ON WEAK LATTICE AND FRAME HOMOMORPHISMS

BERNHARD BANASCHEWSKI AND ALEŠ PULTR

Dedicated to the memory of Günter Bruns

Abstract. In the context of distributive lattices, frames, or σ-frames, a join homomorphism preserving the unit and those binary meets which are zero often preserves all binary meets. This paper analyzes this phenomenon.

In many situations concerning frames it happens that, although the join-compltness is used in full, the finitary meet only comes into play in the form of the top element and the relation $a \wedge b = 0$. As a result, it seems natural to ask how much the standard homomorphisms differ from the maps between frames which preserve just this part of the structure, that is, the top-preserving complete join homomorphisms $f : L \to M$ for which $f(a) \wedge f(b) = 0$ whenever $a \wedge b = 0$. This paper presents a study of this and related questions.

It turns out that the latter maps $L \to M$ are already homomorphisms for a fairly extensive class of frames, namely the L introduced by Isbell [6] as the strongly Hausdorff frames. By way of contrast, any frame free on more than one generator does not have this property. On the other hand, in the more general context of (bounded) distributive lattices, we have a complete characterization: for any such L, the corresponding weak homomorphisms are actually homomorphisms iff L is relatively normal.

In addition to these results we derive a number of others, involving the downset frames of partially ordered sets and the ideal frames of distributive lattices, as well as σ-frames. Further, we consider the weak spectrum of frames arising in this setting.

Thanks go to the National Sciences and Engineering Research Cuncil of Canada for continuing support of the first author by means of a research grant, and to the Ministry of Education of the Czech Republic for the support of the second author by the project LN 00A056.

In the case of lattices, it turns out that, for any given L, the weak homomorphisms $L \to M$ are homomorphisms iff this holds for $M = \mathbf{2}$, the two-element lattice; on the other hand, there are good reasons for expecting this not to be the case for frames but we have not been able to settle this question. Further, in somewhat different direction, our results concerning lattices and their ideal frames make use of the Prime Ideal Theorem, and we do not know whether these still hold without the use of any choice principles.

1. Preliminaries

1.1. We will be concerned with three types of structures.

- *Distributive lattices* (always understood as bounded). In this context, a *homomorphism* is a lattice homomorphism preserving the zero 0 and the unit 1, and the corresponding category will be denoted by

$$\mathbf{DLat}.$$

- *Frames*, that is, complete lattices L satisfying the distribution law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

  for any $a \in L$ and $S \subseteq L$. In this case, a *homomorphism* is a frame homomorphism, that is a mapping preserving all joins and all finitary meets. The resulting category will be denoted by

$$\mathbf{Frm}.$$

- *σ-frames*, that is, lattices with countable suprema satisfying the distribution law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

  for any $a \in L$ and countable $S \subseteq L$.

  In this case, a *homomorphism* preserves all (at most) countable joins and finite meets. The resulting category will be denoted by

$$\sigma\mathbf{Frm}.$$

The two-element Boolean algebra $\{0, 1\}$, appearing in all the three contexts will be denoted by

$$\mathbf{2}.$$

For general information concerning frames the reader may consult Johnstone [7] or Vickers [13].

1.2. In any of the three contexts we will speak of *weak homomorphisms* $f : L \to M$, replacing the requirement of preserving finitary meets by the conditions

(W1) $f(1) = 1$,
(W2) $a \wedge b = 0 \quad \Rightarrow \quad f(a) \wedge f(b) = 0$.

In the context of frames, the second condition can be obviously reformulated as

$$f(a^*) \le f(a)^* \text{ for every } a \in L,$$

where a^* is the pseudocomplement $\bigvee\{x \mid x \wedge a = 0\}$ of a.

The resulting categories of frames and σ-frames will be denoted by

$$\mathbf{WFrm} \quad \text{and} \quad \mathbf{W}\sigma\mathbf{Frm},$$

respectively.

Note. The category **Frm** is the Eilenberg-Moore category of the down-set monad in the category **SLat** of bounded meet-semilattices with the corresponding homomorphisms. Analogously, we can obtain **WFrm** on the basis of **WSLat**, the category of bounded meet-semilattices with the morphisms preserving 0 and 1, and otherwise subject only to the condition that $f(x) \wedge f(y) = 0$ whenever $x \wedge y = 0$. This category, however, does not seem to be of much interest and hence we have not pursued this line any further.

1.3. A distributive lattice (frame, σ-frame) is said to have the property (W) if every weak homomorphism $f : L \to M$ is a homomorphism; similarly, it is said to have the property (W_2) if every weak homomorphism $f : L \to M$ is a homomorphism.

As an easy example we note that any Boolean algebra, Boolean σ-algebra, or complete Boolean algebra L has property (W) in the category **DLat**, σ**Frm**, or **Frm**, respectively: any weak homomorphism $f : L \to M$ preserves complements whenever they exist, and the De Morgan law then ensures that it also preserves binary meets in the present case. In the same vein, but even more trivially, the totally ordered objects in any of these categories have property (W).

