least $1/4$ to $f$. If $v_1$ is 3-vertex, then its neighbor opposite to $f$ is big, and it sends charge of $5/16$ to $f$ through $v_1$. Similarly, we can infer that $f$ receives charge of at least $1/4$ from (or through) the vertices $v_2$, $v_3$ and $v_4$. Hence, $f$ receives charge of at least one unit from the incident vertices and its final charge is non-negative.

**The face $f$ is incident to five $(\geq 3)$-vertices.** The face $f$ sends out charge of $3/2$ units to the incident 2-vertices. Thus it is enough to show that the face $f$ receives charge of at least $1/2$ from incident $(\geq 3)$-vertices. If $f$ is incident to a big vertex, then $f$ receives charge of at least $1/2$ from it and the final charge is non-negative. We assume in the rest that $f$ is incident only to small vertices.

Let $v$ be a 2-vertex incident to $f$. Since $f$ is incident to no big vertex, both the neighbors $v^-$ and $v^+$ of $v$ are $(\geq 3)$-vertices by Lemma 2. If $v^-$ is a $(\geq 4)$-vertex, it sends charge of at least $1/4$ to $f$. And if $v^-$ is a 3-vertex, then its neighbor opposite to $f$ is big by Lemma 7 and it sends through $v^-$ to $f$ charge of $5/16$. Similarly, $f$ receives charge of at least $1/4$ from (or through) $v^+$. Hence, $f$ receives charge of at least $1/2$ in total from the two neighbors of $v$ and the final charge of $f$ is non-negative.

**The face $f$ is incident to six or more $(\geq 3)$-vertices.** Since the face $f$ is incident to at most two 2-vertices, it sends out charge of at most one unit and the final charge of $f$ is non-negative.

□

Finally, we analyze the case of $(\geq 9)$-faces:

**Lemma 16.** *The final charge of each biconnected $(\geq 9)$-face $f$ of a D-minimal graph is non-negative.*

*Proof.* Since $f$ does not contain a 4-thread by Lemma 3, the face $f$ is incident to at least three $(\geq 3)$-vertices. The initial charge of $f$ is $\ell/2-3$ where $\ell$ is the length of $f$. We distinguish four cases according to the number of $(\geq 3)$-vertices incident to $f$:

**The face $f$ is incident to three $(\geq 3)$-vertices.** The face $f$ sends charge of $(\ell-3)/2$ to the incident 2-vertices. It is enough to show that $f$ receives charge of at least $3/2$ from the incident vertices. If $f$ has a 3-thread, then the 3-thread is delimited by two big vertices. Both of them send charge of at least $3/4$ to $f$ by Rule R6. Therefore, if the total charge received by $f$ is less than $3/2$, then $f$ has no 3-thread. Consequently, the length of $f$ is nine and its type is $(2,2,2)$. By Lemma 2, at least two of the $(\geq 3)$-vertices are big and $f$ receives charge of at least $3/2$ from them by Rule R6 in this case.

**The face $f$ is incident to four $(\geq 3)$-vertices.** The face $f$ sends charge of $(\ell-4)/2$ to the incident 2-vertices. It is enough to show that $f$ receives charge of at least 1 from the incident vertices. If $f$ has a 3-thread, then the 3-thread is delimited by two big vertices (by Lemma 4) and each of them sends charge of at least $1/2$ to $f$ by Rule R6. If $f$ has at least three 2-threads, then these threads are delimited by at least two different big vertices by Lemma 2, and $f$ receives charge of at least $1/2$ from each of them by Rule R6. If none of the above cases holds, i.e., $f$ has no 3-thread and at most two 2-threads, then its type must be one of the following: $(2,2,1,0)$, $(2,1,2,0)$, $(2,1,1,1)$, $(2,2,1,1)$, and $(2,1,2,1)$—see Figure 3.

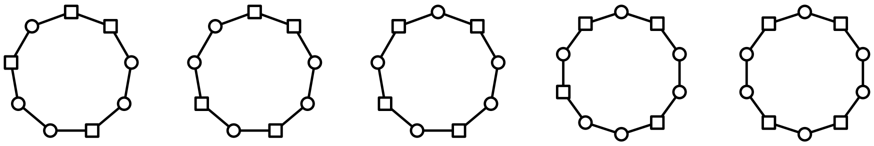

Figure 3: Possible types of a 9-face or a 10-face with no 3-thread and at most two 2-threads. The $(\geq 3)$-vertices are represented by squares and the 2-vertices by circles.

Assume that the type of $f$ is one of the five types listed at the end of the previous paragraph. Since $f$ has a 2-thread, it must be incident to a big vertex $v$ by Lemma 2. Let $v'$, $v''$ and $v'''$ be the remaining $(\geq 3)$-vertices incident to $f$. The face $f$ receives charge of at least $1/2$ from the vertex $v$ by Rule R6. If at least one of $v'$, $v''$ and $v'''$ is big, then it sends additional charge of at least $1/2$ to $f$ by Rule R6, and the total amount of charge received by $f$ is at least one. Let us assume in the rest that all the vertices $v'$, $v''$ and $v'''$ are small.

Observe that in this case the type of $f$ is $(2, 2, 1, 0)$, $(2, 1, 1, 1)$ or $(2, 2, 1, 1)$. Note also that if $v'$ is a 3-vertex, then it satisfies the assumptions of Lemma 7. Similar statements hold for $v''$ and $v'''$.

If $v'$ is a $(\geq 4)$-vertex, $f$ receives charge of at least $1/4$ from $v'$ by Rule R2 or Rule R3. If $v'$ is a 3-vertex, its neighbor opposite to $f$ is big by Lemma 7 and it sends through $v'$ to $f$ charge $5/16$ by Rule R4. Similarly, $f$ receives charge of at least $1/4$ from $v''$ and $v'''$. We conclude that the total charge received by $f$ is at least one.

**The face $f$ is incident to five $(\geq 3)$-vertices.** The face $f$ sends out charge of $(\ell - 5)/2$ to the incident 2-vertices. It is enough to show that $f$ receives charge of at least $1/2$ from the incident vertices. If $f$ is incident to a big vertex, then it receives charge of at least $1/2$ by Rule R6 from this vertex. Assume in the rest that $f$ is incident only to small vertices. In particular, the length of every 2-thread of $f$ is one by Lemma 2. Let $v$ be a 2-vertex incident to $f$ and $v^-$ and $v^+$ the $f$-neighbors of $v$. Note that both $v^-$ and $v^+$ are $(\geq 3)$-vertices. If $v^-$ is a $(\geq 4)$-vertex, then $f$ receives charge of at least $1/4$ from $v^-$ by Rule R2 or Rule R3. If $v^-$ is a 3-vertex, then its neighbor opposite to $v$ is big by Lemma 7 and the face $f$ receives charge of $5/16$

from it through $v$. Similarly, $f$ receives charge of at least $1/4$ from (or through) $v^+$. Altogether, $f$ receives charge of at least $1/2$ as required.

