Mixed Hypergraphs and Other Coloring Problems

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Dedicated to the memory of Heinz-Jürgen Voss, a friend with whom I enjoyed exploring the surprising realm of mixed hypergraphs.

Abstract

A mixed hypergraph is a triple $(V, \mathcal{C}, \mathcal{D})$ where V is the vertex set and $\mathcal{C}$ and $\mathcal{D}$ are sets of subsets of V called $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively. A proper coloring of a mixed hypergraph $(V, \mathcal{C}, \mathcal{D})$ is a coloring of its vertices such that no $\mathcal{C}$-edge is polychromatic and no $\mathcal{D}$-edge is monochromatic. We show that mixed hypergraphs can be used to efficiently model several graph coloring problems including homomorphisms of simple graphs and multigraphs, circular colorings, $(H, C, \leq K)$-colorings, (H, C, K)-colorings, locally surjective, locally bijective and locally injective homomorphisms, $L(p, q)$-labelings, the channel assignment problem, T-colorings and generalized T-colorings.

1 Introduction

A *mixed hypergraph* is a triple $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ where V is its vertex set and $\mathcal{C}$ and $\mathcal{D}$ are sets of subsets of V, i.e., $\mathcal{C}, \mathcal{D} \subseteq 2^V$. The elements of $\mathcal{C}$ are called *$\mathcal{C}$-edges* and the elements of $\mathcal{D}$ are called *$\mathcal{D}$-edges.* A coloring c of vertices of $\mathcal{H}$ is *proper* if each $\mathcal{C}$-edge contains two vertices with a $\mathcal{C}$ommon color and

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each $\mathcal{D}$-edge contains two vertices with $\mathcal{D}$istinict colors. The smallest number of colors which can be used in a proper coloring of a mixed hypergraph $\mathcal{H}$ is called the *(lower) chromatic number* of $\mathcal{H}$ and it is denoted by $\chi(\mathcal{H})$.

Mixed hypergraphs were introduced by Voloshin [27]. Since then they have attracted considerable amount of attention of different researchers as it is witnessed by a recent monograph on the subject by Voloshin [26]. We recall their basic properties and summarize known results on mixed hypergraphs in Section 2.

Our work is motivated by the following proposition on relation of mixed hypergraphs and list colorings of graphs from a recent survey on list colorings by Kratochvíl, Tuza and Voigt [18]. Recall that in a list coloring, given a graph G and a list assignment L of colors to the vertices, the goal is to find a vertex-coloring c of G such that no two adjacent vertices receive the same color and every vertex v receives a color from its list, i.e., $c(v) \in L(v)$. A coloring from such lists is also called an *L-coloring*. We include a short proof of the proposition for the sake of completeness:

Proposition 1 *Let G be a graph and L an assignment of lists of colors to the vertices of G. Then, there exists a mixed hypergraph $\mathcal{H}$ whose proper colorings one-to-one correspond to proper L-colorings of G.*

Proof: Let K be the number of colors in the union of all lists $L(v)$ for $v \in V(G)$. The vertices of the mixed hypergraph $\mathcal{H}$ are vertices of G together with additional K vertices. Each additional vertex is identified with one of the K colors. The mixed hypergraph $\mathcal{H}$ contains the following edges:

1. a $\mathcal{D}$-edge $\{v, v'\}$ for any two adjacent vertices v and v' of G,
2. a $\mathcal{D}$-edge $\{w, w'\}$ for each pair of vertices w and w' corresponding to colors and
3. a $\mathcal{C}$-edge $\{v\} \cup L(v)$ for each vertex v of G where $L(v)$ represents the vertices of $\mathcal{H}$ corresponding to the colors from $L(v)$.

In any proper coloring of $\mathcal{H}$, all vertices corresponding to the colors from the lists must get distinct colors because of $\mathcal{D}$-edges (2). Hence, we may assume that $\mathcal{H}$ is colored so that each of these vertices is colored with the color to which it corresponds. Each vertex $v \in V(G)$ can be colored only by a color from its list $L(v)$ because of $\mathcal{C}$-edges (3). The coloring of the vertices of $v \in V(G)$ is proper (in the usual sense) because of the $\mathcal{D}$-edges (1). Thus, the coloring of the vertices of $V(G)$ can be viewed as an L-coloring of G.

On the other hand, for each proper list coloring of G, there can be constructed a proper coloring of the mixed hypergraph $\mathcal{H}$ in an analogous way.

$\blacksquare$

In this paper, we show that mixed hypergraphs actually model much more general graph coloring problems. All our proofs are efficient in the sense that the corresponding mixed hypergraph can be constructed in time polynomial in the size of input parameters.

Namely, we show that mixed hypergraphs can be used to model the following graph coloring problems:

- **List homomorphisms between simple graphs and multigraphs** A *homomorphism* from a simple graph G to a simple graph H is a mapping $\varphi : V(G) \to V(H)$ such that if vv' is an edge of G, then $\varphi(v)\varphi(v')$ is an edge of H. A *homomorphism* from a multigraph G to a multigraph H is a mapping $\varphi : V(G) \to V(H)$ such that whenever vv' is an edge or a loop in G, $\varphi(v)\varphi(v')$ is an edge or a loop of H. Note that an edge of G can be mapped to a loop of H. Let us mention at this point that all graphs in the rest of this paper are *simple* unless stated otherwise, i.e., they contain neither parallel edges nor loops. Throughout the paper, a *multigraph* means a graph which may contain loops but no parallel edges. In list homomorphisms, the graph G is equipped in addition with a list assignment $L : V(G) \to 2^{V(H)}$ and it is required that the homomorphism φ satisfies $\varphi(v) \in L(v)$ for each vertex $v \in V(G)$. Graph homomorphisms are generalization of usual graph coloring in the sense that a graph G is n-colorable iff G has an homomorphism to the complete graph K_n of order n. We remark that the full characterization of polynomial and NP-complete cases of graph homomorphisms was given by Hell and Nešetřil in [10]. In particular, they proved that a desicion problem whether a graph has homomorphism to a graph H can be solved in polynomial time if H is bipartite, and it is NP-complete otherwise. The reader is also referred to a recent survey [23] by Nešetřil on graph homomorphisms and their applications.

- **Circular colorings**

A k/l-*circular coloring* of a graph G is a coloring of its vertices by numbers $1, \ldots, k$ such that any two adjacent vertices receive numbers κ_1 and κ_2 such that $l \le |\kappa_1 - \kappa_2| \le k - l$. Circular colorings were introduced by Vince [25] under the name of "star colorings". The reader is referred to a recent survey [29] for details on circular colorings which we omit here. The least ratio k/l for which there exists a k/l-circular coloring of G is called the *circular chromatic number* of G and it is denoted by $\chi_c(G)$. Other equivalent definitions of the circular chromatic number can be found in [29]. It is well-known that $\chi(G) < \chi_c(G) \le \chi(G)$ for every graph G. It is easy to see that a graph has a k/l-circular coloring iff G has a homomorphism to the complement of the $(l-1)$-th power of the cycle of length k. The l-*th power* of a graph G is the graph on the same vertex set as G where two vertices are joined by an edge if their distance in G is at most l. In particular, the 0-th power of a graph has no edges at all.

- **$(H, C, \le K)$-colorings and (H, C, K)-colorings of graphs**
  The notions of $(H, C, \le K)$-colorings and (H, C, K)-colorings have recently been introduced by Díaz, Serna and Thilikos [2, 3, 4]. A triple (H, C, K) is called a *partial weighted assignment* if H is a multigraph, $C \subseteq V(H)$ and K is a mapping from the set C to positive integers. A homomorphism φ from G to H is an $(H, C, \le K)$-*coloring* of G if it satisfies in addition to being a homomorphism that the number of preimages of each $w \in C$ in φ is at most $K(w)$. An $(H, C, \le K)$-coloring of G is an (H, C, K)-*coloring* if the number of preimages of each $w \in C$ is exactly $K(w)$. Again, it is possible to consider a list version where each vertex $v \in G$ is assigned a list $L(v)$ of vertices of H and it is required that $\varphi(v) \in L(v)$.

- **$L(p, q)$-labelings of graphs**
  $L(p, q)$-labelings of graphs are important model for assigning frequencies to radio transmitters [8, 9]. An $L(p, q)$-*labeling* is a mapping $\sigma : V(G) \to \{1, \ldots, K\}$ which satisfies that $|\sigma(v) - \sigma(v')| \ge p$ for each $vv' \in E(G)$ and $|\sigma(v) - \sigma(v')| \ge q$ for any two vertices v and v' at distance at most two. The smallest K for which such a labeling exists is called the *span* and it is denoted by $\lambda_{p,q}(G)$. We remark that it is more common that the numbers assigned by σ are $0, \ldots, K$ and in such case, the span of the problem is smaller by one than compared to the span as defined here. We decided to use this less common definition because of compatibility with the channel assignment problem

also studied in this paper.

- **Locally constrained homomorphisms and graph covers**
  Graph homomorphisms between simple graphs with additional local constraints play a special role among all graph homomorphisms as it is witnessed by their intensive study both from the algorithmic and combinatorial point of view [6, 7, 16, 17, 19]. A homomorphism φ from a graph G to a graph H is called *locally injective*, if it is injective restricted to the neighborhood $N_G(v)$ of each vertex $v \in V(G)$, i.e., there are no two vertices v and v' at distance two such that $\varphi(v) = \varphi(v')$. We remark that in this paper, $N_G(v)$ always denotes the set of vertices adjacent to v, in particular, $v \notin N_G(v)$.

