Planar graph coloring avoiding monochromatic subgraphs: trees and paths make things difficult *

Hajo Broersma
Department of Applied Mathematics
Faculty of Electrical Engineering,
Mathematics and Computer Science
University of Twente
7500 AE Enschede, The Netherlands
broersma@math.utwente.nl

Fedor V. Fomin
Department of Informatics
University of Bergen
N-5020 Bergen, Norway
fomin@ii.uib.no

Jan Kratochvíl
Department of Applied Mathematics and
Institute of Theoretical Computer Science
Faculty of Mathematics and Physics
Charles University
118 00 Prague, Czech Republic
honza@kam.ms.mff.cuni.cz

Gerhard J. Woeginger
Department of Applied Mathematics
Faculty of Electrical Engineering,
Mathematics and Computer Science
University of Twente
7500 AE Enschede, The Netherlands
g.j.woeginger@math.utwente.nl

Abstract

We consider the problem of coloring a planar graph with the minimum number of colors such that each color class avoids one or more forbidden graphs as subgraphs. We perform a detailed study of the computational complexity of this problem.

We present a complete picture for the case with a single forbidden connected (induced or non-induced) subgraph. The 2-coloring problem is NP-hard if the forbidden subgraph is a tree with at least two edges, and it is polynomially solvable in all other cases. The 3-coloring problem is NP-hard if the forbidden subgraph is a nontrivial path, and it is polynomially solvable in all other cases. We also derive results for several forbidden sets of cycles.

Keywords: graph coloring; graph partitioning; forbidden subgraph; planar graph; computational complexity.

AMS Subject Classifications: 05C15,05C85,05C17

1 Introduction

We denote by $G = (V, E)$ a finite undirected and simple graph with $|V| = n$ vertices and $|E| = m$ edges. For any non-empty subset $W \subseteq V$, the

*An extended abstract was presented at SWAT2002, see LNCS 2368, 160-169. The work of HJB and FVF is sponsored by NWO-grant 047.008.006. Part of the work was done while FVF was visiting the University of Twente, and while he was a visiting postdoc at DIMATIA-ITI (supported by GAČR 201/99/0242 and by the Ministry of Education of the Czech Republic as project LN00A056). FVF acknowledges support by EC contract IST-1999-14186: Project ALCOM-FT (Algorithms and Complexity - Future Technologies). JK acknowledges support by the Czech Ministry of Education as project LN00A056. GJW acknowledges support by the START program Y43-MAT of the Austrian Ministry of Science.

subgraph of G induced by W is denoted by $G[W]$. A *clique* of G is a non-empty subset $C \subseteq V$ such that all the vertices of C are mutually adjacent. A non-empty subset $I \subseteq V$ is *independent* if no two of its elements are adjacent. An r-*coloring* of the vertices of G is a partition $V_1, V_2, \dots, V_r$ of V; the r sets V_j are called the *color classes* of the r-coloring. An r-coloring is *proper* if every color class is an independent set. The *chromatic number* $\chi(G)$ is the minimum integer r for which a proper r-coloring exists.

Evidently, an r-coloring is proper if and only if for every color class V_j, the induced subgraph $G[V_j]$ does not contain a subgraph isomorphic to P_2. This observation leads to a number of interesting generalizations of the classical graph coloring concept. One such generalization was suggested by Harary [21]: Given a graph property π, a positive integer r, and a graph G, a π r-coloring of G is a (not necessarily proper) r-coloring in which each subgraph induced by a color class has property π. This generalization has been studied for the cases where the graph property π is being acyclic, or planar, or perfect, or a path of length at most k, or a clique of size at most k. We refer the reader to the work of Brown & Corneil [7, 6, 8], Chartrand *et al.* [10, 11, 12], and Sachs [26] for more information on these variants.

In this paper, we will investigate graph colorings where the property π can be defined via some (maybe infinite) list of forbidden induced subgraphs. This naturally leads to the notion of $\mathcal{F}$-free colorings. Let $\mathcal{F} = \{F_1, F_2, \dots\}$ be the set of so-called forbidden graphs. Throughout the paper we will assume that the set $\mathcal{F}$ is non-empty, and that all graphs in $\mathcal{F}$ are connected and contain at least one edge. Moreover, to avoid technical difficulties in the proofs we will assume that no graph of $\mathcal{F}$ is a proper subgraph of another graph of $\mathcal{F}$. For a graph G, a (not necessarily proper) r-coloring with color classes $V_1, V_2, \dots, V_r$ is called *weakly $\mathcal{F}$–free*, if for all $1 \le j \le r$, the graph $G[V_j]$ does not contain any graph from $\mathcal{F}$ as an *induced* subgraph. Similarly, we say that an r-coloring is *strongly $\mathcal{F}$–free* if $G[V_j]$ does not contain any graph from $\mathcal{F}$ as an (induced or non-induced) subgraph. The smallest possible number of colors in a weakly (respectively, strongly) $\mathcal{F}$-free coloring of a graph G is called the *weakly* (respectively, *strongly*) *$\mathcal{F}$-free chromatic number*; it is denoted by $\chi^W(\mathcal{F}, G)$ (respectively, by $\chi^S(\mathcal{F}, G)$).

In the cases where $\mathcal{F} = \{F\}$ consists of a single graph F, we will sometimes simplify the notation and not write the curly brackets: We will write F-free short for $\{F\}$-free, $\chi^W(F, G)$ short for $\chi^W(\{F\}, G)$, and $\chi^S(F, G)$ short for $\chi^S(\{F\}, G)$. With this notation $\chi(G) = \chi^S(P_2, G) = \chi^W(P_2, G)$

holds for every graph G, and hence also

$$\chi^{W}(\mathcal{F}, G) \;\leq\; \chi^{S}(\mathcal{F}, G) \;\leq\; \chi(G).$$

It is easy to construct examples where both inequalities are strict. For instance, for $\mathcal{F} = \{P_3\}$ (the path on three vertices) and $G = C_3$ (the cycle on three vertices) we have $\chi(G) = 3$, $\chi^{S}(P_3, G) = 2$, and $\chi^{W}(P_3, G) = 1$.

Our main concern in the paper are planar graphs. Recall that a graph is *planar* if it can be drawn in the (Euclidean) plane without intersections of edges. Such a drawing is referred to as a *plane* graph. Hence a graph G is planar if and only if there exists a plane graph isomorphic to G.

1.1 Previous results

The literature contains quite a number of papers on weakly and strongly $\mathcal{F}$-free colorings of graphs. The most general result is due to Achlioptas [1]: For any graph F with at least three vertices and for any $r \geq 2$, the problem of deciding whether a given input graph has a weakly F-free r-coloring is NP-hard. We often use weakly (strongly) $\mathcal{F}$-free r-coloring as shorthand for the corresponding decision problem.

The special case of weakly P_3-free coloring is known as the *subcoloring problem* in the literature. It has been studied by Broere & Mynhardt [4], by Albertson, Jamison, Hedetniemi & Locke [2], by Fiala, Jansen, Le & Seidel [16], and by Broersma, Fomin, Nešetřil & Woeginger [5].

Proposition 1.1. [Fiala, Jansen, Le & Seidel [16]]
Weakly P_3-free 2-coloring is NP-hard for triangle-free planar graphs.

A $(1,2)$-*subcoloring* of G is a partition of V into two sets S_1 and S_2 such that S_1 induces an independent set and S_2 induces a subgraph consisting of a matching and some (possibly no) isolated vertices. Le & Le [24] proved that recognizing $(1,2)$-subcolorable cubic graphs is NP-hard, even for triangle-free planar graphs.