**The face $f$ is incident to six or more ($\geq 3$)-vertices.** The face $f$ sends out charge of at most $(\ell - 6)/2$ by Rule R1. Since the initial charge of $f$ is $\ell/2 - 3$ and $\ell \geq 9$, the final charge is non-negative.

□

### 3.6 Final step

We now combine our observations from the previous subsections together:

**Theorem 17.** *If $G$ is a planar graph of maximum degree $\Delta \geq 190 + 2p$, $p \geq 1$, and the girth of $G$ is at least seven, then $G$ has a proper $L(p, 1)$-labeling with span $2p + \Delta - 2$.*

*Proof.* Consider a possible counterexample $G$ and set $D = \Delta$. Since $G$ is not $D$-good, there exists a $D$-minimal graph $G'$. Assign charge to the vertices and faces of $G'$ as described in Subsection 3.2. Apply the rules given in Subsection 3.3 to $G'$. By Proposition 9, the sum of the amounts of initial charge assigned to the vertices and edges of $G'$ is $-6$. On the other hand, the final amounts of charge of every vertex (Lemmas 10–12) and every face (Lemmas 13–16) are non-negative. However, this is impossible since the total amount of charge is preserved by the rules. □

We use an argument applied in [27] to derive the following result for $L(p, q)$-labelings:

**Corollary 18.** *If $G$ is a planar graph of maximum degree $\Delta \geq 190+2\lceil p/q \rceil$, $p, q \geq 1$, and girth at least seven, then $G$ has a proper $L(p, q)$-labeling with span $2p + q\Delta - 2$.*

*Proof.* Let $p' = \lceil p/q \rceil$. By Theorem 17, the graph $G$ has a proper $L(p', 1)$-labeling $c'$ with span $2p' + \Delta - 2$. Define a labeling $c$ by setting $c(v) = qc'(v)$ for each vertex $v$. The labeling $c$ is a proper $L(p'q, q)$-labeling. Therefore, it is also a proper $L(p, q)$-labeling of $G$. The span of $c$ is at most the following:

$$q(2p' + \Delta - 2) = 2\left(p' - \frac{q-1}{q}\right) q + q\Delta - 2 \leq 2p + q\Delta - 2.$$

□

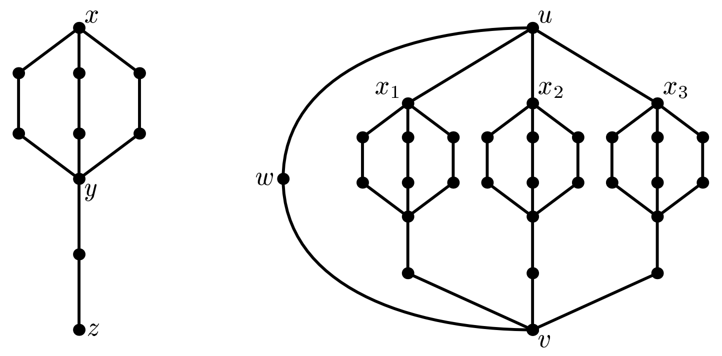


Figure 4: The graphs $G'_4$ and $G_4$.

# 4 Planar graphs of girth six

In this section, we show that Conjecture 2 is not true for $g \in \{5,6\}$, i.e., for each $\Delta \geq 3$, we construct a planar graph $G$ of girth six and maximum degree $\Delta$ such that the chromatic number of its square is $\Delta + 2$. Therefore, the numbers $M(5)$ and $M(6)$ from Conjecture 2 do not exist. On the other hand, Conjecture 2 becomes true for $g = 6$ when the bound on the chromatic number is relaxed to $\Delta + 2$.

We start by constructing a counterexample to Conjecture 2. Let $G'_\Delta$ be a graph of order $2\Delta + 2$ formed by two vertices $x$ and $y$ joined by $(\Delta - 1)$ 2-threads and a vertex $z$ joined to $y$ by a 1-thread. Let $G_\Delta$ be a graph obtained by taking $\Delta - 1$ copies of $G'_\Delta$, identifying all the vertices $z$ of the copies into a single vertex $v$, and adding a vertex $u$ joined to $v$ by a 1-thread and by an edge to the vertex $x$ of each copy of $G'_\Delta$ (see Figure 4). Clearly, the girth of $G_\Delta$ is six and the maximum degree of $G_\Delta$ is $\Delta$. The chromatic number of $G_\Delta$ is determined in the next proposition:

**Proposition 19.** *The chromatic number of the square of the graph $G_\Delta$ is $\Delta + 2$ for every $\Delta \geq 2$.*

*Proof.* It is easy to construct a coloring of $G^2_\Delta$ by $\Delta + 2$ colors. We focus on showing that it cannot be colored by $\Delta + 1$ colors.

We first show that in any proper coloring of the square of $G'_\Delta$, the colors assigned to $x$ and $z$ are distinct. Suppose for contradiction that there exists a proper coloring of $G'^2_\Delta$ by the colors $0, \ldots, \Delta$ such that the colors of both

$x$ and $z$ are the same, say 0. Since the vertex $y$ has degree $\Delta$, either $y$ or one of its neighbors must have color 0. This is impossible because each of these vertices is at distance at most two from $x$ or $z$.

Suppose now that the graph $G_\Delta$ can be colored by the colors $0, \ldots, \Delta$. Let $x_1, \ldots, x_{\Delta-1}$ be the vertices of the copies of $G'_\Delta$ adjacent to the vertex $u$. Let $w$ be the vertex adjacent to $u$ and distinct from all $x_i$, $1 \le i < \Delta$. We may assume that the color of $v$ is 0. By the observation from the previous paragraph, the color of each vertex $x_i$ is distinct from 0. The vertex $u$ has degree $\Delta$. Therefore, either $u$ or one of its neighbors has color 0. This is impossible since the colors of vertices $x_i$ are distinct from 0 and both $u$ and $w$ are at distance at most two from the vertex $v$. We conclude that there is no proper coloring of $G_\Delta^2$ with $\Delta + 1$ colors. $\square$

We now show that $\Delta + 2$ colors suffice to color the square of any planar graph of girth 6 and maximum degree $\Delta$, for $\Delta \ge 8821$; the bound on the maximum degree can be further improved, but we do not provide details in order to keep the proof simpler.

