Sub-Ramsey numbers for Arithmetic Progressions and Schur Triples

Jacob Fox* Veselin Jungić† Radoš Radoičić‡

Abstract

For a given positive integer k, $sr(m,k)$ denotes the minimal positive integer such that every coloring of $[n]$, $n \geq sr(m,k)$, that uses each color at most k times, yields a rainbow $AP(m)$; that is, an m-term arithmetic progression, all of whose terms receive different colors. We prove that $\frac{17}{8}k + O(1) \leq sr(3,k) \leq \frac{15}{7}k + O(1)$ and $sr(m,2) > \lfloor \frac{m^2}{2} \rfloor$, improving the previous bounds of Alon, Caro, and Tuza from 1989. Our new lower bound on $sr(m,2)$ immediately implies that for $n \leq \frac{m^2}{2}$, there exists a mapping $\phi : [n] \to [n]$ without a fixed point such that for every $AP(m)$ $\mathcal{A}$ in $[n]$, the set $\mathcal{A} \cap \phi(\mathcal{A})$ is not empty. We also propose the study of sub-Ramsey–type problems for linear equations other than $x + y = 2z$. For a given positive integer k, we define $ss(k)$ to be the minimal positive integer n such that every coloring of $[n]$, $n \geq ss(k)$, that uses each color at most k times, yields a rainbow solution to the Schur equation $x + y = z$. We prove that $ss(k) = \lfloor \frac{5k}{2} \rfloor + 1$.

Key words: rainbow arithmetic progressions, sub-Ramsey

1 Introduction

Let $\mathbb{N}$ denote the set of positive integers, and for $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A k-term arithmetic progression, $k \in \mathbb{N}$, is a set of the form

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, E-mail: licht@mit.edu

†Department of Mathematics, Simon Fraser University, Burnaby, B.C., V5A 2R6, Canada, E-mail: vjungic@sfu.ca

‡Department of Mathematics, Rutgers, The State University of New Jersey, New Brunswick, NJ, E-mail: rados@math.rutgers.edu

$\{a+(i-1)\,d : i \in [k]\}$, for some $a, d \in \mathbb{N}$, and will be abbreviated as $AP(k)$ throughout. The classical result of van der Waerden [vW27, GRS90] states that for all natural numbers m and k there is an integer $n_0 = n_0(m,k)$, such that every k-coloring of $[n]$, $n \geq n_0$, contains a monochromatic $AP(m)$. This statement was further generalized to sets of positive upper density in the celebrated work of Szemerédi [Sz75]. Canonical versions of van der Waerden's theorem were discovered by Erdős and others [E87].

Given a coloring of $\mathbb{N}$, a set $S \subseteq \mathbb{N}$ is called *rainbow* if all elements of S are colored with different colors. In [JL+03], for the first time in the literature, Jungić et al. considered a rainbow counterpart of van der Waerden's theorem, and proved that every 3-coloring of $\mathbb{N}$ with the upper density of each color greater than $1/6$ contains a rainbow $AP(3)$. Improving on their methods and some extensions [JR03], Axenovich and Fon-Der-Flaass [AF04] proved the following "finite" version of this result.

Theorem 1 (Conjectured in [JL+03], proved in [AF04].) Given $n \geq 3$, every partition of $[n]$ into three color classes $\mathcal{R}$, $\mathcal{G}$, and $\mathcal{B}$ with $\min(|\mathcal{R}|, |\mathcal{G}|, |\mathcal{B}|) > r(n)$, where

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod 6 \\ (n+4)/6 & \text{if } n \equiv 2 \pmod 6 \end{cases} \tag{1}$$

contains a rainbow $AP(3)$.

The following coloring of $\mathbb{N}$:

$$c(i) := \begin{cases} B & \text{if } i \equiv 1 \pmod 6 \\ G & \text{if } i \equiv 4 \pmod 6 \\ R & \text{otherwise} \end{cases}$$

contains no rainbow $AP(3)$. The restriction of this coloring to $[n]$ shows that Theorem 1 is the best possible (for the case $n \equiv 2 \pmod 6$, one has to use a different coloring, see [JL+03]). It is interesting to note that similar statements about the existence of rainbow $AP(k)$ in k-colorings of $[n]$, $k \geq 4$, do not hold [AF04, CFJR].

In lay terms, Axenovich and Fon-Der-Flaass showed that sufficiently *large* color classes in a 3-coloring imply the existence of a rainbow $AP(3)$. In this paper, we are interested in conditions that guarantee the existence of a rainbow $AP(3)$ when color classes have *small* cardinality. The notable distinction between these two approaches is that in the later case the number of colors can be greater than three.

This setup was first studied by Alon, Caro and Tuza in [ACT89], where for a given $k \in \mathbb{N}$, they defined sub-k-colorings as colorings in which every color class has size at most k. For given k, $m \in \mathbb{N}$, they introduced the sub-k-Ramsey number $sr(m,k)$ as the minimum integer $n_0 = n_0(m,k)$ such that every sub-k-coloring of $[n]$, $n > n_0$, yields a rainbow $AP(m)$. They proved that for every $m \geq 3$, $k \geq 2$,

$$\frac{1}{6}\frac{(k-1)m(m-1)}{\log{(k-1)m}} - k + 1 \leq$$

$$sr(m,k) \leq (1+o(1))\frac{24}{13}(k-1)(m-1)^2 \log{(k-1)(m-1)},$$

where the factor of $1 + o(1)$ approaches 1 as $m \to \infty$.

For $k = 2$, we improve on their lower bound by constructing a coloring that has already been used in [JL+03] to prove a lower bound for a related problem concerning rainbow arithmetic progressions in equinumerous colorings.