The case of weakly P_4-free coloring has been investigated by Gimbel & Nešetřil [19] who study the problem of partitioning the vertex set of a graph into induced cographs. Since cographs are exactly the graphs without an induced P_4, the graph parameter studied in [19] equals the weakly P_4-free chromatic number of a graph. In [19] it is proved that the problems of deciding $\chi^{W}(P_4, G) \leq 2$, $\chi^{W}(P_4, G) = 3$, $\chi^{W}(P_4, G) \leq 3$ and $\chi^{W}(P_4, G) = 4$ all are NP-hard and/or coNP-hard for planar graphs. The work of Hoàng &

Le [22] on weakly P_4-free 2-colorings was motivated by the Strong Perfect Graph Conjecture. Among other results, they show that weakly P_4-free 2-coloring is NP-hard for comparability graphs.

A notion that is closely related to strongly F-free r-coloring is the so-called *defective* graph coloring. A defective (k, d)-coloring of a graph is a k-coloring in which each color class induces a subgraph with maximum degree at most d. Defective colorings have been studied for instance by Archdeacon [3], by Cowen, Cowen & Woodall [14], and by Frick & Henning [17]. Cowen, Goddard & Jesurum [13] have shown that the defective $(3, 1)$-coloring problem and the defective $(2, d)$-coloring problem for any $d \geq 1$ are NP-hard even for planar graphs. We observe that for any k, defective $(k, 1)$-coloring is equivalent to strongly P_3-free k-coloring, and hence we derive the following proposition.

Proposition 1.2. [Cowen, Goddard & Jesurum [13]]
(i) Strongly P_3-free 2-coloring is NP-hard for planar graphs.
(ii) Strongly P_3-free 3-coloring is NP-hard for planar graphs.

1.2 Our results

We perform a complexity study of weakly and strongly $\mathcal{F}$-free coloring problems for *planar* graphs. By the Four Color Theorem (4CT), every planar graph G satisfies $\chi(G) \leq 4$. Consequently, every planar graph also satisfies $\chi^w(\mathcal{F}, G) \leq 4$ and $\chi^s(\mathcal{F}, G) \leq 4$, and we may concentrate on 2-colorings and on 3-colorings. For the case of a single forbidden subgraph, we obtain the following results for 2-colorings:

- If the forbidden (connected) subgraph F is not a tree, then *every* planar graph is strongly and hence also weakly F-free 2-colorable. Hence, the corresponding decision problems are trivially solvable.
- If the forbidden subgraph $F = P_2$, then F-free 2-coloring is equivalent to proper 2-coloring. It is well-known that this problem is polynomially solvable.
- If the forbidden subgraph is a tree T with at least two edges, then both weakly and strongly T-free 2-coloring are NP-hard for planar input graphs. Hence, these problems are intractable.

For 3-colorings with a single forbidden subgraph, we obtain the following results:

- If the forbidden (connected) subgraph F is not a path, then *every* planar graph is strongly and hence also weakly F-free 3-colorable. Hence, the corresponding decision problems are trivially solvable.
- For every path P with at least one edge, both weakly and strongly P-free 3-coloring are NP-hard for planar input graphs. Hence, these problems are intractable.

Moreover, we derive several results for 2-colorings with certain forbidden sets of cycles.

- For the forbidden set $\mathcal{F}_{345} = \{C_3, C_4, C_5\}$, both weakly and strongly $\mathcal{F}_{345}$-free 2-coloring are NP-hard for planar input graphs. In fact for any finite set $\mathcal{F}_{\geq 345} \supseteq \{C_3, C_4, C_5\}$ of cycles, both weakly and strongly $\mathcal{F}_{\geq 345}$-free 2-coloring are NP-hard for planar input graphs.
- Also for the forbidden set $\mathcal{F}_{cycle}$ of all cycles, both weakly and strongly $\mathcal{F}_{cycle}$-free 2-coloring are NP-hard for planar input graphs.
- For the forbidden set $\mathcal{F}_{odd}$ of all cycles of odd lengths, *every* planar graph is strongly and hence also weakly $\mathcal{F}_{odd}$-free 2-colorable. This follows from (in fact, is equivalent to) the Four Color Theorem.

2 The machinery for establishing NP-hardness

Throughout this section, let $\mathcal{F}$ denote some fixed set of forbidden planar subgraphs. We assume that all graphs in $\mathcal{F}$ are connected and contain at least two edges. We also assume that no graph of $\mathcal{F}$ is a (not necessarily induced) proper subgraph of another graph from $\mathcal{F}$. We will develop a generic NP-hardness proof for certain types of weakly and strongly $\mathcal{F}$-free 2-coloring problems. The crucial concept is the so-called *equalizer* gadget. Before we define this gadget, let us introduce the following technical concept of crossing graphs. We note that we distinguish between planar graphs and plane graphs (the latter being particular nonintersecting drawings of abstract planar graphs), but we use the same notation for a plane graph and its underlying abstract (planar) graph. When talking about more than one graph, we use subscripts to distinguish their vertex and edge sets (i.e., V_G and E_G denote the vertex and edge sets of a graph G).

Definition 2.1. Given a plane graph G with outer face C and a set $S \subseteq V_G$ of vertices on the boundary of C (referred to as *contact points*), we say that another plane graph H is *crossing* G if the following assertions hold:

1. $G \cup H$ is a plane graph (i.e., no edge of G crosses any edge of H in the simultaneous drawing of G and H),
2. all edges of $E_H \setminus E_G$ are drawn in C,
3. no edge of $E_H \setminus E_G$ is incident with a vertex of $V_G \setminus S$,
4. $V_H \cap (V_G \setminus S) \neq \emptyset$.

If G is a plane graph with a set S of contact points, we say that a planar graph H *may cross* G if some nonintersecting planar drawing of H is crossing G.

Definition 2.2. (Equalizer)
An (a, b)-*equalizer* for $\mathcal{F}$ is a plane graph $\mathcal{E}$ with two nonadjacent contact points a and b on the boundary of the outer face, which satisfies the following properties:

(i) In every weakly $\mathcal{F}$-free 2-coloring of $\mathcal{E}$, a and b receive the same color.

(ii) There exists a strongly $\mathcal{F}$-free 2-coloring of $\mathcal{E}$ such that a and b receive the same color, whereas no monochromatic copy of a graph in $\mathcal{F}$ may cross $\mathcal{E}$. Such a coloring is called a *good* 2-coloring of $\mathcal{E}$.

To better understand the definition of an equalizer $\mathcal{E}$, let us remark right away that if $\mathcal{F}$ contains a graph with a leaf, then an $\mathcal{F}$-free 2-coloring of $\mathcal{E}$ is good if and only if all neighbors of the contact points in $\mathcal{E}$ have the color that is not assigned to the contact points.

The rest of this section is devoted to the proof of the following (technical) main theorem. This theorem is going to generate a number of NP-hardness statements in the subsequent sections of the paper.

Theorem 2.3. *(Technical main result of the paper)*
Let $\mathcal{F}$ be a set of connected planar graphs that all contain at least two edges, such that no graph of $\mathcal{F}$ is a proper subgraph of another graph of $\mathcal{F}$. Assume that

- *$\mathcal{F}$ contains a graph on at least four vertices with a cut vertex, or a 2-connected graph with a planar embedding with at least five vertices on the boundary of the outer face;*

- *there exists an (a,b)-equalizer for $\mathcal{F}$.*

Then deciding weakly $\mathcal{F}$-free 2-colorability and deciding strongly $\mathcal{F}$-free 2-colorability are NP-hard problems for planar input graphs.

We postpone the proof of Theorem 2.3 to Section 2.2, but first introduce some additional tools.

2.1 Gadgets for the NP-hardness proof

We will design a series of gadgets that all use the equalizer gadget as an atomic component. The only connections between an equalizer and the rest of the graph will always be via the contact points. The use of the equalizer gadget is justified (and motivated) by the following lemma.