  As to support our claim of a special role of locally injective homomorphisms, we remark that a graph G has a locally injective homomorphism to the complement of the $(p-1)$-th power of path comprised by K vertices iff $\lambda_{p,1}(G) \le K$.

  A homomorphism φ from a graph G to a graph H is called *locally surjective*, if it is surjective restricted to the neighborhood $N_G(v)$ in G and to the neighborhood $N_H(\varphi(v))$ for each vertex $v \in V(G)$, i.e., for each vertex $v \in V(G)$ and each neighbor w' of $\varphi(v)$ in H, there is a neighbor v' of v in G such that $\varphi(v') = w'$. Locally surjective homomorphisms found their application in sociology study as role-assignment models.

  A homomorphism φ from a graph G to a graph H is called *locally bijective* if it is both locally injective and locally surjective. Finally, a homomorphism φ from a graph G to a graph H is a *covering projection* of G to H if φ is surjective and locally bijective. If such a homomorphism φ exists, then G is said to be a *cover* of H.

- **The channel assignment problem**
  The channel assignment problem is a more general model for assigning frequencies to radio transmitters than $L(p,q)$-labelings. The *channel assignment problem* is a pair (G,w) where G is a simple graph and w is a function which assigns each edge of G a positive integer. The goal is to find a labeling $\sigma : V(G) \to \{1,\dots,K\}$ such that $|\sigma(v) - \sigma(v')| \ge w(vv')$ for each edge $vv' \in E(G)$. The least number K for which such a labeling σ exists is called the *span* of the problem (G,w). Some related results on the channel assignment problem can be found, e.g., in [15, 21, 22].

- **T-colorings**
  A *T-coloring* of a graph G for a finite set $T \subseteq \{0, 1, \ldots\}$ with $0 \in T$ is a labeling σ of the vertices of G with numbers $1, \ldots, K$ such that $|\sigma(v) - \sigma(v')| \notin T$ for each edge $vv' \in E(G)$. The smallest K for which such a labeling σ exists is called the *T-chromatic number* of G and it is denoted by $\chi_T(G)$. T-colorings form an important model for frequency assignment problems, e.g., the choice $T = \{0, 7, 14, 15\}$ gives a model for interferences of UHF television transmitters [20]. This model was originally suggested by Hale [9] and we refer the reader to a survey by Roberts [24] for more detailed introduction to the problem.

- **Generalized T-colorings**
  The final problem considered in this paper is the *generalized T-coloring* which originally appeared in Hale's paper [9] under the name of *frequency constrained channel assignment problem.* The problem is described by the pair (G, t) where G is a graph and t is a function assigning each edge a finite set of non-negative integers such that $0 \in t(e)$ for each edge e of G. If t is a mapping constantly equal to T_0, then the problem is just a T_0-coloring problem. The goal of the problem is to find a labeling $\sigma : V(G) \to \{1, \ldots, K\}$ with K as small as possible such that $|\sigma(v) - \sigma(v')| \notin t(vv')$ for each edge vv' of G. The smallest K for which such a labeling exists is denoted by $\chi_t(G)$.

We show that all the just described problems can be efficiently modeled by mixed hypergraphs. In the case of list homomorphisms and locally injective and bijective homomorphisms, proper colorings of a constructed mixed hypergraph correspond one-to-one to appropriate homomorphisms between the (multi)graphs. Note that locally injective homomorphisms include the case of $L(p, q)$-labelings of graphs and general homomorphisms include the case of circular colorings. In the case of $(H, C, \leq K)$-colorings, (H, C, K)-colorings, locally surjective homomorphisms and graph covers, a constructed mixed hypergraph is colorable if and only if the corresponding coloring, the homomorphism or the covering projection exists. Finally, in the case of $L(p, q)$-labelings, the channel assignment problem, T-colorings and generalized T-colorings, the lower chromatic number of a constructed mixed hypergraph is equal to the span, the T-chromatic number, respectively, of the corresponding problem.

2 Mixed Hypergraphs

In this section, we remind some definitions and results related to mixed hypergraphs. The *feasible set* of mixed hypergraph $\mathcal{H}$, denoted by $\mathcal{F}(\mathcal{H})$, is the set of all integers k such that there exists a proper coloring of $\mathcal{H}$ using exactly k colors. The smallest number contained in $\mathcal{F}(\mathcal{H})$ is the lower chromatic number of $\mathcal{H}$ and the largest number, denoted by $\bar{\chi}(\mathcal{H})$, is the *upper chromatic number* of $\mathcal{H}$. In the rest, an edge of a hypergraph $\mathcal{H}$ is said to be *monochromatic*, if all its vertices have the same color, and it is said to be *polychromatic*, if all its vertices have mutually different colors.

The first result showing that mixed hypergraphs have really "miraculous" properties is the following result from [11]:

Theorem 2 *A set $\mathcal{F}$ of integers is a feasible set of a mixed hypergraph $\mathcal{H}$ if and only if either it is an interval or $1 \notin \mathcal{F}$.*

Note that any set of positive integers avoiding 1 is a feasible set of some mixed hypergraph, e.g., there exists a mixed hypergraph $\mathcal{H}$ with $\mathcal{F}(\mathcal{H}) = \{2, 4\}$. We remark that feasible sets of certain mixed hypergraphs, such as, e.g., interval mixed hypergraphs [11], mixed hypertrees [13], circular mixed hypergraphs [28] and mixed strong hypercacti [14], are known to be always intervals.

Theorem 2 was further generalized by the author in [12]:

Theorem 3 *Let τ be a function assigning each number $1, \ldots, K$ a non-negative integer such that $\tau(1) = 0$ and $\tau(K) \neq 0$. Then, there exists a mixed hypergraph $\mathcal{H}$ of order $O(K + ||\tau||)$ with $\bar{\chi}(\mathcal{H}) = K$ such that the number of proper colorings of $\mathcal{H}$ using exactly k colors, $1 \leq k \leq K$, is precisely $\tau(k)$.*

Here and in the rest of the paper if f is a function to integers, then $||f||$ is the sum of all values of f taken over the whole domain of f. Similarly, if f is a function to sets, then $||f||$ is the sum of cardinalities of images by f again taken over the whole domain of f. We remark that two colorings of a mixed hypergraph $\mathcal{H}$ are considered to be the same if they are the same upto renaming the colors. Theorem 3 essentially says that not only the feasible set of a mixed hypergraph $\mathcal{H}$ can be almost any set of positive integers, but the behavior of the number of proper colorings by certain numbers of colors can also be arbitrary.

Let G and H be two graphs and let $\mathcal{H}$ be a mixed hypergraph such that $V(G) \cup V(H) \subseteq V(\mathcal{H})$. Throughout the paper, $V(G)$ denotes the vertex set of G and $V(H)$ and $V(\mathcal{H})$ are used in an analogous way. We also write n_G, n_H, for the order of G, H, respectively (recall that the order of a graph is the number of its vertices). If c is a coloring of the vertices of $\mathcal{H}$, the morphism $\varphi^c_{\mathcal{H}}$ from G to H is defined as $\varphi^c_{\mathcal{H}}(v) = w$ for $v \in V(G)$ where $w \in V(H)$ is such that $c(v) = c(w)$. If there is no such vertex w or more such vertices for a particular vertex $v \in V(G)$, then $\varphi^c_{\mathcal{H}}(v)$ is undefined.

3 Graph Homomorphisms

We first state the result for the case of simple graphs. Since the construction presented in the next theorem is used in Section 5, we decided to include both this version and the more general version of the construction presented later in Theorem 6.

Theorem 4 *Let G and H be two graphs and let L be a mapping from $V(G)$ to $2^{V(H)}$. There exists a mixed hypergraph $\mathcal{H}$ of order $n_G + n_H$ whose proper colorings one-to-one correspond to L-homomorphisms from G to H. In particular, $\mathcal{H}$ is colorable iff there is an L-homomorphism from G to H. Moreover, the mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and n_H.*

Proof: Let $v_1, \ldots, v_{n_G}$ and $w_1, \ldots, w_{n_H}$ be vertices of G and H, respectively. The vertex set of $\mathcal{H}$ is formed by the $n_G + n_H$ vertices of the graphs G and H. The edges of the mixed hypergraph $\mathcal{H}$ are the following:

1. There is a $\mathcal{D}$-edge $\{w_k, w_l\}$ for each $1 \le k < l \le n_H$.
2. There is a $\mathcal{C}$-edge $\{v_i\} \cup L(v_i)$ for each $1 \le i \le n_G$.
3. There is a $\mathcal{D}$-edge $\{v_i, v_j\}$ for $1 \le i < j \le n_G$ if the edge $v_i v_j$ is contained in the graph G, i.e., $v_i v_j \in E(G)$.
4. There is a $\mathcal{C}$-edge $(V(H) \setminus \{w_k, w_l\}) \cup \{v_i, v_j\}$ for each $w_k w_l \notin E(H)$ and each $v_i v_j \in E(G)$.

Clearly, such a mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and n_H.