The proof is again based on the discharging method. We redefine the terms of a *$D$-good* graph and a *$D$-minimal* graph as well as the definitions of *big* and *small* vertices for the course of our proofs in this section. For an integer $D \ge 8821$, a graph $G$ is called *$D$-good* if its maximum degree is at most $D$ and the chromatic number of $G^2$ is at most $D + 2$. A planar graph $G$ of girth at least 6 and maximum degree at most $D$ is *$D$-minimal* if $G$ is not $D$-good but every proper subgraph of $G$ is $D$-good. If $G$ is a $D$-minimal graph, then $G$ is connected. Observe that $G$ is also 2-connected: otherwise, color the blocks of $G$ separately and afterwards permute the colors so that the colors of the cut-vertices match and the colors of their neighbors are pairwise distinct. In particular, the minimum degree of a $D$-minimal graph is at least two.

A vertex is said to be *small* if its degree is at most 1763, and it is said to be *big* otherwise.

In the following, we show that there is no $D$-minimal graph. We assume that there is a $D$-minimal graph and assign charge to its vertices and its faces. The total amount of initial charge will be negative. We then redistribute charge in two phases as determined by the rules presented in Sections 4.2 and 4.3. We eventually obtain contradiction with our assumption that there exists a $D$-minimal graph by showing that the total final amount of charge is non-negative.

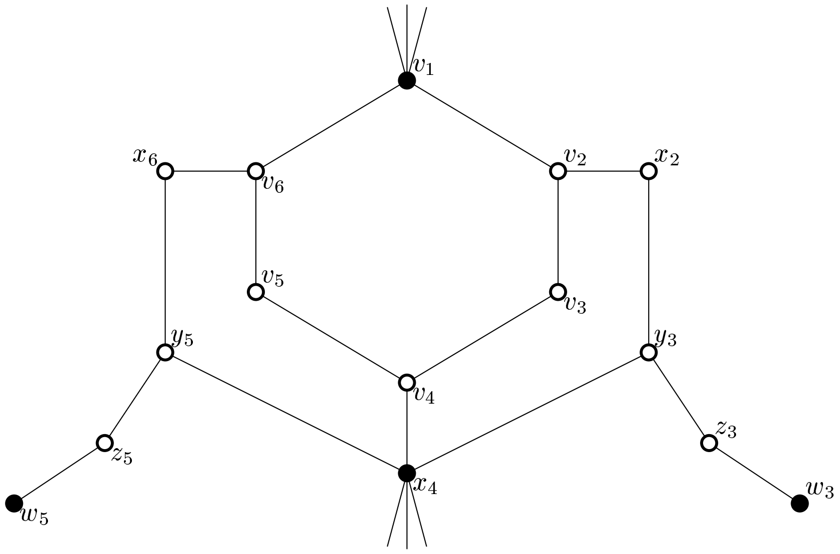


Figure 5: The reducible configuration from Lemma 20(5). The vertices that are not removed in the proof are represented by full circles.

## 4.1 Reducible configurations

Let us first describe several configurations that cannot appear in a $D$-minimal graph. Such a configuration is called *reducible.*

**Lemma 20.** *The following configurations are reducible:*

1. *A small vertex $u$ and a vertex $v$ joined by a 2-thread.*
2. *Vertices $u$ and $v$ joined by two 2-threads.*
3. *A small vertex $v$ joined by a 1-thread to a vertex $u$ of degree at most six, such that all the neighbors of $u$ are small.*
4. *Two adjacent 3-vertices $u$ and $v$ such that all the neighbors of $u$ and $v$ are small and at least one of the neighbors of $u$ is a 2-vertex.*
5. *The configuration in Figure 5, where $v_2$, $v_4$, $v_6$, $y_3$ and $y_5$ are 3-vertices, $v_3$, $v_5$, $x_2$, $x_6$, $z_3$ and $z_5$ are 2-vertices, and $w_3$ and $w_5$ are small vertices (there is no restriction on the degrees of $v_1$ and $x_4$).*

*Proof.* Let $G$ be a $D$-minimal graph, in particular, $\chi(G^2) > D+2$. We deal with the configurations separately. In each of the cases, we first assume that $G$ contains the configuration described in the statement of the lemma and we obtain contradiction by showing that $G$ is not $D$-minimal.

1. Let $x$ and $y$ be the vertices of the 2-thread, where $x$ is the vertex adjacent to $u$. Consider the graph $G' = G \setminus \{x, y\}$. Since $G$ is $D$-minimal, the square of $G'$ is $(D+2)$-colorable. Since the degree of $v$ in $G'$ is at most $D-1$, there are at least two colors distinct from the colors of $v$ and its neighbors. At least one of them (call it $\gamma$) is distinct from the color of $u$. Assign the color $\gamma$ to the vertex $y$. Since $u$ is small, the degree of $x$ in $G^2$ is at most $1763+3 < D$. Therefore, we can choose a color distinct from colors of $u$, its neighbors in $G'$, $v$ and $y$ for $x$. We obtained a proper coloring of $G^2$ by $(D+2)$ colors. This contradicts the $D$-minimality of $G$.

2. Let the vertices of the 2-threads be $x_1$, $x_2$, $y_1$ and $y_2$ where $x_i$ is adjacent to $y_i$ and $u$ for $i = 1, 2$. The square of the graph $G' = G \setminus \{x_1, x_2, y_1, y_2\}$ is $(D+2)$-colorable by the $D$-minimality of $G$. Fix a coloring of $G'$ with $D+2$ colors. Let $C_u$ and $C_v$ be the sets of the colors which are assigned to no vertex in the closed neighborhood of $u$ and $v$, respectively. Since the degrees of $u$ and $v$ in $G'$ are at most $D-2$, both $C_u$ and $C_v$ have sizes at least three. Let $c_u$ and $c_v$ be the colors of $u$ and $v$, respectively. Let $C'_u = C_u \setminus \{c_v\}$ and $C'_v = C_v \setminus \{c_u\}$. Assign the list $C'_u$ to the vertices $x_1$ and $x_2$ and the list $C'_v$ to the vertices $y_1$ and $y_2$. The subgraph of $G^2$ induced by $\{x_1, x_2, y_1, y_2\}$ is a 4-cycle. This graph is 2-choosable. Therefore, its vertices can be colored from the assigned lists. The coloring obtained by extending the coloring of $G'$ to $G$ in this way is a proper coloring of $G^2$ with $D+2$ colors that contradicts our assumption that $G$ is $D$-minimal.