Lemma 2.4. *Consider a nontrivial planar graph H and an edge $xy \in E_H$. Let the graph H^+ result from H by adding a vertex-disjoint copy $\mathcal{E}$ of an (a,b)-equalizer to H and then identifying vertex x with contact point a, and vertex y with contact point b.*

Then H^+ is a planar graph, and H^+ has a strongly/weakly $\mathcal{F}$-free 2-coloring if and only if H has a strongly/weakly $\mathcal{F}$-free 2-coloring in which x and y both receive the same color.

Proof. Since $\mathcal{E}$ is a plane graph with a and b on the boundary of the outer face, H^+ is also planar and it has a nonintersecting drawing such that all edges of H are drawn in the outer face of $\mathcal{E}$. For the proof of the 'only if' part, observe that every strongly/weakly $\mathcal{F}$-free 2-coloring of H^+ induces a strongly/weakly $\mathcal{F}$-free 2-coloring of H. By property (i) in Definition 2.2, this induced coloring assigns the same color to x and y. For the proof of the 'if' part, we construct a strongly/weakly $\mathcal{F}$-free 2-coloring of H^+: We use the strongly/weakly $\mathcal{F}$-free 2-coloring for the subgraph H, and we color the (a,b)-equalizer $\mathcal{E}$ using a good coloring in the sense of property (ii) in Definition 2.2. $\square$

The negator gadget. An (a,b)-*negator* for $\mathcal{F}$ is a plane graph $\mathcal{N}$ with two nonadjacent contact points a and b on the boundary of the outer face, which satisfies the following properties:

(i) In every weakly $\mathcal{F}$-free 2-coloring of $\mathcal{N}$, a and b receive different colors.

(ii) There exists a strongly $\mathcal{F}$-free 2-coloring of $\mathcal{N}$ such that a and b receive different colors, whereas no monochromatic copy of a graph in $\mathcal{F}$ may cross $\mathcal{N}$. Such a coloring is called a *good* 2-coloring of $\mathcal{N}$.

Next we show how to construct such an (a, b)-negator from (a, b)-equalizers. We choose an arbitrary graph $F \in \mathcal{F}$, and take some fixed planar embedding of F to form the so-called *skeleton* of the negator. Let a' and b' denote two vertices on the boundary of the outer face of F. We partition V_F into two disjoint sets V_1 and V_2 in such a way that both $F[V_1]$ and $F[V_2]$ (the subgraphs of F induced by V_1 and V_2) are connected, and so that $a' \in V_1$ and $b' \in V_2$. For every edge $xy \in E_{F[V_1]} \cup E_{F[V_2]}$, we add an equalizer between x and y exactly in the way we described in Lemma 2.4. We create a new vertex a and connect it by an equalizer to a'; we create a new vertex b and connect it by an equalizer to b'. This completes the construction of $\mathcal{N}$. To see that (i) is fulfilled, consider some weakly $\mathcal{F}$-free 2-coloring of $\mathcal{N}$. Suppose that a and b receive the same color. Then the equalizers enforce that this color propagates to all vertices in the skeleton, and this yields a monochromatic induced copy of F, a contradiction. To see that (ii) is fulfilled, we may color $\{a\} \cup V_1$ with one color, and $\{b\} \cup V_2$ with the other color. The vertices inside the equalizer gadgets may be colored using a good coloring in the sense of Definition 2.2.(ii). Any monochromatic copy of a graph $F' \in \mathcal{F}$ would either contain some edges of some equalizer gadget (which is impossible by the goodness of the equalizer coloring) or be a subgraph of $F[V_1]$ or $F[V_2]$ (which is impossible by the assumption we made on $\mathcal{F}$).

In our constructions, the negator gadget will be used similarly as the equalizer gadget as described in Lemma 2.4. While the equalizer gadget can be used to enforce that a pair of vertices receives the same color, with the help of the negator gadget we can enforce that a pair of adjacent vertices in some planar graph must receive different colors in any weakly $\mathcal{F}$-free 2-coloring. We omit the details since the counterpart of Lemma 2.4 with respect to negators and its proof are straightforward variations on Lemma 2.4 and its proof.

For our NP-hardness proof (of Theorem 2.3) we need two additional gadgets.

The clause gadget with four contact points. The gadget $\mathcal{C}_4(a, b, c, d)$ is a plane graph $\mathcal{C}$ with pairwise nonadjacent contact points a, b, c and d that lie in this (cyclic) ordering on the boundary of the outer face of $\mathcal{C}$. It has the following properties:

(i) In every weakly $\mathcal{F}$-free 2-coloring of $\mathcal{C}$, not all four contact points receive the same color.

(ii) Any 2-coloring of the four contact points that uses both colors, can be extended to a strongly $\mathcal{F}$-free 2-coloring of the whole gadget $\mathcal{C}$, in such a way that no monochromatic copy of a graph in $\mathcal{F}$ may cross $\mathcal{C}$. Such a coloring is called a *good* 2-coloring of $\mathcal{C}$.

Now let us construct such a clause gadget $\mathcal{C}_4(a, b, c, d)$. Suppose we are in the case assumed in Theorem 2.3. Hence the set $\mathcal{F}$ contains some graph F that can be planarly embedded such that there are four vertices a', b', c', d' on the boundary of the outer face. We choose this plane graph F to form the skeleton of the clause gadget. We create four new vertices a, b, c, and d. Each of these new vertices is connected by an equalizer to its corresponding primed vertex on the outer face of the skeleton. The vertices in the skeleton are partitioned into four components (with connecting edges between them) such that a', b', c', d' end up in different components. Within each component, we introduce equalizers along every edge in the way we described in Lemma 2.4. This completes the construction.

By now it is routine to verify that the construction indeed fulfills the two properties (i) and (ii). We leave the details to the reader.

The clause gadget with five contact points. The gadget $\mathcal{C}_5(a, b, c, d_1, d_2)$ is a plane graph $\mathcal{C}$ with pairwise nonadjacent contact points a, b, c, d_1, and d_2 that lie in the (cyclic) ordering $a - b - d_1 - c - d_2$ on the boundary of the outer face of $\mathcal{C}$. It has the following properties:

(i) In every weakly $\mathcal{F}$-free 2-coloring of $\mathcal{C}$, the vertices d_1 and d_2 receive the same color, and at least one of the vertices a, b, c receives the opposite color.

(ii) Any 2-coloring of the five contact points that assigns the same color to d_1 and d_2, and the opposite color to at least one of a, b, c, can be extended to a strongly $\mathcal{F}$-free 2-coloring of the whole gadget $\mathcal{C}$, in such a way that no monochromatic copy of a graph in $\mathcal{F}$ may cross $\mathcal{C}$. Such a coloring is called a *good* 2-coloring of $\mathcal{C}$.

Suppose we are in the case assumed in Theorem 2.3, hence the set $\mathcal{F}$ contains a graph on at least four vertices with a cut vertex, or a 2-connected graph with a planar embedding with at least five vertices on the boundary of the outer face.

Let us first discuss the case of a graph $F \in \mathcal{F}$ with a cut vertex d'. The skeleton of $\mathcal{C}_5(a, b, c, d_1, d_2)$ is formed by a planar embedding of F where d' is on the boundary of the outer face. Choose three vertices a', b', c' that all lie on the boundary of the outer face, and that do not belong to the same component of $F - d'$, such that we can move around the boundary of the outer face starting at a', then moving to b', then to d', then to c', then to d' again, and then returning to a' (maybe meeting other vertices, including d' and b', in between). For example, if $F = K_{1,k}$ with $k \geq 3$, we choose d' as the center, and a', b' and c' as three successive end vertices in a cyclic ordering in a planar embedding of F. We can move around the boundary of the outer face from a' (via d') to b', then to d' and to c', and back to a' via d' (and alternating between d' and the possible other end vertices if $k \geq 3$). The other cases are similar. We create five new vertices a, b, c, d_1, and d_2, and we connect them by equalizers to a', b', c', d', and d', respectively, at the place where we hit the primed vertices in the above ordering $a' - b' - d' - c' - d'$ while moving around the boundary of the outer face in the way we described. The vertices in the skeleton are partitioned into four components such that a', b', c', d' end up in different components. Within each component, we introduce equalizers along every edge in the way we described in Lemma 2.4. This completes the construction for the first case.