Theorem 2 *For $m \geq 3$, $sr(m,2) > \lfloor \frac{m^2}{2} \rfloor$.*

Motivated by [EH58] and [AC86], Caro [C87] proved that for every positive integer m, there is a minimum integer $n = n_0(m)$ such that for every $\phi : [n] \to [n]$ without a fixed point, there is an $AP(m)$ $\mathcal{A}$ satisfying: $\phi(i) \notin \mathcal{A}$ for $i \in \mathcal{A}$. Moreover, he showed that $\frac{c_1 m^2}{\log m} \leq n_0(m) \leq m^2 (\log m)^{\frac{c_2 \log m}{\log \log m}}$ for some absolute constants c_1 and c_2. In [ACT89], Alon et al. applied the same methods they had used to bound $sr(m,k)$ to drastically improve the earlier bounds on $n_0(m)$. They proved that for every m,

$$\frac{m(m-1)}{3\log m} + O(1) \leq sr(m,3) - 1 \leq n_0(m) \leq (1+o(1))\frac{48}{13}m^2 \log m.$$

Since $sr(m,k)$ is an increasing function in both m and k, then in particular, $sr(m,2) \leq sr(m,3)$. Therefore, Theorem 2 implies the following significant improvement on the lower bound for $n_0(m)$ for all m:

Corollary 1 *For all positive integers m, $n_0(m) \geq \lfloor \frac{m^2}{2} \rfloor$.*

If m is fixed and k grows, Alon et al.[ACT89] proved that $sr(m,k)$ is linear in k, namely

$$sr(m,k) \leq (1+o(1))\frac{1}{2}m(m-1)^2(k-1).$$

The exact determination of the asymptotic behavior of $sr(m,k)$ appears to be difficult. In the case of $AP(3)$, i.e. for $m = 3$, the previous inequality yields $sr(3,k) \le (1+o(1))6k$. They provided a sharper estimate:

$$\text{as } k \text{ grows, } 2k \le sr(3,k) \le (4.5 + o(1))k.$$

In what follows, we use $sr(k)$ to denote the sub-k-Ramsey number $sr(3,k)$. Using methods developed in [JL+03, AF04], we almost determine the rate of growth of $sr(k)$.

Theorem 3 *As k grows, $\frac{17}{8}k + O(1) \le sr(k) \le \frac{15}{7}k + O(1)$.*

A set $\{x < y < z\}$ of integers is an arithmetic progression of length three if and only if $x + z = 2y$. Hence, one can define sub-Ramsey problems for other linear equations. A classical candidate is the Schur equation $x+y = z$ [S16]. Arguably, the first result in Ramsey theory is due to Schur, who, in 1916, proved that for every k and sufficiently large n, every k-coloring of $[n]$ contains a monochromatic solution to equation $x + y = z$. More than seven decades later, building up on the previous work of Alekseev and Savchev, E. and G. Szekeres (see [JL+03] and references therein), Schönheim [S90] proved the following rainbow counterpart, which is clearly an analogue of Theorem 1.

Theorem 4 ([S90]) For every $n \ge 3$, every partition of $[n]$ into three color classes $\mathcal{R}$, $\mathcal{G}$, and $\mathcal{B}$ with $\min(|\mathcal{R}|, |\mathcal{G}|, |\mathcal{B}|) > n/4$, contains a rainbow solution to equation $x + y = z$. Term $n/4$ cannot be improved.

For a given positive integer k, let $ss(k)$ denote the minimal number such that every coloring of $[n]$, $n \ge ss(k)$, that uses each color at most k times, yields a rainbow solution to equation $x + y = z$. We prove the following theorem.

Theorem 5 *For all positive integers k, $ss(k) = \lfloor \frac{5k}{2} \rfloor + 1$.*

The paper is organized as follows. In Section 2, we construct a coloring that settles Theorem 2 and hence Corollary 1. In Section 3, we use Theorem 1 and prove a somewhat surprising claim that, in order to prove good bounds on $sr(k)$, it suffices to only consider sub-k-colorings with three colors. Furthermore, we relate our problem to the problem of finding good bounds on $\sigma(n)$, the minimum integer k such that there is a sub-k-coloring of $[n]$ with three colors and no rainbow $AP(3)$. In Section 4, we provide lower and upper bounds on $\sigma(n)$, which in turn imply Theorem 3. In Section 5, we prove lemmata that together imply Theorem 5. In Section 6, we conjecture the exact rate of growth of $sr(k)$. We also propose new sub-Ramsey–type problems, while surveying the current stage of rainbow Ramsey theory (see [JRN]).

2 Proof of Theorem 2

We construct a coloring c of $[\lfloor \frac{m^2}{2} \rfloor]$ that uses each color exactly twice and prove that it does not contain a rainbow $AP(m)$. Define a *j-block* B_j ($j \in \mathbb{N}$) to be the sequence $12\ldots j12\ldots j$, where the *left half* and the *right half* of the block are naturally defined. For $a \in \mathbb{Z}$, let $B_j + a$ be the sequence $(a+1)(a+2)\ldots(a+j)(a+1)(a+2)\ldots(a+j)$. Define $B_j^- = B_j - \binom{j+1}{2}$ and $B_i^+ = B_i + \binom{i}{2}$. If $m = 2l+1$ is odd, define the coloring c of $[2l^2+2l]$ in the following way (bars denote endpoints of the blocks):

$$|B_l^-|\ldots|B_j^-|\ldots|B_2^-||B_1^-||B_1^+||B_2^+|\ldots|B_i^+|\ldots|B_l^+|.$$

If $m = 2l$ is even, define the coloring c of $[2l^2]$ in the following way (bars denote endpoints of the blocks):

$$|B_{l-1}^-|\ldots|B_j^-|\ldots|B_2^-||B_1^-||B_1^+||B_2^+|\ldots|B_i^+|\ldots|B_l^+|.$$

We only show the proof of Theorem 2 in the case when m is odd (since the case when m is even is essentially the same). Note that the coloring c uses each of the l^2+l colors exactly twice (the colors are integers from the interval $[1-\binom{l+1}{2}, \binom{l+1}{2}]$). Now, we show that the coloring c of $[2l^2+2l]$ contains no rainbow $AP(2l+1)$. The key observation is that a rainbow AP with common difference d cannot contain elements from opposite halves of any block B_j^- or B_j^+ where d is a factor of j. Fix a longest rainbow AP $\mathcal{A}$ and let d denote its common difference. If $d > l$, then the length of $\mathcal{A}$ is $\leq 2l$. If $d \leq l$, then $\mathcal{A}$ is one of the following three types:
(1) $\mathcal{A}$ is contained in

$$|B_d^-|\ldots|B_j^-|\ldots|B_2^-||B_1^-||B_1^+||B_2^+|\ldots|B_i^+|\ldots|B_d^+|.$$