3. Let $x$ be the 2-vertex of the 1-thread. The square of the graph $G' = G \setminus \{x\}$ is $(D+2)$-colorable. Fix such a coloring. The degree of $u$ in $G'^2$ is at most $5 \cdot 1763 + 5 < D$. Therefore, we can modify the coloring by changing the color of $u$ so that it is distinct from the color of $v$ as well as from the colors of the neighbors if $u$ in $G'^2$. The degree of $x$ in $G^2$ is at most $1763 + 7 < D$. Hence, we can extend this coloring to $x$. This contradicts the $D$-minimality of $G$.

4. Let $x$ be a 2-vertex adjacent to $u$. Let $y$ be the vertex adjacent to $x$ distinct from $u$. Let $w$ be the neighbor of $u$ distinct from $x$ and $v$. By the $D$-minimality of $G$, the square of the graph $G' = G \setminus \{x, u\}$ is $(D+2)$-colorable. Fix such a coloring. The vertex $y$ has degree at most $D-1$ in $G'$, therefore at least two colors are unused on closed neighborhood of $y$ in $G'$. Choose a color for $x$ from the unused colors so that it is distinct from the color of $w$. The degree of $v$ in $G'^2$ is at most $2 \cdot 1763 + 2 < D$. Therefore, it is possible to change the color of $v$ so that it is distinct from the colors of $x$ and $w$. Finally choose a color for $u$: its degree is at most $1763 + 6 < D$ in $G^2$. Therefore, it is always possible. This contradicts the $D$-minimality of $G$.

5. The square of the graph $G' = G \setminus \{v_2, v_3, v_4, v_5, v_6, x_2, x_6, y_3, y_5, z_3, z_5\}$ is $(D+2)$-colorable (the removed vertices are marked by empty circles in Figure 5). Fix a coloring of $G'$ with $D+2$ colors. Since the degree of $x_4$ in $G'$ is at most $D-3$, there are at least four colors which are not assigned to a vertex of the closed neighborhood of $x_4$ in $G'$. Let $L_4$ be the set of the unused colors. The degree of $v_1$ in $G'$ is at most $D-2$, therefore the set $L_1$ of colors that do not appear on closed neighborhood of $v_1$ has size at least three. Let $c_5$ be the color of $w_5$ and $c_3$ the color of $w_3$. Assign the list $L_1$ to vertices $v_2$ and $v_6$, the list $L_4$ to the vertex $v_4$, the list $L_4 \setminus \{c_5\}$ to the vertex $y_5$ and the list $L_4 \setminus \{c_3\}$ to the vertex $y_3$. All 2-vertices of the configuration are adjacent only to small vertices. Therefore, if we were able to color the subgraph $G''$ of $G^2$ induced by $\{v_2, v_4, v_6, y_3, y_5\}$ from the lists, we could choose colors for the 2-vertices of the configuration carefully and extend the coloring to the coloring of the whole graph $G^2$. This would eventually contradict the $D$-minimality of $G$.

   However, such a coloring of $G''$ always exists. Choose a color for $v_4$ from $L_4$ arbitrarily, and remove this color from the lists of the remaining four vertices. The graph $G'' \setminus \{v_4\}$ is a 4-cycle. Since it is 2-choosable, the remaining vertices of $G''$ can be colored from the assigned lists.

□

### 4.2 Initial charge and the first discharging phase

We assign initial charge to the vertices and the faces of a graph in the same way as in Section 3.2. In particular, the sum of initial charges is negative by Proposition 9. Let $\varepsilon = 1/588$. The goal of the first phase is that each 2-vertex receives $2\varepsilon$ units of charge and the amount of charge of other vertices and faces is not decreased too much.

If $u$ is a 2-vertex, an edge $e = uv$ is *void* if either $d(v) \in \{2, 4, 5, 6\}$, or $v$ is a 3-vertex and all its neighbors are small. Intuitively, the void edges are those through which it may be impossible to send any charge to $u$.

In order to simplify the analysis of final charge of big vertices, we send all charge transfered from a big vertex through the edges incident to it. Each rule that deals with big vertices specifies through which edge the charge is (considered to be) sent. The value of $\varepsilon$ and the bound on the degree of big vertices was chosen in such a way that a big vertex is able to send $1 - \varepsilon$ units of charge through each edge incident to it, and its final charge is still non-negative.

If $v$ is a big vertex, we call an edge $uv$ *red* if one of the following conditions holds:

- the vertex $u$ is a 2-vertex, $e \neq uv$ is the other edge incident to $u$, and $e$ is void, or
- the vertex $u$ is a 3-vertex, $x_1$ and $x_2$ are the neighbors of $u$ distinct from $v$, both $x_1$ and $x_2$ are 2-vertices, and all the neighbors of $x_1$ and $x_2$ are small.

The edges incident to big vertices which are not red are called *green*. Intuitively, the green edges are those through which the big vertex does not need to send "too much" charge and the red ones are those through which almost one unit of charge has to be sent.

In order to simplify the description of the rules, we define the following operation: if $f$ is a 6-face and $F$ is the set containing $f$ and all the 6-faces sharing an edge with $f$, a 6-face $f$ is *boosted* from a vertex or face $z$ when $3\varepsilon$ units of charge are transferred from $z$ to each face of $F$. Note that the charge of $z$ decreases by at most $21\varepsilon$.

The discharging rules of the first phase are the following:

**F1** Each $(\geq 7)$-face boosts all the 6-faces sharing an edge with it.

**F2** If $v$ is a big vertex, $e$ is a green edge incident to it and $f$ is a 6-face incident to $e$, then the vertex $v$ boosts $f$. The charge is sent through the edge $e$.

