On the Chromatic Number of the Visibility Graph of a Set of Points in the Plane*

Jan Kára[†] Attila Pór[‡] David R. Wood[§]

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Abstract

The visibility graph $\mathcal{V}(P)$ of a point set $P \subseteq \mathbb{R}^2$ has vertex set P, such that two points $v, w \in P$ are adjacent whenever there is no other point in P on the line segment between v and w. We study the chromatic number of $\mathcal{V}(P)$. We characterise the 2- and 3-chromatic visibility graphs. It is an open problem whether the chromatic number of a visibility graph is bounded by its clique number. Our main result is a super-polynomial lower bound on the chromatic number (in terms of the clique number).

1 Introduction

Let $P \subseteq \mathbb{R}^2$ be a set of points in the plane. Let $\overline{vw}$ denote the closed line-segment between points $v \in \mathbb{R}^2$ and $w \in \mathbb{R}^2$. Two distinct points $v, w \in P$ are *visible* with respect to P if $P \cap \overline{vw} = \{v, w\}$. The *visibility graph* $\mathcal{V}(P)$ of P has vertex set P, where two distinct points $v, w \in P$ are adjacent if and only if they are visible with respect to P.

A k*-colouring* of a graph $G = (V, E)$ is a function $f : V \to C$ for some set C of k *colours*, such that $f(v) \neq f(w)$ for every edge $vw \in E$. We say G is k*-colourable*. The *chromatic number* $\chi(G)$ is the minimum k such that G is k-colourable. The *clique number* $\omega(G)$ is the maximum k such that G has a k-clique.

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[†] Department of Applied Mathematics, Charles University, Prague, Czech Republic (kara@kam.mff.cuni.cz).

[‡] Department of Mathematics, Case Western University, Cleveland, USA (attila.por@case.edu).

[§] School of Computer Science and School of Mathematics and Statistics, McGill University, Montréal, Canada (wood@cs.mcgill.ca).

This paper studies the chromatic number of visibility graphs. We begin with an interesting example.

Proposition 1. *Let $P = \{(x, y) : x, y \in \mathbb{Z}\}$ be the integer lattice. Then $\chi(\mathcal{V}(P)) = 4$.*

Proof. Let $f((x, y)) = (x \bmod 2, y \bmod 2)$ for all $(x, y) \in P$. For any two points (x_1, y_1) and (x_2, y_2) in P for which $f((x_1, y_1)) = f((x_2, y_2))$, both $|x_1 - x_2|$ and $|y_1 - y_2|$ are even. Thus the midpoint of the segment $\overline{(x_1, y_1)(x_2, y_2)}$ is in P, and (x_1, y_1) and (x_2, y_2) are not visible. Hence f is a 4-colouring of $\mathcal{V}(P)$, as illustrated in Figure 1. There is no 3-colouring since $\{(0, 0), (1, 0), (1, 1), (0, 1)\}$ is a 4-clique. Therefore $\chi(\mathcal{V}(P)) = 4$. □

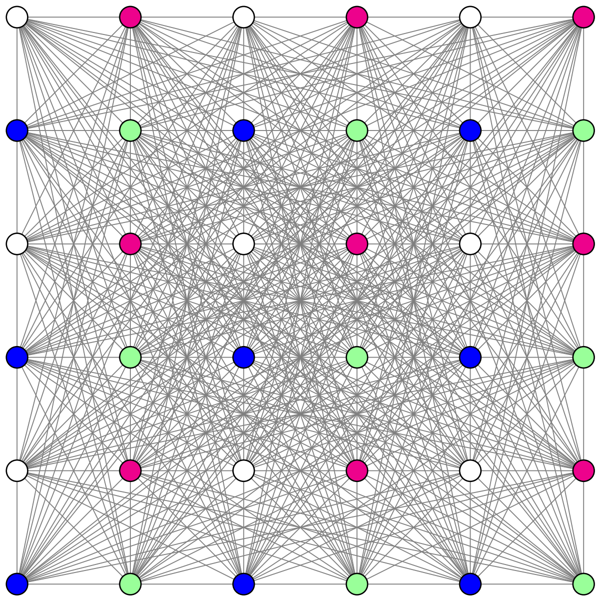

Figure 1: 4-colouring of the visibility graph of the integer lattice.

