Separation properties of two convex polyhedral sets with RHS-parameters

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Abstract

Separation of convex sets is widely used in many branches of mathematics. Often in practice input data are known only approximately and it is advisable to deal with parameters. We initiate a combining of these two principles – separation and parametrization – together. Important properties (existence, description, stability etc.) of separating hyperplanes of two convex polyhedral sets depending on right-hand-side parameters, is dealt with in this article.

Keywords: *separating hyperplane, separating supporting hyperplane, RHS-parameters, convex polyhedra, solution set, stability set.*

1 Introduction

For the purpose of this paper we introduce the following two kinds of separability (cf. [9]).

Definition 1. Sets $X, Y \subset \mathbb{R}^n$ are called *weakly separable* if X, Y are nonempty and there exists a hyperplane $\mathcal{R}$ such that $X \subseteq \overline{\mathcal{R}^-}, Y \subseteq \overline{\mathcal{R}^+}$ holds*. Sets $X, Y \subset \mathbb{R}^n$ are called *strongly separable* if X, Y are weakly

*If $\mathcal{R} = \{\mathbf{x} \mid \mathbf{r}^T\mathbf{x} = s\}$, then $\overline{\mathcal{R}^+} = \{\mathbf{x} \mid \mathbf{r}^T\mathbf{x} \geq s\}$, $\overline{\mathcal{R}^-} = \{\mathbf{x} \mid \mathbf{r}^T\mathbf{x} \leq s\}$.

separable and $dimX = dimY = n$ holds.
If sets $X, Y \subset \mathbb{R}^n$ are nonempty and there exists a hyperplane $\mathcal{R}$ with the property $X \subseteq \overline{\mathcal{R}^-}, Y \subseteq \overline{\mathcal{R}^+}$, then $\mathcal{R}$ is called *the separating hyperplane* of the sets X, Y.

Simply, if sets X, Y are strongly separable, then they are weakly separable with the same system of separating hyperplanes.

In this article we deal with two families of convex polyhedral sets ($\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$):

$$M_1(\boldsymbol{\lambda}) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda}\}, \tag{1}$$
$$M_2(\boldsymbol{\nu}) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \leq \boldsymbol{\nu}\}, \tag{2}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\nu} \in \mathbb{R}^l$ are vectors of parameters. Parametrization is inspired by [1, 2, 10]. The proposed combination of separation and parametrization has never been presented before, except some basic results in [7]. In Sections 2–3 we derive basic separation properties of the families (1), (2). We will define so called solution set and in the sequel so called stability sets. For definition of stability sets we use the explicit description of all separating hyperplanes of two fixed convex polyhedral sets from [3, 4, 5]. Section 5 is concerned with some special cases of families (1), (2). In the last section we give some remarks and examples for supporting separation of the families (1), (2).

Let us introduce some notation. Given a matrix $\mathbf{M}$, the expressions $\mathbf{M}_{i,\cdot}$, $\mathbf{M}_{\cdot,j}$ denote i-th row and j-th column of the matrix $\mathbf{M}$, respectively. Given vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$, the expression $\mathbf{a} < \mathbf{b}$ means $a_i < b_i\ \forall i$. For any set $\mathcal{X}$, let us denote by $\overline{\mathcal{X}}$, $int\mathcal{X}$, $dim\mathcal{X}$ and $conv\mathcal{X}$ the closer, the interior, the dimension, and the convex hull of $\mathcal{X}$, respectively.

2 Solution set

Definition 2. *The solution set* is the set of all $(\boldsymbol{\lambda},\ \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ for which convex polyhedral sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ are strongly separable.

Let us introduce

$$\mathcal{P}_1 \equiv \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid dim M_1(\boldsymbol{\lambda}) = n\}, \tag{3}$$

$$\mathcal{P}_2 \equiv \{\boldsymbol{\nu} \in \mathbb{R}^l \mid dim M_2(\boldsymbol{\nu}) = n\}, \tag{4}$$

$$\mathcal{P} \equiv \mathcal{P}_1 \times \mathcal{P}_2 = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid dim M_1(\boldsymbol{\lambda}) = dim M_2(\boldsymbol{\nu}) = n\}, \tag{5}$$

$$\mathcal{U} \equiv \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid int M_1(\boldsymbol{\lambda}) \cap int M_2(\boldsymbol{\nu}) \neq \emptyset\}. \tag{6}$$

Theorem 1. *The set $\mathcal{P} \setminus \mathcal{U}$ forms the solution set.*

Proof. It follows from the basic separation theorem, see e.g. [8]. □

Now we derive the description of the set $\mathcal{P}_1$. Descriptions the of sets $\mathcal{P}_2$, $\mathcal{P}$ are quite similar. From now on we suppose that matrices $\mathbf{A}$, $\mathbf{C}$ do not contain the zero row.

Theorem 2. *The set $\mathcal{P}_1$ has the following description*

$$\mathcal{P}_1 = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T \boldsymbol{\lambda} > 0 \quad \forall i \in I\}, \tag{7}$$

where $\mathbf{h}_i$, $i \in I$ are non-negative vectors in directions of all edges of the convex polyhedral cone

$$\mathcal{N} = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{A}^T \mathbf{y} = \mathbf{0},\ \mathbf{y} \geq \mathbf{0}\}.$$

Proof. $\mathcal{P}_1$ is the set of all $\boldsymbol{\lambda} \in \mathbb{R}^m$ such that $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} < \boldsymbol{\lambda}\} \neq \emptyset$, which means $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} - \boldsymbol{\varepsilon}\} \neq \emptyset$ for an infinitesimal vector $\boldsymbol{\varepsilon} > \mathbf{0}$. It is true if and only if the problem

$$\max \{\mathbf{0}^T \mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} - \boldsymbol{\varepsilon}\}$$

has an optimal solution. It follows from the theory of duality of linear programming that this is equivalent to the condition, that the problem

$$\min \{(\boldsymbol{\lambda} - \boldsymbol{\varepsilon})^T \mathbf{y} \mid \mathbf{y} \in \mathcal{N}\}, \tag{8}$$

has an optimal solution. The set $\mathcal{N}$ represents a convex polyhedral cone with a vertex in the origin and it is included in the first orthant. Vectors $\mathbf{h}_i \geq \mathbf{0}$, $i \in I$ are vectors in directions of all edges of $\mathcal{N}$. It follows from the theory of polar cones (see [11]), that the problem (8) has an optimal solution if and only if

$$\mathbf{h}_i^T (\boldsymbol{\lambda} - \boldsymbol{\varepsilon}) \geq 0 \quad \forall i \in I \text{ and an infinitesimal vector } \boldsymbol{\varepsilon} > \mathbf{0}$$

holds. Hence the description of $\mathcal{P}_1$ is as follows

$$\begin{aligned}\mathcal{P}_1 &= \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T \boldsymbol{\lambda} \geq \mathbf{h}_i^T \boldsymbol{\varepsilon} \ \ \forall i \in I \text{ and an infinitesimal } \boldsymbol{\varepsilon} > \mathbf{0}\} = \\ &= \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T \boldsymbol{\lambda} > 0 \ \ \forall i \in I\}.\end{aligned}$$

□

Theorem 3. *It holds*

$$\overline{\mathcal{P}_1} = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid M_1(\boldsymbol{\lambda}) \neq \emptyset\}.$$

Proof. The proof is similar to the proof of Theorem 2. The description of $\overline{\mathcal{P}_1}$ is

$$\overline{\mathcal{P}_1} = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T \boldsymbol{\lambda} \geq 0 \quad \forall i \in I\}, \tag{9}$$

where vectors $\mathbf{h}_i,\ i \in I$ have the same meaning as in the proof of Theorem 2.

□

Analogical statements hold for sets $\overline{\mathcal{P}_2}$, $\overline{\mathcal{P}}$.

Theorem 4. *The set* $\mathcal{U}$ *has the description*

$$\mathcal{U} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid \mathbf{g}_i^T (\boldsymbol{\lambda}, \boldsymbol{\nu}) > 0 \quad \forall i \in I'\}, \tag{10}$$

where $\mathbf{g}_i \geq \mathbf{0},\ i \in I'$ *are vectors in directions of all edges of the convex polyhedral cone*

$$\mathcal{N}' = \{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{m+l} \mid \mathbf{A}^T\mathbf{y} + \mathbf{C}^T\mathbf{z} = \mathbf{0},\ \mathbf{y} \geq \mathbf{0},\ \mathbf{z} \geq \mathbf{0}\}. \tag{11}$$

Proof. $\mathcal{U}$ is a set of all $\boldsymbol{\lambda}$, $\boldsymbol{\nu}$ such that $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} < \boldsymbol{\lambda}, \mathbf{C}\mathbf{x} < \boldsymbol{\nu}\} \neq \emptyset$. If we proceed similarly as in the proof of Theorem 2, we obtain the following result

$$\mathcal{U} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid \mathbf{g}_i^T (\boldsymbol{\lambda}, \boldsymbol{\nu}) > 0 \quad \forall i \in I'\}.$$

□

Assertion 1. *Let* $M_1(\boldsymbol{\lambda}^0)$ *be nonempty and bounded for certain* $\boldsymbol{\lambda}^0 \in \mathbb{R}^m$. *Then* $M_1(\boldsymbol{\lambda})$ *is bounded for all* $\boldsymbol{\lambda} \in \mathbb{R}^m$.

Proof. See [12], $M_1(\boldsymbol{\lambda})$ has the same characteristic cone for all $\boldsymbol{\lambda} \in \mathbb{R}^m$. □

It can be also easily proven that $\mathcal{U} \subseteq \mathcal{P}$. Moreover, either the set $\overline{\mathcal{P} \setminus \mathcal{U}}$ is empty or $dim(\mathcal{P} \setminus \mathcal{U}) = n$. The following theorem states that the set $\overline{\mathcal{P} \setminus \mathcal{U}}$ form a solution set for weak separation.