Note. For a distributive lattice L, a weak homomorphism $f : L \to \mathbf{2}$ can be viewed as a generalization of a prime filter F in L since it corresponds to a subset F of L such that

if $a \in F$ and $a \leq b$ then $b \in F$
if $a \vee b \in F$ then $a \in F$ or $b \in F$, and $0 \notin F$
if $a, b \in F$ then $a \wedge b \neq 0$.

Similarly, if L is a σ-frame or a frame, the second condition becomes

if $\bigvee S \in F$ then $S \cap F \neq \emptyset$

for any countable or arbitrary $S \subseteq L$, generalizing the notion of σ-prime or completely prime filter, respectively.

In particular, in the case of frames, there is an analogy of the properties (W_2) and (W) with Cauchy completeness and completeness of a uniform frame. There, too, one has a generalization of completely prime filters, namely the regular Cauchy filters (which satisfy the standard filter condition while the join-property is relaxed) and correspondingly the regular Cauchy maps $L \to M$ extending the notion of regular Cauchy filter in the same way the weak homomorphisms $L \to M$ extend the notion of the above $F \subseteq L$. Thus, (W_2) is analogous to Cauchy completeness, which requires that every regular Cauchy filter be a completely prime filter, and (W) is analogous to completeness. We refer to [2] for the details.

1.4. We will say that elements a, b of a distributive lattice L are *normally separated* if there are u, v such that

(1.4.1) $$u \wedge v = 0, \quad a \leq u \vee b \quad \text{and} \quad b \leq a \vee v.$$

Note this is the same as requiring that there exist disjoint u, v such that $u \vee b = a \vee b = a \vee v$.

A lattice L is called *relatively normal* if any two $a, b \in L$ are normally separated. By the standard definition of normality, L is *normal* if any two $a, b \in L$ such that $a \vee b = 1$ are normally separated. Thus,

L is relatively normal iff $\downarrow c$ is normal for each $c \in L$.

In particular, then, a frame is relatively normal if each of its open quotients is normal. Similarly, we call a frame *hereditarily normal* if *all* its homomorphic images are normal and use the same terminology for lattices.

1.5. Following Isbell [6], a distributive lattice is called *subfit* if

(Subfit) $$a \not\leq b \quad \Rightarrow \quad \exists y \;\; b \vee y \neq 1 = a \vee y,$$

and *fit* if

(Fit) $\quad a \not\leq b \quad \Rightarrow \quad \exists x \exists y,\ a \vee y = 1,\ x \wedge y \leq b$ and $x \not\leq b.$

Further, it is *regular* if

(Reg) $\quad a \not\leq b \quad \Rightarrow \quad \exists x \exists y,\ a \vee y = 1,\ \ x \wedge y = 0$ and $x \not\leq b.$

Note. The condition of regularity is obviously equivalent to saying that, for each $a \in L$,

$$a = \bigvee\{x \mid \exists y \ \ x \wedge y = 0,\ a \vee y = 1\},$$

which is also expressed as

$$a = \bigvee\{x \mid x \prec a\}$$

where $x \prec a$ (x is *rather below*, or *well inside* a) means that there exists y such that $x \wedge y = 0$ and $a \vee y = 1$.

This is also how the regularity is defined for frames in which case the latter condition is equivalent to $x^* \vee a = 1$.

In the context of σ-frames, however, regularity means there is a *countable* $S \subseteq \{x \mid x \prec a\}$ such that $a = \bigvee S$, and consequently

> *a σ-frame can be regular as a lattice without being regular as a σ-frame.*

2. Three easy facts

The following hold for any of the three categories considered here.

2.1. Proposition. *Any homomorphic image of L with the property* (W) *or* (W$_2$) *has the same property.*

Proof. If $h : L \to M$ is any onto homomorphism and $f : M \to K$ a weak homomorphism then $g = fh$ is a weak homomorphism and hence a homomorphism. Since h is a homomorphism onto, f is a homomorphism. $\square$

2.2. Proposition. *A weak homomorphism preserves the meets of all normally separated couples.*

Proof. Choose the u, v as in (1.4.1). Then

$$a \leq u \vee (a \wedge b) \quad \text{and} \quad b \leq v \vee (a \wedge b).$$

Thus,

$$f(a) \wedge f(b) \leq (f(u) \wedge f(v)) \vee f(a \wedge b) = f(a \wedge b)$$

since $f(u) \wedge f(v)$ has to be 0. $\square$

2.3. Proposition. *For any weak homomorphism $f : L \to M$, if $x \prec a$ and $y \prec b$ then $f(x) \wedge f(y) \leq f(a \wedge b)$.*

Proof. Take $\overline{x}$ and $\overline{y}$ such that

$$x \wedge \overline{x} = 0 = y \wedge \overline{y}$$

and

$$a \vee \overline{x} = 1 = b \vee \overline{y}.$$

Then we have $\overline{x} \vee \overline{y} \vee (a \wedge b) = 1$ so that $f(\overline{x}) \vee f(\overline{y}) \vee f(a \wedge b) = 1$ by (W1), and hence

$$f(x) \wedge f(y) = (f(x) \wedge f(y)) \wedge (f(\overline{x}) \vee f(\overline{y}) \vee f(a \wedge b)) \leq f(a \wedge b),$$

the latter by (W2). $\square$

3. Distributive lattices and σ-frames.

3.1. Proposition. *For a distributive lattice L, the following statements are equivalent.*

(1) *L is hereditarily normal.*
(2) *L is relatively normal.*
(3) *L has property* (W).
(4) *L has property* (W_2).