While the visibility graph of the integer lattice has a quadratic number of edges, Proposition 1 proves that it has small chromatic number. Also note that Proposition 1 generalises to prove that the visibility graph of the d-dimensional[1] integer lattice is 2^d. In this case, the chromatic number and the clique number coincide[2]. Whether there is a similar relationship for all visibility graphs is a fundamental open problem.

Conjecture 1. Visibility graphs are χ-bounded. That is, is there a function f such that $\chi(\mathcal{V}(P)) \leq f(\omega(\mathcal{V}(P)))$ for every finite point set P?

[1]Note that the visibility graph of a set of points in $\mathbb{R}^d$, by a suitable projection, is also a visibility graph of some set of points in $\mathbb{R}^2$.

[2]The visibility graph of the integer lattice is not perfect. For example, $((2, 5)(1, 3)(5, 8)(8, 3)(5, 1))$ is an induced 5-cycle.

In Section 2 we make some observations about visibility graphs, and give an elementary bound on their chromatic number. In Section 3 we prove that in Conjecture 1, we can take $f(2) = 2$ and $f(3) = 3$. In fact we characterise the finite point sets whose visibility graph has chromatic number 2 or 3. The next interesting case is $\omega(\mathcal{V}(P)) = 4$. Figure 2 shows a visibility graph with $\omega(\mathcal{V}(P)) = 4$, for which it is easily seen that $\chi(\mathcal{V}(P)) = 5$. It is an open problem whether every visibility graph with $\omega(\mathcal{V}(P)) \leq 4$ has $\chi(\mathcal{V}(P)) \leq 5$.

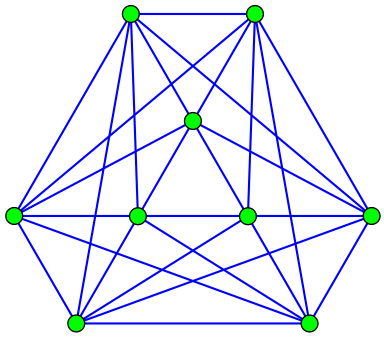

Figure 2: A 5-chromatic visibility graph with maximum clique size 4.

The main result of this paper, presented in Section 4, is a super-polynomial lower bound on the chromatic number in terms of the clique number, for a certain family of visibility graphs.

Note that visibility graphs of polygons are well studied (see [1, 4, 7, 8, 18] for example), but even here, it is an open problem whether the chromatic number is bounded by the clique number. For results and open problems regarding the χ-boundedness of other graph families, especialling those arising in a geometric context, see [10, 11, 12, 13, 15, 16, 17] for example.

2 Observations

The following is a fundamental observation regarding visibility graphs.

Proposition 2. *For every finite point set $P \subset \mathbb{R}^2$, the diameter of the visibility graph $\mathcal{V}(P)$ is*

$$\begin{cases} 1 & \text{if } P \text{ is in general position,} \\ |P| - 1 & \text{if } P \text{ is collinear,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. The diameter is one if and only if $\mathcal{V}(P)$ is complete, which occurs if and only if P is in general position. If P is collinear, then $\mathcal{V}(P)$ is a path, which has diameter $|P| - 1$.

Thus it suffices to prove that if P is not in general position and not collinear, then the diameter of $\mathcal{V}(P)$ is two. Consider two non-visible points $v, w \in P$. Two such points exist, since P is not in general position. Let L be the line containing v and w. Let x be a point in P not on L, such that the perpendicular distance from x to L is minimised. There is such a point x as P is finite, and not all the points in P are collinear. Then v and x are visible and w and x are visible, as otherwise there is a point in P closer to L than x. Thus the distance from v to w in $\mathcal{V}(P)$ is two. Hence the diameter of $\mathcal{V}(P)$ is two. $\square$

Here is one way to colour $\mathcal{V}(P)$.

Proposition 3. *If a point set $P \subseteq \mathbb{R}^2$ can be covered by k lines, then $\chi(\mathcal{V}(P)) \leq 2k$.*

Proof. Associate each point $v \in P$ with one of the k lines that contain v. The subgraph of $\mathcal{V}(P)$ induced by the set of points assigned to any one line is a collection of disjoint paths, and is thus 2-colourable. Using a different pair of colours for each line we obtain a $2k$-colouring of $\mathcal{V}(P)$. $\square$

Corollary 1. *For every point set $P \subseteq \mathbb{R}^2$, $\chi(\mathcal{V}(P))$ is at most twice the minimum degree of $\mathcal{V}(P)$.*

Proof. The result follows from Proposition 3, since P can clearly be covered by $\deg(v)$ lines for any point $v \in P$. $\square$

We conclude this section with the following Ramsey-type conjecture; see [2, 3, 9] for related research.