We claim that a coloring c of the mixed hypergraph $\mathcal{H}$ is proper iff the mapping $\varphi^c_{\mathcal{H}}$ is an L-homomorphism from G to H. Assume first that c is a proper coloring of $\mathcal{H}$. Then, all the vertices of $w_1, \ldots, w_{n_H}$ are colored by different colors because of the $\mathcal{D}$-edges (1) and each vertex v_i, $1 \le i \le n_G$, receives a color from its list $L(v_i)$ because of the $\mathcal{C}$-edges (2). The $\mathcal{D}$-edges (3) now force that $\varphi^c_{\mathcal{H}}$ does not assign to two adjacent vertices the same color. If $\varphi^c_{\mathcal{H}}$ were not an L-homomorphism from G to H, then there would exist vertices v_i and v_j with $w_k = \varphi^c_{\mathcal{H}}(v_i)$ and $w_l = \varphi^c_{\mathcal{H}}(v_j)$ such that v_iv_j is an edge in G but w_kw_l is not an edge of H. In such case, the $\mathcal{C}$-edge $(V(H) \setminus \{w_kw_l\}) \cup \{v_iv_j\}$ would consist of n_H vertices of mutually different colors which is impossible. Hence, $\varphi^c_{\mathcal{H}}$ is indeed an L-homomorphism from G to H.

On the other hand, if φ is an L-homomorphism from G to H, then consider the coloring c of the vertices of $\mathcal{H}$ defined as $c(w_k) = k$ for $1 \le k \le n_H$ and $c(v_i) = k$ for a vertex v_i, $1 \le i \le n_G$, with $\varphi(v_i) = w_k$. It is easy to see that c is a proper coloring of $\mathcal{H}$. In particular, if $w_kw_l \notin E(H)$ and $v_iv_j \in E(G)$, then $\varphi(v_i), \varphi(v_j) \not\subseteq \{w_k, w_l\}$ (this implies that $\mathcal{C}$-edges (4) are colored properly). Hence, proper colorings of $\mathcal{H}$ one-to-one correspond to L-homomorphisms from G to H.

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As noted in Section 1, circular colorings of graphs can be viewed as special types of graph homomorphisms and hence we have the following corollary of Theorem 4:

Corollary 5 *Let G be a graph and $k \ge l$ fixed positive integers. There exists a mixed hypergraph $\mathcal{H}$ such that its proper colorings one-to-one correspond to k/l-circular colorings of G. In particular, $\mathcal{H}$ is colorable iff the circular chromatic number of G is at most k/l. Moreover, the mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G, k and l.*

Proof: Apply Theorem 4 for the graph G and the graph H where H is the complement of the $(l-1)$-th power of the cycle of length k.

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Next, we generalize Theorem 4 to multigraphs:

Theorem 6 *Let G and H be two multigraphs and let L be a mapping from $V(G)$ to $2^{V(H)}$. There exists a mixed hypergraph $\mathcal{H}$ of order $n_G + n_H$ whose proper colorings one-to-one correspond to L-homomorphisms from G to H. In particular, $\mathcal{H}$ is colorable iff there is an L-homomorphism from G to H. Moreover, the mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and n_H.*

Proof: Let $v_1, \ldots, v_{n_G}$ and $w_1, \ldots, w_{n_H}$ be the vertices of the graphs G and H, respectively. Let W be a set of vertices of H with loops. First, we shorten the list $L(v_i)$ for each vertex of the graph G with a loop to $L(v_i) \cap W$ (this does clearly not affect L-homomorphisms from G to H). Now, we can describe the mixed hypergraph H. The vertex set of $\mathcal{H}$ is again formed by the $n_G + n_H$ vertices of the graphs G and H and the edges of $\mathcal{H}$ are the following:

1. There is a $\mathcal{D}$-edge $\{w_k, w_l\}$ for each $1 \leq k < l \leq n_H$.
2. There is a $\mathcal{C}$-edge $\{v_i\} \cup L(v_i)$ for each $1 \leq i \leq n_G$.
3. There is a $\mathcal{D}$-edge $\{v_i, v_j, w_k\}$ for each edge $v_i v_j$ of G, $1 \leq i < j \leq n_G$, and for each vertex w_k, $1 \leq k \leq n_H$, without a loop.
4. There is a $\mathcal{C}$-edge $(V(H) \setminus \{w_k w_l\}) \cup \{v_i v_j\}$ for each $w_k w_l \notin E(H)$ and each edge $v_i v_j$ of G.

Clearly, the order of $\mathcal{H}$ is $n_G + n_H$ and it can be constructed in time polynomial in n_G and n_H.

We show that a coloring c of the mixed hypergraph $\mathcal{H}$ is proper iff the mapping $\varphi^c_{\mathcal{H}}$ is an L-homomorphism from G to H. Let φ be a homomorphism from G to H and set $c(w_k) = k$ for $1 \leq k \leq n_H$ and $c(v_i) = k$ where $\varphi(v_i) = w_k$ for $1 \leq i \leq n_G$. Clearly, $\varphi^c_{\mathcal{H}} = \varphi$. All $\mathcal{D}$-edges (1) are clearly colored properly. $\mathcal{C}$-edges (2) are colored properly because $\varphi(v_i) \in L(v_i)$ for each $1 \leq i \leq n_G$. Consider now a vertex w_k, $1 \leq k \leq n_H$, without a loop. If $c(v_i) = c(v_j) = c(w_k)$ for some edge $v_i v_j$ of G, $1 \leq i < j \leq n_G$, then $\varphi(v_i) = \varphi(v_j) = w_k$ which is impossible. Hence, $\mathcal{D}$-edges (3) are also colored properly. Similarly as in Theorem 4, $\mathcal{C}$-edges (4) are colored properly as well.

Consider now a proper coloring c of the mixed hypergraph $\mathcal{H}$ and we aim to show that $\varphi^c_{\mathcal{H}}$ is an L-homomorphism from G to H. $\mathcal{D}$-edges (1) forces all the vertices w_k, $1 \leq k \leq n_H$, have n_H mutually different colors. $\mathcal{C}$-edges (2) forces that each vertex v_i, $1 \leq i \leq n_G$, receives a color $c(w_k)$ for some $w_k \in L(v_i)$. Hence, $\varphi^c_{\mathcal{H}}(v_i) \in L(v_i)$ and the mapping $\varphi^c_{\mathcal{H}}$ is defined for all vertices v_i of G. Assume now for the sake of contradiction that $\varphi^c_{\mathcal{H}}$ is not a homomorphism from G to H. Then, either a vertex v_i with a loop is mapped to a vertex w_k without a loop or an edge $v_i v_j$ is mapped to non-edge $w_k w_l$ or both end-vertices v_i and v_j of an edge $v_i v_j$ are mapped to the same vertex w_k without a loop. The first is impossible, since

$L(v_i)$ contains only vertices with loops whenever v_i has a loop. The second is impossible because otherwise the $\mathcal{C}$-edge $(V(H) \setminus \{w_k w_l\}) \cup \{v_i v_j\}$ is polychromatic. The third is impossible because otherwise the $\mathcal{D}$-edge $\{v_i, v_j, w_k\}$ is monochromatic. Hence, we may conclude that the mapping $\varphi^c_{\mathcal{H}}$ is indeed an L-homomorphism from G to H.

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4 $(H, C, \leq K)$-colorings and (H, C, K)-colorings

In order to prove our results, we adopt a notion of equivalence of partial weighted assignments from [4]. A partial weighted assignment (H, C, K) is *equivalent* to a partial weighted assignment (H', C', K') if a graph G is (H, C, K)-colorable iff it is (H', C', K')-colorable. Similarly, (H, C, K) is *$\leq$-equivalent* to (H', C', K') if a graph G is $(H, C, \leq K)$-colorable iff it is $(H', C', \leq K')$-colorable. A partial weighted assignment (H, C, K) is said to be *list-equivalent* to a partial weighted assignment (H', C', K') if there exists a list mapping function $\Lambda : V(H) \to 2^{V(H')}$ with the following property: If G is a graph with a list assignment $L : V(G) \to 2^{V(H)}$, then G is L-(H, C, K)-colorable iff G is L'-(H', C', K')-colorable where $L'(v) = \bigcup_{w \in L(v)} \Lambda(w)$ for $v \in V(G)$. Similarly, we can define a notion of being *list-$\leq$-equivalent.*

We prove a simple lemma about list-equivalent and list-$\leq$-equivalent (H, C, K)-colorings and $(H, C, \leq K)$-colorings:

Lemma 7 *Let (H, C, K) be a partial weighted assignment. Then, there exists a partial weighted assignment (H', C', K') such that (H, C, K) is list-equivalent and list-$\leq$-equivalent to (H', C', K') and K' is identically equal to one. In addition, the order of H' is $n_H + ||K|| - |C|$ and it can be constructed from H in time polynomial in n_H and $||K||$ together with the list mapping function.*

Proof: Let $w_1, \ldots, w_{n_H}$ be the vertices of H. Replace each vertex $w_k \in C$ with $K(w_k)$ vertices $w_k^1, \ldots, w_k^{K(w_k)}$. These $K(w_k)$ vertices form an independent set. Otherwise, they do not have loops if w_k does not have a loop in H and they form a clique and they have loops. The resulting graph H' contains an edge between two vertices w_k^α and $w_{k'}^{\alpha'}$ whenever $w_k w_{k'} \in E(H)$ for $1 \leq k, k' \leq n_H$, $1 \leq \alpha \leq K(w_k)$ and $1 \leq \alpha' \leq K(w_{k'})$. Similarly, there is an edge between vertices w_k^α and $w_{k'}$ if $w_k w_{k'} \in E(H)$ for $1 \leq k, k' \leq n_H$ and $1 \leq \alpha \leq K(w_k)$. This finishes the construction of the multigraph H'.