Then $\mathcal{A}$ does not intersect either the left half of B_d^- or the right half of B_d^+. Therefore, the length of $\mathcal{A}$ is at most $1 + \frac{2d^2-1}{d} < 2d+1 \leq 2l+1$.
(2) $\mathcal{A}$ is contained in

$$|B_{(j+1)d}^-||B_{(j+1)d-1}^-|\ldots|B_{jd}^-| \textit{or in} |B_{jd}^+||B_{jd+1}^+|\ldots|B_{(j+1)d}^+|,$$

where $(j+1)d \le l$. Assume the first case occurs (both cases are handled the same way). Then $\mathcal{A}$ does not intersect either the left half of $B^-_{(j+1)d}$ or the right half of B^-_{jd}. Therefore, the length of $\mathcal{A}$ is at most

$$1 + \frac{(2j+1)d^2 - 1}{d} < (2j+1)d + 1 \le 2l + 1.$$

(3) $\mathcal{A}$ is contained in

$$|B^-_l||B^-_{l-1}|\dots|B^-_{jd+1}||B^-_{jd}| or\ in |B^+_{jd}||B^+_{jd+1}|\dots|B^+_{l-1}||B^+_l|,$$

where $l - jd < d$. Assume the first case occurs (both cases are handled the same way). Then $\mathcal{A}$ does not intersect the right half of B^-_{jd}. Therefore, the length of $\mathcal{A}$ is less than

$$1 + \frac{1}{d}(2ld - 1) < 2l + 1.$$

3 A reduction to 3-colorings

As we mentioned in the introduction, number of colors in a sub-k-coloring can be greater than three. In the following lemma we show that it is enough to consider only sub-k-colorings with three colors.

Lemma 1 *Let n, k, $r \in \mathbb{N}$ be such that $n/6 < k < n/2$ and $r \ge 3$. For every sub-k-coloring c of $[n]$ with r colors and no rainbow $AP(3)$ there exists a sub-k-coloring $\overline{c}$ of $[n]$ with three colors and no rainbow $AP(3)$, such that for all $i, j \in [n]$*

$$c(i) = c(j) \Rightarrow \overline{c}(i) = \overline{c}(j).$$

Proof: Let $C_1, C_2, \dots, C_r$ be the color classes of a sub-k-coloring c of $[n]$ with $n/6 < k < n/2$ and $r \ge 3$. Suppose that c contains no rainbow $AP(3)$. Without loss of generality, assume that for all $i, j \in [r]$, $i < j \Rightarrow |C_i| \ge |C_j|$. Then Theorem 1 implies that $|C_3| \le n/6$. Indeed, otherwise $|C_1| \ge |C_2| > n/6$ and $|\cup_{i=3}^r C_i| > n/6$ imply that there is an $AP(3)$ with terms from C_1, C_2, and C_i for some $i \in [3, r]$.

Suppose $|C_2| \le n/6$. Let $s = \min\left\{j : \left|\cup_{i=1}^j C_i\right| > n/6\right\}$. If $s = 1$, then $|\cup_{i=1}^s C_i| = |C_1| \le k < n/2$, and if $s > 1$, then $|\cup_{i=1}^s C_i| = \left|\cup_{i=1}^{s-1} C_i\right| + |C_s| \le n/6 + n/6 = n/3$. In either case, we have $|\cup_{i=1}^s C_i| < n/2$. Let

$t = \min\left\{j : \left|\cup_{i=s+1}^{j} C_i\right| > n/6\right\}$. Since $t \geq 2$ and $|C_2| \leq n/6$, we have $\left|\cup_{i=s+1}^{t} C_i\right| \leq n/3$. It follows that $|[n] \setminus \cup_{i=1}^{t} C_i| > n - n/2 - n/3 = n/6$. Therefore, by Theorem 1, the 3-coloring with color classes $\cup_{i=1}^{s} C_i$, $\cup_{i=s+1}^{t} C_i$, and $[n] \setminus (\cup_{i=1}^{t} C_i)$ yields a rainbow $AP(3)$, that clearly implies the existence of a rainbow $AP(3)$ in the original coloring c. This contradicts our assumptions.

Therefore, $k \geq |C_1| \geq |C_2| > n/6$ and $|C_i| \leq n/6 < k$ for all $i \geq 3$. Then, we define $\overline{c}$ of $[n]$ to be the 3-coloring given by color classes C_1, C_2, and $\cup_{i=3}^{r} C_i$. Clearly, $\overline{c}$ is a sub-k-coloring with no rainbow $AP(3)$, as required. □

For $n \in \mathbb{N}$, we define $\sigma(n)$ as the minimum positive integer k such that there is a sub-k-coloring of $[n]$ with three colors and no rainbow $AP(3)$. Next, we show that $\sigma(n)$ and $sr(k)$ are closely related.

Lemma 2 *Let a, $b > 1$. Then, $ak < sr(k) < bk$ for all k if and only if $\frac{n}{b} < \sigma(n) < \frac{n}{a}$ for all n.*

Proof: Let $n \in \mathbb{N}$ and let a, $b > 1$ be such that $ak < sr(k) < bk$ for all $k \in \mathbb{N}$. Since there is sub-$\sigma(n)$-coloring of $[n]$ with no rainbow $AP(3)$, it follows that $n < sr(\sigma(n)) < b\sigma(n)$. Since $ak < sr(k)$, then $n < sr(n/a)$. Thus, there is a sub-(n/a)-coloring of $[n]$ with no rainbow AP(3). Therefore, $\sigma(n) < n/a$.

Now, suppose $\frac{n}{b} < \sigma(n) < \frac{n}{a}$ and let $k \in \mathbb{N}$. From $k < \sigma(bk)$, it follows that there is a sub-k-coloring of $[bk]$ with a rainbow $AP(3)$. Thus $sr(k) < bk$. From $\sigma(ak) < k$ it follows that there is a sub-k-coloring of $[ak]$ with no rainbow $AP(3)$. Hence $ak < sr(k)$. □

From Lemmata 1 and 2, it is clear that our problem of finding good bounds on $sr(k)$ reduces to the problem of bounding $\sigma(n)$.