**F3** If $v$ is a small vertex of degree at least 4, then it boosts all the incident 6-faces.

**F4** If $v$ is a 2-vertex and $f$ is a face incident to $v$, then $f$ sends $\varepsilon$ units of charge to $v$.

Note that no charge is sent through a red edge in the first phase. We now analyze the amounts of charge after the first phase:

**Lemma 21.** *Let $G$ be a $D$-minimal graph. After the first phase of discharging, the following claims hold:*

1. *at most $1/8$ units of charge was sent through each green edge,*
2. *the charge of a small vertex of degree $d \geq 4$ has decreased by at most $d/16$,*
3. *the charge of each 2-vertex is $2\varepsilon - 1$, and*
4. *the charge of each face is non-negative.*

*Proof.* We prove each claim separately:

1. Charge is sent through green edges only by Rule F2. Each green edge $e$ is incident to at most two 6-faces and thus the total amount of charge sent through $e$ is at most $42\varepsilon \leq 1/8$.
2. Charge is sent from small vertices only by Rule F3. A $d$-vertex is incident to at most $d$ 6-faces. Therefore, the total amount of sent charge is at most $21\varepsilon d \leq d/16$.
3. Each 2-vertex receives $\varepsilon$ units of charge from both the incident faces by Rule F4. Therefore, its charge becomes $2\varepsilon - 1$.
4. Charge is sent from faces by Rules F1 and F4. A $d$-face $f$ shares an edge with at most $d$ 6-faces. Therefore, the total amount of charge sent from $f$ by Rule F1 is at most $21\varepsilon d$. Since at most $d$ 2-vertices are incident to $f$, at most $\varepsilon d$ units of charge are sent by Rule F4. In total, at most $22\varepsilon d$ units of charge are sent from $f$.

   The charge of a $d$-face with $d \geq 7$ after the first phase is at least

$$\frac{d}{2} - 3 - 22\varepsilon d = \left(\frac{1}{2} - 22\varepsilon\right) d - 3 \geq \frac{3}{7}d - 3 \geq 0.$$

Hence, if $f$ is a $(\geq 7)$-face, its final charge is non-negative.

It remains to consider the case when $f$ is a 6-face. Let $k$ be the number of 2-vertices incident to $f$. Observe that $k$ does not exceed 3: otherwise $f$ contains at least four 2-vertices and it thus contains either a 3-thread or two vertices connected by two 2-threads. Both configurations are reducible by Lemma 20.

Initial charge of $f$ is zero and $f$ sends out charge of $k\varepsilon$ by Rule F4. If $k = 0$, the final charge of $f$ is non-negative. Assume that $k > 0$. It is sufficient to prove that $f$ receives at least $3\varepsilon$ units of charge by Rules F1, F2 and F3. We show that $f$ or one of the 6-faces incident to $f$ is boosted during the first phase.

If $f$ shares an edge with a $(\geq 7)$-face, $f$ is incident to a small vertex of degree at least 4, or $f$ is incident to a green edge, then $f$ itself is boosted. Therefore, we may assume that no edge incident to $f$ is green, all the vertices incident to $f$ are either big or have degree 2 or 3, and all the faces sharing an edge with $f$ are 6-faces.

Let $v_1, \ldots, v_6$ be the vertices of $f$ in a cyclic order around the face.

Suppose first that $f$ is incident to at least two big vertices. Assume that $v_1$ is a big vertex. The second big vertex of $f$ is $v_4$: otherwise, the two big vertices are either $f$-neighbors or share an $f$-neighbor and at least one of the edges of $f$ is green. If all the $f$-neighbors of $v_1$ and $v_4$ were 2-vertices, then $v_1$ and $v_4$ would be joined by two 2-threads, which is impossible by Lemma 20. Therefore at least one of the big vertices is adjacent to a 3-vertex. Assume that $v_2$ is a 3-vertex. But since $v_4$ is big, the edge $v_1v_2$ is green regardless of the degree of $v_3$. Therefore, the face $f$ is boosted.

If $f$ is incident to no big vertex, then no two 2-vertices of $f$ are adjacent by Lemma 20(1). Assume that $v_2$ is a 2-vertex. Therefore, $v_1$ and $v_3$ are 3-vertices. Let $x_1$ and $x_3$ be the neighbors of $v_1$ and $v_3$ not incident to $f$. Since $v_6$ and $v_4$ are small, both $x_1$ and $x_3$ are big by Lemma 20(3). Let $f'$ be the 6-face incident to $v_2$ distinct from $f$. Note that both $x_1$ and $x_3$ belong to the 6-face $f'$ and share a common $f'$-neighbor. Hence, at least one of the edges incident to $f'$ is green. Consequently, $f'$ is boosted and $f$ receives the charge of $3\varepsilon$ units.

It remains to consider the case when $f$ contains exactly one big vertex, say $v_1$. If $v_4$ were a 2-vertex, we could use a similar argument as in the previous paragraph to show that the other face incident to $v_4$ is boosted. Therefore, we can assume that $v_4$ is a 3-vertex. In addition, either $v_2$ or $v_3$ is a 2-vertex, since the edge $v_1v_2$ is not green.

First suppose that $v_2$ is a 2-vertex. Hence $v_3$ is a 3-vertex. Let $x_3$ and $x_4$ be the neighbors of $v_3$ and $v_4$ not incident to $f$. If $x_3$ is big, then the edge $v_1v_2$ is green. And, if $x_4$ is big, then the edge $x_4v_4$ is green. In both the cases, $f$ receives the required charge. If both $x_3$ and $x_4$ are small, the configuration is reducible by Lemma 20(4). The case that $v_6$ is a 2-vertex is symmetrical.

Suppose now that both $v_2$ and $v_6$ are 3-vertices and $v_3$ is a 2-vertex. We may assume that the neighbors of $v_2$ and $v_6$ (including $v_5$) distinct from $v_1$ are 2-vertices: otherwise, one of the edges $v_1v_2$ and $v_1v_6$ would be green. Let $x_2$, $x_4$ and $x_6$ be the vertices adjacent to $v_2$, $v_4$ and $v_6$ and not incident to $f$. By Lemma 20(3), the vertex $x_4$ is big. Let $f_3$ and $f_5$ be the faces incident to $v_3$ and $v_5$ and distinct from $f$. Let $y_5$ be the remaining vertex of $f_5$ distinct from $x_6$, $x_4$, $v_4$, $v_5$ and $v_6$. Let $y_3$ be the remaining vertex of $f_3$ distinct from $x_2$, $x_4$, $v_2$, $v_3$ and $v_4$. The degrees of both $y_3$ and $y_5$ must be 3: they cannot be two by Lemma 20(1) and if one of them were greater than 3, then one of the edges $y_3x_4$ and $y_5x_4$ would be green and $f$ would receive charge because of boosting from $f_3$ or $f_5$. Let $z_3$ and $z_5$ be the neighbors of $y_3$ and $y_5$ distinct from $x_6$, $x_4$ and $x_2$. Both $z_3$ and $z_5$ must be 2-vertices and all their neighbors must be small, since otherwise one of edges $y_3x_4$ or $y_5x_4$ is green. However, the resulting configuration is reducible by Lemma 20(5). This finishes the proof of the claim.