The set C' contains all vertices w_k^α with $w_k \in C$ and $1 \leq \alpha \leq K(w_k)$ and the weight function K is set to be identically one. The list mapping Λ is defined as follows:

$$\Lambda(w_k) = \begin{cases} \{w_k^1, \dots, w_k^{K(w_k)}\} & \text{if } w_k \in C, \\ w_k & \text{otherwise.} \end{cases}$$

Clearly, the order of H' is $n_H + ||K|| - |C|$ and both H' and Λ can be constructed from H in time polynomial in n_H and $||K||$. We verify that the partial weighted assignment (H, C, K) is list-equivalent and list-$\leq$-equivalent to (H', C', K') with the list mapping function Λ.

Fix H, C, K and L together with the corresponding list L' and let φ be an L-$(H, C, \leq K)$-coloring of G. Define $\varphi'(v_i)$ for $v_i \in V(G)$ to be $\varphi(v_i)$ if $\varphi(v_i) \notin C$ and to be w_k^α for some α if $\varphi(v_i) = w_k \in C$. Since each vertex $w_k \in C$ has at most $K(w_k)$ preimages in the mapping φ, it is possible to construct φ' such that for each $1 \leq \alpha \leq K(w_k)$ there exists at most one vertex $v_i \in V(G)$ mapped to w_k^α by φ'. In addition, we have $\varphi'(v_i) \in L'(v_i)$ for each $v_i \in V(G)$. Hence, φ' is an L'-$(H', C', \leq K')$-coloring of G. Analogically, if φ is an L-(H, C, K)-coloring of G, then φ' is an L'-(H', C', K')-coloring of G. On the other hand, if φ' is an L'-$(H', C', \leq K')$-coloring of G, then φ defined as $\varphi(v_i) = w$ for the unique vertex $w_k \in V(H)$ such that $\varphi'(v_i) \in \Lambda(w)$ is an L-$(H, C, \leq K)$-coloring of G. An analogical statement holds for φ' being an L'-(H', C', K')-coloring of G. Hence, the partial weighted assignment (H, C, K) is list-equivalent and list-$\leq$-equivalent to (H', C', K').

■

We are now ready to prove the first of two theorems contained in this section:

Theorem 8 *Let (H, C, K) be a partial weighted assignment, G a multigraph and L a list function $V(G) \to 2^{V(H)}$. Then, there exists a mixed hypergraph $\mathcal{H}$ of order $n_G + n_H + ||K|| - |C|$ which is colorable if and only if G is L-$(H, C, \leq K)$-colorable. In addition, such a mixed hypergraph H can be constructed in time polynomial in n_G, n_H and $||K||$.*

Proof: Let (H', C', K') be a partial weighted assignment to which the partial weighted assignment (H, C, K) is list-$\leq$-equivalent and such that K' is identically equal to one. By Lemma 7, there exists such a partial assignment with the order of H' equal to $n_H + ||K|| - |C|$. Let L' be the list function

$V(G) \to 2^{V(H')}$ corresponding L. Let $\mathcal{H}_0$ be the mixed hypergraph from Theorem 6 for graphs G and H' and the list assignment L'. We know that L'-homomorphisms φ from G to H' one-to-one correspond to proper colorings c of $\mathcal{H}_0$ in the following sense: For each L'-homomorphism φ, there exists a unique proper coloring c of $\mathcal{H}_0$ with $\varphi^c_{\mathcal{H}_0} = \varphi$ and on the other hand, if c is a proper coloring of $\mathcal{H}_0$, then $\varphi^c_{\mathcal{H}_0}$ is an L'-homomorphism from G to H'.

It easily follows from the definition that an L'-homomorphism φ from G to H is an L'-$(H', C', \leq K')$-coloring of G if and only if the preimage of each vertex from C is a set consisting of at most one-element. Hence, to such L'-homomorphisms φ correspond precisely those proper colorings c of $\mathcal{H}_0$ with no vertices $v_i, v_{i'} \in V(G)$ and a vertex $w_k \in C$ satisfying $c(v_i) = c(v_{i'}) = c(w_k)$. Add now to the mixed hypergraph $\mathcal{H}_0$ a $\mathcal{D}$-edge $\{v_i, v_{i'}, w_k\}$ for each $1 \leq i < i' \leq n_G$ and $w_k \in C$ and let $\mathcal{H}$ be the resulting mixed hypergraph. Observe now that proper colorings of $\mathcal{H}$ are precisely those proper colorings of $\mathcal{H}_0$ with $\varphi^c_{\mathcal{H}} = \varphi^c_{\mathcal{H}_0}$ being an L'-$(H', C', \leq K')$-coloring of G. Hence, proper colorings of the mixed hypergraph $\mathcal{H}$ one-to-one correspond to L'-$(H', C', \leq K')$-colorings of G. In particular, $\mathcal{H}$ is colorable if and only if there is an L-$(H, C, \leq K)$-coloring of G. We remark that the correspondence between proper colorings of $\mathcal{H}$ and L'-(H', C', K')-colorings is one-to-one, but we miss one-to-one correspondence between L-$(H, C, \leq K)$-colorings and L'-(H', C', K')-colorings of G in order to get one-to-one correspondence between L-$(H, C, \leq K)$-colorings of G and proper colorings of $\mathcal{H}$.

Note that the mixed hypergraph $\mathcal{H}_0$ can be constructed in time polynomial in n_G, n_H and $||K||$ by Theorem 6 and Lemma 7. Thus, the mixed hypergraph $\mathcal{H}$ can also be constructed in time polynomial in n_G, n_H and $||K||$.

■

Next, a construction of auxiliary "color selecting" gadgets is presented. Note that the gadget $\mathcal{H}_p$ "selects" one out of the (at most) p colors of the vertices $\alpha_1, \ldots, \alpha_p$ for the vertex γ (providing that the color of β is different from the color of each of the vertices $\alpha_1, \ldots, \alpha_p$). Let us mention at this point that the *size* of a hypergraph is the sum of sizes of all its edges.

Lemma 9 *For each p, there exists a mixed hypergraph $\mathcal{H}_p$ with $p + 2$ distinguished vertices $\alpha_1, \ldots, \alpha_p$, β and γ with the following property:*

1. *Every precoloring c of the $p+2$ distinguished vertices $\alpha_1, \ldots, \alpha_p$, β and*

γ of $\mathcal{H}$ such that $c(\beta) \notin \{c(\alpha_1), \ldots, c(\alpha_n)\}$ and $c(\gamma) \in \{c(\alpha_1), \ldots, c(\alpha_n)\}$ can be extended to the whole mixed hypergraph $\mathcal{H}_p$.

2. *If c is a proper coloring of $\mathcal{H}_p$ with $c(\beta) \notin \{c(\alpha_1), \ldots, c(\alpha_n)\}$, then $c(\gamma) \in \{c(\alpha_1), \ldots, c(\alpha_n)\}$.*

Both the order and the size of the mixed hypergraph $\mathcal{H}_p$ is $O(p)$ and $\mathcal{H}_p$ can be constructed in time linear in p.

Proof: The proof proceeds by induction on p. This also suggests a simple linear-time iterative algorithm for construction of $\mathcal{H}_p$. If $p = 1$, then $\mathcal{H}_1$ consists just of the three vertices α_1, β and γ and a single $\mathcal{C}$-edge $\{\alpha_1, \gamma\}$.

Assume that $p \geq 2$ and consider the already constructed mixed hypergraph $\mathcal{H}_{p-1}$. Vertices $\alpha_1, \ldots, \alpha_{p-1}$ and β are identified with the corresponding special vertices of $\mathcal{H}_{p-1}$. Let γ' be the remaining special vertex of $\mathcal{H}_{p-1}$. Let us now introduce a new vertex x and form $\mathcal{D}$-edges $\{\gamma', \alpha_p, x\}$ and $\{\beta, \gamma\}$ and $\mathcal{C}$-edges $\{\alpha_p, \gamma, \gamma'\}$, $\{x, \gamma, \gamma'\}$ and $\{x, \beta, \gamma'\}$. We verify that the obtained mixed hypergraph, denoted by $\mathcal{H}_p$, has the desired properties.

Assume that the vertices $\alpha_1, \ldots, \alpha_p$, β and γ are precolored by a precoloring c as described in (1). If $c(\gamma) = c(\alpha_p)$, then color the vertex γ' by the color $c(\alpha_{p-1})$. If $c(\gamma) \neq c(\alpha_p)$, then color the vertex γ' by the color $c(\gamma)$. In both cases, the color of the vertex γ' is one of the colors $\alpha_1, \ldots, \alpha_{p-1}$ and hence $\mathcal{H}_{p-1}$ can be colored by the induction hypothesis. Finally, the vertex x is colored by the color $c(\beta)$ if $c(\alpha_p) = c(\gamma')$ and it is colored by the color $c(\gamma')$ otherwise. It is straightforward to verify that the $\mathcal{D}$-edges $\{\gamma', \alpha_p, x\}$ and $\{\beta, \gamma\}$ and the $\mathcal{C}$-edges $\{\alpha_p, \gamma, \gamma'\}$, $\{x, \gamma, \gamma'\}$ and $\{x, \beta, \gamma'\}$ are colored properly. Hence, we have constructed an extension of the precoloring c to the whole mixed hypergraph $\mathcal{H}_p$ as desired.