Theorem 5. *It holds*

$$\overline{\mathcal{P} \setminus \mathcal{U}} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid M_1(\boldsymbol{\lambda}),\ M_2(\boldsymbol{\nu}) \text{ are weakly separable}\}.$$

Proof. Inclusion $\subseteq$: Let us have $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \overline{\mathcal{P} \setminus \mathcal{U}}$. Then there exists a sequence $(\boldsymbol{\lambda}_n, \boldsymbol{\nu}_n) \in \mathcal{P} \setminus \mathcal{U}$, $(\boldsymbol{\lambda}_n, \boldsymbol{\nu}_n) \to (\boldsymbol{\lambda}, \boldsymbol{\nu})$ and separating hyperplanes $\mathcal{R}_n = \{\mathbf{x} \mid \mathbf{r}_n^T \mathbf{x} = s_n\}$, $\|(\mathbf{r}_n, s_n)\| = 1$ for which $M_1(\boldsymbol{\lambda}_n) \subseteq \overline{\mathcal{R}_n^-}$, $M_2(\boldsymbol{\nu}_n) \subseteq \overline{\mathcal{R}_n^+}$ hold. We can assume without the loss of generality that $(\mathbf{r}_n, s_n) \to (\mathbf{r}, s)$ (otherwise we can choose a convergent subsequence). From the continuity of linear constraints it follows that $M_1(\boldsymbol{\lambda}) \subseteq \overline{\mathcal{R}^-}$, $M_2(\boldsymbol{\nu}) \subseteq \overline{\mathcal{R}^+}$ for $\mathcal{R} = \{\mathbf{x} \mid \mathbf{r}^T \mathbf{x} = s\}$.
Inclusion $\supseteq$: Let us have $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ and a hyperplane $\mathcal{R} = \{\mathbf{x} \mid \mathbf{r}^T \mathbf{x} = s\}$ such that $\emptyset \neq M_1(\boldsymbol{\lambda}) \subseteq \overline{\mathcal{R}^-}$, $\emptyset \neq M_2(\boldsymbol{\nu}) \subseteq \overline{\mathcal{R}^+}$. Consider a convex polyhedral set $M_1(\boldsymbol{\lambda} + \boldsymbol{\varepsilon}) = \{\mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} + \boldsymbol{\varepsilon}\}$, where $\boldsymbol{\varepsilon} > \mathbf{0}$ is infinitesimal. Then $dim M_1(\boldsymbol{\lambda} + \boldsymbol{\varepsilon}) = n$. A problem $\max\{(\mathbf{r}^T \mathbf{x} - s) \mid \mathbf{x} \in M_1(\boldsymbol{\lambda} + \boldsymbol{\varepsilon})\}$ has an optimal solution (characteristic cones for $M_1(\boldsymbol{\lambda})$ and $M_1(\boldsymbol{\lambda} + \boldsymbol{\varepsilon})$ are identical according to [12]). Denote by $h(\boldsymbol{\varepsilon})$ the optimal value of this problem. It is obvious that the function $h(\boldsymbol{\varepsilon})$ is continuous. Therefore the set $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}(\mathbf{x} + \frac{h(\boldsymbol{\varepsilon})}{\mathbf{r}^T \mathbf{r}}.\mathbf{r}) \leq \boldsymbol{\lambda} + \boldsymbol{\varepsilon}\} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} + \boldsymbol{\varepsilon} - \frac{h(\boldsymbol{\varepsilon})}{\mathbf{r}^T \mathbf{r}}.\mathbf{A}\mathbf{r}\}$ has dimension n, is the subset of the halfspace $\overline{\mathcal{R}^-}$ and for $\boldsymbol{\varepsilon} \to \mathbf{0}$ converges to $M_1(\boldsymbol{\lambda})$.
Analogously for $M_2(\boldsymbol{\nu})$. □

Let us introduce

$$\mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \equiv$$

$$\left\{(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathbb{R}_+^{m+l+2} \mid \begin{pmatrix} \mathbf{A}^T & \mathbf{0} & \mathbf{C}^T & \mathbf{0} \\ \boldsymbol{\lambda}^T & 1 & \boldsymbol{\nu}^T & 1 \\ \mathbf{1}^T & 1 & \mathbf{1}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ u_{m+1} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix} \right\}. \tag{12}$$

With help of the convex polytope $\mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu})$ we give the explicit description of all separating hyperplanes of the convex polyhedral sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ for all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathcal{P} \setminus \mathcal{U}$.

Theorem 6. *Let us have* $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$.

(i) For an arbitrary $(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}(\mathbf{b}, \mathbf{d})$*, for which* $\mathbf{u}^T\mathbf{A} \neq \mathbf{0}^T$*, the set*

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = u_{m+1}\}$$

represents a separating hyperplane of $M_1(\mathbf{b}),\ M_2(\mathbf{d})$.

(ii) An arbitrary separating hyperplane $\mathcal{R}$ *of two convex polyhedral sets* $M_1(\mathbf{b}),\ M_2(\mathbf{d})$ *has the description*

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = u_{m+1}\},$$

where $(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}(\mathbf{b}, \mathbf{d}),\ \mathbf{u}^T\mathbf{A} \neq \mathbf{0}^T$.

Proof. It follows from Theorem 2 of [4]. □

Assertion 2. *It holds*

$$\mathcal{P} \setminus \mathcal{U} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathcal{P} \mid \mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \neq \emptyset\}.$$

Proof. It follows from Theorem 5 of [4]. □

From now on we assume without the loss of generality that $rank(\mathbf{A}) = rank(\mathbf{C}) = n$. Otherwise if for instance $rank(\mathbf{A}) < n$, then there exists i_0-th column, $i_0 \in \{1, \ldots, n\}$ of matrix $\mathbf{A}$, which is linearly dependent on other columns of matrix $\mathbf{A}$. Each separating hyperplane of convex polyhedral sets $M_1(\boldsymbol{\lambda}),\ M_2(\boldsymbol{\nu}),\ (\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$, has its normal vector $\mathbf{u}^T\mathbf{A}$ for certain $\mathbf{u} \in \mathbb{R}^m$ (see Teorem 6). Hence i_0-th component of this normal vector is uniquely determined by other components. Therefore for the purpose of separability of convex polyhedral sets $M_1(\boldsymbol{\lambda}),\ M_2(\boldsymbol{\nu}),\ (\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ we can extract i_0-th column from matrix $\mathbf{A}$ and from matrix $\mathbf{C}$ as well (since $\mathbf{u}^T\mathbf{A} = \mathbf{v}^T\mathbf{C}$ for certain $\mathbf{v} \in \mathbb{R}^l$). In principle it is a linear transformation of families $M_1(\boldsymbol{\lambda}),\ M_2(\boldsymbol{\nu})$ to the subspace of $\mathbb{R}^n$.

It follows directly from the previous assumption that $m \geq n$, and $l \geq n$.

3 Stability sets

In this section we deal with so called stability sets. Stablity sets are defined in the similar way in [10]. It is natural to define stability sets as sets of all $(\boldsymbol{\lambda},\boldsymbol{\nu}) \in \mathcal{P} \setminus \mathcal{U}$ such that all the sets $\mathcal{Q}(\boldsymbol{\lambda},\boldsymbol{\nu})$ have the same system of feasible bases (see Definition 3). But for the sake of simplicity we will use the set $\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu})$ from (14) instead of $\mathcal{Q}(\boldsymbol{\lambda},\boldsymbol{\nu})$. Next we will without the loss of generality assume that

$$rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{pmatrix} = n+1. \tag{13}$$

Otherwise it would occur any of the following possibilities:

(i) If $rank \left(\begin{smallmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{smallmatrix}\right) = rank \left(\begin{smallmatrix} \mathbf{A}^T & \mathbf{C}^T \end{smallmatrix} \right)$, then the set $\mathcal{N}'$ from (11) has a description $\mathcal{N}' = \{\mathbf{0}\}$ and therefore $\mathcal{U} = \mathbb{R}^{m+l}$ and the solution set $\mathcal{P} \setminus \mathcal{U} = \emptyset$.

(ii) If $rank \left(\begin{smallmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{smallmatrix}\right) > rank \left(\begin{smallmatrix} \mathbf{A}^T & \mathbf{C}^T \end{smallmatrix} \right)$, then $rank \left(\begin{smallmatrix} \mathbf{A}^T & \mathbf{C}^T \end{smallmatrix} \right) < n$ and in the description of $\mathcal{Q}(\boldsymbol{\lambda},\boldsymbol{\nu})$ there are linear dependent equations, which we can remove.

Let us introduce

$$\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu}) \equiv \left\{ (\mathbf{u},\mathbf{v},v_{l+1}) \in \mathbb{R}_+^{m+l+1} \mid \mathbf{Z}(\boldsymbol{\lambda},\boldsymbol{\nu}) \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \mathbf{z} \right\}, \tag{14}$$

where

$$\mathbf{Z}(\boldsymbol{\lambda},\boldsymbol{\nu}) \equiv \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T & \mathbf{0} \\ \boldsymbol{\lambda}^T & \boldsymbol{\nu}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix}, \quad \mathbf{z} \equiv \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix}.$$

In Definition 3 we introduce the term "basis" of the convex polyhedral set $M_1(\boldsymbol{\lambda})$ (and analogously for $M_2(\boldsymbol{\nu})$) and $\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu})$. For the sake of simplicity we use the following notion: the expression $\mathbf{A}_B$ means the restriction of the matrix $\mathbf{A}$ to the basic rows, the expression $\mathbf{Z}_B(\boldsymbol{\lambda},\boldsymbol{\nu})$ means the restriction of the matrix $\mathbf{Z}(\boldsymbol{\lambda},\boldsymbol{\nu})$ to the basic columns.

Definition 3. *The basis of convex polyhedral set* $M_1(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \mathbb{R}^m$, from (1) is any vector $B \in \{1,\ldots,m\}^n$, for which $rank(\mathbf{A}_B) = n$ holds. A basis B of $M_1(\boldsymbol{\lambda})$ is called *feasible*, if $\mathbf{A}_B^{-1}\boldsymbol{\lambda}_B \in M_1(\boldsymbol{\lambda})$.

The basis of convex polyhedral set $\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu})$, $(\boldsymbol{\lambda},\boldsymbol{\nu}) \in \mathbb{R}^{m+l}$, from (14) is an arbitrary vector $B \in \{1,\ldots,m+n+1\}^{n+2}$, for which $rank(\mathbf{Z}_B(\boldsymbol{\lambda},\boldsymbol{\nu})) = n+2$ holds. A basis B of $\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu})$ is *feasible*, if $(\mathbf{Z}_B(\boldsymbol{\lambda},\boldsymbol{\nu}))^{-1}\mathbf{z} \geq \mathbf{0}$.

Definition 4. Let an arbitrary $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$ be given. *The stability set* corresponding to the point $(\mathbf{b}, \mathbf{d})$ is the closure of the set of all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ under which all feasible bases of the convex polytope $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$ remain feasible for $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$.

In Definition 4 we use closure of the mentioned set for simplicity of description of stability sets. Note also that any stability set is uniquelly determined by a system of bases which it preserves. The point $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$ in Definition 4 is used only for computational purposes.

Clearly, there is a finite number of stability sets for fixed matrices $\mathbf{A}$, $\mathbf{C}$. The following assertion states that for the description of all separating hyperplanes of two convex polyhedral sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ we can use the set $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ instead of $\mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu})$.