4 Proof of Theorem 3

For a given 3-coloring $c : [n] \to \{R, B, G\}$ let $\mathcal{R}$, $\mathcal{B}$, and $\mathcal{G}$ denote sets of elements of $[n]$ colored with R, G, and B, respectively. First, we determine an upper bound for $\sigma(n)$.

Proposition 1 *For all $n \in \mathbb{N}$, $\sigma(n) \leq \frac{8n}{17} + 9$.*

Proof: We define a 3-coloring $c : \mathbb{N} \to \{R, G, B\}$ by

$$c(n) = \begin{cases} G & \text{if } n \equiv 0 \pmod{17} \\ R & \text{if } n \equiv 1, 2, 4, 8, 9, 13, 15, 16 \pmod{17} \\ B & \text{if } n \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}. \end{cases}$$

The coloring c is periodic with a period 17. We claim that c contains no rainbow $AP(3)$. Otherwise, let $\{i, j, k\}$ be an $AP(3)$ with $i + k = 2j$. If $c(j) = G$, then $i + k \equiv 0 \pmod{17}$, which implies $c(i) = c(k)$. If $c(i) = G$, then $2j \equiv k \pmod{17}$. It is not difficult to check that in this case $c(j) = c(2j) = c(k)$.

Let $n \in \mathbb{N}$, $n = 17m + k$, where $m \in \mathbb{N} \cup \{0\}$ and $k \in \{0\} \cup [16]$. Let $c^{(n)}$ be the restriction of c on $[n]$. For $c^{(n)}$ and $\mathcal{X} \in \{\mathcal{R}, \mathcal{B}\}$ we have

$$|\mathcal{X}| \leq 8m + k = \frac{8(n-k)}{17} + k = \frac{8n}{17} + \frac{9k}{17} \leq \frac{8n}{17} + 9.$$

Therefore, for every $n \in \mathbb{N}$ there is a sub-$(\frac{8n}{17} + 9)$-coloring with no rainbow $AP(3)$, which implies that $\sigma(n) \leq \frac{8n}{17} + 9$ for all n. $\square$

Next, we prove a lower bound for $\sigma(n)$. We will do so through a sequence of lemmata. We start with some definitions from [JL+03, JR03]. Given a 3-coloring $c : [n] \to \{R, B, G\}$, we say that $X \in \{R, B, G\}$ is a *dominant color* if for every two consecutive elements of $[n]$ that are colored with different colors, one of them is colored with X. We say that $Y \in \{R, B, G\}$ is a *recessive color* if there are no two consecutive elements of $[n]$ colored with Y.

Lemma 3 ([JR03]) In every 3-coloring $c : [n] \to \{R, B, G\}$ with no rainbow $AP(3)$, one of the colors must be dominant and another color must be recessive.

Without loss of generality, let R be a dominant color and let G be a recessive color. The set $g_1 < g_2 < \ldots < g_s$ of all elements of $[n]$ colored by G divide $[n]$ naturally into subsegments, called *blocks*, of the form $I_i = [g_i, g_{i+1} - 1]$, for $1 \leq i \leq s - 1$, $I_s = [g_s, n]$, and, if $g_1 \neq 1$, $I_0 = [1, g_1 - 1]$. Clearly, each block I_i, $1 \leq i \leq s$, contains a single element colored by G.

Our goal is to show the following.

Proposition 2 *Any 3-coloring of $[n]$ with no rainbow $AP(3)$ has a color class of size greater than or equal to $\frac{7n}{15} - O(1)$.*

If B is a recessive color, then, since R is dominant and G is recessive, we have $2|\mathcal{G}| \le |\mathcal{R}|$ and $2|\mathcal{B}| \le |\mathcal{R}|$. This implies $|\mathcal{R}| \ge \frac{n}{2} > \frac{7n}{15}$. Therefore, in the rest of the proof of Proposition 2, we can assume that B is not a recessive color.

Next, we prove that G, as the (unique) recessive color, is indeed sparse.

Lemma 4 $g_{i+1} - g_i > 3$ *for* $1 \le i \le s-1$.

Proof: Suppose there exists $i \in [s-1]$ such that $g_{i+1} = g_i + 2$. Note that the fact that G is recessive and R is dominant implies $c(g_i+1) = R$. Since B is not recessive there exists $j \in [n]$ such that $c(j) = c(j+1) = B$. Fix j so that there is no other occurrence of consecutive elements colored with B between $j+1$ and g_i, if $j+1 < g_i$; or between g_{i+1} and j if $j > g_{i+1}$.

If $g_i \equiv j \pmod 2$, then the following $AP(3)$s: $\{g_i, \frac{g_i+j}{2}, j\}$, $\{g_i+1, \frac{g_i+j}{2}+1, j+1\}$, and $\{g_i+2, \frac{g_i+j}{2}+1, j\}$ are not rainbow, so $c\left(\frac{g_i+j}{2}\right) \in \{G, B\}$ and $c\left(\frac{g_i+j}{2}+1\right) = B$. This contradicts either our choice of j or our assumption that R is the dominant color. If $g_i \not\equiv j \pmod 2$, then the following $AP(3)$s: $\{g_i, \frac{g_i+j+1}{2}, j+1\}$, $\{g_i+1, \frac{g_i+1+j}{2}, j\}$, and $\{g_i+2, \frac{g_i+j+3}{2}, j+1\}$ are not rainbow, so we have that $c\left(\frac{g_i+j+1}{2}\right) = B$ and $c\left(\frac{g_i+j+3}{2}\right) \in \{G, B\}$, which, as above, contradicts our assumptions.

Therefore, $g_{i+1} - g_i > 2$ for all i.

Now, suppose there is $i \in [s-1]$ such that $g_{i+1} = g_i + 3$. Since R is dominant and c has no rainbow $AP(3)$, we have $c(g_i+1) = c(g_i+2) = R$. As above, we choose j with $c(j) = c(j+1) = B$, that is the closest to either g_i from the left or g_{i+1} from the right.