Let us now prove the property (2). Consider a proper coloring c of $\mathcal{H}_p$. By the induction hypotheses, the color of the vertex γ' is one of the colors $c(\alpha_1), \ldots, c(\alpha_{p-1})$. Hence, it is enough to show that the color $c(\gamma)$ is the color $c(\alpha_p)$ or $c(\gamma')$. If $c(\alpha_p) \neq c(\gamma')$, then the color $c(\gamma)$ is either $c(\alpha_p)$ or $c(\gamma')$ because of the $\mathcal{C}$-edge $\{\alpha_p, \gamma, \gamma'\}$. Next, we consider the case that $c(\alpha_p) = c(\gamma')$. In this case, the color of the vertex x must be $c(\beta)$ because the $\mathcal{D}$-edge $\{\gamma', \alpha_p, x\}$ and the $\mathcal{C}$-edge $\{x, \beta, \gamma'\}$. Then, the $\mathcal{D}$-edge $\{\beta, \gamma\}$ and the $\mathcal{C}$-edge $\{x, \gamma, \gamma'\}$ forces the vertex γ to have the same color as γ' and α_p.

■

Theorem 10 *Let (H, C, K) be a partial weighted assignment, G a multigraph and L a list function $V(G) \to 2^{V(H)}$. Then, there exists a mixed hypergraph $\mathcal{H}$ of order $O(n_G + n_H + n_G||K||)$ which is colorable if and only if G is L-(H, C, K)-colorable. In addition, such a mixed hypergraph H can be constructed in time polynomial in n_G, n_H and $||K||$.*

Proof: Similarly as in the proof of Theorem 8, let (H', C', K') be a partial weighted assignment to which the partial weighted assignment (H, C, K) is list-$\leq$-equivalent and such that K' is identically equal to one. Let further $\mathcal{H}_0$ be the mixed hypergraph of order $n_G + n_{H'} = n_G + n_H + ||K|| - |C|$ obtained in Theorem 8 whose proper colorings one-to-one correspond to L'-$(H', C', \leq K')$-colorings of G. Recall that for each L'-$(H', C', \leq K')$-coloring φ, there exists a unique proper coloring c of $\mathcal{H}_0$ with $\varphi^c_{\mathcal{H}_0} = \varphi$ and on the other hand, if c is a proper coloring of $\mathcal{H}_0$, then $\varphi^c_{\mathcal{H}_0}$ is an L'-$(H', C', \leq K')$-coloring of G.

We now prune proper colorings of $\mathcal{H}_0$ corresponding to L'-$(H', C', \leq K')$-colorings of G which are not L'-(H', C', K')-colorings. Proper colorings c of $\mathcal{H}_0$ which should survive are those such that for each vertex $w_k \in C'$, there exists a vertex $v_i \in V(G)$ with $c(v_i) = c(w_k)$, while all other colorings should be pruned. Recall that such a vertex v_i can exist at most one because proper colorings c of $\mathcal{H}_0$ correspond to L'-$(H', C', \leq K')$-colorings of G. First add a new special vertex w_0 to the mixed hypergraph $\mathcal{H}_0$ and join it by an edge to each vertex $w_k \in V(H)$. By the construction of $\mathcal{H}_0$, only the colors used to color some vertex from $V(H)$ can be used to color the vertices from $V(G)$ and thus each proper coloring of $\mathcal{H}_0$ can be uniquely extended to the new mixed hypergraph by assigning a completely new color to the vertex w_0.

The final mixed hypergraph $\mathcal{H}$ is formed by adding a copy of the mixed hypergraph $\mathcal{H}_{n_G}$ from Lemma 9 for each $w_k \in C'$ and identifying the vertices α_i with v_i for $1 \leq i \leq n_G$, the vertex β with w_0 and the vertex γ with w_k. If for a vertex $w_k \in C'$, there is a vertex $v_i \in V(G)$ with $c(v_i) = c(w_k)$, then a proper coloring c of the vertices of $\mathcal{H}_0$ can be extended to the copy of $\mathcal{H}_{n_G}$ for the vertex w_k by Lemma 9. Hence, if φ is an L'-$(H', C', \leq K')$-coloring, there is a proper coloring c of the whole mixed hypergraph $\mathcal{H}$. On the other hand, any proper coloring c of $\mathcal{H}$ satisfies that the vertex w_0 has a color which is not used by c to color any other vertex and hence by Lemma 9 for each $w_k \in C'$, there must be a vertex $v_i \in V(G)$ with $c(v_i) = c(w_k)$. We may conclude that proper colorings of $\mathcal{H}$ are those whose restriction to $\mathcal{H}_0$ is a proper coloring of $\mathcal{H}_0$ corresponding to an L'-(H', C', K')-coloring of G and each proper coloring of $\mathcal{H}_0$ corresponding to an L'-(H', C', K')-coloring of G can be extended to $\mathcal{H}$. In particular, $\mathcal{H}$ is colorable if and only if G is L'-(H', C', K')-colorable and this is the case if and only if G is L-(H, C, K)-colorable as desired.

Since the order of each of $|C'| = ||K||$ copies of $\mathcal{H}_{n_G}$ added to $\mathcal{H}_0$ is $O(n_G)$, the order of the constructed mixed hypergraph $\mathcal{H}$ is $O(n_G + n_H + n_G||K||)$. The mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G, n_H and $||K||$ by Theorem 8 and Lemma 9.

■

5 Locally Constrained Homomorphisms

We consider each kind of locally constrained homomorphisms separately:

5.1 Locally Injective Homomorphisms

Theorem 11 *Let G and H be two graphs and let L be a mapping from $V(G)$ to $2^{V(H)}$. There exists a mixed hypergraph $\mathcal{H}$ of order $n_G + n_H$ whose proper colorings one-to-one correspond to locally injective L-homomorphisms from G to H. In particular, $\mathcal{H}$ is colorable iff there is a locally injective L-homomorphism from G to H. Moreover, the mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and n_H.*

Proof: Consider the mixed hypergraph $\mathcal{H}$ constructed in Theorem 4 and let us keep the notation from the proof of Theorem 4. Add an $\mathcal{D}$-edge between any two vertices v_i and v_j of G in distance two, i.e., those which have a common neighbor. Observe now that if c is a proper coloring of $\mathcal{H}$, then the mapping $\varphi^c_{\mathcal{H}}$ is locally injective and vice versa, i.e., for each locally injective homomorphism φ, there is a proper coloring c with $\varphi^c_{\mathcal{H}} = \varphi$. Thus, proper colorings of $\mathcal{H}$ one-to-one correspond to locally injective L-homomorphisms from G to H.

■

The immediate corollary of the preceding theorem is the following result on $L(p, 1)$-labelings of graphs:

Corollary 12 *Let $p \geq 1$ and $K \geq 1$ be fixed integers. For any given graph G, it is possible to construct in time polynomial in n_G and K a mixed* *hypergraph $\mathcal{H}$ of order $n_G + K$ such that proper colorings of $\mathcal{H}$ one-to-one correspond to proper $L(p,1)$-labelings of G with the span K.*

Proof: As noted in the beginning of the paper, an $L(p,1)$-labeling of a graph G with the span K can be viewed as a locally injective homomorphism to the complement of the $(p-1)$-th power of the path comprised by K vertices. Realizing this, the claim directly follows from Theorem 11.

■

5.2 Locally Bijective Homomorphisms

Theorem 13 *Let G and H be two graphs and let L be a mapping from $V(G)$ to $2^{V(H)}$. There exists a mixed hypergraph $\mathcal{H}$ of order $(n_G+1)\cdot(n_H+1)$ whose proper colorings one-to-one correspond to locally bijective L-homomorphisms from G to H. In particular, $\mathcal{H}$ is colorable iff there is a locally bijective L-homomorphism from G to H. Moreover, the mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and n_H.*

Proof: Let $\mathcal{H}_0$ be the mixed hypergraph for G, H and L from Theorem 11. First, add a new vertex w_0 to $\mathcal{H}_0$ and join it by a $\mathcal{D}$-edge with each vertex w_i, $1 \le i \le n_H$. Since each proper coloring of $\mathcal{H}_0$ uses exactly n_H colors (recall the proof of Theorem 4), each proper coloring of $\mathcal{H}_0$ can be uniquely extended to the vertex w_0 by assigning a completely new color to it.

We now add to the mixed hypergraph a vertex $u_{i,k}$ together with the following edges for each $1 \le i \le n_G$ and $1 \le k \le n_H$:

1. a $\mathcal{D}$-edge $\{v_i, w_k, u_{i,k}\}$,
2. a $\mathcal{C}$-edge $\{v_i, w_k, u_{i,k}\}$,
3. a $\mathcal{C}$-edge $\{w_0, w_k, u_{i,k}\}$ and
4. a $\mathcal{C}$-edge $\{w_k, u_{i,k}, w_l\} \cup N_G(v_i)$ for each neighbor w_l of w_k in H (recall that $v_i \notin N_G(v_i)$).

This completes the description of the mixed hypergraph $\mathcal{H}$. Clearly, the order of $\mathcal{H}$ is $n_G n_H + n_G + n_H + 1 = (n_G+1)\cdot(n_H+1)$ and it can be constructed in time polynomial in n_G and n_H. We show that for each locally bijective L-homomorphism φ, there is a proper coloring c with $\varphi^c_{\mathcal{H}} = \varphi$ and that for each proper coloring c of $\mathcal{H}$, the homomorphism $\varphi^c_{\mathcal{H}} = \varphi$ is locally bijective.