Next we discuss the case of a 2-connected planar graph $F \in \mathcal{F}$ that has a planar embedding with at least five vertices on the boundary of the outer face. We use such an embedding as the skeleton of $\mathcal{C}_5(a, b, c, d_1, d_2)$. Consider the cycle C that forms the boundary of the outer face. Choose five vertices $v_0 - v_1 - v_2 - v_3 - v_4$ in this order along C. Because all these v-vertices are on the outer face, only two subcases are possible.

- (Subcase 1) *There is a face D inside C that touches all these v-vertices.* Then we choose two nonadjacent vertices d_1' and d_2' from these five and three additional appropriate vertices a', b', c' from C such that the cyclic ordering along the cycle C is $a' - b' - d_1' - c' - d_2'$. Then we connect d_1' and d_2' by an equalizer that is put inside D. Notice that in the graph $F - \{d_1', d_2'\}$ the vertex c' is in a component different from the component containing a' or b'.

- (Subcase 2) *There is an i and a path P (possibly just one edge) internally-disjoint from C that connects two vertices v_i and v_{i+3}*

(where the indices are taken modulo 5). We put $d'_1 = v_i$, $d'_2 = v_{i+3}$ and call the remaining three v-vertices a', b', c' in such a way that the cyclic ordering along C is $a' - b' - d'_1 - c' - d'_2$. For every edge of P we connect its incident vertices by an equalizer. Again notice that in the graph $F - V_P$ the vertex c' is in a component different from the component containing a' or b'.

In either subcase, we create five new vertices a, b, c, d_1, d_2, and connect them by equalizers to their corresponding primed vertices on the outer face of the skeleton. Finally, we partition the vertices of the skeleton into five connected subgraphs, each containing one of the vertices a', b', c', d'_1, d'_2, and we introduce equalizers along the edges of these subgraphs as in Lemma 2.4. This completes the construction.

It can be verified that this construction in both cases and subcases indeed fulfills the two properties (i) and (ii).

2.2 The NP-hardness argument

Now we prove Theorem 2.3. The proof will be done by a reduction from an NP-hard variant of the 3-satisfiability problem: Let Φ be a Boolean formula in conjunctive normal form over a set X of logical variables; every clause in Φ contains exactly three variables. With Φ we associate a graph Q_Φ. The vertices of Q_Φ are the clauses and the variables in Φ. There are two types of edges in Q_Φ. The first type belongs to a cycle that spans all the clauses in some ordering. The second type connects a variable $x \in X$ to a clause $\phi \in \Phi$ if and only if x or $\bar{x}$ occurs as a literal in ϕ. We call a formula Φ *planar* if and only if for some choice of the cycle spanning all the clauses of Φ the associated graph Q_Φ is planar. Fellows *et al.* [15] proved that the restriction of the 3-satisfiability problem to planar formulae is NP-hard. (To be precise, they only show the NP-hardness for formulas with *at most* three literals per clause. One may achieve exactly three literals per clause by dropping the requirement of distinctness of literals per clause. Since variable-clause incidences will later be replaced by gadgets with nonadjacent contact points, our final graph will have no multiple edges anyway.)

Consider an arbitrary planar formula Φ as described above, and let Q_Φ be an associated planar graph. We will construct in polynomial time a planar graph G_Φ which has the following two properties: If formula Φ is satisfiable, then G_Φ has a strongly $\mathcal{F}$-free 2-coloring. If G_Φ has a weakly $\mathcal{F}$-free 2-coloring, then formula Φ is satisfiable. This clearly will prove

Theorem 2.3.

Fix a planar embedding of Q_Φ. The cycle through the clause vertices divides the plane into a bounded and an unbounded region. Variables in X that are embedded in the unbounded region are called *outer* variables, and variables in the bounded region are called *inner* variables. We thicken the edges and the vertices in the planar embedding of Q_Φ such that they become streets and squares; this yields a map into which we will put our gadgets. For every variable $x \in X$, we put a vertex $v(x)$ into the square corresponding to x. For every clause $\phi \in \Phi$, we put a corresponding clause gadget into the square corresponding to ϕ.

- If all three literals in clause ϕ belong to inner variables, then the clause gadget for ϕ is a clause gadget $\mathcal{C}_4(a, b, c, d)$ with four contact points. The contact point d lies in the center of the square, and the contact points a, b, c lie at the beginning of the streets leading to these three inner variables.

- If two literals in clause ϕ belong to inner variables and one literal belongs to an outer variable, then the clause gadget for ϕ is a clause gadget $\mathcal{C}_5(a, b, c, d_1, d_2)$ with five contact points. The contact points d_1 and d_2 lie at the beginning of the streets that lead to the left and the right neighbors of the clause ϕ on the clause cycle. The contact points a and b lie at the beginning of the streets that lead to the two inner variables. The contact point c lies at the beginning of the street that leads to the outer variable.

- The case of three outer variables, and the case of one inner and two outer variables are handled symmetrically to the above two cases.

If the variable x occurs un-negated (respectively, negated) in the clause ϕ, then we put an equalizer (respectively, a negator) from $v(x)$ to the corresponding contact point in the clause gadget for ϕ. Finally, we put an equalizer gadget between the d-vertices into every street that connects a clause square to another clause square, and thus connect all clause gadgets into a ring. These equalizer gadgets connect contact points d from clause gadgets $\mathcal{C}_4(a, b, c, d)$, and contact points d_1 and d_2 from clause gadgets $\mathcal{C}_5(a, b, c, d_1, d_2)$ in an appropriate way. This completes the construction of the graph G_Φ which is easily seen to be planar.

Assume that formula Φ is satisfiable, and consider a satisfying truth assignment. Intuitively speaking, color 1 will correspond to TRUE and color 0 will correspond to FALSE. Color all contact points d, d_1, and d_2 of clause gadgets by color 0. If variable x is TRUE, then color the vertex $v(x)$ by color 1. If x is FALSE, then color $v(x)$ by 0. The equalizers and negators propagate the colors (respectively opposite colors) of the variables to the corresponding contact points a, b, c in the clause gadgets. Since the truth assignment is a satisfying truth assignment, in every clause gadget at least one of the contact points a, b, c is colored 1. Moreover, in every clause gadget the contact points d (respectively, d_1 and d_2) are colored 0. Therefore, we can use property (ii) of the clause gadgets to get a strongly $\mathcal{F}$-free 2-coloring of all used clause gadgets. Altogether, this yields a strongly $\mathcal{F}$-free 2-coloring for the graph G_Φ.

Now assume that G_Φ has a weakly $\mathcal{F}$-free 2-coloring. Because of the ring of equalizer gadgets that connect the clause gadgets to each other and property (i) of the $\mathcal{C}_5$-gadgets, all contact points d, d_1, d_2 of clause gadgets must receive the same color; without loss of generality we assume that this color is 0. We construct the following truth assignment for X: If $v(x)$ is colored 1, then x is set to TRUE. If $v(x)$ is colored 0, then x is set to FALSE. Suppose for the sake of contradiction that some clause ϕ in Φ is not satisfied by this truth setting. Then the three literals in ϕ all are FALSE, and hence all three contact points a, b, c in the corresponding clause gadget are colored 0. But then *all* contact points of this clause gadget are colored 0, and by property (i) of the clause gadgets the coloring cannot be a weakly $\mathcal{F}$-free 2-coloring. This contradiction shows that Φ is satisfiable.

This completes the proof of Theorem 2.3.

3 Tree-free 2-colorings of planar graphs

The main result of this section is an NP-hardness result for weakly and strongly T-free 2-colorings of planar graphs for the case where T is a tree with at least two edges (see Theorem 3.3). The proof of this result is based on an inductive argument over the number of edges in T. The following two propositions are used as the base case of the induction.