Conjecture 2. For all integers $k, \ell \geq 2$ there is an $n = n(k, \ell)$ such that every set P of at least n points in the plane contains ℓ collinear points or k pairwise visible points (that is, $\omega(\mathcal{V}(P)) \geq k$).

Note that $n(k, \ell) > (\ell - 1)^{\log_2(k-1)}$ since the projection of the d-dimensional $(\ell - 1) \times (\ell - 1) \times \cdots \times (\ell - 1)$ integer lattice has no set of ℓ collinear points and no k pairwise visible points for $k = 2^d + 1$.

3 The 2- and 3-Chromatic Visibility Graphs

In what follows we characterise the finite point sets whose visibility graph has chromatic number 2 or 3.

Theorem 1. *Let P be a finite point set. Then the following are equivalent:*
(a) $\chi(\mathcal{V}(P)) \leq 2$,
(b) *all the points in P are collinear,*
(c) $\mathcal{V}(P)$ *has no K_3 subgraph.*

Proof. That (a) implies (c) is immediate. If all the points in P are collinear, then $\mathcal{V}(P)$ is a path, which is obviously 2-colourable. Thus (b) implies (a). It remains to prove that (c) implies (b). Suppose that not all the points in P are collinear. Let $\{u, v, w\}$ be a set of three non-collinear points in P such that the triangle uvw has minimum area. If there is a distinct point $x \in P \cap \overline{uv}$, then $\{x, v, w\}$ are non-collinear and the triangle xvw has less area than uvw, which is a contradiction. Thus u and v are visible. Similarly u and w are visible, and v and w are visible. Hence $\{u, v, w\}$ induce K_3 in $\mathcal{V}(P)$. □

Before characterising the 3-colourable visibility graphs, consider when $\mathcal{V}(P)$ is planar. In $\mathcal{V}(P)$ there is a line-segment between every pair of vertices (which may be comprised of many edges). Eppstein [6] characterised those planar graphs in which there is a line-segment between every pair of vertices. He called these the *dilation-free* planar graphs, as illustrated in Figure 3.

Lemma 1 ([6]). *Let P be a point set. Then $\mathcal{V}(P)$ is planar if and only if at least one of the following conditions hold:*
(a) *all the points in P are collinear,*
(b) *all the points in P, except for one, are collinear,*
(c) *all the points in P are collinear, except for two non-visible points,*
(d) *all the points in P are collinear, except for two points $v, w \in P$, such that the line-segment $\overline{vw}$ does not intersect the line-segment that contains $P \setminus \{v, w\}$,*
(e) $\mathcal{V}(P)$ *is the drawing of the octahedron shown in Figure 3(e).*

Theorem 2. *Let P be a finite point set. Then the following are equivalent:*
(*i*) $\chi(\mathcal{V}(P)) \leq 3$,
(*ii*) *P satisfies conditions* (a)*,* (b)*,* (c) *or* (e) *in Lemma 1,*
(*iii*) $\mathcal{V}(P)$ *has no K_4 subgraph.*

Proof. That (i) implies (iii) is immediate. It is easy to construct a 3-colouring of a visibility graph that satisfies conditions (a), (b), (c) or (e) in Lemma 1. Thus (ii) implies (i). It remains to prove that (iii) implies (ii). Suppose that $\mathcal{V}(P)$ has no K_4 subgraph. Develin *et al.* [5] proved that a visibility graph is planar or contains K_4. (This result applies to a broad range of visibility graphs that includes visibility graphs of point sets.) Thus $\mathcal{V}(P)$ is planar. Lemma 1 describes all the planar visibility graphs. Of these only those satisfying condition (d) contain K_4. □