If $g_i \equiv j \pmod 2$, then the following $AP(3)$s: $\{g_i, \frac{g_i+j}{2}, j\}$, $\{g_i+1, \frac{g_i+j}{2}+1, j+1\}$, and $\{g_i+3, \frac{g_i+j}{2}+2, j+1\}$ cannot be rainbow, so we have $c\left(\frac{g_i+j}{2}\right) \in \{G, B\}$, $c\left(\frac{g_i+j}{2}+2\right) \in \{G, B\}$, and $c\left(\frac{g_i+j}{2}+1\right) = R$.[1] Since there are no two elements colored with G that are one place apart and since c has no rainbow AP(3), we have that $c\left(\frac{g_i+j}{2}\right) = \left(\frac{g_i+j}{2}+2\right) = B$.

If $g_i \equiv \frac{g_i+j}{2} \pmod 2$, then from the fact that $\left\{g_i, \frac{g_i+(g_i+j)/2}{2}+1, \frac{g_i+j}{2}+2\right\}$ and $\left\{g_i+2, \frac{g_i+(g_i+j)/2}{2}+1, \frac{g_i+j}{2}\right\}$ are not rainbow, it follows that $c\left(\frac{g_i+(g_i+j)/2}{2}+1\right) = B$. At the same time, since

[1] Here, we have also used the definition of j.

$\left\{g_i, \frac{g_i+(g_i+j)/2}{2}, \frac{g_i+j}{2}\right\}$ is not rainbow, then $c\left(\frac{g_i+(g_i+j)/2}{2}\right) \in \{G, B\}$.

However, $\left\{c\left(\dfrac{g_i+(g_i+j)/2}{2}\right), c\left(\dfrac{g_i+(g_i+j)/2}{2}+1\right)\right\} \subseteq \{G, B\}$

contradicts our choice of j and our assumption that R is the dominant color.

If $g_i \not\equiv \frac{g_i+j}{2} \pmod 2$, then the fact that the following $AP(3)$s: $\left\{g_i+3, \frac{g_i+(g_i+j)/2+1}{2}+1, \frac{g_i+j}{2}\right\}$ and $\left\{g_i+3, \frac{g_i+(g_i+j)/2+1}{2}+2, \frac{g_i+j}{2}+2\right\}$ are not rainbow implies that

$$\left\{c\left(\frac{g_i+(g_i+j)/2+1}{2}+1\right), c\left(\frac{g_i+(g_i+j)/2+1}{2}+2\right)\right\} \subseteq \{G, B\},$$

which is a contradiction as above.

If $g_i \not\equiv j \pmod 2$, then the AP(3)s: $\{g_i, \frac{g_i+j+1}{2}, j+1\}$, $\{g_i+1, \frac{g_i+1+j}{2}, j\}$, and $\{g_i+3, \frac{g_i+j+1}{2}+1, j\}$ are not rainbow, so we have $c\left(\frac{g_i+j+1}{2}\right) = B$ and $c\left(\frac{g_i+j+1}{2}+1\right) \in \{G, B\}$, which again contradicts our assumptions.

Therefore, $g_{i+1} - g_i > 3$ for all i. □

Now, we have the following corollaries.

Corollary 2 *If* $\{c(k), c(k+2)\} \subseteq \{B, G\}$ *for some* $k \in [n-2]$*, then* $c(k) = c(k+2) = B$.

Corollary 3 *Each block* I_i*,* $1 \le i \le s-1$*, is of length of at least four.*

In the rest of the proof of Lemma 2, we discuss two cases.

Case 1. Each block I_j, $1 \le j \le s-1$, contains two consecutive elements colored with B.

An easy computation shows that if a block I_j, $1 \le j \le s-1$, has two consecutive numbers colored by B, then the length of I_j satisfies $|I_j| \ge 15$. As an illustration of the omitted case analysis, we show that $|I_j| \ne 14$.

Suppose there exists an $i \in [s]$ such that $g_{i+1} = g_i + 14$. Then $c(g_i + 1) = c(g_i + 2) = c(g_i + 12) = c(g_i + 13) = R$. Since c has no rainbow $AP(3)$, 3-term arithmetic progressions $\{g_i, g_i+2, g_i+4\}$, $\{g_i, g_i+6, g_i+12\}$, $\{g_i+2, g_i+8, g_i+14\}$, and $\{g_i+10, g_i+12, g_i+14\}$ imply $c(g_i+4) = R$, $c(g_i+6) = R$, $c(g_i+8) = R$, and $c(g_i+10) = R$. Hence, if $|I_j| = 14$ then it is impossible to have two consecutive numbers in I_j colored with B.

Since $|I_j| \ge 15$ for all $1 \le j \le s-1$, we have $s = |\mathcal{G}| < n/15$. Therefore,

$$\max\{|\mathcal{R}|, |\mathcal{B}|\} \ge \frac{n - n/15}{2} = \frac{7n}{15}.$$

Case 2. There is a block with no two consecutive numbers colored with B.

Suppose I_j, $0 \leq j \leq s$, is the first block that contains two consecutive elements colored with B. Let $m \in I_j$ denote the smallest number k in I_j such that $c(k) = c(k+1) = B$. Next, we show that there cannot be three elements colored with G both before and after m.

Lemma 5 *If $m > g_3$, then $m > g_{s-2}$.*

Proof: Supose this is not true and let $g_3 < m < g_{s-2}$. Then, there are u, v, x, and y such that $g_u < g_v < m < g_x < g_y$, $g_u \equiv g_v \pmod 2$, and $g_x \equiv g_y \pmod 2$.

If $2m - g_v + 2 < n$, then $\{g_v, m, 2m - g_v\}$ and $\{g_v, m+1, 2m - g_v + 2\}$ are $AP(3)$s that are not rainbow, and we have $\{c(2m - g_v), c(2m - g_v + 2)\} \subseteq \{G, B\}$. From Corollary 2 it follows that $c(2m - g_v) = c(2m - g_v + 2) = B$. Since $\{g_u, (2m - g_v + g_u)/2, 2m - g_v\}$ and $\{g_u, (2m - g_v + g_u + 2)/2, 2m - g_v + 2\}$ are $AP(3)$s that are not rainbow, it follows that $c((2m - g_v + g_u)/2) = c((2m - g_v + g_u)/2 + 1) = B$. However, since $g_u < g_v$, we have that $(2m - g_v + g_u)/2 < m$, which contradicts our choice of m. Therefore, $2m - g_v + 2 > n$.