□

### 4.3 The second phase of discharging

In this phase we redistribute the charge so that the final charge of all vertices is non-negative. The following rules are used during this phase:

**S1** If $v$ is a big vertex adjacent to a 2-vertex $u$, then $v$ sends $1-\varepsilon$ units of charge to $u$ if $uv$ is red and it sends $3/4$ units of charge to $u$ if $uv$ is green. The charge is sent through the edge $uv$.

**S2** If $v$ is a big vertex adjacent to a 3-vertex $u$ and the edge $uv$ is red, then $v$ sends $(1-\varepsilon)/2$ units of charge to both the 2-vertices adjacent to $u$. The charge is sent through the edge $uv$.

**S3** Suppose that $v$ is a big vertex adjacent to a 3-vertex $u$, the edge $uv$ is green, and $x$ is a 2-vertex adjacent to $u$. If $x$ has a big neighbor, then $v$ sends charge of $1/4$ to $x$. Otherwise, $v$ sends charge of $1/2$ to $x$. The charge is sent through the edge $uv$.

**S4** If $v$ is a big vertex adjacent to a $d$-vertex $u$, $4 \leq d \leq 6$, then the vertex $v$ sends $3/4$ units of charge to $u$. The charge is sent through the edge $uv$.

**S5** If $v$ is a $d$-vertex, $4 \leq d \leq 6$, adjacent to a 2-vertex $u$, and if $v$ has at least one big neighbor, then $v$ sends $1/2$ units of charge to $u$.

**S6** If $v$ is a small vertex of degree $d > 6$ adjacent to a 2-vertex $u$, then $v$ sends $1/2$ units of charge to $u$.

We now analyze the amounts of charge sent during the second phase:

**Lemma 22.** *Let $G$ be a $D$-minimal graph. The following claims hold:*

1. *at most $3/4$ units of charge was sent through each green edge during the second phase,*
2. *at most $1-\varepsilon$ units of charge was sent through each red edge during the second phase, and*
3. *the charge of each vertex is non-negative after performing the first and the second phase.*

*Proof.* We prove each claim separately:

1. At most one of Rules S1, S3 and S4 applies to each green edge. At most $3/4$ units of charge is sent through such an edge by any of the rules. The only case in which this is not obvious is the case of Rule S3. However, there can be at most one vertex $x$ without a big neighbor that satisfies the assumptions of the rule: otherwise the edge $uv$ is red.
2. At most one of Rules S1 and S2 applies to each red edge and the charge sent through such an edge is exactly $1-\varepsilon$ by any of the rules.
3. Let $v$ be a $d$-vertex of $G$. We consider several cases regarding the degree of the vertex $v$:

$d = 2$**:** Let $x$ and $y$ be the neighbors of $v$. It suffices to show that $v$ received at least $1 - \varepsilon$ units of charge during the second phase because charge of $v$ was at least $2\varepsilon$ after the first phase by Lemma 21.

Suppose first that $x$ is big. If the edge $vy$ is void, then the edge $xv$ is red and $v$ received charge of $1 - \varepsilon$ from $x$ by Rule S1. Assume that the edge $vy$ is not void and that the edge $xv$ is green. Consequently, $v$ received $3/4$ units of charge by Rule S1. Additionally, since $vy$ is not void, then either $y$ is a 3-vertex and has a big neighbor $w$, or $y$ is a $(\geq 7)$-vertex. In the former case, $v$ receives $1/4$ units of charge from $w$ by Rule S3. In the latter case, $y$ sends $1/2$ units of charge to $v$ by Rules S1 or S6. In both the cases, the total charge received by $v$ is at least 1.

The final case is that both $x$ and $y$ are small. By Lemma 20(1), neither $x$ nor $y$ has degree 2. We show that $v$ receives at least $(1-\varepsilon)/2$ units of charge through $x$. Note that by symmetry $v$ also receives at least $(1-\varepsilon)/2$ units of charge through $y$, i.e., $v$ receives $1 - \varepsilon$ units of charge in total. Let $d'$ be the degree of $x$. If $3 \leq d' \leq 6$, at least one neighbor of $x$ must be big by Lemma 20(3). Consequently, $v$ receives at least $(1-\varepsilon)/2$ by one of Rules S2, S3 and S5. If $d' \geq 7$, then $v$ receives $1/2$ from $x$ by Rule S6.

$d = 3$**:** None of the discharging rules changes the charge of a vertex of degree three. Therefore, the final charge of $v$ is zero.

$4 \leq d \leq 6$**:** The $d$-vertex $v$ sent charge of at most $d/16$ units during the first phase by Lemma 21(2). If $v$ is not adjacent to a big vertex, then it does not send anything during the second phase. Otherwise, it sends at most $(d-1)/2$ units of charge by Rule S5 and receives charge of at least $3/4$ units by Rule S4. Therefore, the final charge of $v$ is

$$d - 3 - \frac{d}{16} - \frac{d-1}{2} + \frac{3}{4} = \frac{7d}{16} - \frac{7}{4} \geq 0.$$

$d \geq 6$ **and** $v$ **is small:** The vertex $v$ sends at most $d/16$ units of charge during the first phase by Lemma 21(2) and at most $d/2$ units of charge during the second phase by Rule S6. Therefore, the final charge of $v$ is at least

$$d - 3 - \frac{d}{16} - \frac{d}{2} = \frac{7d}{16} - 3 > 0.$$

$v$ **is big:** All the charge sent out from the big vertex $v$ was sent through some of the edges incident to it. Charge is sent through a red edge $e$ only in the second phase and the total amount of such charge is at most $1-\varepsilon$ by the previous claim of this lemma. At most $1/8$ units of charge is sent through a green edge $e$ in the first phase by Lemma 21 and at most $3/4$ units in the second phase, thus in total $7/8 < 1-\varepsilon$. Therefore, $v$ has the final charge of at least $d-3-(1-\varepsilon)d = \varepsilon d - 3 \geq 0$ (recall that $v$ is a $d$-vertex with $d > 1763$).