Assertion 3. *Let* $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$, $(\mathbf{u}^*, \mathbf{v}^*, v^*_{l+1}) \in \mathcal{Q}^*(\mathbf{b}, \mathbf{d})$, $\mathbf{u}^{*T}\mathbf{A} \neq \mathbf{0}^T$ *and* $\eta \in \langle 0, v^*_{l+1}\rangle$ *be arbitrarily given. Then the set*

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^{*T}(\mathbf{A}\mathbf{x} - \mathbf{b}) = \eta\} \tag{15}$$

represents a separating hyperplane of convex polyhedral sets $M_1(\mathbf{b})$, $M_2(\mathbf{d})$. *Conversely, an arbitrary separating hyperplane* $\mathcal{R}$ *of convex polyhedral sets* $M_1(\mathbf{b})$, $M_2(\mathbf{d})$ *can be written in the form* (15) *for certain point* $(\mathbf{u}^*, \mathbf{v}^*, v^*_{l+1}) \in \mathcal{Q}^*(\mathbf{b}, \mathbf{d})$, $\mathbf{u}^{*T}\mathbf{A} \neq \mathbf{0}^T$ *and certain* $\eta \in \langle 0, v^*_{l+1}\rangle$.

Proof.

(i) Let $(\mathbf{u}^*, \mathbf{v}^*, v^*_{l+1}) \in \mathcal{Q}^*(\mathbf{b}, \mathbf{d})$, $\mathbf{u}^{*T}\mathbf{A} \neq \mathbf{0}^T$, $\eta \in \langle 0, v^*_{l+1}\rangle$ be arbitrarily given. Then define for $\alpha = \sum_{i=1}^m u^*_i + \sum_{j=1}^l v^*_j + v^*_{l+1} > 0$

$$\mathbf{u} \equiv \frac{\mathbf{u}^*}{\alpha}, \quad \mathbf{v} \equiv \frac{\mathbf{v}^*}{\alpha}, \quad u_{m+1} = \frac{\eta}{\alpha}, \quad v_{l+1} \equiv \frac{v^*_{l+1}}{\alpha} - u_{m+1}.$$

Hence $(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}(\mathbf{b}, \mathbf{d})$ and $\mathcal{R} = \{\mathbf{x} \mid \mathbf{u}^{*T}(\mathbf{A}\mathbf{x} - \mathbf{b}) = \eta\} = \{\mathbf{x} \mid \mathbf{u}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = u_{m+1}\}$ represents a separating hyperplane of $M_1(\mathbf{b})$, $M_2(\mathbf{d})$ from Theorem 6.

(ii) It follows from Theorem 6 that an arbitrary separating hyperplane of $M_1(\mathbf{b})$, $M_2(\mathbf{d})$ can be written as $\mathcal{R}' = \{\mathbf{x} \mid \mathbf{u}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = u_{m+1}\}$, where $(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}(\mathbf{b}, \mathbf{d})$, $\mathbf{u}^T\mathbf{A} \neq \mathbf{0}^T$. Since $u_{m+1} + v_{l+1} < 1$, we can define for $\alpha = 1 - u_{m+1} - v_{l+1} > 0$

$$\mathbf{u}^* \equiv \frac{\mathbf{u}}{\alpha}, \quad \mathbf{v}^* \equiv \frac{\mathbf{v}}{\alpha}, \quad v^*_{l+1} \equiv \frac{u_{m+1} + v_{l+1}}{\alpha}.$$

Hence $(\mathbf{u}^*, \mathbf{v}^*, v^*_{l+1}) \in \mathcal{Q}^*(\mathbf{b}, \mathbf{d})$. Since $\frac{u_{m+1}}{\alpha} \in \langle 0, v^*_{l+1}\rangle$, the description of the separating hyperplane $\mathcal{R}' = \{\mathbf{x} \mid \mathbf{u}^{*T}(\mathbf{A}\mathbf{x} - \mathbf{b}) = \frac{u_{m+1}}{\alpha}\}$ has the form of (15).

□

Now we derive the description of a stability set. Let $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$ be given arbitrarily. Denote by B an arbitrary feasible basis of the convex polytope $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$ and define $\mathbf{D} \equiv \mathbf{Z}_B(\mathbf{b}, \mathbf{d})$. The basis B will be preserved for all $(\boldsymbol{\lambda},\ \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ such, that

$$(\mathbf{Z}_B(\boldsymbol{\lambda}, \boldsymbol{\nu}))^{-1}\mathbf{z} \geq \mathbf{0}. \tag{16}$$

Let us introduce vectors $\mathbf{p} \in \mathbb{R}^{n+2}$, $\widetilde{\mathbf{q}}, \mathbf{q} \in \mathbb{R}^{m+l+1}$ in this way:

$$\mathbf{p} \equiv \mathbf{e}_{n+1} = \begin{pmatrix} \mathbf{0} \\ 1 \\ 0 \end{pmatrix}, \quad \widetilde{\mathbf{q}} \equiv \begin{pmatrix} \boldsymbol{\lambda} - \mathbf{b} \\ \boldsymbol{\nu} - \mathbf{d} \\ 0 \end{pmatrix}, \quad \mathbf{q} \equiv \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\nu} \\ 0 \end{pmatrix}.$$

Assume

$$1 + \widetilde{\mathbf{q}}_B^T \mathbf{D}^{-1}\mathbf{p} > 0, \tag{17}$$

where $\widetilde{\mathbf{q}}_B$ is the restriction of the vector $\widetilde{\mathbf{q}}$ to the basic elements. For the choice $\boldsymbol{\lambda} = \mathbf{b},\ \boldsymbol{\nu} = \mathbf{d}$ the condition (17) holds (see also Remark 1). Then it follows from the well known Sherman–Morrison formula:

$$(\mathbf{Z}_B(\boldsymbol{\lambda}, \boldsymbol{\nu}))^{-1} = (\mathbf{Z}_B(\mathbf{b}, \mathbf{d}) + \mathbf{p}\widetilde{\mathbf{q}}_B^T)^{-1} = \mathbf{D}^{-1} - \frac{\mathbf{D}^{-1}\mathbf{p}\widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}}{1 + \widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}\mathbf{p}}.$$

Now rearrange expression (16) under our assumption (17):

$$\begin{aligned} \left(\mathbf{D}^{-1} - \frac{\mathbf{D}^{-1}\mathbf{e}_{n+1}\widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}}{1 + \widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}\mathbf{e}_{n+1}}\right)\mathbf{e}_{n+2} \geq \mathbf{0}, \\ \mathbf{D}^{-1}_{\cdot,n+2} + \mathbf{D}^{-1}_{\cdot,n+2}(\widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) \geq \mathbf{0}. \end{aligned} \tag{18}$$

It holds for the vector $\widetilde{\mathbf{q}}_B^T$:

$$\widetilde{\mathbf{q}}_B^T = \begin{cases} \mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot}, & \text{if } (m+l+1) \notin B, \\ \mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot} + \mathbf{e}_k^T, & \text{if } (m+l+1) = B_k \text{ for certain } k \in \{1, \ldots, n+2\}. \end{cases}$$

(i) If $(m+l+1) \notin B$, then the expression (18) is equivalent to

$$\begin{aligned} \mathbf{D}^{-1}_{\cdot,n+2} + \mathbf{D}^{-1}_{\cdot,n+2}\left((\mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot})\mathbf{D}^{-1}_{\cdot,n+1}\right) - \\ \mathbf{D}^{-1}_{\cdot,n+1}\left((\mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot})\mathbf{D}^{-1}_{\cdot,n+2}\right) \geq \mathbf{0}, \\ \mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) \geq \mathbf{0}. \end{aligned} \tag{19}$$

(ii) Suppose $(m+l+1) = B_k$ for certain $k \in \{1,\ldots,n+2\}$. Then $\mathbf{D}_{\cdot,k} = \mathbf{e}_{n+1}$, and therefore $\mathbf{D}^{-1}_{\cdot,n+1} = \mathbf{e}_k$. The absolute term of the expression (18) is equal to

$$\mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{e}_k^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{e}_k^T\mathbf{D}^{-1}_{\cdot,n+2}) =$$

$$\mathbf{D}^{-1}_{\cdot,n+2} - \mathbf{e}_k\mathbf{D}^{-1}_{k,n+2} = \begin{cases} 0 & \text{for row } k, \\ > 0 & \text{for row } \neq k. \end{cases}$$

Nonabsolute terms of the expression (18) have the following description (recall that $\mathbf{q}_{B_k} = 0$):

$$\mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) =$$

$$-\mathbf{e}_k(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) = \begin{cases} -\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2} & \text{for row } k, \\ 0 & \text{for row } \neq k. \end{cases}$$

Remark 1. Let us investigate the expression (17).

(i) If $(m+l+1) \notin B$, then the expression (17) can be simplified to

$$\begin{aligned} 1 + (\mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot})\mathbf{D}^{-1}_{\cdot,n+1} > 0, \\ \mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1} > 0. \end{aligned}$$

The sum of inequalities from (19) results to (recall that $\mathbf{1}^T = \mathbf{D}_{n+2,\cdot}$)

$$\begin{aligned} (\mathbf{1}^T\mathbf{D}^{-1}_{\cdot,n+2})(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - (\mathbf{1}^T\mathbf{D}^{-1}_{\cdot,n+1})(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) \geq 0, \\ \mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1} \geq 0. \end{aligned}$$

Since the stability set is always closed (by its definition), the condition (17) is redundant.

(ii) If $(m+l+1) = B_k$ for certain $k \in \{1, \ldots, n+2\}$, then the expression (17) can be simplified to (recall that $\mathbf{D}^{-1}_{\cdot,n+1} = \mathbf{e}_k$, $\mathbf{q}_{B_k} = 0$)

$$1 + (\mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot} + \mathbf{e}_k^T)\mathbf{D}^{-1}_{\cdot,n+1} > 0,$$
$$1 > 0$$

and this inequality holds trivially. Therefore the condition (17) is redundant.

The following remark summarizes the description of stability sets.

Remark 2. Each feasible basis B of the convex polytope $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$ remains feasible on the set described by (16), which forms a linear system of inequalities with respect to the parameters $\boldsymbol{\lambda}$, $\boldsymbol{\nu}$. The description of a given stability set is represented by a union of all these systems of inequalities for all feasible bases of $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$.