Proof. (1)⇒(2) is trivial, (2)⇒(3) by 2.2, and (3)⇒(4) is trivial.

(4)⇒(1): By 2.1 it will be enough to show that any L with the property (W_2) is normal. We show this by using the familiar criterion (Johnstone [7]) that $L \in \mathbf{DLat}$ is normal iff each prime ideal is contained in a unique maximal ideal.

Suppose then that L has a prime ideal P with distinct maximal ideals $P_1, P_2 \supseteq P$. We let $\xi_i : L \to \mathbf{2}$ be the homomorphisms such that $\xi_i(a) = 0$ iff $a \in P_i$, and put $h(a) = \xi_1(a) \vee \xi_2(a)$. Simple calculation then shows that h is a $(0, \vee, 1)$-homomorphism. Further

$$\begin{aligned} h(a) \wedge h(b) &= (\xi_1(a) \vee \xi_2(a)) \wedge (\xi_1(b) \vee \xi_2(b)) = \\ &= h(a \wedge b)(\xi_1(a) \wedge \xi_2(b)) \vee (\xi_2(a) \vee \xi_1(b)) \leq \\ &\leq h(a \wedge b) \vee \xi(a \wedge b) \end{aligned}$$

for the homomorphism $\xi : L \to \mathbf{2}$ corresponding to P, and hence $h(a) \wedge h(b)$ whenever $a \wedge b = 0$.

On the other hand, since P_1 and P_2 are incomparable we have $a_1 \in P_2 \setminus P_1$ and $a_2 \in P_1 \setminus P_2$, and then

$$h(a_1) \wedge h(a_2) = (\xi_1(a_1) \vee \xi_2(a_1))) \wedge (\xi_1(a_2) \vee \xi_2(a_2)) = 1 \wedge 1 = 1$$

while $h(a_1 \wedge a_2) = 0$ since $a_1 \wedge a_2 \in P_1 \cap P_2$. Hence h is a weak homomorphism but not a homomorphism, contradicting (W_2). $\square$

3.2. Corolllary. *Any distributive lattice which is free on more than one generator does not have the property* (W).

Proof. It is obviously sufficient to exhibit a two-generated distributive lattice which is not normal, and the four-element Boolean algebra with a new zero added is clearly of the kind. $\square$

As an immediate consequence of this we note that a coproduct $L \oplus M$ of distributive lattices with property (W) need not have that property: for $L = M$ the three element chain, $L \oplus M$ is free on two generators. Further, the example proves the same for σ-frames and frames.

3.3. A somewhat shorter version of the proof of the above implication (4)⇒(1) can be obtained using Priestley duality. Recall that this is the duality between **DLat** and the category of Priestley spaces (ordered compact spaces X such that whenever $x \neq y$ there is a clopen downset $U \subseteq X$ such that $x \notin U \ni y$) and monotone continuous maps (see Priestley [10], [11]). We will need that in this duality

- a Priestley space corresponds to the lattice of its clopen downsets, and
- the relatively normal lattices are characterized by the fact that the order of the corresponding space is a forest, that is, no two incomparable elements have a common lower bound (this has long been known for spaces, see, e.g., [9]; for the general case see, e.g., [12], [1]).

The proof of (4)⇒(1) can now proceed as follows.

Let X be the Priestley dual of L so that we can view L as the lattice of clopen downsets in X, and suppose there are incomparable $x_1, x_2 \in X$ and an $x < x_1, x_2$. Then $h : L \to \mathbf{2}$ such that

$$h(U) = 1 \quad \text{iff} \quad U \cap \{x_1, x_2\} \neq \emptyset.$$

is a weak homomorphism: trivially $h(X) = 1$ and $h(\emptyset) = 0$, and $h(U \cup V) = h(U) \vee h(V)$ since finite joins coincide with unions; further if $U \cap V = \emptyset$ then we necessarily have $U \cap \{x_1, x_2\} = \emptyset$ or $V \cap \{x_1, x_2\} = \emptyset$, and hence

$h(U) \wedge h(V) = 0$. On the other hand, given x_1, x_2 are incomparable, we have clopen downsets $U_i \ni x_i$ such that $x_1 \notin U_2$ and $x_2 \notin U_1$. Then $h(U_1) = h(U_2) = 1$ and $h(U_1 \cap U_2) = 0$, again contradicting (W_2). $\square$