Let $K_{1,k}$ denote the complete bipartite graph with one vertex in one color class and the other $k \geq 1$ vertices in the other color class.

Proposition 3.1. *For every $k \geq 2$, it is NP-hard to decide whether a planar graph has a weakly (strongly) $K_{1,k}$-free 2-coloring.*

Proof. For $k = 2$, the statement for weakly $K_{1,k}$-free 2-colorings follows

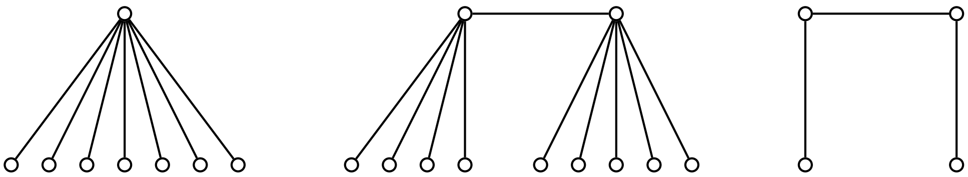

Figure 1: The graph $K_{1,7}$ and the double-stars $X_{4,5}$ and $X_{1,1}$.

from Proposition 1.1, and the statement for strongly $K_{1,k}$-free 2-colorings follows from Proposition 1.2.(i).

For $k \geq 3$, we apply Theorem 2.3. The first condition in this theorem is fulfilled, since for $k \geq 3$, the star $K_{1,k}$ is a graph on at least four vertices with a cut vertex.

For the second condition, we note that the complete bipartite graph $K_{2,2k-1}$ with color classes I with $|I| = 2k-1$, and $\{a,b\}$, is an (a,b)-equalizer for $\mathcal{F} = \{K_{1,k}\}$. This graph satisfies Definition 2.2.(i): In any 2-coloring, at least k of the vertices in I receive the same color, say color 0. If a and b are colored differently, then one of them is colored 0. This would yield an induced monochromatic $K_{1,k}$. A good coloring as required in Definition 2.2.(ii) results from coloring a and b by the same color, and all vertices in I by the opposite color. This coloring has no monochromatic copy of $K_{1,k}$ itself, and since all neighbors of the contact points are colored with the other color than the contact points, no monochromatic copy of $K_{1,k}$ may cross the equalizer. $\square$

As we mentioned in Section 1.1, Cowen, Goddard & Jesurum [13] have shown that the defective $(2,d)$-coloring problem for any $d \geq 1$ is NP-hard (even) for planar graphs. This implies that strongly $K_{1,k}$-free 2-coloring is NP-hard for planar graphs for any $k \geq 2$, so the above proof is needed for the weak case only.

For $1 \leq k \leq m$, a *double-star* $X_{k,m}$ is a tree of the following form: $X_{k,m}$ has $k+m+2$ vertices. There are two adjacent central vertices y_1 and y_2. Vertex y_1 is adjacent to k leaves, and y_2 is adjacent to m leaves.

In other words, the double-star $X_{k,m}$ results from adding an edge between two vertices of maximum degree of $K_{1,k}$ and $K_{1,m}$. See Figure 1 for an illustration. Note that $X_{1,1}$ is isomorphic to the path P_4.

Proposition 3.2. *For every $1 \le k \le m$, it is NP-hard to decide whether a planar graph has a weakly (strongly) $X_{k,m}$-free 2-coloring.*

Proof. We apply Theorem 2.3. The first condition in this theorem is fulfilled, since $X_{k,m}$ is a graph on at least four vertices with a cut vertex. For the second condition, we construct an (a,b)-equalizer.

The (a,b)-equalizer $\mathcal{E} = (V', E')$ consists of $2m+k-1$ independent copies (V^i, E^i) of the double-star $X_{k,m}$ where $1 \le i \le 2m+k-1$. Moreover, there are five special vertices a, b, v_1, v_2, and v_3. We define

$$
\begin{aligned}
V' &= \{v_1, v_2, v_3, a, b\} \cup \bigcup_{1 \le i \le 2m+k-1} V^i \qquad \text{and} \\
E' &= \{v_1v_2, v_2v_3, v_1v_3, av_3, bv_3\} \cup \\
&\quad \bigcup_{1 \le i \le 2m+k-1} E^i \cup \\
&\quad \bigcup_{1 \le i \le m} \{v_1 v \colon v \in V^i\} \cup \\
&\quad \bigcup_{m+1 \le i \le 2m} \{v_2 v \colon v \in V^i\} \cup \\
&\quad \bigcup_{2m+1 \le i \le 2m+k-1} \{v_3 v \colon v \in V^i\}.
\end{aligned}
$$

See Figure 2 for an illustration.

We claim that every 2-coloring of $\mathcal{E}$ with a and b colored differently contains a monochromatic induced copy of $X_{k,m}$: Consider some weakly $X_{k,m}$-free 2-coloring of $\mathcal{E}$. Then each copy (V^i, E^i) of $X_{k,m}$ must have at least one vertex that is colored 0 and at least one vertex that is colored 1. If v_1 and v_2 had the same color, then together with appropriate vertices in V^i, $1 \le i \le 2m$, they would form a monochromatic copy of $X_{k,m}$. Hence, we may assume by symmetry that v_1 is colored 1, and that v_2 and v_3 are colored 0. Suppose for the sake of contradiction that a and b are colored differently. Then one of them would be colored 0, and there would be a monochromatic copy of $X_{k,m}$ with center vertices v_3 and v_2. Thus $\mathcal{E}$ satisfies property (i) in Definition 2.2.

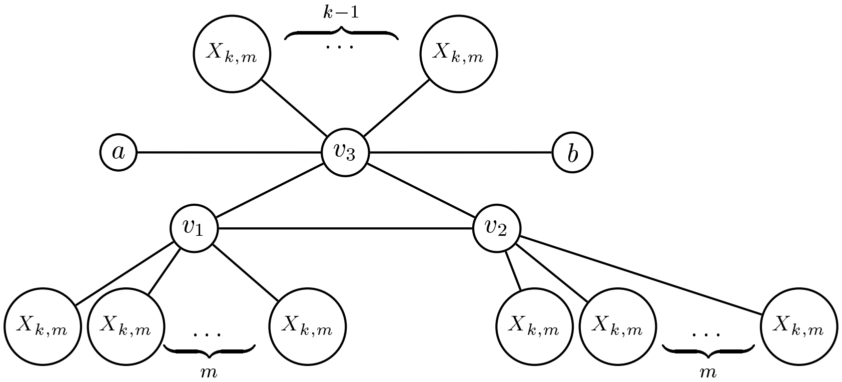

Figure 2: An equalizer for the double-star $X_{k,m}$.

To show that also property (ii) in Definition 2.2 is satisfied, we construct a good 2-coloring: The vertices a, b, v_1 are colored 0, and v_2 and v_3 are colored 1. In every set V^i with $1 \leq i \leq m$, one vertex is colored 0 and all other vertices are colored 1. In every set V^i with $m+1 \leq i \leq 2m+k-1$, one vertex is colored 1 and all other vertices are colored 0. This coloring has no monochromatic copy of $X_{k,m}$, and since vertex v_3 as the only neighbor of the contact points a, b is colored differently than a, b, no monochromatic copy of any tree may cross $\mathcal{E}$. $\square$

Now we are ready to prove the main result of this section.

Theorem 3.3. *Let T be a tree with at least two edges. Then it is NP-hard to decide whether a planar input graph G has a weakly (strongly) T-free 2-coloring.*

Proof. By induction on the number ℓ of edges in T. If T has $\ell = 2$ edges, then $T = K_{1,2}$, and NP-hardness follows by Proposition 1.1. If T has $\ell \geq 3$ edges, then we consider the so-called *shaved* tree T^* of T that results from T by removing all the leaves. If the shaved tree T^* is a single vertex, then T is a star, and NP-hardness follows by Proposition 3.1. If the shaved tree T^* is a single edge, then T is a double-star, and NP-hardness follows by Proposition 3.2.