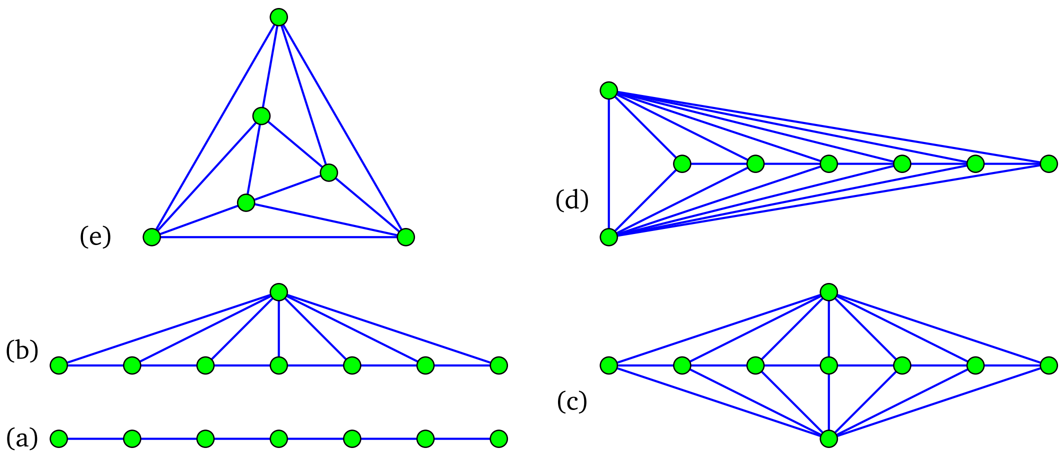

Figure 3: The planar visibility graphs.

4 A Lower Bound

In this section we prove the following super-polynomial lower bound on the chromatic number of a visibility graph.

Theorem 3. *There are constants $c_1, c_2, c_3, c_4 > 0$ and an infinite sequence of visibility graphs $G_0, G_1, G_2, \ldots$, such that $\omega(G_i) \to \infty$ and*

$$\chi(G_i) \geq (c_1 \log \omega(G_i))^{c_2 \log \omega(G_i)} = (c_3\, \omega(G_i))^{c_4 + \log\log \omega(G_i)} .$$

Before proving Theorem 3 we recall two definitions from the literature. Let G and H be graphs. The *lexicographic product* of G by H, denoted by $H[G]$, is the graph with vertex set $V(G) \times V(H)$, where $\{va, wb\}$ is an edge if and only if $ab \in E(H)$, or $a = b$ and $vw \in E(G)$. The *fractional chromatic number* $\chi_f(G)$ of a graph G is the infimum of all fractions a/b such that, to each vertex of G, one can assign a b-element subset of $\{1, 2, \ldots, a\}$ in such a way that adjacent vertices are assigned disjoint subsets. Obviously $\chi_f(G) \leq \chi(G)$. Scheinerman and Ullman [20] proved the following important property about the fractional chromatic number of the lexicographic product.

Lemma 2 ([20]). *For all graphs G and H, $\chi_f(H[G]) = \chi_f(H) \cdot \chi_f(G)$.*

Lemma 3. *For every visibility graph G and for every finite graph H, there is a visibility graph X such that $\omega(H) \cdot \omega(G) \leq \omega(X) \leq \omega(H) \cdot \omega(G) + 2|V(H)|$, and $\chi_f(X) \geq \chi_f(H) \cdot \chi_f(G)$.*

Proof. Suppose $V(H) = \{1, 2, \ldots, n\}$. Let $\{D_1, D_2, \ldots, D_n\}$ be a set of closed unit discs in the plane, whose centres are positioned on the vertices of a sufficiently large regular

n-gon. For all $1 \leq i < j \leq n$, we say a segment with endpoints in D_i and D_j is an ij-*segment*. Here 'sufficiently large' means that for each disc D_i:

(1) the only disc that an ij-segment intersects is D_i and D_j,

(2) there is a line L_i such that every ij-segment crosses L_i, and

(3) whenever an ij-segment crosses an ik-segment ($j \neq k$), the crossing point is on the side of L_i that contains D_i.

Scale G so that its convex hull is enclosed in a unit disc and no vertex is at the centre of the disc. Let $\{G_1, G_2, \dots, G_n\}$ be copies of G, one associated with each vertex of H. Place each G_i in the disc D_i, rotated so that if three points in $\bigcup_i V(G_i)$ are collinear, then they are in a single G_i. This can be achieved by rotating each G_i in turn. At each step, there are only finitely many forbidden rotation angles.