3.4. Proposition. *Each regular σ-frame has the property* (W).
Proof. By regularity and 2.3 we have

$$f(a) \wedge f(b) = \bigvee\{f(x \mid x \in C\} \wedge \bigvee\{f(y) \mid y \in D\} =$$
$$= \bigvee\{f(x) \wedge f(y) \mid (x, y) \in C \times D\} \leq f(a \wedge b)$$

for some countable $C \subseteq \{x \mid x \prec a\}$ and $D \subseteq \{y \mid y \prec b\}$. $\square$

3.5. Note. Since every regular σ-frame is normal (Gilmour [4]) this proposition is in fact an immediate consequence of the implication (1)⇒(3) in 3.1 which clearly holds for σ-frames as well. On the other hand, the above proof shows how easily it follows directly from 2.3; moreover, the same argument applies to frames, where normality is not implied by regularity.

4. Frames

4.1. Using 2.2 again as in 3.1 we immediately obtain

Proposition. *For a frame L consider the following statements.*
(1) *L is relatively normal.*
(2) *L has property* (W).
(3) *L has property* (W_2).
Then (1)⇒(2)⇒(3).

This time, however, the statements are not equivalent. Using the proof of 3.4, now with $C = \{x \mid x \prec a\}$ and $D = \{y \mid y \prec b\}$, we obtain

4.1.1. Fact. *Each regular frame has the property* (W).

Further, for frames regularity does not imply normality (since there are Tychonoff spaces which are not normal) and hence in the above proposition

(2) does not imply (1).

4.2. Fact 4.1.1 is actually a consequence of a stronger result. We shall show that (W) is already implied by the strong Hausdorff property (in the sense of Isbell [6]) which is well known to be weaker than regularity ([6]).

Recall that a frame L is said to be *strongly Hausdorff* if the codiagonal $\nabla : L \oplus L \to L$ is a closed homomorphism. This can be expressed by requiring for $s_L = \bigvee\{x \oplus y \mid x \wedge y = 0\}$ that $(a \oplus 1) \vee s_L = (1 \oplus a) \vee s_L$ which is easily seen to be equivalent to

$$\text{(H)} \qquad \forall a, b \in L, \quad (a \oplus b) \vee s_L = ((a \wedge b) \oplus 1) \vee s_L.$$

4.2.1. Proposition. *Each strongly Hausdorff frame has the property* (W).

Proof. Let $f : L \to M$ be a weak homomorphism with L strongly Hausdorff. By the familiar property of the frame coproduct (as tensor product for complete join homomorphisms – see [8]) we obtain a join-preserving $f \oplus f : L \oplus L \to M \oplus M$ such that $(f \oplus f)(a \oplus b) = f(a) \oplus f(b)$. Since $x \wedge y = 0$ implies that $f(x) \wedge f(y) = 0$ we have

$$(*) \qquad (f \oplus f)(s_L) \subseteq s_M.$$

Now, by (H) we have $(f \oplus f)(((a \wedge b) \oplus 1) \vee s_L) = (f \oplus f)((a \oplus b) \vee s_L)$ and hence by $(*)$ and the formula for $f \oplus f$,

$$(f(a \wedge b) \oplus 1) \vee s_M = (f(a) \oplus f(b)) \vee s_M.$$

Applying the codiagonal ∇_M of M to this equality we obtain that $f(a \wedge b) = f(a) \wedge f(b)$. $\square$

4.2.2. For certain frames, the converse of 4.2.1 also holds. Recall from [7] (p.84) that a frame L is called a *T_U-frame* if, for any pair of frame homomorphisms $f, g : L \to M$, $f \leq g$ implies $f = g$. Then we have

Proposition. *A T_U-frame L has property (W) iff it is strongly Hausdorff.*

Proof. For any $f, g : L \to M$ such that $f(x) \wedge g(y) = 0$ whenever $x \wedge y = 0$, $h : L \to M$ given by $h(x) = f(x) \vee g(x)$ is a weak homomorphism, by easy calculation. Hence if L is a T_U-frame with property (W) then $f = h = g$ since $f, g \leq h$, and applying this to

$$f(x) = (x \oplus 1) \vee s_L \quad \text{and} \quad g(x) = (1 \oplus x) \vee s_L,$$

for s_L as in 4.2, this shows L is strongly Hausdorff. $\square$

4.2.3. Remark. It may be worth noting that the earlier example of the totally ordered frames as frames with property (W) is as disjoint from the strongly Hausdorff frames as possible: a simple argument shows that a totally ordered frame L is of the latter kind iff $L \cong \mathbf{2}$.