Hence, it remains to settle the case where the shaved tree T^* contains at least two edges. In this case we know from the induction hypothesis that weakly (strongly) T^*-free 2-coloring is NP-hard. Consider an arbitrary planar input graph G^* for weakly (strongly) T^*-free 2-coloring. To complete the NP-hardness proof, we will construct in polynomial time a planar graph G that has a weakly (strongly) T-free 2-coloring if and only if G^* has a weakly (strongly) T^*-free 2-coloring: Let Δ be the maximum number of leaves pending on a vertex of T. For every vertex v in G^*, we create Δ independent copies $T_1(v), \ldots, T_\Delta(v)$ of T, and we join v to all vertices of all these copies.

Assume first that G^* is weakly (strongly) T^*-free 2-colorable. We extend this coloring to a weakly (strongly) T-free 2-coloring of G by taking a proper 2-coloring of every subgraph $T_i(v)$ in G, such that for every $v \in V_{G^*}$, exactly one vertex of each $T_i(v)$ receives the same color as v. It can be verified that this extended 2-coloring for G does not contain any monochromatic copy of T.

Now assume that G is weakly (strongly) T-free 2-colorable, and let c be such a 2-coloring. Every subgraph $T_i(v)$ in G must meet both colors. This implies that every vertex v in the subgraph G^* of G has at least Δ neighbors of color 0 and at least Δ neighbors of color 1 in the subgraphs $T_i(v)$. Any monochromatic (induced) copy of T^* in G^* would then extend to a monochromatic (induced) copy of T in G, and hence the restriction of the coloring c to the subgraph G^* is a weakly (strongly) T^*-free 2-coloring. This concludes the proof of the theorem. □

Using the same ideas as in the proofs of the previous theorem and propositions, one can prove the following more general statement about larger sets of forbidden graphs.

Theorem 3.4. *Let $\mathcal{F}$ be a finite set of graphs containing at least one tree with at least two edges. Then both weakly and strongly $\mathcal{F}$-free 2-coloring are NP-hard.*

Proof. Let $T \in \mathcal{F}$ be a tree in $\mathcal{F}$ with a minimum number of edges. If $T = P_3$, then the remaining graphs in $\mathcal{F}$ must be complete (every non-complete connected graph contains P_3 as an induced subgraph), so they may be only K_3 or K_4. The NP-hardness of $\mathcal{F}$-free 2-coloring then follows from Proposition 1.1, since for coloring triangle-free graphs, graphs containing triangles are irrelevant as forbidden subgraphs.

For T being a star (with at least three leaves) or a double-star, the result follows directly from the construction of equalizers in the proofs of Propositions 3.1 and 3.2, since the good colorings presented there are such that the neighbors of the contact points receive a different color than the contact points, and the only connected monochromatic subgraphs of the equalizers are singletons (in case of $T = K_{1,k}$) or smaller double-stars ($X_{k-1,m}$ in case of $T = X_{k,m}$). Since T itself may be used as the skeleton of the negator and the clause gadgets, the good colorings of these gadgets also do not contain a monochromatic copy of any graph of $\mathcal{F}$ (neither induced nor non-induced).

If T is such that the shaved tree T^* has at least two edges, we proceed by induction similarly as in the proof of Theorem 3.3. For any graph $F \in \mathcal{F}$, we denote by F^* the shaved copy of F, i.e., the graph obtained from F by deleting all leaves (vertices of degree one). Let Δ be the maximum number of leaves pending on a vertex of a graph from $\mathcal{F}$. Construct the graph G from a graph G^* as in the proof of Theorem 3.3 and use the fact that G has a weakly (strongly) $\mathcal{F}$-free 2-coloring if and only if G^* has a weakly (strongly) $\mathcal{F}^*$-free 2-coloring, where $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$. □

4 Cycle-free 2-colorings of planar graphs

In this section we turn to the case when the forbidden graph F is not a tree and hence contains a cycle (we assume F is connected).

If F contains an odd cycle, then the Four Color Theorem (4CT) shows that any planar graph G has a strongly F-free 2-coloring: a proper 4-coloring of G partitions V_G into two sets S_1 and S_2 each inducing a bipartite graph. Coloring all the vertices of S_i by color i yields a strongly F-free 2-coloring of G. If we extend the set of forbidden graphs to all cycles of odd length, denoted by $\mathcal{F}_{odd}$, then the converse is also true: In any $\mathcal{F}_{odd}$-free 2-coloring of G both monochromatic subgraphs of G are bipartite, yielding a 4-coloring of G. To summarize we obtain the following.

Proposition 4.1. *The statement* "$\chi^s(\mathcal{F}_{odd}, G) \leq 2$ for every planar graph G" *is equivalent to the 4CT.*

In case F is just the triangle C_3, one can avoid using the heavy 4CT machinery to prove that $\chi^s(C_3, G) \leq 2$ for every planar graph G by applying a result due to Burstein [9]. A brief sketch of the argument is as follows. Prove by induction that in any plane triangulation, any nonmonochromatic precoloring of the outer face (triangle) can be extended to a coloring which avoids monochromatic triangles.

If F contains no triangles, a result of Thomassen [28] can be applied. He proved that the vertex set of any planar graph can be partitioned into two sets each of which induces a subgraph with no cycles of length exceeding 3. Hence every planar graph is strongly $\mathcal{F}_{\geq 4}$-free 2-colorable, where $\mathcal{F}_{\geq 4}$ denotes the set of all cycles of length exceeding 3. The following theorem summarizes the above observations.

Theorem 4.2. *If the forbidden connected subgraph F is not a tree, then every planar graph is strongly and hence also weakly F-free 2-colorable.*

The picture changes if one forbids all cycles, or a combination of cycles including the triangle. A result of Stein [27] states that the vertex set of a plane triangulation G can be partitioned into two sets each inducing a forest if and only if the plane dual of G is hamiltonian. Since deciding hamiltonicity of planar cubic graphs is NP-hard (see [18]), this implies that deciding whether a (maximal) planar graph has a weakly (strongly) cycle-free 2-coloring is NP-hard. We are able to slightly strengthen this statement.

Theorem 4.3. *Let $\mathcal{F}_{345} = \{C_3, C_4, C_5\}$ be the set of cycles of lengths three, four, and five. Then the problem of deciding whether a given planar graph has a weakly (strongly) $\mathcal{F}_{345}$-free 2-coloring is NP-hard.*

We prove this theorem in a more general setting as a corollary of Theorem 4.4. But first let us note that $\{C_3, C_4, C_5\}$ is a minimal set of cycles which determines an NP-hard instance of the $\mathcal{F}$-free 2-coloring problem. Indeed, if $\mathcal{F} \subset \{C_3, C_4, C_5\}$ is a proper subset, then every planar graph is strongly $\mathcal{F}$-free 2-colorable. We have noted this already for $\mathcal{F} \subseteq \{C_3, C_5\}$ and $\mathcal{F} \subseteq \{C_4, C_5\}$, and the last case $\mathcal{F} = \{C_3, C_4\}$ is covered by the result of Kaiser & Škrekovski [23] who proved that every planar graph is strongly $\{C_3, C_4\}$-free 2-colorable.

Theorem 4.4. *Let $\mathcal{F}$ be a finite set of planar 2-connected graphs. If there exists a planar graph which is not weakly (strongly) $\mathcal{F}$-free 2-colorable, then weakly (strongly) $\mathcal{F}$-free 2-coloring is NP-hard for planar input graphs.*

Proof. Consider the graphs of $\mathcal{F}$ with some fixed plane embeddings. If every face of every graph $F \in \mathcal{F}$ is C_3 or C_4, then every planar graph is strongly $\mathcal{F}$-free 2-colorable by the main result in [23]. If not, there is an $F \in \mathcal{F}$ with a face of size at least 5 and the first assumption of Theorem 2.3 is met.