□

### 4.4 Final step

We now combine our claims from the previous subsections:

**Theorem 23.** *If $G$ is a planar graph of maximum degree $\Delta \geq 8821$ and girth at least six, then $G$ has a proper $L(1,1)$-labeling with span $\Delta+1$, i.e., $\chi(G^2) \leq \Delta+2$.*

*Proof.* If the statement of the theorem is false, then there exists a $D$-minimal graph. Consider such a $D$-minimal graph $G$. Assign charge to the vertices and the faces of $G$ as described in Section 3.2. By Proposition 9, the sum of all the charges is negative. Apply the discharging rules of the two phases described in Sections 4.2 and 4.3. The final amount of charge of each face is non-negative after the first phase by Lemma 21 and it is preserved during the second phase, i.e., it is non-negative after the second phase. The final amount of charge of each vertex is non-negative after the second phase by Lemma 22. Therefore, the total final amount of charge is non-negative. We conclude that there is no $D$-minimal graph. □

## 5 Conclusion

One may ask whether the bound proven in Theorem 17 cannot be further improved, e.g., to $2p+\Delta-3$. However, the bound is tight for all considered pairs of $\Delta$ and $p$ as shown in the following proposition (though the next proposition follows from results of [12], see Proposition 25, we include its short proof for the sake of completeness):

**Proposition 24.** *Let $p$ and $\Delta \geq 2p$ be arbitrary integers. There exists a tree $T$ with maximum degree $\Delta$ such that the span of an optimal $L(p,1)$-labeling of $T$ is $2p + \Delta - 2$.*

*Proof.* It can be easily proven by induction on the order of a tree that the span of an optimal labeling of any tree with maximum degree $\Delta$ is at most $2p + \Delta - 2$. Therefore, it is enough to construct a tree with no $L(p,1)$-labeling with span less than $2p + \Delta - 2$. Let us consider the following tree $T$: a vertex $v_0$ is adjacent to $\Delta$ vertices $v_1, \ldots, v_\Delta$ and each of the vertices $v_1, \ldots, v_\Delta$ is adjacent to $\Delta - 1$ leaves. Clearly, the maximum degree of $T$ is $\Delta$.

Assume that $T$ has a proper $L(p,1)$-labeling $c$ of span at most $2p+\Delta-3$. Since $\Delta \geq 2p$, the color of at least one of the vertices $v_0, \ldots, v_\Delta$ is between $p-1$ and $p+\Delta-2$, i.e., $c(v_i) \in \{p-1, \ldots, p+\Delta-2\}$ for some $i$. The color of each neighbor of $v_i$ is either at most $c(v_i) - p$ or at least $c(v_i) + p$. Since there are only $\Delta - 1$ such colors, two of the neighbors of $v_i$ have the same color and the labeling $c$ is not proper. □

One may also ask whether the condition $\Delta \geq 190 + 2p$ in Theorem 17 cannot be further weakened. The answer is positive (we strongly believe that Conjecture 2 holds with $M(7) \approx 50$) but we decided not to try to refine the discharging phase and the analysis in order to avoid adding more pages to the paper. It is also natural to consider $L(p,q)$-labelings of planar graphs with no short cycles for $q > 2$. In such case, the following result of Georges and Mauro [12] comes to use:

**Proposition 25.** *Let $p$ and $q$, $p \geq q$, be two positive integers. There exists a $\Delta_0$ such that the span of an optimal $L(p,q)$-labeling of the infinite $\Delta$-regular tree $T_\Delta$, $\Delta \geq \Delta_0$, is the following:*

$$\lambda_{p,q}(T_\Delta) = \begin{cases} q\Delta + \lfloor\frac{p-1}{q}\rfloor q + p - q & \text{if } \lfloor\frac{p}{q}\rfloor \leq \frac{p}{q} \leq \lfloor\frac{p}{q}\rfloor + \frac{1}{2}, \\ q\Delta + 2\lfloor\frac{p}{q}\rfloor q & \text{otherwise.} \end{cases}$$

We now derive the following theorem on the spans of optimum $L(p,q)$-labelings of planar graphs for $q > 1$ with large girth:

**Theorem 26.** *Let $p$ and $q$, $p \geq q$, be two positive integers. There exists an integer $\Delta_0$ such that the following holds for every planar graph $G$ of maximum degree $\Delta \geq \Delta_0$ and of girth at least 21:*

$$\lambda_{p,q}(G) \leq \begin{cases} q\Delta + \lfloor\frac{p-1}{q}\rfloor q + p - q & \text{if } \lfloor\frac{p}{q}\rfloor \leq \frac{p}{q} \leq \lfloor\frac{p}{q}\rfloor + \frac{1}{2}, \\ q\Delta + 2\lfloor\frac{p}{q}\rfloor q & \text{otherwise.} \end{cases}$$

*The bounds are tight for all $p$, $q$ and $\Delta \geq \Delta_0$.*

*Proof.* Choose $\Delta_0$ to be the maximum of $\Delta_0$ from Proposition 25 and $4 \cdot \frac{p+q-1}{q}$. Let $\Lambda$ be the optimum span of an $L(p,q)$-labeling of the infinite $\Delta$-regular tree $T_\Delta$. Let $G$ be a planar graph of the smallest order such that the maximum degree of $G$ is at most $\Delta$, $G$ contains no cycle of length less than 21 and $\lambda_{p,q}(G) > \Lambda$. Clearly, $G$ is connected. Moreover, $G$ contains a cycle (any tree of maximum degree at most $\Delta$ is a subgraph of $T_\Delta$ and thus it has an $L(p,q)$-labeling of span at most $\Lambda$). Keep now removing vertices of degree one from $G$ until there is no such vertex. Let $G'$ be the obtained (non-empty) subgraph of $G$.

Since $G'$ is a planar graph of minimum degree two with girth at least 21, it contains a 4-thread. Let the 4-thread be comprised of vertices $v_1$, $v_2$, $v_3$ and $v_4$. By our assumption, the graph $G \setminus \{v_2, v_3\}$ has an $L(p,q)$-labeling $c$ of span at most $\Lambda$. The labeling $c$ can be extended to $v_2$ and $v_3$: there are at most $2p-1$ colors forbidden for $v_2$ because of the color assigned to $v_1$ and $2q-1$ colors forbidden because of the other neighbor of $v_1$, and $2q-1$ colors because of the vertex $v_4$. In total, there are at most $2p+4q-3$ forbidden colors for $v_2$. Similarly, there are at most $2p+4q-3$ forbidden colors for $v_3$. Hence, there are at least $\Lambda+1-(2p+4q-3) > q\Delta_0-(2p+4q-3) \geq 2p$ available colors for each of $v_2$ and $v_3$. Consequently, the vertices $v_2$ and $v_3$ can be assigned colors that differ by at least $p$ and the labeling $c$ can be extended to $v$.