Consider a locally bijective L-homomorphism φ from G to H and let us define the following coloring c of the vertices of $\mathcal{H}$:

$$\begin{array}{rcll} c(w_k) & = & k & \text{for } 0 \leq k \leq n_H \\ c(v_i) & = & k & \text{where } \varphi(v_i) = w_k \\ c(u_{i,k}) & = & 0 & \text{if } \varphi(v_i) = w_k \text{ and} \\ c(u_{i,k}) & = & k & \text{if } \varphi(v_i) \neq w_k. \end{array}$$

Since φ is a locally bijective L-homomorphism, in particular, it is locally injective, c colors properly all $\mathcal{C}$-edges and $\mathcal{D}$-edges of $\mathcal{H}_0$. Let us fix $1 \leq i \leq n_G$ and $1 \leq k \leq n_H$ and we show that the edges (1)–(4) are also colored properly. If $\varphi(v_i) \neq w_k$, then $c(v_i) \neq c(w_k) = c(u_{i,k})$ and hence all edges containing the vertex $u_{i,k}$ are colored properly. If $\varphi(v_i) = w_k$, then $c(v_i) = c(w_k) \neq c(u_{i,k})$ and the edges (1), (2) and (3) are colored properly. Since φ is locally bijective, there is for each $w_l \in N_H(w_k)$ a neighbor v_j of v_i in G such that $\varphi(v_j) = w_l$. Thus, for each such w_l, there is a vertex $v_j \in N_G(v_i)$ with $c(w_l) = c(v_j)$ and the $\mathcal{C}$-edges (4) are colored properly.

We now show that for each proper coloring c of the mixed hypergraph $\mathcal{H}$, $\varphi^c_{\mathcal{H}}$ is a locally bijective homomorphism from G to H. In addition we prove, that each coloring of the vertices $v_1, \ldots, v_{n_G}$ and $w_0, w_1, \ldots, w_{n_H}$ uniquely determines colors of the vertices $u_{i,k}$ for $1 \leq i \leq n_G$ and $1 \leq k \leq n_H$.

Fix a proper coloring c of $\mathcal{H}$. If the colors $c(v_i)$ and $c(w_k)$ are different, then the color of the vertex $u_{i,k}$ is either $c(v_i)$ or $c(w_k)$ because of the $\mathcal{C}$-edge (2). This color must actually be the color $c(w_k)$ because of the $\mathcal{C}$-edge (3). On the other hand, if $c(v_i) = c(w_k)$, then the color of $u_{i,k}$ must be different from $c(v_i) = c(w_k)$ because of the $\mathcal{D}$-edge (1) and it must be the color $c(w_0)$ because of the $\mathcal{C}$-edge (3). Hence, colors of the vertices $u_{i,k}$ for $1 \leq i \leq n_G$ and $1 \leq k \leq n_H$ are uniquely determined by the colors of the vertices $v_1, \ldots, v_{n_G}$ and $w_0, w_1, \ldots, w_{n_H}$ as claimed.

If $c(v_i) = c(w_k)$, then $c(u_{i,k}) = c(w_0) = 0$. Observe now that in the $\mathcal{C}$-edge $\{w_k, u_{i,k}, w_l\} \cup N_G(v_i)$, only the vertex w_k is colored by the color k (no neighbor of v_i can have the color k) and only the vertex $u_{i,k}$ is colored by the color 0 (no vertex from $V(G)$ is colored by the color 0). By the construction of $\mathcal{H}_0$, all the vertices $N_G(v_i)$ have different colors and hence the only possibility how the $\mathcal{C}$-edge $\{w_k, u_{i,k}, w_l\} \cup N_G(v_i)$ can be properly colored is that $c(w_l) = c(v_j)$ for some $v_j \in N_G(v_i)$. Then, for each neighbor w_l of w_k in H, there is a neighbor v_j of v_i with $\varphi^c_{\mathcal{H}}(v_j) = w_k$, i.e., $\varphi^c_{\mathcal{H}}$ is locally surjective. It follows from Theorem 11 that the homomorphism $\varphi^c_{\mathcal{H}}$ is also locally injective and hence locally bijective.

■

5.3 Graph Covers

Theorem 14 *Let G and H be two graphs. There exists a mixed hypergraph $\mathcal{H}$ of order $O(n_G n_H)$ such that $\mathcal{H}$ is colorable if and only if there is G is a cover of H. Moreover, the mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and n_H.*

Proof: Let $\mathcal{H}_0$ be the mixed hypergraph from Theorem 13 for graphs G and H. Recall that for each proper coloring c, the mapping $\varphi^c_{\mathcal{H}_0}$ is a locally bijective homomorphism from G to H and on the other hand, for each locally bijective homomorphism φ there is a proper coloring c of $\mathcal{H}_0$ with $\varphi^c_{\mathcal{H}_0} = \varphi$. We remark that if H is connected, then a locally bijective homomorphism from G to H is a covering projection of G to H and vice versa. But since H need not to be connected, we need to assure that there exists a surjective locally bijective homomorphism from G to H.

For each $1 \le k \le n_H$, add a copy of the mixed hypergraph $\mathcal{H}_{n_G}$ from Lemma 9 identifying vertices α_i with v_i for $1 \le i \le n_G$, the vertex β with w_0 and γ with w_k. Let $\mathcal{H}$ be the resulting mixed hypergraph. Clearly, the order of $\mathcal{H}$ is at most $O(n_G n_H)$ and it can be constructed in time polynomial in n_G and n_H by Lemma 9 and Theorem 13. Next, we show that $\mathcal{H}$ has a proper coloring if and only if there exists a covering projection of G to H.

Let φ be a covering projection of G to H and consider a proper coloring c of $\mathcal{H}_0$ such that $\varphi^c_{\mathcal{H}_0} = \varphi$. Since w_0 is the only vertex colored with the color $c(w_0)$ (recall the proof of Theorem 13) and φ is surjective, i.e., for each $1 \le k \le n_H$, there exists $1 \le i \le n_G$ such that $c(v_i) = c(w_k)$, the coloring c can be extended to copies of the mixed hypergraph $\mathcal{H}_{n_G}$ by Lemma 9. Hence, $\mathcal{H}$ has a proper coloring.

On the other hand, if c is a proper coloring of $\mathcal{H}$, then the vertex w_0 has a color different from all the vertices v_i, $1 \le i \le n_G$, and w_k, $1 \le k \le n_H$. Hence, by Lemma 9, for each $1 \le k \le n_H$, there is a vertex v_i, $1 \le i \le n_G$ with $c(v_i) = c(w_k)$. Then, the homomorphism $\varphi^c_{\mathcal{H}} = \varphi^c_{\mathcal{H}_0}$ is surjective and since it is locally bijective by Theorem 13, it must actually be a covering projection of G to H.

■

5.4 Locally Surjective Homomorphisms

Theorem 15 *Let G and H be two graphs and let L be a mapping from $V(G)$ to $2^{V(H)}$. There exists a mixed hypergraph $\mathcal{H}$ of order $O(n_G n_H^2)$ which is* *colorable if and only if there is a locally surjective L-homomorphism from G to H. Moreover, the mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and n_H.*

Proof: Let $\mathcal{H}_0$ be the mixed hypergraph for G, H and L from Theorem 4. First, add a new vertex w_0 to $\mathcal{H}_0$ and join it by a $\mathcal{D}$-edge to each vertex w_i, $1 \le i \le n_H$. Since each proper coloring of $\mathcal{H}_0$ uses exactly n_H colors, each properly coloring of $\mathcal{H}_0$ can be uniquely to the vertex w_0 by assigning a new color to it. Let $\mathcal{H}'$ be the resulting mixed hypergraph $\mathcal{H}_0$.

Next, we add vertices $u_{i,k}$ for each $1 \le i \le n_G$ and $1 \le k \le n_H$ to $\mathcal{H}'$ and place a copy of the hypergraph $\mathcal{H}_p$ from Lemma 9 where p is the degree of the vertex v_i in G identifying the following vertices:

- Vertices $\alpha_{i'}$, $1 \le i \le \mathrm{deg_G(v_i)}$, are identified with neighbors of v_i.
- The vertex β is identified with the vertex w_0.
- The vertex γ is identified with the vertex $u_{i,k}$.

Finally, add a vertex $u'_{i,k,l}$ for each $1 \le i \le n_G$ and $1 \le k, l \le n_H$, $k \ne l$, such that $w_k w_l \in E(H)$ together with the following edges to the mixed hypergraph:

1. a $\mathcal{C}$-edge $\{w_0, w_l, u'_{i,k,l}\}$,
2. a $\mathcal{D}$-edge $\{v_i, w_l, u'_{i,k,l}\}$,
3. a $\mathcal{C}$-edge $\{v_i, w_l, u'_{i,k,l}\}$ and
4. a $\mathcal{C}$-edge $\{w_l, u'_{i,k,l}, u_{i,k}, w_k\}$.

Let $\mathcal{H}$ be the resulting mixed hypergraph. It can be clearly constructed in time polynomial in n_G and n_H and the order of $\mathcal{H}$ is $O(n_G n_H^2)$ by Lemma 9.