Let X_0 be the visibility graph defined by the point set $\bigcup_i V(G_i)$. By property (1) and the choice of orientations, every point in G_i is visible with every point in G_j for all $i \neq j$. Visibility within each G_i is preserved by scaling and rotating. Thus $X_0 = K_n[G]$.

We now introduce *blocker* points to our set, so that the subgraph of the visibility graph induced by $\bigcup_i V(G_i)$ is $H[G]$. For every non-edge ij of H (that is, an edge of $\overline{H}$), and for all vertices $p \in V(G_i)$ and $q \in V(G_j)$, add one blocker point at the intersection of the segment pq and the line L_i, and add another blocker point at the intersection of the segment pq and the line L_j. If two blocker points coincide, then just use one point. This construction is illustrated in Figure 4.

Let X be the visibility graph of the point set obtained. By property (3) above, for every edge $ij \in E(H)$, every vertex in G_i is visible with every vertex in G_j. Thus the subgraph of X induced by $\bigcup_i V(G_i)$ is $H[G]$.

Obviously $\omega(H[G]) = \omega(H) \cdot \omega(G)$. The blocker vertices on each line L_i can add at most two vertices to a maximum clique. Thus $\omega(H) \cdot \omega(G) \leq \omega(X) \leq \omega(H) \cdot \omega(G) + 2|V(H)|$, as claimed. By Lemma 2 and since $H[G]$ is an induced subgraph of X, $\chi_f(X) \geq \chi_f(H[G]) = \chi_f(H) \cdot \chi_f(G)$, as claimed. □

The following result of Larsen *et al.* [14] is based on the famous construction of Mycielski [19].

Lemma 4 ([14]). *For all $k \geq 0$, there is a triangle-free graph M_k on $3 \cdot 2^k - 1$ vertices such that $\chi_f(M_k) \geq \sqrt{2k}$.*

Proof Construction. Let $M_0 = K_2$. Construct M_{k+1} from M_k as follows. Suppose $V(M_k) = \{v_i : 1 \leq i \leq n_k\}$. Let $V(M_{k+1}) = \{x_i, y_i : 1 \leq i \leq n_k\} \cup \{z\}$. Let $E(M_{k+1}) = \{x_i x_j : v_i v_j \in E(M_k)\} \cup \{x_i y_j : v_i v_j \in E(M_k)\} \cup \{y_i z : 1 \leq i \leq n_k\}$. Note

