Füredi–Hajnal Conjecture Implies Stanley–Wilf Conjecture

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Abstract

We show that Stanley–Wilf enumerative conjecture on permutations follows easily from Füredi–Hajnal extremal conjecture on 0-1 matrices. We apply the method of deriving an (almost) exponential bound on number of objects from a (almost) linear bound on their sizes, which was discovered by Alon and Friedgut. They proved by it a weaker form of Stanley–Wilf conjecture. Using bipartite graphs, we give a simpler proof of their result.

Покажем, что гипотеза Стэнли и Вилфа о числе перестановок вытекает простым образом из экстремальной гипотезы Фиреды и Хайнала о 0-1 матрицах. Применяем метод вывода (почти) экспоненциальной оценки числа объектов из (почти) линейной оценки их величин открытый Алоном и Фридгутом. Этим методом они доказали гипотезу Стэнли и Вилфа в ослабленной форме. С помощью двудольных графов получим более простое доказательство их результата.

Stanley–Wilf conjecture asserts that the number of n-permutations not containing a given permutation is exponential in n. Alon and Friedgut [1] proved that it is true provided we have a linear upper bound on lengths of certain words over an ordered alphabet. They also proved a weaker version of it with an almost exponential upper bound. In the present note we want to inform the reader about this interesting development by reproving the latter result in a simpler way. We use bipartite graphs instead of words. We point out that in 1992 Füredi and Hajnal almost made an extremal conjecture on 0-1 matrices that now can easily be seen to imply Stanley–Wilf conjecture. We prove that both extremal conjectures are logically equivalent.

Two finite sequences $u = a_1a_2\ldots a_k \in \mathbf{N}^*$ and $v = b_1b_2\ldots b_l \in \mathbf{N}^*$ of positive integers are *isomorphic* if $k = l$ and, for every i and j, $a_i < a_j$ is equivalent to $b_i < b_j$. We say that v *contains* u if v has a subsequence that is isomorphic to u; we write $v \supset_< u$. Replacing $<$ by $=$, we define in the same way the corresponding isomorphism and containment relation $\supset_=$. For example, $31225345 \supset_= 2121$ but $31225345 \not\supset_< 2121$. The length of a sequence u is $|u|$ and $[n]$ is the set $\{1, 2, \ldots, n\}$.

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Permutations can be represented as elements of $\mathbf{N}^*$, e.g. $p = 32154$. If $|p| = k$, we speak of a *k-permutation*. The following conjecture from 1990 (?) is well known among enumerative combinatorialists.

STANLEY–WILF CONJECTURE. For each permutation p there is a constant $c = c(p)$ such that the number $S_n(p)$ of n-permutations q, $q \not\succ_< p$ satisfies $S_n(p) < c^n$.

An evidence for it is the result of Bóna [2] that it holds for all p of the form $p = I_1 I_2 \dots I_r$ where $I_1 < I_2 < \cdots < I_r$ and each sequence I_i is decreasing.

A sequence $v = b_1 b_2 \dots b_l \in \mathbf{N}^*$ is *k-sparse* if $b_j = b_i, j > i$ implies $j - i \geq k$. In other words, in each interval in v of length $\leq k$ all terms are distinct.

ALON–FRIEDGUT CONJECTURE. For each k-permutation p there is a constant $c = c(p)$ such that if $v \in [n]^*$ is k-sparse and $v \not\succ_< p$ then $|v| < cn$.

In [1] it is shown that if AFC is true then so is SWC. To be precise, AFC is put there in the form of a question rather than a conjecture.

Suppose $M = (m_{ij})$ and $N = (n_{ij})$ are $a \times b$ and $c \times d$ matrices with entries in $\{0, 1\}$. *M contains N* if there are indices $1 \leq i_1 < \cdots < i_c \leq a$ and $1 \leq j_1 < \cdots < j_d \leq b$ such that, for all $r = 1 \dots c$ and $s = 1 \dots d$, $m_{i_r, j_s} = 1$ whenever $n_{r,s} = 1$. So M has a (not necessarily consecutive) submatrix of N's size that has 1s on all the places where N has them and maybe on some others.