We show that $\mathcal{H}$ is colorable if and only if there is a locally surjective L-homomorphism from G to H. First assume that there is a locally surjective L-homomorphism φ from G to H. Consider the following partial coloring of the vertices of the mixed hypergraph $\mathcal{H}$:

$$
\begin{array}{rcll}
c(w_k) & = & k & \text{for } 0 \le k \le n_H \\
c(v_i) & = & k & \text{where } \varphi(v_i) = w_k \\
c(u_{i,k}) & = & k & \text{if } \varphi(v_j) = w_k \text{ for some neighbor } v_j \text{ of the vertex } v_i, \\
 & = & k' & \text{otherwise,} \\
 & & & \text{where } \varphi(v_j) = w_{k'} \text{ for some neighbor } v_j \text{ of the vertex } v_i, \\
c(u'_{i,k,l}) & = & 0 & \text{if } \varphi(v_i) = w_l \text{ and} \\
 & = & l & \text{otherwise.}
\end{array}
$$

Since φ is a homomorphism from G to H, all edges of $\mathcal{H}$ present in $\mathcal{H}_0$ are colored properly. The partial coloring can be extended to the whole copies of $\mathcal{H}_p$ by Lemma 9 because the color of $c(u_{i,k})$ is among the colors of the neighbors of the vertex v_i in G and the color $c(w_0) = 0$ is different from all of these colors. Consider now a fixed triple i, k and l such that $1 \le i \le n_G$ and $1 \le k, l \le n_H$, $k \ne l$. If $\varphi(v_i) \ne w_l$, then $c(u'_{i,k,l}) = c(w_l) = l \ne c(v_i)$ and all the edges (1), (2), (3) and (4) are colored properly. If $\varphi(v_i) = w_l$, then $c(u'_{i,k,l}) = c(w_0) = 0$ and $c(v_i) = c(w_l) = l \ne 0$. Hence, all the edges (1), (2) and (3) are colored properly. Since φ is a locally surjective homomorphism, there is a neighbor v_j of v_i mapped to a neighbor of w_l and we have $c(u_{i,k}) = k$. Thus, the corresponding $\mathcal{C}$-edge (4) is also colored properly.

We now complete the proof by showing that if c is a proper coloring of the mixed hypergraph $\mathcal{H}$, then $\varphi^c_{\mathcal{H}}$ is a locally surjective L-homomorphism from G to H. Since c is a proper coloring of $\mathcal{H}_0$, $\varphi^c_{\mathcal{H}}$ is a homomorphism form G to H by Theorem 4. Next, we prove that $\varphi^c_{\mathcal{H}}$ is locally surjective. Assume for the sake of contradiction that there is a vertex v_i with $\varphi^c_{\mathcal{H}}(v_i) = w_l$ and a neighbor w_k of w_l in H such that no neighbor of v_i is mapped by $\varphi^c_{\mathcal{H}}$ to w_l. Because of the $\mathcal{D}$-edge (2), the color of the vertex $u'_{i,k,l}$ must different from the color $c(v_i) = c(w_l)$. Actually, it must be the color $c(w_0)$ because of the $\mathcal{C}$-edge (1). We can now conclude that the colors $c(u'_{i,k,l}) = c(w_0)$, $c(w_k)$ and $c(w_l)$ are mutually different. By Lemma 9, the color of the vertex $u_{i,k}$ must be equal to a color of a neighbor of v_i. Hence, $c(u_{i,k})$ is neither $c(w_0)$ nor $c(w_k)$ and it must be $c(w_l)$ because of the $\mathcal{C}$-edge (4). But, then a neighbor of v_i is colored by the color $c(w_l)$, i.e., a neighbor of v_i is mapped by $\varphi^c_{\mathcal{H}}$ to the vertex w_l which we assumed not to be the case.

■

6 Generalized T-colorings

We first recall the following construction from [12] (let us remind that the size of a hypergraph is the sum of sizes of its edges):

Lemma 16 *For each $k \ge 2$, there is a mixed hypergraph $\mathcal{H}^0_k$ of order at most $3k-2$ with k distinguished vertices $w_1, \dots, w_k$ with the following properties:*

- $\mathcal{F}(\mathcal{H}^0_k) = \{2, \dots, k\}$ *and*

- *if c is a proper coloring of $\mathcal{H}_k^0$ which uses exactly k' colors, then all the vertices $w_1, \ldots, w_{k'}$ receive distinct colors and $c(w_{k'}) = c(w_{k'+1}) = \ldots = c(w_k)$.*

Moreover, the size of $\mathcal{H}_k^0$ is at most $O(k^3)$ and it can be constructed in time polynomial in k.

Now, we can prove the main theorem of this section from which we later derive results for more specialized coloring problems:

Theorem 17 *Let (G,t) be a generalized T-coloring problem where G is a graph with at least one edge. Then, there exists a mixed hypergraph $\mathcal{H}$ of order $O(n_G||t|| + ||t||^2)$ such that its lower chromatic number is equal to $\chi_t(G)$. Moreover, such a mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and $||t||$. In addition, the order of $\mathcal{H}$ can be reduced to $O(n_G K + ||t|| K)$ if an upper bound K on $\chi_t(G)$ is given.*

Proof: If the upper bound K on $\chi_t(G)$ is not given, set $K = ||t|| + 1$ (this is always an upper bound on $\chi_t(G)$ [5]). Hence, we have that $2 \le \chi_t(G) \le K$. Let $v_1, \ldots, v_{n_G}$ be the vertices of the graph G. The mixed hypergraph $\mathcal{H}$ contains a copy of the mixed hypergraph $\mathcal{H}_K^0$ from Lemma 16. Let $w_1, \ldots, w_K$ be the K distinguished vertices of $\mathcal{H}_K^0$. For each $1 \le i \le n_G$, we add a new vertex v'_i and a copy of the hypergraph $\mathcal{H}_{K-1}$ from Lemma 9 identifying the vertices α_k with w_{k+1} for $1 \le k \le K-1$, the vertex β with the vertex w_1 and the vertex γ with v'_i.

Next, we add vertices $u_{i,i',k,l}$ and $u'_{i,i',k,l}$ for each $1 \le i < i' \le n_G$ with $v_i v_{'i} \in E(G)$ and $1 \le k < l \le K$ with $l - k \in t(v_i v_{i'})$. Finally, we add to the mixed hypergraph under construction the following edges:

1. a $\mathcal{C}$-edge $\{v_i, v'_i, w_1\}$ for each $1 \le i \le n_G$,
2. a $\mathcal{C}$-edge $\{v_i, v_{i'}, u_{i,i',k,l}\}$ for each vertex $u_{i,i',k,l}$,
3. a $\mathcal{D}$-edge $\{w_k, u_{i,i',k,l}\}$ for each vertex $u_{i,i',k,l}$,
4. a $\mathcal{D}$-edge $\{u_{i,i',k,l}, u'_{i,i',k,l}, w_l\}$ for each vertex $u_{i,i',k,l}$,
5. a $\mathcal{D}$-edge $\{u'_{i,i',k,l}, w_{l-1}\}$ for each vertex $u'_{i,i',k,l}$,
6. a $\mathcal{C}$-edge $\{u'_{i,i',k,l}, w_{l-1}, w_l\}$ for each vertex $u'_{i,i',k,l}$ and
7. a $\mathcal{D}$-edge $\{v_i, v_{i'}\}$ for each two vertices v_i and $v_{i'}$ joined by an edge in G.

The order of $\mathcal{H}$ is clearly $O(n_G K + ||t||K)$ and it can be constructed in time polynomial in $||t||$, n_G and $K \leq ||t||$.

We claim that the lower chromatic number of $\mathcal{H}$ is equal to $\chi_t(G)$. We first prove that $\chi(\mathcal{H})$ is at most $\chi_t(G)$. Fix an optimal labeling σ of (G, w) using numbers $1, \ldots, \chi_t(G)$. We construct a proper coloring of $\mathcal{H}$ using $\chi_t(G)$ colors. First color the vertices of $\mathcal{H}^0_K$ by colors $1, \ldots, \chi_t(G)$. Recall that $2 \leq \chi_t(G)$. By Lemma 16, it can be assumed that the vertex w_i is colored by the color i for $1 \leq i \leq \chi_t(G)$ and the vertices w_i for $i > \chi_t(G)$ are colored by the color $\chi_t(G)$. Next, color the copies of $\mathcal{H}_{K-1}$ so that the color of v'_i is $\sigma(v_i)$ if $\sigma(v_i) \geq 2$ and it is 2 if $\sigma(v_i) = 1$. This is possible by Lemma 9. Finally, color the vertex v_i by the color $\sigma(v_i)$ for each $1 \leq i \leq n_G$.

It now remains to color the vertices $u_{i,i',k,l}$ and $u'_{i,i',k,l}$ of $\mathcal{H}$. Let $1 \leq i < i' \leq n_G$ be such that $v_i v_{i'}$ is an edge of G and let $1 \leq k < l \leq K$ such that $l - k \in t(v_i v_{i'})$. If $\chi_t(G) \leq k < l$, then color the vertex $u'_{i,i',k,l}$ with the color 1 and the vertex $u_{i,i',k,l}$ with the color $\sigma(v_i) = c(v_i)$ or $\sigma(v_{i'}) = c(v_{i'})$ so that $c(u_{i,i',k,l}) \neq c(w_k) = \chi_t(G)$ (recall that $\sigma(v_i) \neq \sigma(v_{i'})$ since $0 \in t(v_i v_{i'})$). If $k < \chi_t(G) < l$, then color the vertex $u'_{i,i',k,l}$ with the color $c(w_k) = k$ and the vertex $u_{i,i',k,l}$ with the color $\sigma(v_i) = c(v_i)$ or $\sigma(v_{i'}) = c(v_{i'})$ so that $c(u_{i,i',k,l}) \neq c(w_k) = k$. The final case is that $k < l \leq \chi_t(G)$. Color the vertex $u'_{i,i',k,l}$ with the color $c(w_l) = l$. Observe now that since σ is a proper labeling of G, we have $|\sigma(v_{i'}) - \sigma(v_i)| = |c(v_{i'}) - c(v_i)| \neq l - k$ and thus one of $c(v_i)$ and $c(v_{i'})$ is neither k nor l. Color now the vertex $u_{i,i',k,l}$ by one of the colors $c(v_i)$ and $c(v_{i'})$ so that $c(u_{i,i',k,l}) \neq k, l$.