Next we show how to construct an equalizer. Let H' be a smallest (by the number of edges) planar graph which is not weakly (strongly) $\mathcal{F}$-free 2-colorable. Take an edge $xy \in E_{H'}$ and denote by H the graph obtained from H' by deleting this edge. Then H is weakly (strongly) $\mathcal{F}$-free 2-colorable, and in every weakly (strongly) $\mathcal{F}$-free 2-coloring of H the vertices x and y receive the same color. We construct an equalizer $\mathcal{E}$ by concatenating sufficiently many copies of H. More formally, choose a number k to be larger than the order of any graph in $\mathcal{F}$. The copies of H will be $H_i = (V_i, E_i)$ with $V_i = \{v_i : v \in V_H\}$ and $E_i = \{u_i v_i : uv \in E_H\}$, for $i = 1, 2, \ldots, k$. For $i = 1, 2, \ldots, k-1$, we identify y_i with x_{i+1}, and we set $a = x_1$ and $b = y_k$ to be the contact points.

Clearly $\mathcal{E}$ is planar and in every weakly (strongly) $\mathcal{F}$-free 2-coloring of $\mathcal{E}$ the vertices x_i, y_i for $i = 1, 2, \ldots, k$, and hence also a and b, receive the same color.

Let c be a weakly (strongly) $\mathcal{F}$-free 2-coloring of H. Color $\mathcal{E}$ using c on every H_i. Consider a graph $F \in \mathcal{F}$. No copy of F which lies entirely in $\mathcal{E}$ is monochromatic, since the 2-connectedness of F implies that such a copy of F lies entirely within one of the H_i's. Therefore this 2-coloring of $\mathcal{E}$ is weakly (strongly) F-free. It also follows from the 2-connectedness of F that every copy of F which crosses $\mathcal{E}$ contains a path from a to b through $\mathcal{E}$. But every such path has more vertices than F. Hence the 2-coloring of $\mathcal{E}$ is good. $\square$

To conclude the proof of Theorem 4.3, it would suffice to construct a planar graph which is not weakly $\mathcal{F}_{345}$-free 2-colorable. It is, however, equally simple to describe an equalizer for $\mathcal{F}_{345}$.

Let $\Theta(x, y)$ be the graph depicted in Figure 3. This graph has the following important property: In any weakly $\mathcal{F}_{345}$-free 2-coloring of $\Theta(x, y)$, the vertices x and y have different colors (we leave the simple proof of this fact to the reader). The (a, b)-equalizer is constructed from a graph $\Theta(a, x)$ and a graph $\Theta(b, y)$ by identifying the two vertices x and y. A good 2-coloring of the (a, b)-equalizer is induced by the 2-coloring indicated in Figure 3.

The following statement is now a direct corollary of Theorem 4.4 and Theorem 4.3.

Corollary 4.5. *For any finite set $\mathcal{F}_{\geq 345} \supseteq \{C_3, C_4, C_5\}$ of cycles, both weakly and strongly $\mathcal{F}_{\geq 345}$-free 2-coloring are NP-hard for planar input graphs.*

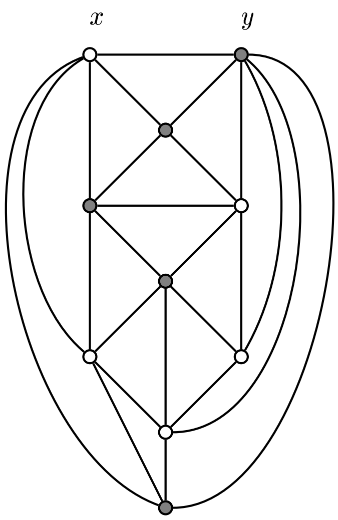

Figure 3: A gadget for the forbidden set $\mathcal{F}_{345}$ in Theorem 4.3.

5 3-colorings of planar graphs

A *linear forest* is a disjoint union of paths (some of which may be trivial). The following result was proved independently in [20] and [25].

Proposition 5.1. *[Goddard [20] and Poh [25]]*
Every planar graph has a partition of its vertex set into three subsets such that every subset induces a linear forest.

This result immediately implies that if a connected graph F is not a path, then $\chi^w(F, G) \leq 3$ and $\chi^s(F, G) \leq 3$ hold for *all* planar graphs G. Hence, these coloring problems are trivially solvable in polynomial time.

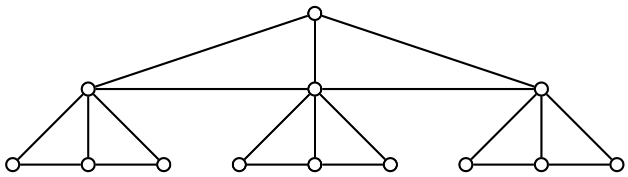

Figure 4: Example for the graph Y_3.

We now turn to the remaining cases of F-free 3-coloring for planar graphs where the forbidden graph F is a path. We start with a technical lemma that will yield a gadget for the NP-hardness argument.

Lemma 5.2. *For every $k \geq 2$, there exists an outerplanar graph Y_k that satisfies the following properties.*

(i) *Y_k is not weakly P_k-free 2-colorable.*

(ii) *There exists a strongly P_k-free 3-coloring of Y_k, in which one of the colors is only used on an independent set of vertices.*

Proof. The skeleton of the graph Y_k is formed by a regular tree, in which every inner vertex has exactly k children, and all paths from the root to a leaf have exactly k vertices. Additionally to the edges in this regular tree, the children of every inner vertex are connected by a path. See Figure 4 for an illustration.

Suppose there was a P_k-free 2-coloring of Y_k. Since the children of any vertex induce a path P_k, both colors must show up at the children. Consequently, for every inner vertex v at least one of its children must get the same color as v. But this yields a monochromatic induced P_k running from the root to some leaf. This contradiction proves property (i). Property (ii) is straightforward to prove: The graph Y_k is outerplanar, and hence properly 3-colorable. $\square$

Theorem 5.3. *For any path P_k with $k \geq 2$, it is NP-hard to decide whether a planar input graph has a weakly (strongly) P_k-free 3-coloring.*

Proof. We will use induction on k. The basic cases are $k = 2$ and $k = 3$. For $k = 2$, weakly and strongly P_2-free 3-coloring is equivalent to proper 3-coloring which is well-known to be NP-hard for planar graphs.

Next, consider the case $k = 3$. Proposition 1.2.(ii) yields NP-hardness of strongly P_3-free 3-coloring for planar graphs. For weakly P_3-free 3-coloring, we sketch a reduction from proper 3-coloring of planar graphs. As a gadget, we use the outerplanar graph Z depicted in Figure 5. The crucial property of Z is that it does not allow a weakly P_3-free 2-coloring, as is easily checked. Now consider an arbitrary planar graph G. From G we construct the planar graph G': For every vertex v in G, create a copy $Z(v)$ of Z, and add all possible edges between v and $Z(v)$. Since $Z(v)$ is outerplanar, the new graph G' is planar. It can be verified that $\chi(G) \leq 3$ if and only if $\chi^W(P_3, G') \leq 3$.

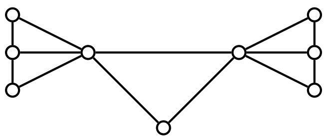

Figure 5: The graph Z in the proof of Theorem 5.3.

