Genus of 2-Isomorphic Graphs

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Abstract

We prove that two graphs with isomorphic graphic matroids have the same genus. We present a new constructive proof of the fact that twisting operation does not effect the genus of a graph.

1 Introduction

Two isomorphic graphs clearly have isomorphic graphic matroids. But there are non-isomorphic graphs with isomorphic matroids. These graphs were characterized by Whitney (see [5]). Following Whitney we call graphs with isomorphic graphic matroids *2-isomorphic*.

Galluccio and Loebl described in [4] the way how to efficiently calculate some polynomials associated with a given graph as a linear combination of 4^g Pfaffians, where g is the genus of the graph. Since these polynomials can be derived from the Tutte polynomial, they do not distinguish the graphs with isomorphic graphic matroids. So the question whether 2-isomorphic graphs have the same genus is very natural in this context.

The positive answer to this question simply follows from results of Battle, Harary, Kodama and Youngs in [7] and [1] and results of Decker , Glover and Huneke in [2], [3]. In papers [7], [1] is proved that the genus of a graph is the sum of genera of its blocks. In papers [2] and [3] Decker, Glover and Huneke proved that the genus of 2-amalgamation of G_1 and G_2 depends only on G_1 and G_2. From this it follows that twisting keeps the genus. In this paper we present a new constructive proof. We show how to modify some minimal embedding of a graph G in to an embedding of twisted G of the same genus.

The paper contains a part of my master thesis (see [6]). After finishing it, prof. Archdeacon draw our attention to the papers of Decker, Glower and Huneke [2] and [3].

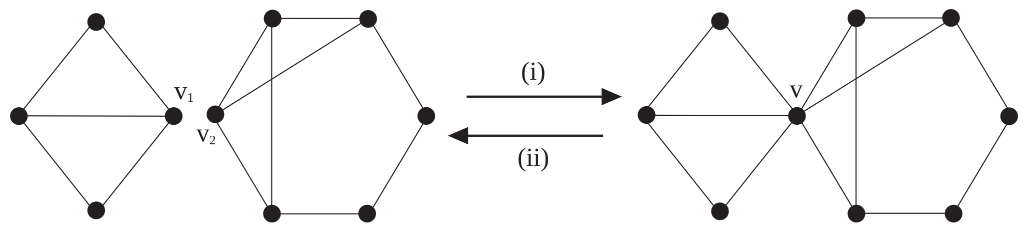
2 2-Isomorphism

Let G be a graph. Then we will denote by $\mathcal{M}(G)$ the corresponding graphic matroid. First we introduce three operations on graphs which will be used to define 2-isomorphism :

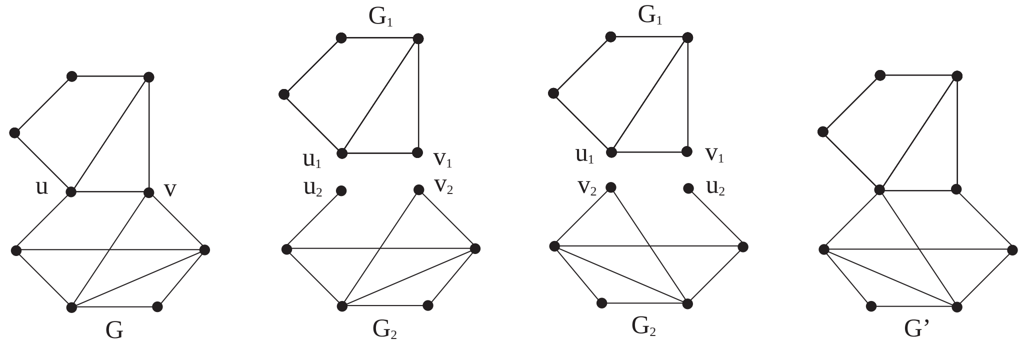
Definition 2.1 *Let G be a graph. Then we define the following operations on G.*