We now show that c is a proper coloring of the mixed hypergraph $\mathcal{H}$. By the construction of c, all the edges contained in $\mathcal{H}^0_K$ and in the copies of $\mathcal{H}_{K-1}$ are properly colored. Hence, we have to show that the edges (1), (2), $\ldots$, (7) are also colored properly. In each $\mathcal{C}$-edge (1), either the vertex w_1 or the vertex v'_i have the same color as the vertex v_i for $1 \leq i \leq n_G$. Hence, each such $\mathcal{C}$-edge is colored properly. Since the vertex $u_{i,i',k,l}$ is colored either with the color $c(v_i)$ or with the color $c(v_{i'})$, each $\mathcal{C}$-edge (2) is also colored properly. Let us now fix i, i', k and l such that $\mathcal{H}$ contains the vertices $u_{i,i',k,l}$ and $u'_{i,i',k,l}$. We distinguish three cases as in the definition of the coloring c:

$\chi_t(G) \leq k < l$ The $\mathcal{D}$-edge (3) $\{w_k, u_{i,i',k,l}\}$ is colored properly because the color $c(u_{i,i',k,l})$ was chosen so that $c(u_{i,i',k,l}) \neq c(w_k)$. The $\mathcal{D}$-edges (4) and (5) are colored properly because $c(u'_{i,i',k,l}) = 1$ and $c(w_{l-1}) = c(w_l) = \chi_t(G) > 1$. And, the $\mathcal{C}$-edge (6) is colored properly because $c(w_{l-1}) = c(w_l) = \chi_t(G)$ by Lemma 16.

$k < \chi_t(G) < l$ Once more, the $\mathcal{D}$-edge (3) $\{w_k, u_{i,i',k,l}\}$ is colored properly because the color $c(u_{i,i',k,l})$ was chosen so that $c(u_{i,i',k,l}) \neq c(w_k) = k$. The $\mathcal{D}$-edges (4) and (5) are colored properly because $c(u'_{i,i',k,l}) = k$ and $c(w_{l-1}) = c(w_l) = \chi_t(G) > k$. And, the $\mathcal{C}$-edge (6) is colored properly because of the pair of the vertices w_{l-1} and w_l with $c(w_{l-1}) = c(w_l) = \chi_t(G)$.

$k < l \leq \chi_t(G)$ Since $c(u_{i,i',k,l}) \neq k, l$, $c(w_k) = k$ and $c(w_l) = l$, both the $\mathcal{D}$-edges (3) and (4) are colored properly. Since $c(u'_{i,i',k,l}) = l$ and $c(w_{l-1}) = l - 1 \neq c(w_l) = l$, both the $\mathcal{D}$-edge (5) and the $\mathcal{C}$-edge (6) are colored properly.

So, it remains to check that $\mathcal{D}$-edges (7) are colored properly. This directly follows from the fact that $0 \in t(v_i v_{i'})$ for each edge $v_i v_{i'} \in E(G)$ and hence $\sigma(v_i) = c(v_i) \neq \sigma(v_{i'}) = c(v_{i'})$.

Next, we prove the opposite inequality, i.e., $\chi_t(G) \leq \chi(\mathcal{H})$. Fix a proper coloring c of the mixed hypergraph $\mathcal{H}$ using $\chi(H)$ colors. Let ℓ be the number of colors used by c on $\mathcal{H}^0_K$. In the rest, we construct a proper labeling σ of G using numbers $1, \ldots, \chi(\ell)$. Note that $\ell \geq 2$ by Lemma 16 and we may assume that $c(w_k) = k$ for $1 \leq k \leq \ell$ and $c(w_k) = \ell$ for $k > \ell$. By Lemma 9, the color of the vertex v'_i is one of the colors $2, \ldots, \ell$. The $\mathcal{C}$-edges (1) now forces that each vertex v_i is colored by the color 1 or the color $c(v'_i) \in \{2, \ldots, \ell\}$. Let us define now $\sigma(v_i) = c(v_i)$ for all $1 \leq i \leq n_G$. Note that the labeling σ uses only the numbers $1, \ldots, \ell$ and $\ell \leq \chi(H)$.

In this paragraph, we prove that the labeling σ is proper. Because of the $\mathcal{D}$-edges (7), any two adjacent vertices of G receive distinct labels. If σ is not a proper labeling, then there must be two adjacent vertices v_i and $v_{i'}$, $1 \leq i < i' \leq n_G$, with $0 < |\sigma(v_i) - \sigma(v_{i'})| = |c(v_i) - c(v_{i'})| \in t(v_i v_{i'})$. We show that this is impossible and hence the labeling σ is proper as claimed. Let $k = \min\{c(v_i), c(v_{i'})\}$ and $l = \max\{c(v_i), c(v_{i'})\}$. Observe that $1 \leq k < l \leq \ell$ and thus $c(w_{l-1}) \neq c(w_l)$. The color of the vertex $u'_{i,i',k,l}$ must be l because of the $\mathcal{D}$-edge (5) and the $\mathcal{C}$-edge (6). Because of the $\mathcal{C}$-edge (2), the color $c(u_{i,i',k,l})$ is either k or l. On the other hand, $c(u_{i,i',k,l})$ can be neither k because of the $\mathcal{D}$-edge (3) nor l because of the $\mathcal{C}$-edge (4) — contradiction. This shows that $|\sigma(v_i) - \sigma(v_{i'})| \notin t(v_i v_{i'})$ for each edge $v_i v_{i'}$ of G and hence σ is a proper labeling.

■

In each of the three following subsections, we reformulate Theorem 17 to special types of generalized T-colorings:

6.1 The channel assignment problem

Theorem 18 *Let (G, w) be a channel assignment problem with at least one edge. Then, there exists a mixed hypergraph $\mathcal{H}$ of order $(n_G||w|| + ||w||^2)$ such that its lower chromatic number is equal to the span of (G, w). Moreover, such a mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and $||w||$. In addition, the order of $\mathcal{H}$ can be reduced to $O(n_G K + ||w||K)$ if an upper bound K on the span of (G, w) is given.*

Proof: Observe that the channel assignment problem (G, w) corresponds to the generalized T-coloring problem (G, t) with $t(v_i v_{i'}) = \{0, \ldots, w(v_i v_{i'}) - 1\}$ for each $v_i v_{i'} \in E(G)$ and apply Theorem 17 to (G, t).

■

6.2 $L(p, q)$-labelings

Theorem 19 *Let G be a given graph with at least one edge and let $p \geq q \geq 1$ be integers. Then, there exists a mixed hypergraph $\mathcal{H}$ of order $O(p^2 n_G^4)$ such that the lower chromatic number of $\mathcal{H}$ is equal to $\lambda_{p,q}(G)$. Moreover, such a mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G, p and q.*

Proof: As stated in Section 1, the $L(p, q)$-labeling problem of a graph G can be viewed as a special channel assignment problem in the second power G^2 of the graph G. Let (G^2, w) be such a channel assignment problem corresponding to the $L(p, q)$-labeling problem for the graph G. Observe that $\lambda_{p,q}(G) \leq p(n_G - 1) + 1$ and $||w|| \leq O(pn_G^2)$ in (G^2, w). The statement now follows readily from Theorem 18 applied for (G^2, w).

■

6.3 T-colorings

Theorem 20 *Let G be a graph and T a finite set of non-negative integers with $0 \in T$. Then, there exists a mixed hypergraph $\mathcal{H}$ of order $O(n_G^3 |T|^2)$ such that its lower chromatic number is equal to the T-chromatic number of G. Moreover, such a mixed hypergraph $\mathcal{H}$ can be constructed in time polynomial in n_G and $|T|$.*

Proof: Observe that the T-chromatic number of G is at most $(n_G - 1)|T| + 1$ and apply Theorem 17.

■

7 Conclusion

We have proved that mixed hypergraphs can model coloring problems such as homomorphisms of simple graphs and multigraphs, circular colorings, $(H, C, \leq K)$-coloring, (H, C, K)-coloring, locally surjective, locally bijective and locally injective homomorphisms, $L(p, q)$-labeling, the channel assignment problem, T-coloring and generalized T-coloring. In all cases, it is possible to construct the mixed hypergraph which models the coloring problem in polynomial time. However, in some cases, we managed to prove that proper colorings of the constructed mixed hypergraph one-to-one correspond to proper colorings in the original problem, but in some cases, we did not manage to prove such a result. This motivates the following open problems:

Problem 21 *Find a polynomial-time construction of a mixed hypergraph $\mathcal{H}$ for a given multigraph G and a partial weighted assignment (H, C, K) such that proper colorings of $\mathcal{H}$ one-to-one correspond to (H, C, K)-colorings, $(H, C, \leq K)$-colorings, respectively, of G.*

Problem 22 *Find a polynomial-time construction of a mixed hypergraph $\mathcal{H}$ for given graphs G and H such that proper colorings of $\mathcal{H}$ one-to-one correspond to locally surjective homomorphisms from G to H, covering projections of G to H, respectively.*

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