Let $p = \max\{k \in [m, g_x] : c(k) = c(k+1) = B\}$. From

$$2p - g_x \geq 2m - g_x \geq n + g_v - 1 - g_x = n - 1 - (g_x - g_v) \geq n - 1 - (n-2) \geq 1$$

and the fact that $\{2p - g_x, p, g_x\}$ and $\{2p - g_x + 2, p+1, g_x\}$ are not rainbow, it follows that $c(2p - x) = c(2p - x + 2) = B$. This implies that

$$c\left(\frac{2p - g_x + g_y}{2}\right) = c\left(\frac{2p - g_x + g_y}{2} + 1\right) = B \text{ and } \frac{2p - g_x + g_y}{2} > p,$$

which contradicts our choice of p. □

Case 2 naturally breaks into two subcases: (1) $m > g_3$, and (2) $m < g_3$. First we deal with (1).

Let $m > g_3$. The following lemma shows that B, although a non-recessive color, is sparse after m.

Lemma 6 *If $m > g_3$, then for every $k \in [m+2, n-3]$, $\{c(k), c(k+1), c(k+2), c(k+3)\} \cap \{R, G\} \neq \emptyset$.*

Proof: Suppose there exists $k \in [m+2, n-3]$ such that $c(k) = c(k+1) = c(k+2) = c(k+3) = B$. Let $k' \in \{k, k+1\}$ be such that $g_3 \equiv k' \pmod 2$.

Then $c\left(\frac{g_3+k'}{2}\right) = c\left(\frac{g_3+k'+1}{2}+1\right) = B$. From the proof of Lemma 5, we have $2m - g_3 + 2 > n$. From $k' \leq n-3 < 2m - g_3 + 2 - 3$, it follows that $\frac{g_3+k'}{2} < m$, which contradicts our choice of m. $\square$

Although we do not need the following lemma in the proof of the lower bound on $\sigma(n)$, we present it as a version of a more complicated analysis still to come.

Lemma 7 *If $m > g_3$, then c has a color class of size greater than $\frac{5n}{12} - O(1)$.*

Proof: Let $m > g_3$. Then, by Lemma 5, we have $m > g_{s-2}$. By Corollary 3, each block I_k, $1 \leq k \leq s-3$, is of length at least four, so $m \geq 4(s-3)$. Since R is the dominant color and since there are no two consecutive elements in $[m-1]$, colored both with B or both with G, we have $|\{k \in [1, m-1] : c(k) = R\}| \geq \frac{m-1}{2}$. Moreover, from Lemma 6, it follows that at least one of every four consecutive elements of $[m+2, n]$ is colored with R. Hence $|\{k \in [m+2, n] : c(k) = R\}| \geq \frac{n-m-2}{4}$. It follows that

$$|\mathcal{R}| \geq \frac{m-1}{2} + \frac{n-m-2}{4} = \frac{n+m}{4} - 1 \geq \frac{n}{4} + s - 4.$$

Then, $\max\{|\mathcal{R}|, |\mathcal{B}|\} \geq |\mathcal{R}| \geq \frac{n}{4} + |\mathcal{G}| - 4 \geq \frac{n}{4} + (n - 2\max\{|\mathcal{R}|, |\mathcal{B}|\}) - 4$ implies $3\max\{|\mathcal{R}|, |\mathcal{B}|\} \geq \frac{5n}{4} - 4$. Hence, we obtain $\max\{|\mathcal{R}|, |\mathcal{B}|\} \geq \frac{5n}{12} - \frac{4}{3}$. So, if $m > g_3$, then one of the classes has size at least $5n/12 - O(1)$. $\square$

In order to prove the lower bound on $\sigma(n)$, claimed in Proposition 2, we need to dig deeper into the structure of coloring c.

Lemma 8 $m \geq 2g_j - 1$.

Proof: Suppose $m < 2g_j - 1$. Then, $2g_j - m$, $2g_j - m - 1 \in [m]$, and $c(2g_j - m) \neq R \neq c(2g_j - m - 1)$. Since R is dominant and G is recessive, we have $c(2g_j - m) = c(2g_j - m - 1) = B$, which is impossible because of our choice of m. $\square$

Lemma 9 $|\{k \in [g_j, 2g_j - 1] : c(k) = R\}| \geq |\{k \in [g_j - 1] : c(k) = R\}|$.

Proof: For every $k \in [g_j - 1]$ with $c(k) = R$, an element $2g_j - k$ of $[g_j, 2g_j - 1]$ is colored with R, since the $AP(3)$ $\{k, g_j, 2g_j - k\}$ is not rainbow. $\square$

Since R is dominant and G is recessive, the definition of m implies $|\{k \in [2g_j - 1, m-1] : c(k) = R\}| \geq \frac{m-2g_j+1}{2}$. Furthermore, from Lemma 6, it follows that $|\{k \in [m+2, n] : c(k) = R\}| \geq \frac{n-(m+2)+1}{4}$. Now, using Lemma 9, we get:

$$|\mathcal{R}| \geq 2|\{k \in [g_j - 1] : c(k) = R\}| + \frac{m - 2g_j + 1}{2} + \frac{n - m - 1}{4},$$

which by Lemma 8 becomes:

$$|\mathcal{R}| \geq 2|\{k \in [g_j - 1] : c(k) = R\}| + \frac{n}{4} - \frac{g_j}{2}.$$

By Corollary 3, each block I_i, $1 \leq i \leq j-1$, has length at least four. Moreover, each block starts and ends with the string GRR, since c has no rainbow $AP(3)$. Now, the definition of m implies

$$|\{k \in I_i : c(k) = R\}| \geq \frac{|I_i|}{2} + 1,$$

for all $i \in [j-1]$, where $|I_i|$ denotes the length of the block I_i. Similarly, since $m > g_3$, $|\{k \in I_0 : c(k) = R\}| \geq \frac{|I_0|}{2}$. Summing up these inequalities, we get

$$|\{k \in [g_j - 1] : c(k) = R\}| = \sum_{i=0}^{j-1} |\{k \in I_i : c(k) = R\}| \geq \frac{g_j - 1}{2} + (j-1),$$

since $\sum_{i=0}^{j-1} |I_i| = g_j - 1$. Therefore,

$$|\mathcal{R}| \geq \frac{n}{4} + \frac{g_j}{2} + 2j - 3.$$

Since each block I_i, $1 \leq i \leq j-1$, has length at least four, we have $g_j \geq 4(j-1)$. Thus, $|\mathcal{R}| \geq \frac{n}{4} + 4j - 5$. By Lemma 5, we have $j \geq s - 2$ and $|\mathcal{R}| \geq \frac{n}{4} + 4s - 13$. Hence,

$$\max\{|\mathcal{R}|, |\mathcal{B}|\} \geq |\mathcal{R}| \geq \frac{n}{4} + 4|\mathcal{G}| - 13 \geq \frac{n}{4} + 4(n - 2\max\{|\mathcal{R}|, |\mathcal{B}|\}) - 13.$$