We now show that the bound from the statement is tight. By the compactness principle and Proposition 25, there exists a finite tree $T$ of maximum degree $\Delta$ with $\lambda_{p,q}(T) = \Lambda$. Therefore, the bounds from the statement of the theorem cannot be improved. $\square$

Note that Theorem 26 holds for any minor-closed class of graphs. We think that the assumption on the girth from Theorem 26 can be weakened to seven:

**Problem 1.** *Is it true that for every positive integers $p$ and $q$, $p \geq q$, there exists an integer $\Delta_0$ such the following holds for every planar graph $G$ of maximum degree $\Delta \geq \Delta_0$ and of girth at least seven:*

$$\lambda_{p,q}(G) \leq \begin{cases} q\Delta + \lfloor \frac{p-1}{q} \rfloor q + p - q & \text{if } \lfloor \frac{p}{q} \rfloor \leq \frac{p}{q} \leq \lfloor \frac{p}{q} \rfloor + \frac{1}{2}, \\ q\Delta + 2\lfloor \frac{p}{q} \rfloor q & \text{otherwise.} \end{cases}$$

In this paper, we have settled Conjecture 2. However, it remains open whether the following weakened version of it is true:

**Problem 2.** *Is it true that there exists an integer $M$ such that the square of every planar graph $G$ with maximum degree $\Delta \geq M$ and girth at least 5 is $(\Delta + 2)$-colorable?*

# References


[1] G. Agnarsson, R. Greenlaw, M. M. Halldórsson, *Powers of chordal graphs and their coloring*, to appear in Congr. Numer.

[2] G. Agnarsson, M. M. Halldórsson, *Coloring powers of planar graphs*, Proc. SODA'00, SIAM press, 2000, 654–662.

[3] G. Agnarsson, M. M. Halldórsson, *Coloring powers of planar graphs*, SIAM J. Discrete Math. 16(4) (2003), 651–662.

[4] N. Alon, B. Mohar: *The chromatic number of graph powers*, Combin. Probab. Comput. 11 (2002), 1–10.

[5] H. L. Bodlaender, T. Kloks, R. B. Tan, J. van Leeuwen, *$\lambda$-coloring of graphs*, G. Goos, J. Hartmanis, J. van Leeuwen, eds., Proc. STACS'00, LNCS Vol. 1770, Springer, 2000, 395–406.

[6] O. Borodin, H. J. Broersma, A. Glebov, J. van den Heuvel, *Stars and bunches in planar graphs. Part II: General planar graphs and colourings*, CDAM Reserach Report 2002-05, 2002.

[7] G. J. Chang, W.-T. Ke, D. D.-F. Liu, R. K. Yeh, *On $L(d,1)$-labellings of graphs*, Discrete Math. 3(1) (2000), 57–66.

[8] G. J. Chang, D. Kuo, *The $L(2,1)$-labeling problem on graphs*, SIAM J. Discrete Math. 9(2) (1996), 309–316.

[9] P. Erdős, A. L. Rubin, H. Taylor, *Choosability in graphs*, Congress. Numer. 26 (1980), 122–157.

[10] J. Fiala, J. Kratochvíl, T. Kloks, *Fixed-parameter complexity of $\lambda$-labelings*, Discrete Appl. Math. 113(1) (2001), 59–72.

[11] D. A. Fotakis, S. E. Nikoletseas, V. G. Papadopoulou, P. G. Spirakis, *NP-Completeness results and efficient approximations for radiocoloring in planar graphs*, B. Rovan, ed., Proc. MFCS'00, LNCS Vol. 1893, Springer, 2000, 363–372.

[12] J. P. Georges, D. W. Mauro, *Labeling trees with a condition at distance two*, Discrete Math. 269 (2003), 127–148.

[13] J. R. Griggs, R. K. Yeh, *Labeling graphs with a condition at distance 2*, SIAM J. Discrete Math. 5 (1992), 586–595.

[14] J. van den Heuvel, S. McGuiness, *Colouring of the square of a planar graph*, J. Graph Theory 42 (2003), 110–124.

[15] S. Jendrol', Z. Skupien, *Local structures in plane maps and distance colourings*, Discrete Math. 236 (2001), 167–177.

[16] S. Jendrol', H.-J. Voss, *Light subgraphs of graphs embedded in the plane and in the projective plane—a survey*, IM Preprint series A, No. 1/2004, Pavol Jozef Šafárik University, Slovakia, 2004.

[17] T. R. Jensen, B. Toft, *Graph coloring problems*, John-Wiley and Sons, New York, 1995.

[18] T. K. Jonas, *Graph coloring analogues with a condition at distance two: $L(2,1)$-labelings and list $\lambda$-labelings*, Ph.D. thesis, University of South Carolina, SC, 1993.

[19] J.-H. Kang, *$L(2,1)$-labeling of 3-regular Hamiltonian graphs*, submitted for publication.

[20] J.-H. Kang, *$L(2,1)$-labelling of 3-regular Hamiltonian graphs*, Ph.D. thesis, University of Illinois, Urbana-Champaign, IL, 2004.

[21] D. Kráľ, *An exact algorithm for channel assignment problem*, Discrete Appl. Math. 145(2) (2004), 326–331.

[22] D. Kráľ, R. Škrekovski, *A theorem about channel assignment problem*, SIAM J. Discrete Math., 16(3) (2003), 426–437.

[23] D. Kráľ, *Coloring powers of chordal graphs*, SIAM J. Discrete Math. 18(3) (2004), 451–461.

[24] K.-W. Lih, W.-F. Wang, *Coloring the square of an outerplanar graph*, technical report, Academia Sinica, Taiwan, 2002.

[25] T. Madaras, A. Marcinová, *On the structural result on normal plane maps*, Discuss. Math. Graph Theory 22(2002), 293–303.

[26] C. McDiarmid, *On the span in channel assignment problems: bounds, computing and counting*, Discrete Math. 266 (2003), 387–397.

[27] M. Molloy, M. R. Salavatipour, *A bound on the chromatic number of the square of a planar graph*, to appear in J. Combin. Theory Ser. B.

[28] M. Molloy, M. R. Salavatipour, *Frequency channel assignment on planar networks*, R. H. Möhring, R. Raman, eds., Proc. ESA'02, LNCS Vol. 2461, Springer, 2002, 736–747.

[29] D. Sakai, *Labeling chordal graphs: distance two condition*, SIAM J. Discrete Math. 7 (1994), 133–140.

[30] V. G. Vizing, *Colouring the vertices of a graph in prescribed colours* (in Russian), Diskret. Anal. 29 (1976), 3–10.

[31] W.-F. Wang, K.-W. Lih, *Labeling planar graphs with conditions on girth and distance two*, SIAM J. Discrete Math. 17(2) (2003), 264–275.

[32] G. Wegner, *Graphs with given diameter and a coloring problem*, technical report, University of Dortmund, Germany, 1977.

[33] S. A. Wong, *Colouring graphs with respect to distance*, M.Sc. thesis, University of Waterloo, Canada, 1996.