For $k \geq 4$, we will give a reduction from weakly (strongly) P_{k-2}-free 3-coloring to weakly (strongly) P_k-free 3-coloring. Consider an arbitrary planar graph G, and construct the following planar graph G': For every vertex v in G, create a copy $Y_k(v)$ of the graph Y_k from Lemma 5.2, and add all possible edges between v and $Y_k(v)$. Since Y_k is outerplanar, the new graph G' is planar. If G has a weakly (strongly) P_{k-2}-free 3-coloring, then this can be extended to a weakly (strongly) P_k-free 3-coloring of G' by coloring the subgraphs $Y_k(v)$ according to Lemma 5.2.(ii). And if G' has a weakly (strongly) P_k-free 3-coloring, then by Lemma 5.2.(i) this induces a weakly (strongly) P_{k-2}-free 3-coloring for G. $\square$

6 Concluding remarks and open problems

6.1 Triangle-free graphs. By modifying the gadgets for the equalizers in such a way that the planar graph G_Φ constructed in the proof of Theorem 2.3 becomes triangle-free, one might be able to prove complexity results for weakly (strongly) F-free 2-coloring restricted to triangle-free planar graphs. In fact, it is not difficult to apply this method to prove that for $F = K_{1,k}$ with $k \geq 2$, weakly (strongly) F-free 2-coloring remains NP-hard for triangle-free planar graphs.

Problem. Is the following true? For every connected graph F of diameter at least 3 and for every triangle-free planar graph G, there is a weakly F-free 2-coloring of G.

6.2 Monotonicity. All our NP-hardness techniques are such that hardness proofs for $\mathcal{F}$-free 2-colorability extend naturally to NP-hardness of $\mathcal{F}'$-free 2-colorability for any finite $\mathcal{F}' \supseteq \mathcal{F}$. This raises the following question.

Problem. For finite sets of graphs $\mathcal{F}' \supseteq \mathcal{F}$, is it true that $\mathcal{F}' - F - 2 - CPG \propto \mathcal{F} - F - 2 - CPG$? ($\mathcal{F} - F - 2 - CPG$ standing for $\mathcal{F}$-Free-2-Coloring-Planar-Graphs)

Note that this is not necessarily true for infinite sets of forbidden graphs. The infinite set $\mathcal{F}_{cycle}$ of all cycles has uncountably many subsets, and if each of these defines a different problem, infinitely many of them will have to be undecidable, whereas deciding the existence of an $\mathcal{F}_{cycle}$-free 2-coloring is surely in NP.

6.3 Forbidden sets of cycles. It would be interesting to characterize for which particular (finite) sets of forbidden cycles the $\mathcal{F}$-free 2-coloring problem on planar graphs is feasible and for which it is hard. In particular, for two cycles this question remains open if one of them is the triangle and the other one is an even cycle of length greater than 4.

Problem. For which $k > 2$ does there exist a planar graph which is not $\{C_3, C_{2k}\}$-free 2-colorable?

6.4 Equalizers. Despite our inductive proof of NP-hardness for forbidden trees, it would be interesting to know whether one can use the equalizer gadget machinery directly.

Problem. Does there exist an equalizer for any tree T?

Acknowledgments

We are grateful to Oleg Borodin, Alesha Glebov, Sasha Kostochka, and Carsten Thomassen for fruitful discussions on the topic of this paper.

References

[1] D. Achlioptas, *The complexity of G-free colorability*, Discrete Math., 165/166 (1997), pp. 21–30. Graphs and combinatorics (Marseille, 1995).

[2] M. O. Albertson, R. E. Jamison, S. T. Hedetniemi, and S. C. Locke, *The subchromatic number of a graph*, Discrete Math., 74 (1989), pp. 33–49.

[3] D. Archdeacon, *A note on defective colorings of graphs in surfaces*, J. Graph Theory, 11 (1987), pp. 517–519.

[4] I. Broere and C. M. Mynhardt, *Generalized colorings of outer-planar and planar graphs*, in Graph theory with applications to algorithms and computer science (Kalamazoo, Mich., 1984), Wiley, New York, 1985, pp. 151–161.

[5] H. J. Broersma, F. V. Fomin, J. Nešetřil, and G. J. Woeginger, *More about subcolorings*, Computing, 69 (2002), pp. 187–203.

[6] J. I. Brown and D. G. Corneil, *On generalized graph colorings*, J. Graph Theory, 11 (1987), pp. 87–99.

[7] J. I. Brown and D. G. Corneil, *Graph properties and hypergraph colorings*, Discrete Math., 98 (1991), pp. 81–93.

[8] J. I. Brown and D. G. Corneil, *On uniquely $-G$ k-colorable graphs*, Quaestiones Math., 15 (1992), pp. 477–487.

[9] M. I. Burstein, *The bi-colorability of planar hypergraphs*, Sakharth. SSR Mecn. Akad. Moambe, 78 (1975), pp. 293–296.

[10] G. Chartrand, D. P. Geller, and S. Hedetniemi, *A generalization of the chromatic number*, Proc. Cambridge Philos. Soc., 64 (1968), pp. 265–271.

[11] G. Chartrand and H. V. Kronk, *The point-arboricity of planar graphs*, J. London Math. Soc., 44 (1969), pp. 612–616.

[12] G. Chartrand, H. V. Kronk, and C. E. Wall, *The point-arboricity of a graph*, Israel J. Math., 6 (1968), pp. 169–175.

[13] L. J. Cowen, W. Goddard, and C. E. Jesurum, *Defective coloring revisited*, J. Graph Theory, 24 (1997), pp. 205–219.

[14] L. J. Cowen, R. H. Cowen, and D. R. Woodall, *Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency*, J. Graph Theory, 10 (1986), pp. 187–195.

[15] M. R. Fellows, J. Kratochvíl, M. Middendorf, F. Pfeiffer, *The complexity of induced minors and related problems*, Algorithmica 13 (1995), 266-282

[16] J. Fiala, K. Jansen, V. B. Le, and E. Seidel, *Graph subcoloring: Complexity and algorithms*, in Graph-theoretic concepts in computer science, WG 2001, Springer, Berlin, 2001, pp. 154–165.

[17] M. Frick and M. A. Henning, *Extremal results on defective colorings of graphs*, Discrete Math., 126 (1994), pp. 151–158.

[18] M. R. Garey, D. S. Johnson, and R. E. Tarjan, *The planar Hamiltonian problem is NP-complete*, Siam J. Comput., 5 (1976), pp. 704–714.

[19] J. Gimbel and J. Nešetřil, *Partitions of graphs into cographs*, Technical Report 2000-470, KAM-DIMATIA, Charles University, Czech Republic, 2000.

[20] W. Goddard, *Acyclic colorings of planar graphs*, Discrete Math., 91 (1991), pp. 91–94.

[21] F. Harary, *Conditional colorability in graphs*, in Graphs and applications (Boulder, Colorado, 1982), Wiley, New York, 1985, pp. 127–136.

[22] C. T. HOÀNG AND V. B. LE, *P_4-free colorings and P_4-bipartite graphs*, Discrete Math. Theor. Comput. Sci., 4 (2001), pp. 109–122 (electronic).

[23] T. KAISER, R. ŠKREKOVSKI, *Planar graph colorings without short monochromatic cycles*, University of Ljubljana, Department of Mathematics; PP 817; submitted to J. Graph Theory.

[24] H.-O. LE AND V. B. LE, *The NP-completeness of $(1, r)$-subcoloring of cubic graphs*, Information Proc. Letters, 81 (2002), pp. 157–162.

[25] K. S. POH, *On the linear vertex-arboricity of a planar graph*, J. Graph Theory, 14 (1990), pp. 73–75.

[26] H. SACHS, *Finite graphs (Investigations and generalizations concerning the construction of finite graphs having given chromatic number and no triangles).*, in Recent Progress in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968), Academic Press, New York, 1969, pp. 175–184.

[27] S. K. STEIN, *B-sets and planar graphs*, Pac. J. Math., 37 (1971), pp. 217–224.

[28] C. THOMASSEN, *Decomposing a planar graph into degenerate graphs*, J. Combin. Theory Ser. B 65 (1995), pp. 305–314.