(i) If $(m+l+1) \notin B$ then the system of inequalities corresponding to the basis B has the description

$$\mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+2}) \geq \mathbf{0}.$$

(ii) If $(m+l+1) \in B$ then we obtain only one inequality corresponding to the basis B with the description

$$-\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+2} \geq 0.$$

Each basis of convex polyhedral set $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ determines certain vertex of $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ (but a vertex can be determined by more than one basis). All vertices of $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ on a given stability set follow:

(i) If $(m+l+1) \notin B$, then the vertex of $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ corresponding to the basis B has the following description:

$$(\mathbf{u}, \mathbf{v}, v_{l+1})_B = \frac{\mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+2})}{\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+1}},$$

$$(\mathbf{u}, \mathbf{v}, v_{l+1})_N = \mathbf{0}.$$

(ii) If $(m+l+1) = B_k$ for certain $k \in \{1, \dots, n+2\}$, then the vertex of $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ corresponding to the basis B has the following description:

$$(\mathbf{u}, \mathbf{v}, v_{l+1})_B = \mathbf{D}^{-1}_{\cdot, n+2} - \mathbf{e}_k \mathbf{D}^{-1}_{k, n+2} - \mathbf{e}_k (\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot, n+2}), \ (\mathbf{u}, \mathbf{v}, v_{l+1})_N = \mathbf{0}.$$

Another usefull properties on stability sets can be derived. For instance if S is a stability set corresponding to certain $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$, then always $int(S) \subseteq \mathcal{P}$. It also holds, that the intersection of two different stability sets is a set of the dimension less than n (for more details see [7]).

4 Examples

Example 1. Given matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we will compute solution set and all stability sets. First, we compute sets $\mathcal{P}$, $\mathcal{U}$. They have the following description:

$$\begin{aligned} \mathcal{P}_1 &= \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid \lambda_1 + \lambda_2 + \lambda_3 > 0\}, \ \mathcal{P}_2 = \mathbb{R}^2, \\ \mathcal{P} &= \mathcal{P}_1 \times \mathcal{P}_2 = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 > 0\}, \\ \mathcal{U} &= \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \nu_1 > 0, \ \lambda_2 + \nu_2 > 0, \ \lambda_1 + \lambda_2 + \lambda_3 > 0\}. \end{aligned}$$

1. Choose $(\mathbf{b}^1, \mathbf{d}^1) \in \mathcal{P} \setminus \mathcal{U}$, e.g. $\mathbf{b}^1 = (0, 0, 2)^T$, $\mathbf{d}^1 = (1, -1)^T$. All feasible bases of $\mathcal{Q}^*(\mathbf{b}^1, \mathbf{d}^1)$ with the corresponding systems of inequalities are the following:

Basis $(1, 2, 4, 5) : -\lambda_2 - \nu_2 \geq 0, \ \lambda_1 + \nu_1 \geq 0.$
Basis $(1, 2, 5, 6) : -\lambda_2 - \nu_2 \geq 0.$
Basis $(2, 3, 5, 6) : -\lambda_2 - \nu_2 \geq 0.$
Basis $(2, 4, 5, 6) : -\lambda_2 - \nu_2 \geq 0.$
Basis $(1, 2, 3, 5) : -\lambda_2 - \nu_2 \geq 0, \ \lambda_1 + \lambda_2 + \lambda_3 \geq 0.$

The first stability set (denote it as S_1) is described as follows: $\lambda_1 + \nu_1 \geq 0, \ -\lambda_2 - \nu_2 \geq 0, \ \lambda_1 + \lambda_2 + \lambda_3 \geq 0.$

2. Choose $(\mathbf{b}^2, \mathbf{d}^2) \in \mathcal{P} \setminus \mathcal{U} \setminus S_1$, e.g. $\mathbf{b}^2 = (0,0,2)^T$, $\mathbf{d}^2 = (-1,-1)^T$. The second stability set (denote it as S_2) has the following description: $-\lambda_1 - \nu_1 \geq 0$, $-\lambda_2 - \nu_2 \geq 0$, $\lambda_1 + \lambda_2 + \lambda_3 \geq 0$.

3. Analogously the last stability set. Choose $(\mathbf{b}^3, \mathbf{d}^3) \in \mathcal{P} \setminus \mathcal{U} \setminus S_1 \setminus S_2$ e.g. $\mathbf{b}^3 = (0,0,2)^T$, $\mathbf{d}^3 = (-1,1)^T$. The third stability set has the following description: $-\lambda_1 - \nu_1 \geq 0$, $\lambda_2 + \nu_2 \geq 0$, $\lambda_1 + \lambda_2 + \lambda_3 \geq 0$.

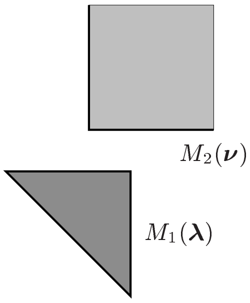

Figure 1: The 1^{th} stability set.

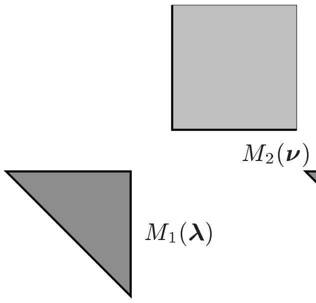

Figure 2: The 2^{nd} stability set.

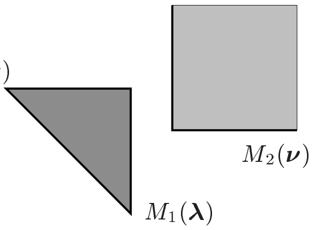

Figure 3: The 3^{rd} stability set.

We have obtained three stability sets (except degenerated stability sets, which have a dimension less than n) the solution set $\mathcal{P} \setminus \mathcal{U}$ consists of. The geometric interpretation of the three stability sets is illustrated by Figure 1 – 3; each stability set determines a special position of the sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$.

The Tables 1 and 2 contain further results obtained on `PC (x86), Pentium 4, 2.6 GHz, 512 MB RAM, Gentoo Linux`. Our source code is written in MATLAB (R12.1). In each of the mentioned tables, there are for given matrices $\mathbf{A}$, $\mathbf{C}$ written down the number of stability sets and the aproximate computing time (in minutes and seconds). For the Table 2, the input data of matrices $\mathbf{A}$, $\mathbf{C}$ were generated pseudorandomly. With the increase of m, l, n the number of stability sets and the computing time increases very rapidly.

Table 1: Examples in $\mathbb{R}^2$.

matrix $\mathbf{A}$	matrix $\mathbf{C}$	number of stability sets	computing time
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$	6	1 s
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$	7	1 s
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}$	29	6 s
$\begin{pmatrix} 1 & 0 \\ -3 & 5 \\ -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -4 & -1 \\ 3 & -4 \\ 2 & 3 \end{pmatrix}$	20	2 s
$\begin{pmatrix} -3 & -1 \\ 6 & -7 \\ -9 & 7 \end{pmatrix}$	$\begin{pmatrix} -4 & 5 \\ 3 & -4 \\ -9 & -1 \\ 6 & 8 \end{pmatrix}$	123	2 min 38 s
$\begin{pmatrix} 1 & -5 \\ -5 & 7 \\ -1 & 9 \\ 3 & 8 \end{pmatrix}$	$\begin{pmatrix} 5 & 5 \\ 9 & 2 \\ -8 & 7 \\ -3 & -4 \end{pmatrix}$	260	3 min 5 s

5 Special cases

All special kinds of postoptimality analyses (parametric analysis, sensitivity analysis or tolerance analysis – see e.g. [2]) can be applied to the stability sets, defined in the Section 3. But this is not the subject of this paper. In this section we will discuss some special cases of families of convex polyhedral sets (1), (2).

Table 2: Examples in $\mathbb{R}^3,\ \mathbb{R}^4$ – pseudorandom data.

matrix $\mathbf{A}$	matrix $\mathbf{C}$	stability sets	computing time
$\begin{pmatrix} -2 & 5 & 3 \\ -1 & -3 & -3 \\ 4 & -2 & 3 \\ 1 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 3 & 1 \\ 0 & 3 & 5 \\ 0 & 2 & -3 \end{pmatrix}$	103	1 min 25 s
$\begin{pmatrix} -2 & -5 & 9 \\ 9 & -2 & -7 \\ -7 & -2 & 5 \\ 1 & -3 & -4 \end{pmatrix}$	$\begin{pmatrix} 9 & -6 & 2 \\ -8 & 6 & 6 \\ -4 & -4 & -3 \\ -5 & 9 & -5 \end{pmatrix}$	1147	63 min 47 s
$\begin{pmatrix} 5 & 3 & 1 & -1 \\ -3 & 0 & 3 & 4 \\ -5 & 4 & -2 & 0 \\ -4 & 0 & 3 & 2 \\ 3 & 3 & 3 & -4 \end{pmatrix}$	$\begin{pmatrix} 5 & 1 & -2 & 5 \\ -1 & -2 & 5 & -2 \\ 4 & 2 & -3 & -4 \\ 0 & -4 & 3 & 3 \\ -2 & 4 & -3 & -3 \end{pmatrix}$	1666	91 min 42 s

5.1 One parametr

We deal with two families of convex polyhedral sets

$$M_1(\lambda) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \begin{pmatrix} \lambda \\ \mathbf{b} \end{pmatrix}\},$$
$$M_2 = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \leq \mathbf{d}\},$$

where $\mathbf{A} \in \mathbb{R}^{m\times n},\ \mathbf{C} \in \mathbb{R}^{l\times n},\ \mathbf{b} \in \mathbb{R}^{m-1},\ \mathbf{d} \in \mathbb{R}^l$ and λ is a real parametr. Using (7) and (10) we obtain the description of the solution set $\mathcal{P} \setminus \mathcal{U}$ as an interval in $\mathbb{R}$. Therefore the solution set for this case is always convex.

From Remark 2 we can directly obtain the description of stability sets. Note that generally, there exists more than one stability set.

5.2 Parameter with fixed direction

We deal with two families of convex polyhedral sets

$$M_1(\lambda) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b} + \lambda\mathbf{b}'\},$$
$$M_2(\lambda) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \leq \mathbf{d} + \lambda\mathbf{d}'\},$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{C} \in \mathbb{R}^{l \times n}$, $\mathbf{b}, \mathbf{b}' \in \mathbb{R}^m$, $\mathbf{d}, \mathbf{d}' \in \mathbb{R}^l$ and λ is a real parametr. Using (7) and (10) we obtain the description of the solution set $\mathcal{P} \setminus \mathcal{U}$ as a difference of two intervals in $\mathbb{R}$. Therefore the solution set for this case is not generally convex.

The description of stability sets we obtain directly from Remark 2 (in general, there exists more than one stability set).