It follows from here that

$$\max\{|\mathcal{R}|, |\mathcal{B}|\} \geq \frac{17n}{36} - \frac{13}{9} \geq \frac{7n}{15} - \frac{13}{9}.$$

Finally, we deal with the remaining subcase (2).

Let $m < g_3$. Let $t = \max\{k : c(k) = c(k+1) = B\}$. If $t < g_{s-2}$, then we apply the argument for the previous subcase to the coloring $\overline{c} : [n] \to \{R, B, G\}$ defined by $\overline{c}(i) = c(n+1-i)$. Let $r \in [s-2, s]$ be the greatest integer with the property that $t \geq g_r$. We need the following lemma.

Lemma 10 *Suppose $c(u) = c(u+1) = B$, $c(v) = c(x) = G$, and $c(y) = c(y+1) = B$, where $u < v < x < y$ are integers in $[n]$. Then, there are two consecutive elements in $[v+1, x-1]$ colored with B.*

Proof: Let $u' = \max\{k < v : c(k) = c(k+1) = B\}$, and $y' = \min\{k > x : c(k) = c(k+1) = B\}$. Note that $u' \geq u$ and $y' \leq y$. Without loss of generality, we can assume that $v - u' - 1 \leq y' - x$. Clearly, arithmetic progressions $\{u', v, 2v - u'\}$ and $\{u' + 1, v, 2v - u' - 1\}$ are not rainbow, so $c(2v - u' - 1) = c(2v - u') = B$. If $2v - u' < x$, we have completed the proof. Otherwise, we have $2v - u' = (v - u' - 1) + (v + 1) \leq (y' - x) + x = y'$, which contradicts our definition of y'. $\square$

Thus, given two blocks, both with pairs of consecutive numbers colored with B, there is a block between them with a pair of consecutive numbers colored with B. This immediately implies that each of the blocks $I_j, I_{j+1}, \ldots, I_r$ contains a pair of consecutive numbers colored with B. As in Case 1, we conclude that each of these blocks has length at least 15. From $|\mathcal{G}| \leq 1 + (r' - r + 1) + 2 \leq 4 + \frac{n}{15}$, we get

$$\max\{|\mathcal{R}|, |\mathcal{B}|\} \geq \frac{n - n/15 - 4}{2} = \frac{7n}{15} - 2.$$

We have completed our proof of Proposition 2, which combined with Proposition 1 gives Theorem 3.

5 Proof of Theorem 5

We call a coloring of $[n]$ *rainbow Schur-free* if it does not contain any rainbow solutions to equation $x + y = z$. In order to show the lower bound $ss(k) > \lfloor \frac{5k}{2} \rfloor$, we define the coloring $c : [n] \to \{R, B, G\}$ as follows:

$$c(i) := \begin{cases} R & \text{if } i \equiv 1 \text{ or } 4 \pmod 5 \\ B & \text{if } i \equiv 2 \text{ or } 3 \pmod 5 \\ G & \text{if } i \equiv 0 \pmod 5 \end{cases}$$

Clearly, c is rainbow Schur-free and each color class has at most $\lceil \frac{2n}{5} \rceil$ elements.

Now, let c denote an arbitrary rainbow Schur-free coloring of $[n]$. In the rest of the section, we establish properties of c that imply that one of the color classes has size at least $\frac{2n}{5}$. The tight upper bound $ss(k) \leq \lfloor \frac{5k}{2} \rfloor + 1$ immediately follows. Recall that in a coloring of $[n]$, a color X is called dominant if for every two consecutive integers with different colors, one of them is colored with X. Note that in every coloring that uses at least three colors, there is at most one dominant color. Also, recall that a color Y is called recessive if no two consecutive elements of $[n]$ receive color Y.

By the pigeonhole principle, we may assume that c uses at least three colors; so there is at most one dominant color. In fact, it is easy to conclude that color $R := c(1)$ is the unique dominant color. Indeed, if $c(1)$ is not dominant, then there exist integers i and $i+1$ such that the colors $c(1)$, $c(i)$, and $c(i+1)$ are all different. However, the set $\{1, i, i+1\}$ is then a rainbow solution to $x + y = z$, which contradicts our assumption on c. Furthermore, if all the colors that are not dominant are recessive, then for every pair of consecutive integers $1 \leq j < j+1 \leq n$, we have $c(j) = R$ or $c(j+1) = R$. Hence, the there are at least $\frac{n}{2} > \frac{2n}{5}$ elements colored with (the dominant color) R. Therefore, we may assume that at least one of the non-dominant colors in c is not recessive. As the following lemma shows, this color is necessarily unique as well.

Lemma 11 *All but at most one non-dominant color in c is recessive.*

Proof: Suppose there are (at least) two colors in c that are not dominant and not recessive. Let $i, i+1, \ldots, i+k$ be the longest string of consecutive integers colored with such a color, which we denote by Y. Let $j, j+1$ be a string of two consecutive elements colored with Z, where Z denotes a non-dominant and non-recessive color other than Y. There are two possible cases depending on which of these two monochromatic strings comes first.

If $i + k < j$, then none of the integers in the string $j-i-k, j-i-k+1, \ldots, j-i+1$ can receive the dominant color R. Hence, all of them receive the same color, which is not dominant and is not recessive. However, the length of this string is $k+2$, which contradicts our choice of the string $i, i+1, \ldots, i+k$.