Füredi and Hajnal [3] investigated the extremal function $f(m, n; N)$ that is defined as the maximum number of 1s in an $m \times n$ 0-1 matrix M that does not contain N; $f(n; N) = f(n, n; N)$. They looked systematically at all 37 substantially distinct Ns with four 1s and no zero row or column. (We leave to the interested reader to prove as an exercise that if N has at most three 1s then $f(m, n; N) = O(m + n)$. For four 1s the situation is much more complicated.) In the concluding section of [3] they ask if for each permutation matrix N we have $f(n; N) = O(n)$: "Is it true that the complexity of all permutation configurations are linear?" We take the liberty to formulate it as a conjecture.

FÜREDI–HAJNAL CONJECTURE. For each permutation matrix N we have $f(n; N) = O(n)$.

We show, using the ideas of [1], that FHC implies SWC. We prefer to think of the matter in terms of bipartite graphs. Let G and H be two simple bipartite graphs with parts $[a], [b]'$ and $[c], [d]'$. Here $[b]' = \{1', 2', \dots, b'\}$ and all the parts are linearly ordered in the natural way. We say that *G contains H* if H is a (ordered!) subgraph of G, that is there exist *increasing* injections $f : [c] \to [a]$ and $g : [d]' \to [b]'$ such that if $\{i, j'\}$ is an edge of H then $\{f(i), g(j')\}$ is an edge of G. Besides being a sequence and a 0-1 matrix, each permutation $p = a_1 a_2 \dots a_k$ is also a bipartite graph G_p with the parts $[k], [k]'$ and the edges $\{a_i, i'\}, i \in [k]$. FHC then says that each bipartite graph on $[n], [n]'$ not containing G_p has $O(n)$ edges.

Theorem 1 *If FHC is true then so is SWC.*

Proof. Let p be a permutation and $M(n)$ the set of simple bipartite graphs on $[n], [n]'$ that do not contain G_p. We assume FHC — there is a constant c such that $|E(G)| < cn$ for each $G \in M(n)$. Let $n > 1$. For a $G \in M(n)$ we define a G_1 on $[m], [m]'$, where $m = \lceil n/2 \rceil$, by

$$\{i, j'\} \in E(G_1) \iff \exists\, e \in E(G) : \ e \cap \{2i-1, 2i\} \neq \emptyset \,\&\, e \cap \{2j'-1, 2j'\} \neq \emptyset.$$

Clearly, $G_1 \in M(m)$. Also, one G_1 arises from at most $15^{|E(G_1)|} < 15^{cm}$ graphs G because there are 15 possibilities for a nonempty restriction of G to $\{2i-1, 2i\}, \{2j'-1, 2j'\}$. Hence,

$$|M(n)| < 15^{c\lceil n/2 \rceil} \cdot |M(\lceil n/2 \rceil)|.$$

Iterating the inequality until $|M(1)| = 2$, we obtain an upper bound on $|M(n)|$ that is exponential in n. But $S_n(p) \leq |M(n)|$ because for each n-permutation $q, q \not\succ_< p$ we have $G_q \in M(n)$. Thus

$$S_n(p) \leq |M(n)| < 15^{2cn}.$$

□

Alon and Friedgut prove the weaker form of SWC with an almost exponential bound by means of the following result due to Klazar [5]. Suppose $u \in [k]^*$ is given. If $v \in [n]^*$ is k-sparse and $v \not\succ_= u$ — notice that now we use the weaker containment, then

$$|v| < nc^{\alpha(n)^d}, \qquad (*)$$

where $c, d > 1$ are moderate constants depending only on u and $\alpha(n)$ is the inverse of the Ackermann function $A(n)$ known from the recursion theory.