5.3 A fixed separating hyperplane

Let us have a fixed hyperplane

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} = s\}.$$

We will find out for which values of paramters $\boldsymbol{\lambda}, \boldsymbol{\nu}$ the convex polyhedral sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ from (1), (2) are strongly separable by this hyperplane $\mathcal{R}$, i.e.

$$M_1(\boldsymbol{\lambda}) \subseteq \overline{\mathcal{R}^+},\ M_2(\boldsymbol{\nu}) \subseteq \overline{\mathcal{R}^-},\ dim M_1(\boldsymbol{\lambda}) = dim M_2(\boldsymbol{\nu}) = n, \tag{20}$$

hold. The description of the set of all parameters $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ satisfying (20) is as follows

$$\mathcal{A}_1 \times \mathcal{A}_2,$$

where $\mathcal{A}_1$ is the solution set for sets $M_1(\boldsymbol{\lambda})$, $\overline{\mathcal{R}^-}$ with the description

$$\mathcal{A}_1 = \mathcal{P}_1 \setminus \mathcal{U}_1,$$

where $\mathcal{P}_1$ has the form of (3) with its description from (7) (because $\overline{\mathcal{R}^-}$ is always full dimensional). The description of $\mathcal{U}_1$ follows easily from Theorem 4, applied to convex polyhedral sets $M_1(\boldsymbol{\lambda})$, $\overline{\mathcal{R}^-}$. Analogously $\mathcal{A}_2$ forms the solution set for $M_2(\boldsymbol{\nu})$, $\overline{\mathcal{R}^+}$ with the following description

$$\mathcal{A}_2 = \mathcal{P}_2 \setminus \mathcal{U}_2.$$

The set $\mathcal{A}_1$ does not need to be convex. For example $\mathcal{A}_1$ is not convex for the hyperplane $\mathcal{R} = \{x \in \mathbb{R} \mid x = 1\}$ and

$$M_1(\boldsymbol{\lambda}^1) = M_1(1,3) = \{x \in \mathbb{R} \mid x \leq 1,\ x \leq 3\},$$
$$M_1(\boldsymbol{\lambda}^2) = M_1(3,1) = \{x \in \mathbb{R} \mid x \leq 3,\ x \leq 1\}.$$

For the convex combination $\boldsymbol{\lambda}^{12} = \frac{1}{2}\boldsymbol{\lambda}^1 + \frac{1}{2}\boldsymbol{\lambda}^2 = \frac{1}{2}(1,3) + \frac{1}{2}(3,1) = (2,2)$ the convex polyhedral set $M_1(\boldsymbol{\lambda}^{12})$ intersects the hyperplane $\mathcal{R}$.

5.4 A permanent separating hyperplane

Given $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ from (1), (2) with the property that $\boldsymbol{\lambda} \in \mathcal{Z}_1$, $\boldsymbol{\nu} \in \mathcal{Z}_2$, where $\mathcal{Z}_1 \subset \mathbb{R}^m$, $\mathcal{Z}_2 \subset \mathbb{R}^l$ are convex polytopes. Without the loss of generality assume that $\mathcal{Z}_1 \subset \overline{\mathcal{P}_1}$ and $\mathcal{Z}_2 \subset \overline{\mathcal{P}_2}$ (otherwise we restrict the sets to $\mathcal{Z}_1 \cap \overline{\mathcal{P}_1}$, $\mathcal{Z}_2 \cap \overline{\mathcal{P}_2}$ respectively). The problem we study in this section consists in answering the question whether there exists a separating hyperplane $\mathcal{R}$ such that:

$$M_1(\boldsymbol{\lambda}) \subseteq \overline{\mathcal{R}^-} \ \forall \boldsymbol{\lambda} \in \mathcal{Z}_1, \quad M_2(\boldsymbol{\nu}) \subseteq \overline{\mathcal{R}^+} \ \forall \boldsymbol{\nu} \in \mathcal{Z}_2. \tag{21}$$

The hyperplane $\mathcal{R}$ satysfying (21) is called *permanent*. Note that a permanent separating hyperplane may not exists even though for $\forall \boldsymbol{\lambda} \in \mathcal{Z}_1$ and $\forall \boldsymbol{\nu} \in \mathcal{Z}_2$ the convex sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ are strongly or weakly separable. We check the existence of a permanent separating hyperplane by the following process: Compute the convex hull of sets $\cup_{\boldsymbol{\lambda} \in \mathcal{Z}_1} M_1(\boldsymbol{\lambda})$ and $\cup_{\boldsymbol{\nu} \in \mathcal{Z}_2} M_2(\boldsymbol{\nu})$ and check separability of these convex hulls. By this method we also enumerate all permanent separating hyperplanes. Let us now concentrate on calculation of *conv* $(\cup_{\boldsymbol{\lambda} \in \mathcal{Z}_1} M_1(\boldsymbol{\lambda}))$.

Lemma 1. *Let a basis B of the convex polyhedral set $M_1(\boldsymbol{\lambda})$ be given. Denote $N \equiv \{1, \dots, m\} \setminus B$. Then $\mathcal{S}_B^1$, the set of all $\boldsymbol{\lambda} \in \mathbb{R}^m$ for which the basis B is feasible for $M_1(\boldsymbol{\lambda})$, has the following description*

$$\mathcal{S}_B^1 = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T(\boldsymbol{\lambda}_B, \boldsymbol{\lambda}_N) \geq 0 \quad \forall i \in I_B\}, \tag{22}$$

where $\mathbf{h}_i$, $i \in I_B$ are vectors in directions of all edges of the convex polyhedral cone

$$\mathcal{N}_B \equiv \{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \mid \mathbf{A}_B^T \mathbf{y} + \mathbf{A}_N^T \mathbf{z} = \mathbf{0},\ \mathbf{z} \geq \mathbf{0}\}.$$

Proof. The basis B remain feasible for all $\boldsymbol{\lambda} \in \mathbb{R}^m$, for which the set

$$M_B(\boldsymbol{\lambda}) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}_N \mathbf{x} \leq \boldsymbol{\lambda}_N,\ \mathbf{A}_B \mathbf{x} = \boldsymbol{\lambda}_B\}$$

is nonempty, i.e. the problem

$$\max \{\mathbf{0}^T \mathbf{x} \mid \mathbf{x} \in M_B(\boldsymbol{\lambda})\}$$

has an optimal solution. It follows from the duality of linear programming, that this problem is equivalent to the situation, that the problem

$$\min \{\boldsymbol{\lambda}_B^T \mathbf{y} + \boldsymbol{\lambda}_N^T \mathbf{z} \mid (\mathbf{y}, \mathbf{z}) \in \mathcal{N}_B\},$$

has an optimal solution. We know that $\mathbf{h}_i$, $i \in I_B$ are vectors in all edges directions of the convex polyhedral cone $\mathcal{N}_B$ (if $\mathcal{N}_B$ contains a nontrivial subspace, then we take its base vectors with both positive and negative sign). It follows from the theory of polar cones (see [11])

$$\mathcal{S}_B^1 = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T(\boldsymbol{\lambda}_B, \boldsymbol{\lambda}_N) \geq 0 \quad \forall i \in I_B\}.$$

□

Let $\mathbf{b} \in \mathcal{Z}_1$ hold and let B be a feasible basis of the convex polyhedral set $M_1(\mathbf{b})$. The basis B remains feasible for all $\boldsymbol{\lambda} \in \mathcal{S}_B^1$ from (22). Hence the set

$$\mathcal{Z}_1(\mathfrak{S}) \equiv \mathcal{Z}_1 \cap \left(\cap_{B\in\mathfrak{S}}\, \mathcal{S}_B^1\right),$$

where $\mathfrak{S}$ is the set of all feasible bases of $M_1(\mathbf{b})$, forms the set of all $\boldsymbol{\lambda} \in \mathcal{Z}_1$ for which all feasible bases of $M_1(\mathbf{b})$ remain fesible for $M_1(\boldsymbol{\lambda})$. In this way we can represent the set $\mathcal{Z}_1$ as the union of disjoint stability-like sets $\mathcal{Z}_1(\mathfrak{S}_i)$, $i \in I$, where I is an index set of a finite cardinality.

Lemma 2. *We have*

$$conv\Big(\bigcup_{\boldsymbol{\lambda}\in\mathcal{Z}_1(\mathfrak{S})} M_1(\boldsymbol{\lambda})\Big) = conv\Big(\bigcup_{\boldsymbol{\lambda}\in\{vertices\ of\ \mathcal{Z}_1(\mathfrak{S})\}} M_1(\boldsymbol{\lambda})\Big).$$

Proof. Let $\mathbf{b}^1$, $\mathbf{b}^2 \in \mathcal{Z}_1(\mathfrak{S})$ be given. We prove that for each convex combination $c_1\mathbf{b}^1 + c_2\mathbf{b}^2$, $c_1, c_2 \geq 0$, $c_1 + c_2 = 1$, it $M_1(c_1\mathbf{b}^1 + c_2\mathbf{b}^2) \subseteq conv(M_1(\mathbf{b}^1) \cup M_1(\mathbf{b}^2))$ holds. If $\mathbf{x} \in M_1(c_1\mathbf{b}^1 + c_2\mathbf{b}^2)$ is arbitrarily given, then $\mathbf{x}$ can be written as a convex combination of all vertices and vectors in directions of unbounded edges of the convex polyhedral set $M_1(c_1\mathbf{b}^1+c_2\mathbf{b}^2)$. Directions of unbounded edges of convex polyhedral sets $M_1(c_1\mathbf{b}^1 + c_2\mathbf{b}^2)$ and $conv(M_1(\mathbf{b}^1) \cup M_1(\mathbf{b}^2))$ coincide. All vertices of $M_1(c_1\mathbf{b}^1 + c_2\mathbf{b}^2)$ have the description $\mathbf{A}_B^{-1}(c_1\mathbf{b}^1 + c_2\mathbf{b}^2)_B = c_1\mathbf{A}_B^{-1}\mathbf{b}_B^1 + c_2\mathbf{A}_B^{-1}\mathbf{b}_B^2$, $B \in \mathfrak{S}$, which is a convex combination of vertices of $M_1(\mathbf{b}^1)$ and $M_1(\mathbf{b}^2)$. Therefore each point $\mathbf{x} \in M_1(c_1\mathbf{b}^1 + c_2\mathbf{b}^2)$ is equal to a convex combination of vertices of $M_1(\mathbf{b}^1)$ and $M_1(\mathbf{b}^2)$ and vectors in directions of unbounded edges of $conv(M_1(\mathbf{b}^1) \cup M_1(\mathbf{b}^2))$. So it is proven that $M_1(c_1\mathbf{b}^1 + c_2\mathbf{b}^2) \subseteq conv(M_1(\mathbf{b}^1) \cup M_1(\mathbf{b}^2))$. Consequently,

$$conv\Big(\bigcup_{\boldsymbol{\lambda}\in\mathcal{Z}_1(\mathfrak{S})} M_1(\boldsymbol{\lambda})\Big) = conv\Big(\bigcup_{\boldsymbol{\lambda}\in\{\text{vertices of } \mathcal{Z}_1(\mathfrak{S})\}} M_1(\boldsymbol{\lambda})\Big).$$