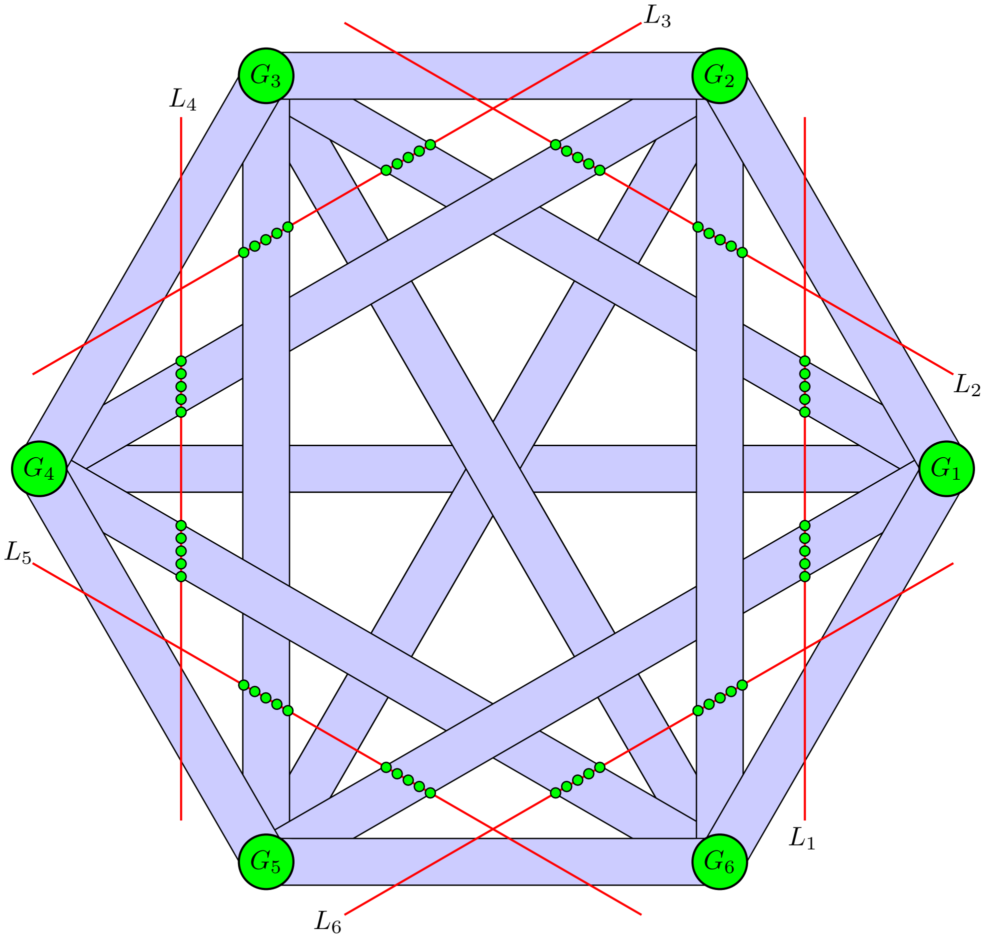

Figure 4: Construction of X from a visibility graph G with $H = K_{3,3}$.

that M_{k+1} has $n_{k+1} = 2n_k + 1$ vertices. Since $n_0 = 2$, it follows that $n_k = 3 \cdot 2^k - 1$. Mycielski [19] proved that M_k is triangle-free and $\chi(M_k) = k + 2$. Larsen *et al.* [14] proved that $\chi_f(M_{k+1}) = \chi_f(M_k) + 1/\chi_f(M_k)$. It follows that $\chi_f(M_k) \geq \sqrt{2k}$ (and this is asymptotically tight; see [14, 20]). □

Proof of Theorem 3. (In what follows we make little effort to optimise the constants c_1 and c_2.) Let $G_0 = K_1$. For all $i \geq 0$, apply Lemma 3 to obtain a visibility graph $G_{i+1}(= X)$ from $G_i(= G)$, where H is the Mycielski graph $M_{k(i)}$, chosen so that

$$3\omega(G_i) \;\leq\; |V(M_{k(i)})| \;=\; 3 \cdot 2^{k(i)} - 1 \;\leq\; 6\,\omega(G_i) \;\; . \tag{1}$$

By Lemma 3 and since $M_{k(i)}$ is triangle-free,

$$2\omega(G_i) \;\leq\; \omega(G_{i+1}) \;\leq\; 2\omega(G_i) + 2|V(M_{k(i)})| \;\leq\; 14\omega(G_i) \;\; ,$$

where the last inequality follows from (1). Since $\omega(G_0) = 1$,

$$2^i \;\leq\; \omega(G_i) \;\leq\; 14^i \;\; . \tag{2}$$

We now prove a lower bound on the chromatic number of G_{i+1}. By Lemmata 3 and 4,

$$\chi_f(G_{i+1}) \;\geq\; \chi_f(M_{k(i)}) \cdot \chi_f(G_i) \;\geq\; \sqrt{2k(i)}\,\chi_f(G_i) \;\; .$$

By (1) and the lower bound in (2), $k(i) > \log_2 \omega(G_i) \geq i$. Hence,

$$\chi_f(G_{i+1}) \;>\; \sqrt{2i}\,\chi_f(G_i) \;\; .$$

Since $\chi_f(G_0) = 1$,

$$\chi_f(G_i) \;\geq\; \sqrt{2^i i!} \;\; .$$

By Stirling's Formula,

$$\chi_f(G_i) \;\geq\; \sqrt{(2i/\boldsymbol{e})^i} \;\; .$$

By the upper bound in (2), $i \geq \log_{14} \omega(G_i)$. Hence

$$\chi_f(G_i) \;\geq\; \left(\frac{2}{\boldsymbol{e}} \log_{14} \omega(G_i)\right)^{\frac{1}{2}\log_{14}\omega(G_i)} \;\; .$$

Obviously $\chi(G_i) \geq \chi_f(G_i)$. Thus for an appropriate choice of constants $c_1, c_2 > 0$,

$$\chi(G_i) \;\geq\; \chi_f(G_i) \;\geq\; (c_1 \log \omega(G_i))^{c_2 \log \omega(G_i)} \;\; ,$$

as claimed. □

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