4.3. Since subfitness combined with normality implies regularity (Banaschewski [2], also Banaschewski-Pultr [3]), and since normality alone does not imply (W) or (W_2) (add a new top to a general frame), the question naturally arises whether the subfitness may not suffice alone. However, this is not the case, as the example of the cofinite topology $\mathrm{Cof}(X)$ on an infinite set X shows. Here, $U \cap V = \emptyset$ iff $U = \emptyset$ or $V = \emptyset$, and hence any join-preserving $f : \mathrm{Cof}(L) \to M$ such that $f(X) = 1$ is a weak homomorphism. Consider, then, $f : \mathrm{Cof}(X) \to \mathbf{2}$ such that $f(V) = 0$ iff $V \subseteq U$ for some fixed $U \in \mathrm{Cof}(X)$ which misses at least two elements of X. This is clearly of this kind but not a homomorphism since there exist $V, W \nsubseteq U$ in $\mathrm{Cof}(X)$ such that $V \cap W \subseteq U$.

Of course, subfitness is a very weak property so that this is no surprise. Comparing (Reg) and (Fit) in 1.5 we see that fitness seems to be much closer to regularity, and since we have seen that the strong Hausdorff condition, also slightly weaker than regularity, does imply (W), we naturally ask whether this may imply (W) or (W_2). The answer is negative, even for a condition somewhat stronger and still closer to regularity.

One of the characterizations of fitness involves the following sequence of conditions.

For an element a of a frame L and an ordinal α set

$$a_0 = 0 \quad a_{\alpha+1} = \bigvee\{u \mid \exists x,\ x \vee a = 1 \ \&\ u \wedge x \leq a_\alpha\}$$
$$a_\alpha = \bigvee\{a_\beta \mid \beta < \alpha\} \text{ for limit } \alpha.$$

A frame is said to have the property (SRα) if

(SRα) $\qquad$ for all $a \in L, \quad a_\alpha = a.$

Note that

(SR1) *coincides with regularity.*

Since we will be interested in particular in (SR2), here is a more explicit description: it requires that

(SR2) $\qquad a = \bigvee\{u \mid \exists x,\ x \vee a = 1 \ \&\ u \wedge x \leq \bigvee\{v \mid v \prec a\}\}$

for all $a \in L$.

In [5] it has been proved that

- if $\alpha < \beta$ then (SRα) is strictly stronger than (SRβ),
- a frame L is fit iff for every $a \in L$ there is an α such that $a_\alpha = a$.

Proposition. (SR2) *already fails to imply* (W_2).

Proof. Take the disjoint union $X = \omega \cup \{a, b\}$ with distinct a and b, and let θ be the cofinite topology on ω. On the set X define a topology τ by setting

$$U \in \tau \quad \text{iff} \quad U \cap \{a, b\} \neq \emptyset \;\Rightarrow\; U \cap \omega \in \theta.$$

In particular, for each $x \in \omega$, $\{x\}$ and $X \setminus \{x\}$ are in τ and hence, for U_1 as in the definition of (SRα),

$$\text{for any } U \in \tau, \quad U_1 \supseteq U \cap \omega$$

and considering the $W = (\{a, b\} \setminus U) \cup \omega$ for which $X \cap U \subseteq U \cap \omega \subseteq U_1$ and $W \cup U = X$, we see that $U_2 = U$. Thus, τ satisfies (SR2).

Now $f : \tau \to \mathbf{2}$ defined by $f(U) = 1$ iff $U \cap \{a, b\} \neq \emptyset$ obviously preserves the bottom, the top and the joins; further, if $U \cap V = \emptyset$ then either $U \cap \{a, b\} = \emptyset$ or $V \cap \{a, b\} = \emptyset$, showing f is a weak homomorphism. On the other hand, for $U = \omega \cup \{a\}$ and $V = \omega \cup \{b\}$ we have $f(U \cap V) = 0 \neq f(U) \wedge f(V)$. $\square$

4.5. The weak spectrum. For a frame L, let

$$\Lambda L = \{\alpha \mid \alpha : L \to \mathbf{2} \text{ weak homomorphism}\}$$

and $\Lambda_a = \{\alpha \in \Lambda L \mid \alpha(a) = 1\}$ for any $a \in L$. Then obviously

(4.5.1) $$\Lambda_0 = \emptyset, \quad \Lambda_1 = \Lambda L, \quad \Lambda_{\bigvee a_i} = \bigcup \Lambda_{a_i}$$

$$\text{and if } \; a \wedge b = 0 \; \text{ then } \; \Lambda_a \cap \Lambda_b = \emptyset.$$

¿From now on,

ΛL will be equipped with the topology generated by the Λ_a, $a \in L$.

For a weak homomorphism $h : L \to M$ define $\Lambda h : \Lambda M \to \Lambda L$ by setting $\Lambda h(\alpha) = \alpha h$. Since

(16.2) $$(\Lambda h)^{-1}(\Lambda_a) = \{\alpha \mid \alpha h \in \Lambda_a\} = \{\alpha \mid \alpha(h(a)) = 1\} = \Lambda_{h(a)},$$

the mapping Λh is continuous.