□

It follows from Lemma 2 that we can reduce a calculation of the convex hull of infinite number of convex polyhedral sets to calculation of the convex hull of finite number (how to compute a convex hull see e.g. [4]). Analogously we compute convex hulls of $\cup_{\boldsymbol{\lambda}\in\mathcal{Z}_1(\mathfrak{S}_i)} M_1(\boldsymbol{\lambda})$, $i \in I$. Eventually, we obtain

$$conv\Big(\bigcup_{\boldsymbol{\lambda}\in\mathcal{Z}_1} M_1(\boldsymbol{\lambda})\Big) = conv\Big(\bigcup_{i\in I}\ \bigcup_{\boldsymbol{\lambda}\in\{\texttt{vertices of } \mathcal{Z}_1(\mathfrak{S}_i)\}} M_1(\boldsymbol{\lambda})\Big). \qquad (23)$$

Example 2. Given a matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$$

and a family of convex sets

$$M_1(\boldsymbol{\lambda}) = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{A}\mathbf{x} \le \boldsymbol{\lambda}\},$$

where $\boldsymbol{\lambda} \in \mathcal{Z}_1 = \{\mathbf{b} \in \mathbb{R}^5 \mid \mathbf{b} = (0, 10, 2, -8, 14)^T + t.(8, -8, -8, 8, -5)^T$, with $t \in \langle 0, 1\rangle\}$. Convex polyhedral set M_2 is fixed and given as follows

$$M_2 = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 - x_2 \ge 4.9,\ x_2 \ge 2.3\},$$

Choose $\mathbf{b}^1 \in \mathcal{Z}_1$ for instance in the following way: $\mathbf{b}^1 = (0, 10, 2, -8, 14)^T$. The set of all feasible bases of convex polyhedral set $M_1(\mathbf{b}^1)$ is $\mathfrak{S}_1 = \{(1, 2),\ (1, 4),\ (2, 3),\ (3, 4)\}$. Particular bases of $\mathfrak{S}_1$ are preserved on sets (see (22)):

$$\begin{aligned}
\mathcal{S}^1_{(1,2)} &= \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid \lambda_1 + \lambda_3 \ge 0,\ \lambda_2 + \lambda_4 \ge 0,\ -\lambda_1 - \lambda_2 + \lambda_5 \ge 0\},\\
\mathcal{S}^1_{(1,4)} &= \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid \lambda_1 + \lambda_3 \ge 0,\ \lambda_2 + \lambda_4 \ge 0,\ -\lambda_1 + \lambda_4 + \lambda_5 \ge 0\},\\
\mathcal{S}^1_{(2,3)} &= \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid \lambda_1 + \lambda_3 \ge 0,\ \lambda_2 + \lambda_4 \ge 0,\ -\lambda_2 + \lambda_3 + \lambda_5 \ge 0\},\\
\mathcal{S}^1_{(3,4)} &= \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid \lambda_1 + \lambda_3 \ge 0,\ \lambda_2 + \lambda_4 \ge 0,\ \lambda_3 + \lambda_4 + \lambda_5 \ge 0\}.
\end{aligned}$$

The intercsection of these sets is $\cap_{B\in\mathfrak{S}_1}\mathcal{S}^1_B = \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid \lambda_1 + \lambda_3 \ge 0,\ \lambda_2 + \lambda_4 \ge 0,\ -\lambda_1 - \lambda_2 + \lambda_5 \ge 0\}$. Hence the set $\mathcal{Z}_1(\mathfrak{S}_1)$ has the description $\mathcal{Z}_1(\mathfrak{S}_1) = \{\boldsymbol{\lambda} \in \mathcal{Z}_1$ for $t \in \langle 0, \frac{4}{5}\rangle\}$ and consists of two vertices $\mathbf{b}^1$ and

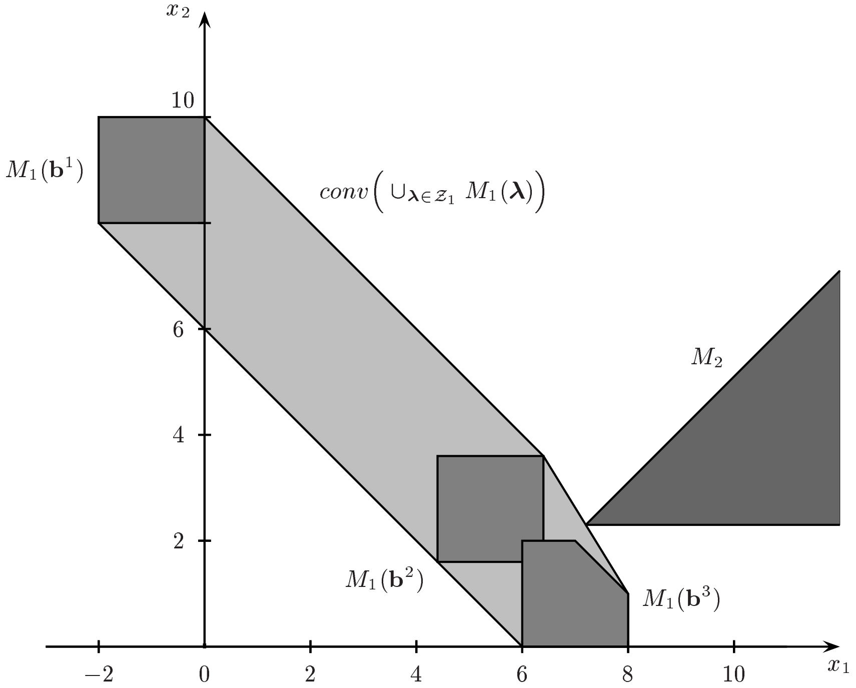

Figure 4: Illustration to Example 2.

$\mathbf{b}^2 = (6.4, 3.6, -4.4, -1.6, 10)^T$. For vector $\mathbf{b}^2$ is the set of all feasible bases of $M_1(\mathbf{b}^2)$ equal to $\mathfrak{S}_2 = \{(1,4),\ (2,3),\ (3,4),\ (1,5),\ (2,5)\}$. Compute:

$$\mathcal{S}^1_{(1,5)} = \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid \lambda_1 + \lambda_3 \geq 0,\ \lambda_1 + \lambda_2 - \lambda_5 \geq 0,\ -\lambda_1 + \lambda_4 + \lambda_5 \geq 0\},$$
$$\mathcal{S}^1_{(2,5)} = \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid \lambda_2 + \lambda_4 \geq 0,\ -\lambda_2 + \lambda_3 + \lambda_5 \geq 0,\ \lambda_1 + \lambda_2 - \lambda_5 \geq 0\}.$$

The intersection $\cap_{B\in\mathfrak{S}_2}\mathcal{S}^1_B = \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid -\lambda_1 + \lambda_4 + \lambda_5 \geq 0,\ -\lambda_2 + \lambda_3 + \lambda_5 \geq 0,\ \lambda_1 + \lambda_2 - \lambda_5 \geq 0\}$. Hence the set $\mathcal{Z}_1(\mathfrak{S}_2)$ can be described as follows $\mathcal{Z}_1(\mathfrak{S}_2) = \{\boldsymbol{\lambda} \in \mathcal{Z}_1 \text{ for } t \in \langle \frac{4}{5}, 1\rangle\}$ and consists of two vertices $\mathbf{b}^2$ and $\mathbf{b}^3 = (8, 2, -6, 0, 9)^T$. Eventually, we obtain (see Figure 4)

$$conv\Big(\bigcup_{\boldsymbol{\lambda}\in\mathcal{Z}_1} M_1(\boldsymbol{\lambda})\Big) = conv\Big(\bigcup_{\boldsymbol{\lambda}\in\{\mathbf{b}^1,\mathbf{b}^2,\mathbf{b}^3\}} M_1(\boldsymbol{\lambda})\Big) =$$

$$\left\{ \mathbf{x} \in \mathbb{R}^2 \mid \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ -1 & -1 \\ 2.6 & 1.6 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 8 \\ 10 \\ 2 \\ 0 \\ 10 \\ -1 \\ 22.4 \end{pmatrix} \right\}.$$

There exists only one permanent separating hyperplane

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^2 \mid 2.6\,x_1 + 1.6\,x_2 = 22.4\}.$$

6 Separating supporting hyperplanes

In this section we deal with separating hyperplanes of convex polyhedral sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$, $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathcal{P} \setminus \mathcal{U}$ from (1), (2), which are their supporting as well. First, we define so called subbasis of convex polyhedral set $M_1(\boldsymbol{\lambda})$ (and analogously of $M_2(\boldsymbol{\nu})$) and sub-stability sets (this notion is used, because sub-stability sets are subsets of stability sets defined in Section 3).

Definition 5. *The sub-basis* of the convex polyhedral set $M_1(\mathbf{b})$, $\mathbf{b} \in \mathbb{R}^m$, is any vector $B^S \in \{1, \ldots, m\}^k$, $1 \leq k \leq m$, for which $B_i^S \neq B_j^S$ for $i \neq j$ holds. The sub-basis B^S is called *feasible*, if $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}_{B^S}\mathbf{x} = \mathbf{b}_{B^S},\ \mathbf{A}_{N^S}\mathbf{x} \leq \mathbf{b}_{N^S}\} \neq \emptyset$ for $N^S = \{1, \ldots, m\} \setminus B^S$. Let B be a basis of the convex polytope $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$, $(\mathbf{b}, \mathbf{d}) \in \mathbb{R}^{m+l}$. Let us introduce the set $B^1 \equiv \{i \in B \mid 1 \leq i \leq m\}$ as *the sub-basis of set $M_1(\mathbf{b})$ created from B*. Analogously $B^2 \equiv \{i \in \{1, \ldots, l\} \mid (i + m) \in B\}$ is called the sub-basis of the set $M_2(\mathbf{d})$ created from B.

All feasible bases of convex polyhedral sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$, $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathcal{P} \setminus \mathcal{U}$ are not generally preserved on stability sets. This reflects the following example:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, \ \mathbf{b}^1 = \begin{pmatrix} -9 \\ 5 \\ -1 \end{pmatrix}, \ \mathbf{d}^1 = \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix}.$$

The corresponding stability set S has the description $S = \{-\lambda_1 - \nu_1 \geq 0,\ \lambda_2 + \nu_2 \geq 0,\ \nu_1 + \nu_2 + \nu_3 \geq 0,\ \lambda_3 + \nu_1 + \nu_2 \geq 0\}$. The basis $(1, 2)$ is feasible for $M_1(\mathbf{b}^1)$. But if $\mathbf{b}^2 = (-9, 5, -6)^T$, $\mathbf{d}^2 = \mathbf{d}^1$, $(\mathbf{b}^2, \mathbf{d}^2) \in S$, then the basis $(1, 2)$ is not feasible for $M_1(\mathbf{b}^2)$.