Similarly, if $i > j+1$, then none of the integers in the string $i-j-1, i-j, \ldots, i-j+k$ can receive the dominant color R. Hence, all of them receive the same color, which is not dominant and is not recessive. However, the length of this string is $k+2$, which again contradicts our choice of the string $i, i+1, \ldots, i+k$. $\square$

Let B denote the unique color in c which is neither dominant nor recessive. Let N_c be the number of elements of $[n]$ that are not colored with R or G. Thus, these integers receive a non-dominant color that is recessive. As in Lemma 1, we can limit our consideration to 3-colorings. Define the 3-coloring $\bar{c}$ by $\bar{c}(i) = c(i)$, if $c(i) = R$ or B, and $c(i) = G$ otherwise. Let $\mathcal{G} = \{g : g \in [n], c(g) = G\}$. Then $\bar{c}$ is a rainbow Schur-free coloring of $[n]$ and $|\mathcal{G}| = N_c$. For $1 \leq i \leq |\mathcal{G}|$, let g_i denote the i^{th} smallest element of $\mathcal{G}$. Let $\mathcal{B} = \{b : b \in [n-1], c(b) = B, c(b+1) = B\}$. For $1 \leq i \leq |\mathcal{B}|$, let b_i denote the i^{th} smallest element of $\mathcal{B}$. If $b_1 > g_1$, then $c(b_1 - g_1) \neq R$ and $c(b_1 + 1 - g_1) \neq R$, so $b_1 - g_1 \in \mathcal{B}$ and $b_1 - g_1 < b_1$, a contradiction. Hence, $b_1 < g_1$. Since $c(g_1 - 1) = R$, then $1 < b_1 < b_1 + 1 < g_1 - 1 < g_1$, so $g_1 \geq 5$.

Next, we show that for $1 \leq i \leq |\mathcal{G}| - 1$, there exists $b' \in \mathcal{B}$ such that $g_i < b < g_{i+1}$. Since $b_1 < g_1 < g_i$, then there exists a largest element $b \in \mathcal{B}$ such that $b < g_i$. Since $c(g_i - b) \neq R$ and $c(g_i - b - 1) \neq R$, then $g_i - b - 1 \in \mathcal{B}$. However, then $c(g_{i+1} - (g_i - b)) \neq R$ and $c(g_{i+1} - (g_i - b - 1)) \neq R$, which implies that $b + g_{i+1} - g_i \in \mathcal{B}$. Since b is the largest element in $\mathcal{B}$ that is less than g_i, we have $b + g_{i+1} - g_i > g_i$. Defining $b' = b + g_{i+1} - g_i$, we obtain $b' \in \mathcal{B}$ such that $g_i < b' < g_{i+1}$.

Now, clearly, $c(g_i + 1) = c(g_{i+1} - 1) = R$, so $g_i < g_i + 1 < b' < b' + 1 < g_{i+1} - 1 < g_{i+1}$. Therefore, $g_{i+1} - g_i \geq 5$ for $1 \leq i \leq |\mathcal{G}| - 1$. Since $g_1 \geq 5$, then $|\mathcal{G}| \leq \frac{n}{5}$. It immediately follows that in the coloring $\bar{c}$, as well as in c, we have at least $\frac{2n}{5}$ elements colored with R or B. We have completed the proof of Theorem 5.

6 Conclusion

Despite the results presented in this paper, exact determination of $sr(3, k)$ remains an open problem. However, Theorem 3 greatly improves the bounds from [ACT89] and leaves only a small gap. We conjecture that the lower bound is the exact order of magnitude (up to an additive constant).

Conjecture 1 *For all* $k \in \mathbb{N}$, $sr(3, k) = \frac{17}{8}k + O(1)$.

Furthermore, we believe that our methods cannot be used for improving the bounds on $sr(m, k)$ in [ACT89], when $m \neq 3$. The main obstacle is the fact that there is no analogue of Theorem 1 for m-term arithmetic progressions, $m \geq 4$ (as shown in [AF04] for $m \geq 5$, and [CFJR] for $m = 4$), that could be used as in Lemma 1.

Fox et al. [FMR] consider yet another partition-regular[2] equation, "the Sidon equation" $x + y = z + w$, which is a classical object in combinatorial number theory. They proved the following.

Theorem 6 ([FMR]) For every $n \geq 4$, every partition of $[n]$ into four color classes $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{Y}$, such that

$$\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|, |\mathcal{Y}|\} > \frac{n+1}{6}$$

contains a rainbow solution of $x + y = z + w$. Moreover, this result is tight.

For a given positive integer k, let $sd(k)$ denote the minimal number such that every coloring of $[n]$, $n \geq sd(k)$, that uses each color at most k times, yields a rainbow solution to equation $x + y = z + w$. We propose the following open problem.

Problem 1 *Determine* $sd(k)$.

We hope one could use Theorem 6 to prove a lemma similar to Lemma 1 and reduce Problem 1 to studying the minimal size of the largest color class in 4-colorings of $[n]$ without rainbow solutions to the above equations. Some structural results about such colorings are already provided in [FMR].

It is interesting to note that there are still no other existential rainbow-type results for partition regular equations other than the ones mentioned above. We are nowhere near the rainbow Rado-type characterization. For numerous open problems concerning the existence of rainbow subsets of integers in appropriate colorings of $[n]$ or $\mathbb{N}$, please refer to the survey [JRN].

Both rainbow-Ramsey and sub-Ramsey problems have received considerable attention in graph theory. The sub-Ramsey number of a graph G, denoted by $sr(G, k)$, is the smallest integer n such that every edge-coloring of K_n, where each color is used at most k times, contains a rainbow subgraph isomorphic to G. Hell and Montellano [HM03] improved the bounds of Alspach et al. [AG+86], and proved that $sr(K_m, k)$ is $O(km^2)$ and $\Omega(m^{3/2})$. Hahn and Thomassen [HT86] show that $sr(P_m, k) = sr(C_m, k) = m$, when m is large enough with respect to k.[3] Results on sub-Ramsey

[2] For the definition of partition regularity, please refer to [GRS90].

[3] P_m and C_m denote the path and the cycle with m vertices, respectively.

number of stars and some other results dealing with existence of rainbow subgraphs in colorings with bounded color classes can be found in [AJMP03, ENR83, FHS87, FR93, LRW96].

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