Thus, we have a contravariant functor

$$\Lambda : \mathbf{WFrm} \to \mathbf{Top}.$$

On the other hand there is the contravariant functor

$$\mathcal{O} = J \cdot \mathfrak{O} : \mathbf{Top} \to \mathbf{WFrm}$$

where $\mathfrak{O} : \mathbf{Top} \to \mathbf{Frm}$ is the standard topology and preimage functor and $J : \mathbf{Frm} \subseteq \mathbf{WFrm}$ is the identical embedding. Further, we have the natural transformations

$$\lambda : \mathrm{Id} \to \mathcal{O}\Lambda, \quad \varepsilon : \mathrm{Id} \to \Lambda\mathcal{O}$$

defined by

$$\lambda_L(a) = \Lambda_a, \quad \varepsilon_X(x) = 1 \text{ iff } x \in U$$

as is seen by straightforward checking.

4.5.1. Proposition. *The contravariant functors $\mathcal{O}$ and Λ are adjoint on the right.*

Proof. We verify the familiar adjunction identities for λ and ε. Thus, for the composite

$$\mathcal{O}X \xrightarrow{\lambda_{\mathcal{O}X}} \mathcal{O}\Lambda\mathcal{O}X \xrightarrow{\mathcal{O}(\varepsilon_X)} \mathcal{O}X,$$

$\mathcal{O}\varepsilon_X(\lambda_{\mathcal{O}X}(U)) = \varepsilon_X^{-1}(\lambda_{\mathcal{O}X}(U)) = \{x \mid \varepsilon_X(x) \in \Lambda_U\} = \{x \mid \varepsilon_X(x)(U) = 1\} = \{x \mid x \in U\} = U$.

Similarly, for

$$\Lambda L \xrightarrow{\varepsilon_{\Lambda L}} \Lambda\mathcal{O}\Lambda L \xrightarrow{\Lambda(\lambda_L)} \Lambda L,$$

$(\Lambda\lambda_L(\varepsilon_{\Lambda L}(\alpha)))(a) = 1$ iff $(\varepsilon_{\Lambda L}(\alpha) \cdot \lambda_L)(a) = 1$ iff $(\varepsilon_{\Lambda L}(\alpha)(\Lambda_a) = 1$ iff $\alpha \in \Lambda_a$ iff $\alpha(a) = 1$. $\square$

4.5.2. As a consequence, we have the following easy criterion for (W_2) in the case of topologies of sober spaces.

Proposition *Let X be a sober space. Then $\mathfrak{O}(X)$ has the property* (W_2) *iff $\varepsilon_X : X \to \Lambda\mathcal{O}X$ is a homeomorphism.*

Proof. ($\Rightarrow$) Here $\Lambda\mathcal{O}(X)$ is just the standard spectrum $\Sigma\mathfrak{O}(X)$ and ε_X coincides with the corresponding unit $X \to \Sigma\mathfrak{O}(X)$, an isomorphism since X is sober.

($\Rightarrow$) Let P be a one-point space and view $\mathbf{2}$ as $\mathcal{O}(P)$. The we have the isomorphisms

$$\mathbf{WFrm}(\mathcal{O}(X), \mathbf{2}) \cong \mathbf{Top}(P, \Lambda\mathcal{O}(X)) \cong \mathbf{Top}(P, X) \cong \mathbf{Frm}(\mathfrak{O}(X), \mathbf{2})$$

the first by 4.5.1, the second given by $\mathbf{Top}(\mathrm{id}_P, \varepsilon_X)$, and the third by the sobriety of X and the adjunction of the standard spectrum functor, showing that every weak homomorphism $\mathcal{O}(X) \to \mathbf{2}$ is a homomorphism. $\square$

Note. In general, the frames $\mathcal{O}\Lambda\mathcal{O}(X)$ do not have property (W_2): otherwise, the spatial frames with the property would be coreflective in the category of all spatial frames with coreflection maps $\mathcal{O}(\varepsilon_X) : \mathcal{O}\Lambda\mathcal{O}(X) \to \mathcal{O}(X)$, but this would imply, for the Sierpinski space S, that

$$\mathcal{O}(S \times S) = \mathcal{O}(S) \oplus \mathcal{O}(S)$$

has property (W_2), which is not the case as we observed in connection with Corollary 3.2.

5. Some special cases

In the following, X will be a partially ordered set and $\mathfrak{D}X$ the frame of its *downsets*, that is, of all $U \subseteq X$ such that $a \leq b$ and $b \in U$ implies $a \in U$. Further, recall that a partially ordered set is called a *forest* whenever $\downarrow a \cap \downarrow b = \emptyset$ for any incomparable elements a and b.

5.1. Proposition. *The following are equivalent for any frame $L = \mathfrak{D}X$.*

(1) *X is a forest.*
(2) *L is relatively normal.*
(3) *L has property* (W).
(4) *L has property* (W_2).

Proof. We only need to show (1)⇒(2) and (4)⇒(1).