In Definition 6 we will introduce sub-stability set as the set of all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in S$ under which some of the feasible sub-bases of $M_1(\mathbf{b})$, $M_2(\mathbf{d})$ are preserved for $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ as well. It is not necessary to require preservation of all feasible sub-bases of $M_1(\mathbf{b})$, $M_2(\mathbf{d})$, because some of them does not participate in separating or supporting separating of the convex polyhedral sets $M_1(\mathbf{b})$, $M_2(\mathbf{d})$.

Definition 6. Let us have $(\mathbf{b},\mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$ and S the corresponding stability set. Assume that B is an arbitrary feasible basis of the convex polytope $\mathcal{Q}^*(\mathbf{b},\mathbf{d})$ for which the corresponding separating hyperplane $\mathcal{R}^B$ is also supporting for $M_1(\mathbf{b})$, $M_2(\mathbf{d})$. Let B^1 be the sub-basis of $M_1(\mathbf{b})$ and B^2 the sub-basis of $M_2(\mathbf{d})$ created from B. Let $(\mathbf{u},\mathbf{v},v_{l+1})$ be the vertex of $\mathcal{Q}^*(\mathbf{b},\mathbf{d})$ corresponding to the basis B. Let us introduce $B_r^1 = \{i \in B^1 \mid (\mathbf{u},\mathbf{v},v_{l+1})_i \neq 0\}$, $B_r^2 = \{i \in B^2 \mid (\mathbf{u},\mathbf{v},v_{l+1})_i \neq 0\}$. Then we define *sub-stability set* S^0 as the set of all $(\boldsymbol{\lambda},\boldsymbol{\nu}) \in S$ for which the sub-basis B_r^1 is feasible for $M_1(\boldsymbol{\lambda})$, the sub-basis B_r^2 is feasible for $M_2(\boldsymbol{\nu})$ and this holds for each feasible basis B from our assumption.

Note that the intersection of two different sub-stability sets may be of full dimension.

Lemma 3. *Let us have $(\mathbf{b},\mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$ and S^0 the corresponding sub-stability set. Suppose that $(\mathbf{b},\mathbf{d}) \in int(S^0)$. Let B be such basis of the convex polytope $\mathcal{Q}^*(\mathbf{b},\mathbf{d})$, for which the corresponding vertex defines the separating supporting hyperplane of convex polyhedral sets $M_1(\mathbf{b})$, $M_2(\mathbf{d})$. Then for all $(\boldsymbol{\lambda},\boldsymbol{\nu}) \in S^0$ the vertex of $\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu})$, corresponding to the basis B, defines the separating supporting hyperplane of $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$.*

Proof. Let $(\mathbf{u},\mathbf{v},v_{l+1})$ be the vertex of $\mathcal{Q}^*(\mathbf{b},\mathbf{d})$ corresponding to the basis B. First we prove that $(m+l+1) \notin B$. For contradiction suppose $(m+l+1) = B_k$ for some $k \in \{1,\dots,n+2\}$. Since the corresponding separating hyperplane $\mathcal{R} = \{\mathbf{x} \mid \mathbf{u}^T(\mathbf{A}\mathbf{x}-\mathbf{b}) = v_{l+1}\}$ from (15) is supporting as well, it follows that $v_{l+1} = 0$. It follows from Remark 2

$$0 = v_{l+1} = (\mathbf{u},\mathbf{v},v_{l+1})_{B_k} = \left(\mathbf{D}^{-1}_{\cdot,n+2} - \mathbf{e}_k\mathbf{D}^{-1}_{k,n+2} - \mathbf{e}_k(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2})\right)_k = \mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}.$$

The right-hand side of this equation is a linear function, which is nonnegative on a neighbourhood of $(\mathbf{b},\mathbf{d})$ and zero at the point $(\mathbf{b},\mathbf{d})$. Hence the linear function is zero on a neighbourhood of $(\mathbf{b},\mathbf{d})$, from which we obtain $\mathbf{D}^{-1}_{\cdot,n+2} = \mathbf{0}$. It is a contradiction with the regularity of $\mathbf{D}^{-1}$.

Next suppose that for certain $k \in \{1, \dots, n+2\}$ it holds

$$0 = (\mathbf{u}, \mathbf{v}, v_{l+1})_{B_k} = \frac{\mathbf{D}^{-1}_{k,n+2}(\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{k,n+1}(\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+2})}{\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+1}},$$

i.e.

$$0 = \mathbf{D}^{-1}_{k,n+2}(\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{k,n+1}(\mathbf{q}_B^T \mathbf{D}^{-1}_{\cdot,n+2}). \tag{24}$$

The right-hand side of the expression (24) is a linear function, which is non-negative on a neighbourhood of $(\mathbf{b}, \mathbf{d})$ and zero at the point $(\mathbf{b}, \mathbf{d})$. Hence the linear function is zero on a neighbourhood of $(\mathbf{b}, \mathbf{d})$, from which we obtain $\mathbf{D}^{-1}_{k,n+1} = \mathbf{D}^{-1}_{k,n+2} = 0$. Therefore the property $0 = (\mathbf{u}, \mathbf{v}, v_{l+1})_{B_k}$ holds on the whole sub-stability set S^0.
Introduce B^1_r as a sub-basis B^1 after extraction indices $i \in B^1$, for which $(\mathbf{u}, \mathbf{v}, v_{l+1})_i = 0$. Analogously introduce B^2_r. The sub-basis B^1_r of $M_1(\mathbf{b})$ is feasible according to the assumption of Lemma 3. The separating hyperplane corresponding to the basis B has for all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in S^0$ the following description

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T(\mathbf{A}\mathbf{x} - \boldsymbol{\lambda}) = 0\} \\
&= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T_{B^1}(\mathbf{A}_{B^1}\mathbf{x} - \boldsymbol{\lambda}_{B^1}) = 0\} \\
&= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T_{B^1_r}(\mathbf{A}_{B^1_r}\mathbf{x} - \boldsymbol{\lambda}_{B^1_r}) = 0\}.
\end{aligned}$$

Similarly for B^2_r the sub-basis of $M_2(\boldsymbol{\nu})$. Since S^0 is the sub-stability set, the sub-bases B^1_r, B^2_r remain feasible for all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in S^0$ and $\mathcal{R}$ represents the separating supporting hyperplane of $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$. $\square$

Remark 3. It follows from Lemma 3 that if there exists a supporting hyperplane separating convex polyhedral sets $M_1(\boldsymbol{\lambda}^0)$, $M_2(\boldsymbol{\nu}^0)$ for certain point $(\boldsymbol{\lambda}^0, \boldsymbol{\nu}^0) \in int(S^0)$, then there exists a separating supporting hyperplane of $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ for all $(\boldsymbol{\lambda}, \boldsymbol{\nu})$ in the sub-stability set S^0 (moreover, this hyperplane supports $M_1(\boldsymbol{\lambda})$ and $M_2(\boldsymbol{\nu})$ at the same faces for all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in S^0$). That is why we introduce so called sub-stability sets of *supporting* and *non-supporting type*.

Remark 4. Now we derive the description of a sub-stability set. Let us have $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$ and S^0 the corresponding sub-stability set. Let B be a basis of the convex polytope $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$ and B^1 the sub-basis of $M_1(\mathbf{b})$, B^2 the sub-basis of $M_2(\mathbf{d})$ created from B. The description of the set $\mathcal{S}^1_{B^1}$ of all $\boldsymbol{\lambda} \in \mathbb{R}^m$ for which the sub-basis B^1 remains feasible for $M_1(\boldsymbol{\lambda})$ follows from Lemma 1 (since we can simply extend Lemma 1 to the case when B^1 is a sub-basis of $M_1(\boldsymbol{\lambda})$):

$$\mathcal{S}^1_{B^1} = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T(\boldsymbol{\lambda}_{B^1}, \boldsymbol{\lambda}_{N^1}) \geq 0 \quad \forall i \in I_{B^1}\}. \tag{25}$$

where $N^1 \equiv \{1, \ldots, m\} \setminus B^1$ and $\mathbf{h}_i$, $i \in I_{B^1}$ are vectors in directions of all edges of convex polyhedral cone

$$\mathcal{N}_{B^1} \equiv \{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \mid \mathbf{A}^T_{B^1}\mathbf{y} + \mathbf{A}^T_{N^1}\mathbf{z} = \mathbf{0},\ \mathbf{z} \geq \mathbf{0}\}.$$

In a similar way denote by $\mathcal{S}^2_{B^2}$ the set of all $\boldsymbol{\nu} \in \mathbb{R}^l$ for which the sub-basis B^2 remains feasible for $M_2(\boldsymbol{\nu})$. Analogously we obtain

$$\mathcal{S}^2_{B^2} = \{\boldsymbol{\nu} \in \mathbb{R}^l \mid \mathbf{g}_j^T(\boldsymbol{\nu}_{B^2}, \boldsymbol{\nu}_{N^2}) \geq 0 \quad \forall j \in J_{B^2}\},$$

where $N^2 \equiv \{1, \ldots, l\} \setminus B^2$ and $\mathbf{g}_j$, $j \in J_{B^2}$ are vectors in directions of all edges of convex polyhedral cone $\{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^l \mid \mathbf{C}^T_{B^2}\mathbf{y} + \mathbf{C}^T_{N^2}\mathbf{z} = \mathbf{0},\ \mathbf{z} \geq \mathbf{0}\}$. Let us introduce

$$\begin{aligned}\mathcal{S}_B = \mathcal{S}^1_{B^1} \cap \mathcal{S}^2_{B^2} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid\, & \mathbf{h}_i^T(\boldsymbol{\lambda}_{B^1}, \boldsymbol{\lambda}_{N^1}) \geq 0 \ \ \forall i \in I_{B^1}, \\ & \mathbf{g}_j^T(\boldsymbol{\nu}_{B^2}, \boldsymbol{\nu}_{N^2}) \geq 0 \ \ \forall j \in J_{B^2}\}.\end{aligned} \tag{26}$$

Let B^i, $i \in I$, be all bases of the convex polytope $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$, for which the corresponding separating hyperplanes are supporting. Suppose the extraction of all indices $j \in B^i$, for which for the corresponding vertex $(\mathbf{u}, \mathbf{v}, v_{l+1})^i$ of $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$ the equality $(\mathbf{u}, \mathbf{v}, v_{l+1})^i_j = 0$ holds. Then the descritpion of the sub-stability set corresponding to $(\mathbf{b}, \mathbf{d})$ is the following

$$S^0 \cap (\cap_{i \in I} \mathcal{S}_{B^i}).$$

There is a finite number of sub-stability sets and each of them forms a convex polyhedral cone with one vertex in the origin. Consequently, the set of all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathcal{P} \setminus \mathcal{U}$ for which there exists a separating supporting hyperplane for $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ is formed by a union of finite number of convex polyhedral cones with vertices in the origin.