We remind the reader the definition of $A(n)$ and $\alpha(n)$. If $F_1(n) = 2n$, $F_2(n) = 2^n$, and in general $F_{i+1}(n) = F_i(F_i(\ldots F_i(1)\ldots))$ with n iterations of F_i, then $A(n) = F_n(n)$ and $\alpha(n) = \min\{m : \ A(m) \geq n\}$. Although $\alpha(n) \to \infty$, in practice $\alpha(n)$ is bounded:

$$\alpha(n) \leq 4 \text{ for } n < 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$$

where the tower has $2^{16} = 65536$ 2s.

Theorem 2 *For each fixed permutation p,*

$$S_n(p) \leq |M(n)| < 225^{n\beta(n)} \quad \textit{with} \quad \beta(n) = c^{\alpha(n)^d},$$

where c and d are moderate constants depending only on $|p|$.

Proof. We use the notation of the previous proof and set $k = |p|$. If instead of $|E(G_1)| < cm$ the bound $|E(G_1)| < m\beta(m)$ with an increasing function $\beta(m)$ is used, we get

$$S_n(p) \leq |M(n)| < 15^{2n\beta(n)} = 225^{n\beta(n)}.$$

Let $G \in M(n)$. It remains to derive a good bound $|E(G)| < n\beta(n)$. Consider the sequence $v = L_1 L_2 \dots L_n \in [n]^*$, where L_i is the list of the neighbours of i' in G, in the increasing order. The sequence v is in general not k-sparse but it is easy to see that by deleting $\leq (k-1)(n-1)$ appropriate elements, $\leq k-1$ from the beginning of each of $L_2, \dots, L_n$, we can obtain a k-sparse subsequence w. It is also not difficult to see that if $v \supset_= u(k)$, where $u(k) = 12\dots k12\dots k\dots 12\dots k$ has k segments $12\dots k$, then G contains G_p, in fact for any k-permutation p. So $w \not\supset_= u(k)$ and we can apply the aforementioned result:

$$|E(G)| = |v| \leq (k-1)(n-1) + |w| < (k-1)(n-1) + nc^{\alpha(n)^d}.$$

$\square$

Our bound is weaker compared to the bound of Alon and Friedgut in [1]. They use a more complicated induction step in which they decrease n more than to ours $\lceil n/2 \rceil$. As we mentioned, they do not work with graphs but in $(\mathbf{N}^*, \supset_<)$ and $(\mathbf{N}^*, \supset_=)$.

We show that both extremal conjectures are equivalent.

Theorem 3 *AFC and FHC are mutually equivalent.*

Proof. We prove first that AFC implies FHC. If $v = a_1 a_2 \dots a_l \in \mathbf{N}^*$, $1 \leq i \leq j < k \leq l$, and $a_j > a_{j+1}$, we say that i and k are in v separated by fall. Let p be a permutation and G a bipartite graph on $[n], [n]'$ not containing G_p. Consider the sequence $v = L_1 L_2 \dots L_n \in [n]^*$ of the previous proof. Recall that each L_i is increasing. It may happen that $v \supset_< p$ because one L_i can contribute to the subsequence isomorphic to p by more than one element. To prevent it, we take a larger permutation $p' \supset_< p$, $|p'| = k'$ such that each two consecutive elements of the subsequence isomorphic to p are in p' separated by fall. Now $v \supset_< p'$ is impossible. As we know, by deleting $< k'n$ elements we can obtain a k'-sparse subsequence w. By AFC for p', we have a linear bound on $|w|$. So $|E(G)| = |v| < k'n + |w| = O(n)$.

We prove that FHC implies AFC. If we are content in AFC with *any* bound, the (unconditional) proof is easy. Suppose $u \in [k]^*$, $v \in [n]^*$ is k-sparse, and $v \not\supset_< u$. Split v into intervals of length k and a remainder of length $< k$. By the pigeonhole principle, there are at most $(|u|-1)n()k$ intervals. Therefore