(1)⇒(2). For any $A, B \in \mathfrak{D}X$ let

$$U = \{x \in X \mid x \leq a \text{ for some } a \in A \setminus B\}$$

and

$$V = \{x \in X \mid x \leq b \text{ for some } b \in B \setminus A\}.$$

Then $A \subseteq B \cup U$ and $B \subseteq A \cup V$, and since any $a \in A \setminus B$ and $b \in B \setminus A$ are incomparable it follows that $U \cap V = \emptyset$ because X is a forest.

(4)⇒(1). Suppose there exist incomparable $a, b \in X$ with $c \in \downarrow a \cap \downarrow b$ and take $h : \mathfrak{D}X \to \mathbf{2}$ such that

$$h(U) = 1 \quad \text{iff} \quad U \cap \{a, b\} \neq \emptyset.$$

Then h evidently preserves all joins and the top. and if $U \cap V = \emptyset$ in $\mathfrak{D}X$ then $U \cap \{a, b\} = \emptyset$ or $V \cap \{a, b\} = \emptyset$ and hence $h(U) \wedge h(V) = 0$. On the other hand, $h(\downarrow a) \wedge h(\downarrow b) = 1$ but since a and b are incomparable $h(\downarrow a \cap \downarrow b) = 0$. Thus h is a weak homomorphism but not a homomorphism. □

If the partially ordered set considered above is a meet-semilattice M, this will be a forest iff M is a chain. Consequently we have

5.2. Corollary. *For any meet-semilattice M, the following are equivalent.*

(1) *M is a chain.*
(2) *$\mathfrak{D}M$ is relatively normal.*

(3) $\mathfrak{D}M$ *has property* (W).
(4) $\mathfrak{D}M$ *has property* (W_2).

Another familiar frame derived from some other entity is the frame $\mathfrak{J}L$ of ideals of a distributive lattice L. In this case we have

5.3. Proposition. *The following are equivalent for any* $L \in \mathbf{DLat}$.
(1) $\mathfrak{J}L$ *has property* (W).
(2) $\mathfrak{J}L$ *has property* (W_2).
(3) L *is relatively normal.*

Proof. (1)⇒(2) is trivial.

(2)⇒(3). Since $\downarrow(\downarrow a) = \mathfrak{J}(\downarrow a)$ will inherit the property (W_2) fom $\mathfrak{J}L$ it is enough to show L is normal. We do this by the same technique used in the proof of Proposition 3.1.

Suppose P is a prime ideal in L with distinct maximal ideals $P_1, P_2 \supseteq P$, and let ξ_1, ξ_2, and ξ now be corresponding *frame* homomorphisms $\mathfrak{J}L \to \mathbf{2}$ such that $\xi_i(J) = 0$ iff $J \subseteq P_i$, and analogously for P and ξ. Now $h(J) = \xi_1(J) \vee \xi_2(J)$ defines a weak frame homomorphism $h : \mathfrak{J}L \to \mathbf{2}$ since again

$$h(I) \wedge h(J) \leq h(I \cap J) \vee \xi(I \cap J);$$

further, if $a_1 \in P_2 \setminus P_1$ and $a_2 \in P_1 \setminus P_2$ by the incomparability of P_1 and P_2 then

$$\xi_1(\downarrow a_1) = 1 = \xi_2(\downarrow a_2) \quad \text{and} \quad \xi_2(\downarrow a_1) = 0 = \xi_1(\downarrow a_2)$$

and hence $h(\downarrow a_1) \wedge h(\downarrow a_2) = 1$ while $h(\downarrow a_1 \cap \downarrow a_2) = 0$, contradicting property (W_2).

(3)⇒(1). Given any weak frame homomorphism $h : \mathfrak{J}L \to M$, the map $L \to M$ taking a to $h(\downarrow a)$ is a weak lattice homomorphism by Proposition 3.1, and consequently we have, for any $I, J \in \mathfrak{J}L$,

$$\begin{aligned} h(I) \wedge h(J) &= \bigvee \{h(\downarrow a) \wedge h(\downarrow b) \mid a \in I, b \in J\} = \\ &= \bigvee \{h(\downarrow(a \cap b)) \mid a \in I, b \in J\} = h(I \cap J). \qquad \square \end{aligned}$$

As a suggestive characterization we then also have the following

5.4. Corollary. *For any* $L \in \mathbf{DLat}$, $\mathfrak{J}L$ *has the property (W) iff* L *has the property (W).*

A further consequence is the following result concerning free frames, based on Corollary 3.2 and the familiar fact that these may be represented as the ideal frames $\mathfrak{J}L$ of the free $L \in \mathbf{DLat}$:

5.5. Corollary. *Any frame which is free on more than one generator does not have the property (W).*

5.6. Remarks. (1) It is easy to see that $L \in \mathbf{DLat}$ is relatively normal whenever $\mathfrak{J}L$ is but we do not know whether the converse holds.

(2) By way of contrast with Proposition 5.3, we note that the *congruence* lattice of any $L \in \mathbf{DLat}$ has property (W) because of its zero-dimensionality.

(3) An obvious rephrasing of Corollary 5.4 would be: a coherent frame has property (W) iff the lattice of its compact elements has property (W).