The following Tables 3 – 4 contain several examples in the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. In these tables the number of stability sets, the number of sub-stability sets (and the number of sub-stability sets of the supporting type) and the aproximate computing time are included for given matrices $\mathbf{A}$, $\mathbf{C}$. Calculations were carried out in MATLAB (see Section 4).

Table 3: Examples in $\mathbb{R}^2$ – pseudorandom data.

matrix $\mathbf{A}$	matrix $\mathbf{C}$	stability sets	sub-stability sets (supporting types)	computing time
$\begin{pmatrix} -9 & -2 \\ 5 & 7 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ -9 & 4 \\ 3 & -1 \end{pmatrix}$	71	94 (58)	3 min 45 s
$\begin{pmatrix} -4 & 5 \\ 5 & 1 \\ -6 & -8 \end{pmatrix}$	$\begin{pmatrix} -4 & -1 \\ 6 & -8 \\ 0 & 1 \\ 7 & -8 \end{pmatrix}$	89	119 (119)	11 min 33 s
$\begin{pmatrix} 0 & -3 \\ -4 & 0 \\ 1 & -4 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 1 \\ 2 & -3 \\ 4 & -3 \\ 3 & -2 \end{pmatrix}$	126	259 (232)	19 min 17 s
$\begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & 3 \\ -2 & 2 \\ 0 & -1 \\ 3 & -2 \\ 1 & -3 \end{pmatrix}$	187	301 (212)	26 min 34 s

Example 3. Let us have B^1 a sub-basis of $M_1(\boldsymbol{\lambda})$ and B^2 a sub-basis of $M_2(\boldsymbol{\nu})$. Denote by $\mathcal{S}_{B^1B^2}$ the set of all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$, for which there exists a separating hyperplane supporting $M_1(\boldsymbol{\lambda})$ in the face determined by the sub-basis B^1 and supporting $M_2(\boldsymbol{\nu})$ in the face determined by the sub-basis B^2. Denote $B = B^1 \cup \{B^2 + m\}$, where $B^2 + m = \{i + m \mid i \in B^2\}$. Clearly $(m + l + 1) \notin B$. Then the set $\mathcal{S}_{B^1B^2}$ has the following description

$$\mathcal{S}_{B^1B^2} = \mathcal{S}_B \cap \mathcal{S}_B^{\mathcal{Q}},$$

where $\mathcal{S}_B$ is from (26) (in order that B^1, B^2 may be feasible) and $\mathcal{S}_B^{\mathcal{Q}}$ is the set of all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ such that B is the feasible sub-basis of the convex polytope $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ (in order that may exist a separating hyperplane). Therefore $\mathcal{S}_B^{\mathcal{Q}}$ is the set of all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$, for which the system of linear equalities and inequalities (see (14))

$$\mathbf{Z}_B(\boldsymbol{\lambda}, \boldsymbol{\nu})\mathbf{w} = \mathbf{z}, \quad \mathbf{w} \geq \mathbf{0}$$

has a solution. From [6] it follows the description $\mathcal{S}_B^{\mathcal{Q}} = \mathbb{R}^{m+l} \setminus (\mathcal{U}_1 \cup \mathcal{U}_2)$,

Table 4: Examples in $\mathbb{R}^3$ – pseudorandom data.

matrix $\mathbf{A}$	matrix $\mathbf{C}$	stability sets	sub-stability sets (supporting types)	computing time
$\begin{pmatrix} 1 & -3 & -4 \\ -4 & -5 & 1 \\ 5 & 5 & -5 \end{pmatrix}$	$\begin{pmatrix} 2 & -4 & 5 \\ 1 & 4 & 3 \\ -5 & 0 & 2 \\ -3 & -4 & 1 \end{pmatrix}$	208	253 (234)	25 min 25 s
$\begin{pmatrix} 1 & 2 & -5 \\ -1 & -5 & -1 \\ 3 & -3 & -4 \end{pmatrix}$	$\begin{pmatrix} -3 & 0 & 4 \\ -1 & 2 & -4 \\ -5 & 2 & 3 \\ -4 & -3 & -4 \\ -2 & 2 & 4 \end{pmatrix}$	145	246 (228)	24 min 32 s

where

$$\mathcal{U}_1 = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid (\boldsymbol{\lambda}^T, \boldsymbol{\nu}^T)_B \mathbf{w}^i > 0 \quad \forall i \in V\},$$
$$\mathcal{U}_2 = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid (\boldsymbol{\lambda}^T, \boldsymbol{\nu}^T)_B \mathbf{w}^i < 0 \quad \forall i \in V\},$$

where $\mathbf{w}^i$, $i \in V$ are all vertices of the convex polytope

$$\mathcal{Q}_B \equiv \{\mathbf{w} \in \mathbb{R}^{|B|} \mid \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{pmatrix}_B \mathbf{w} = \mathbf{e}_{n+1}, \quad \mathbf{w} \geq \mathbf{0}\}. \tag{27}$$

Consider a particular example. Let the same matrices as in Example 1 are given, i.e.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Let us have $B^1 = (1, 2)$, $B^2 = (1, 2)$. Therefore $B = (1, 2, 4, 5)$. First we compute the set $\mathcal{S}_B$. The convex polyhedral cone

$$\mathcal{N}_{B^1} = \left\{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^3 \mid \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \mathbf{z} = \mathbf{0}, \ \mathbf{z} \geq \mathbf{0}\right\}$$

has only one edge in the direction of $\mathbf{h}_1 = (1, 1, 1)^T$. The convex polyhedral cone

$$\mathcal{N}_{B^2} = \left\{\mathbf{y} \in \mathbb{R}^2 \mid \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} = \mathbf{0}\right\}$$

has no edge. The decription of $\mathcal{S}_B$ from (26) is the following

$$\mathcal{S}_B = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 \geq 0\}.$$

The set $\mathcal{Q}_B$ from (27) is described by the system

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{w} \geq \mathbf{0}$$

and contains two vertices $\mathbf{w}^1 = (\frac{1}{2}, 0, \frac{1}{2}, 0)^T$, $\mathbf{w}^2 = (0, \frac{1}{2}, 0, \frac{1}{2})^T$. Hence $\mathcal{S}_B^{\mathcal{Q}}$ has the description

$$\begin{aligned} \mathcal{S}_B^{\mathcal{Q}} &= \mathbb{R}^5 \setminus (\{\lambda_1 + \nu_1 > 0,\ \lambda_2 + \nu_2 > 0\} \cup \{\lambda_1 + \nu_1 < 0,\ \lambda_2 + \nu_2 < 0\}) = \\ &= \{\lambda_1 + \nu_1 \geq 0,\ \lambda_2 + \nu_2 \leq 0\} \cup \{\lambda_1 + \nu_1 \leq 0,\ \lambda_2 + \nu_2 \geq 0\}). \end{aligned}$$

The set $\mathcal{S}_{B^1B^2}$ is not convex and has the description

$$\begin{aligned} \mathcal{S}_{B^1B^2} =& \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 \geq 0,\ \lambda_1 + \nu_1 \geq 0,\ \lambda_2 + \nu_2 \leq 0\} \cup \\ & \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 \geq 0,\ \lambda_1 + \nu_1 \leq 0,\ \lambda_2 + \nu_2 \geq 0\}. \end{aligned}$$

This set corresponds to the union of the stability sets number 1 and 3 from Example 1.

(ii) Let us have $B^1 = (1)$, $B^2 = (1)$. Therefore $B = (1, 4)$. The convex polyhedral cone

$$\mathcal{N}_{B^1} = \left\{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^3 \mid \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{z} = \mathbf{0},\ \mathbf{z} \geq \mathbf{0}\right\}$$

has only one edge in the direction of $\mathbf{h}_1 = (1, 1, 1)^T$. The convex polyhedral cone

$$\mathcal{N}_{B^2} = \left\{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^2 \mid \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mathbf{z} = \mathbf{0},\ \mathbf{z} \geq \mathbf{0}\right\}$$

has no edge. Hence

$$\mathcal{S}_B = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 \geq 0\}.$$

The set $\mathcal{Q}_B$ from (27) is described by

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{w} \geq \mathbf{0}$$

and contains one vertex $\mathbf{w}^1 = (\frac{1}{2}, \frac{1}{2})^T$. Hence

$$\mathcal{S}_B^Q = \mathbb{R}^5 \setminus (\{\lambda_1 + \nu_1 > 0\} \cup \{\lambda_1 + \nu_1 < 0\}) = \{\lambda_1 + \nu_1 = 0\}.$$

The set $\mathcal{S}_{B^1B^2}$ has the description

$$\mathcal{S}_{B^1B^2} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 \geq 0,\ \lambda_1 + \nu_1 = 0\}.$$

(iii) Let us have $B^1 = (1, 2, 3)$, $B^2 = (1, 2)$. Therefore $B = (1, 2, 3, 4, 5)$. Similarly we obtain

$$\begin{aligned} \mathcal{S}_B &= \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 = 0\}, \\ \mathcal{S}_B^Q &= \mathbb{R}^5 \setminus \big(\{\lambda_1 + \nu_1 > 0,\ \lambda_2 + \nu_2 > 0,\ \lambda_1 + \lambda_2 + \lambda_3 > 0\} \cup \\ &\qquad \{\lambda_1 + \nu_1 < 0,\ \lambda_2 + \nu_2 < 0,\ \lambda_1 + \lambda_2 + \lambda_3 < 0\}\big). \end{aligned}$$

Hence the set $\mathcal{S}_{B^1B^2}$ has the description

$$\begin{aligned} \mathcal{S}_{B^1B^2} =&\{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 = 0,\ \lambda_1 + \nu_1 \geq 0,\ \lambda_2 + \nu_2 \leq 0\} \cup \\ &\{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 = 0,\ \lambda_1 + \nu_1 \leq 0,\ \lambda_2 + \nu_2 \geq 0\}. \end{aligned}$$

This set corresponds to the union of the stability sets number 1 and 3 from Example 1 with the property, that the set $M_1(\boldsymbol{\lambda})$ consists of one point only.

$$\begin{aligned} \mathcal{S}_B &= \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 = 0\}, \\ \mathcal{S}_B^Q &= \mathbb{R}^5 \setminus \big(\{\lambda_1 + \nu_1 > 0,\ \lambda_2 + \nu_2 > 0,\ \lambda_1 + \lambda_2 + \lambda_3 > 0\} \cup \\ &\qquad \{\lambda_1 + \nu_1 < 0,\ \lambda_2 + \nu_2 < 0,\ \lambda_1 + \lambda_2 + \lambda_3 < 0\}\big). \end{aligned}$$

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