$$|v| < k((|u|-1)n()k + 1).$$

Now suppose that p is a k-permutation and $v \in [n]^*$ is a k-sparse sequence not containing p. Using FHC, we prove a linear bound on $|v|$. There is a constant c such that $|E(G)| < cn$ holds for each bipartite graph on $[n], [n]'$ not containing G_p. If $k > c$, we set $l = k$ and $w = v$. Else we fix a positive integer $l, l > c$ and take the longest l-sparse subsequence w of v. We show that $|w|$ is proportional to $|v|$. The subsequence w splits v into nonempty intervals disjoint to w. Let I be one of them. Since no term of I can be used to extend w, each of them equals to one of the $\leq l-1$ terms of w

preceding I or following it. So there are only $\leq 2l-2$ distinct numbers in I and, since I is k-sparse and $I \not\succ_< p$, $|I| < k((k-1)2l-2()k+1) = d(k,l)$. Thus

$$|v| < (d(k,l)+1)(|w|+1).$$

We split w into m intervals w_i of length l and a remainder r of length $< l$: $w = w_1 w_2 \dots w_m r$. We define a bipartite graph G on $[n], [m]'$ by

$$\{i, j'\} \in E(G) \iff i \text{ appears in } w_j.$$

Clearly, G does not contain G_p and each j' has degree l. If $m \geq n$, after adding some isolated vertices to $[n]$ G can be regarded as a graph on $[m], [m]'$. But G has $lm > cm$ edges, a contradiction. Hence $m < n$ and we have the bound

$$|v| < (d(k,l)+1)(|w|+1) < (d(k,l)+1)(ln+1) = O(n).$$

$\square$

Conclusion and remarks

In $(*)$ one can set, for $n > n(u)$, $c = 1000k^3$ and $d = |u| - 4$, see [5]. Thus in Theorem 2 one can set, for $n > n(k)$, $c = 1000k^3$ and $d = k^2$. The bound $(*)$ cannot in general be improved to $O(n)$. For example, it is known that if $v_1, v_2 \in [n]^*$ are 2-sparse and have the maximum length with respect to $v_1 \not\succ_= 12121, v_2 \not\succ_= 121212$, then $n\alpha(n) \ll |v_1| \ll n\alpha(n)$ and $n2^{\alpha(n)} \ll |v_2| \ll n2^{\alpha(n)}$. For more information see the book [8] of Sharir and Agarwal. So it is possible that FHC is false. If FHC is true, to prove it seems to require tools far stronger than those in [3].

The paper [1] teaches us that we understand SWC better by viewing permutations as elements of $\mathbf{N}^*$. We saw that we can view them also as bipartite graphs. The situation can be generalized further to the class of ordered hypergraphs, see [6] and [7].

In the end of [4] Gessel mentions the problem to decide if for each permutation p the sequence $\{S_n(p)\}_{n\geq 1}$ is P-recursive. The above results are not relevant to the problem and it appears to be very difficult. On the other hand, a strong optimism in this respect was expressed by Zeilberger [9].

References

[1] N. Alon and E. Friedgut, On the number of permutations avoiding a given pattern, *J. Combinatorial Th. Series A*, to appear.

[2] M. Bóna, The solution of a conjecture of Stanley and Wilf for all layered patterns, *J. Combinatorial Th. Series A* **85** (1999), 96–104.

[3] Z. Füredi and P. Hajnal, Davenport–Schinzel theory of matrices, *Discrete Math.* **103** (1992), 233–251.

[4] I.M. Gessel, Symmetric functions and P-recursiveness, *J. Combinatorial Th. Series A* **53** (1990), 257–285.

[5] M. Klazar, A general upper bound in Extremal theory of sequences, *Comment. Math. Univ. Carolin.* **33** (1992), 737–746.

[6] M. Klazar, Counting pattern-free set partitions I: A generalization of Stirling numbers of the second kind, *Europ. J. Combinatorics*, to appear.

[7] M. Klazar, Counting pattern-free set partitions II: Noncrossing and other hypergraphs, in preparation.

[8] M. Sharir and P.K. Agarwal, *Davenport–Schinzel sequences and their geometric applications,* Cambridge University Press, Cambridge, UK, 1995.

[9] D. Zeilberger, Enumeration schemes, and more importantly, their automatic generation, *Ann. of Combinatorics* **2** (1998), 185-195.

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