We close with a result concerning compactness and the property (W).

5.7. Propoosition. *The following are equivalent for a compact subfit frame L.*

(1) *L is regular.*
(2) *L has property* (W).
(3) *L has property* (W_2).

Proof. We only have to show (3)⇒(1). Furthermore, since any normal subfit frame is regular, as noted earlier, it will be sufficient to prove normality.

Let L be any compact frame with property (W_2) and $a, b \in L$ such that $a \vee b = 1$. Then $F = \{x \wedge y \mid a \vee x = 1 = b \vee y\}$ is a filter in L, and we want to show that $0 \in F$. Now if F is proper then there exists a prime ideal P in L disjoint from F, and for $s = \bigvee P$ it follows that $s \vee a < 1$ and $s \vee b < 1$: if $s \vee a = 1$, say, then also $x \vee a = 1$ for some $x \in P$ by compactness but then $x \in F$, a contradiction. Next, using compactness again, take maximal elements $s_1 \geq s \vee a$ and $s_2 \geq s \vee b$ in L, noting these are necessarily distinsct since $s_1 = s_2$ implies $1 = a \vee b \leq s_1$.

Now let $\xi_i : L \to \mathbf{2}$ be the frame homomorphisms such that $\xi_i(x) = 0$ iff $x \leq s_i$, and put

$$h(x) = \xi_1(x) \vee \xi_2(x).$$

Clearly, h preserves the unit and arbitrary joins including the zero. Further, for any any x and y in L,

$$h(x) \wedge h(y) = h(x \wedge y) \vee (\xi_1(x) \wedge \xi_2(y)) \vee (\xi_2(x) \wedge \xi_1(y)).$$

In particular, let $x \wedge y = 0$. Then $h(x \wedge y) = 0$ as noted already; also, $x \in P$ or$y \in P$ since P is a prime ideal, hence $x \leq s$ or $y \leq s$ so that $x \leq s_1$ or $y \leq s_2$ and simiarly $x \leq s_2$ or $y \leq s_1$. It follows that

$$\xi_1(x) \wedge \xi_2(y) = 0 = \xi_2(x) \wedge \xi_1(y)$$

and consequently $h(x) \wedge h(y) = 0$, showing h is a weak homomorphism.

Finally, $h(s_1) = 1 = h(s_2)$ since $s_1 \neq s_2$ and therefore

$$\xi_2(s_1) = 1 = \xi_1(s_2);$$

on the other hand, however, $h(s_1 \wedge s_2) = 0$ so that h is not a homomorphism – a contradiction. $\square$

5.8. Remark. Parts of the above proof and some simple additional arguments establish the following general result.

A compact frame L is normal iff no prime ideal of L lies below two distinct maximal elements of L.

References

[1] R.N. Ball and A. Pultr, *Forbidden Forests in Priestley Spaces,* to appear in Cahiers de Top. et Géom. Diff. Cat.

[2] B. Banaschewski, *Uniform completion,* in: "Topological and Algebraic Strucures in Fuzzy Sets", eds. S.E. Rodabaugh and E.P. Klement. Kluwer Academic Publishers, Dordercht, 2003, 1-30.

[3] B. Banaschewski and A. Pultr, *Cauchy points of uniform and nearness frames,* Quaestiones Math. **19**(1-2) (1996), 101-127.

[4] C.R.A. Gilmour, *Realcompact spaces and regular σ-frames,* Math. Proc. Cambridge Phil. Soc. **96** (1984), 73-79.

[5] H. Herrlich and A. Pultr, *Nearness, subfitness and sequential regularity,* Appl. Cat. Structures **8** (2000), 67-80.

[6] J.R. Isbell, *Atomless parts of spaces,* Math. Scand. **31** (1972), 5-32.

[7] P.T. Johnstone, *Stone Spaces,* Cambridge Sudies in Advanced Math. no 3 (Camb. Univ. Press 1983).

[8] A. Joyal and M. Tierney, *An extension of the Galois theory of Grothendieck,* Memoirs of the AMS **51**, Number 309 (1984).

[9] A. Monteiro, *L'arithmetique des filtres et les espaces topologiques, I, II,* Notas Lógica Mat. (1974), 29-30.

[10] H.A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces,* Bull. London Math. Soc. **2** (1970), 186-190.

[11] H.A. Priestley, *Ordered topological spaces and the representation of distributive lattices,* Proc. London Math. Soc. **324** (1972), 507-530.

[12] J.T. Snodgrass and C. Tsinakis, *The finite basis theorem for relatively normal lattices,* Algebra Universalis **33** (1995), 40-67.

[13] S. Vickers, *"Topology via Logic",* Cambridge Tracts in Th. Comp. Sci., No. 5, Cambridge University Press, Cambridge, 1985.

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1, Canada

Department of Applied Mathematics and ITI, MFF, Charles University, CZ 11800 Praha 1, Malostranské nám. 25, Czech Republic
E-mail address: pultr@kam.ms.mff.cuni.cz