- *(i) Vertex identification. Let v_1 and v_2 be two vertices of distinct components of G. We modify G by identifying v_1 and v_2 into a new vertex v.*
- *(ii) Vertex cleaving. This is the reverse operation of vertex identification so that a graph can only be cleft at a cut-vertex.*
- *(iii) Twisting. Suppose that G is obtained from two disjoint graphs G_1 and G_2 by identifying the vertices u_1 of G_1 and u_2 of G_2 into the vertex u of G, and identifying vertices v_1 of G_1 and v_2 of G_2 into the vertex v of G. Then the twisted G is a graph which is constructed by identifying, instead, u_1 with v_2 and v_1 with u_2. Graphs G_1 and G_2 are called pieces of the twisting and vertices u and v are called twisting vertices.*

The following figures show examples of these operations. The first one is an example of vertex identification and vertex cleaving operation :



The second figure shows an example of twisting operation on a graph :



Definition 2.2 *Graph G is 2-isomorphic to graph H if H can be transformed into a graph isomorphic to G by a sequence of operations (i), (ii) and (iii).*

Evidently 2-isomorphism is an equivalence relation. Since the operations (i), (ii) and (iii) don't effect the cycles of a given graph, if G is 2-isomorphic to H, then $\mathcal{M}(G) \cong \mathcal{M}(H)$. The next theorem shows that this implication from graphs to matroids is the equivalence. It was first proved by Whitney in 1933.

Theorem 2.3 (Whitney's 2-Isomorphism Theorem) *Let G and H be graphs without isolated vertices. Then $\mathcal{M}(G)$ and $\mathcal{M}(H)$ are isomorphic if and only if G and H are 2-isomorphic.*

3 Embeddings of Graphs

The origin of definitions and results described in this section is in papers of J. W. T. Youngs [7] and Joseph Battle, Frank Harary, Yukihiro Kodama and J. W. T. Youngs [1].

We will use common topological terminology of manifolds. A two dimensional manifold will be called shortly *2-manifold*.

Definition 3.1 *Let M be an arbitrary 2-manifold. The Euler characteristic of M denoted $\chi(M)$ is defined from any cellular decomposition of M by*

$$\chi(M) = \alpha^0 - \alpha^1 + \alpha^2$$

where α^i is the number of $i-$cells in this decomposition, $i = 0, 1, 2$. Moreover, $\chi(M) \leq 2$.

Definition 3.2 *An orientable 2-manifold without boundary in Euclidean space* $\mathbb{E}_3$ *will be called surface. Let* S *be a surface, then* $\gamma(S) = [2 - \chi(S)/2]$ *is called a genus of* S.

Since the Euler characteristic of any surface S is even, $\gamma(S)$ is a nonnegative integer. The basic example of a surface is the sphere. Genus of the sphere is zero. Any other connected surface is homeomorphic to the sphere with 'handles', where the number of 'handles' is equal to the genus of this surface. So the torus is homeomorphic to the sphere with one 'handle'.

Definition 3.3 *Let* X *be a general topological space. An arc is a homeomorphic image of the closed unit interval. A loop is also an image of the close unit interval satisfying that image of any closed interval* $[s_1; s_2] \subset [0; 1]$, $[s_1, s_2] \neq [0; 1]$ *is an arc but the image of* 0 *and* 1 *is the same point of* X. *An open arc is an arc without its two endpoints, images of* 0 *and* 1. *The endpoint of a loop is the point which is the image of* 0 *and also the image of* 1.

Definition 3.4 *Let* X *be a topological space. Let* G *be a graph,* $V(G) = \{v_1, v_2, \ldots, v_n\}$ *and* $E(G) = \{e_1, e_2, \ldots, e_m\}$. *Then an embedding of* G *in* X *is a subspace* $G(X)$ *of* X *such that*

$$G(X) = \bigcup_{i=1}^{n} v_i(X) \cup \bigcup_{j=1}^{m} e_j(X)$$

where

(i) $v_1(X), v_2(X), \ldots, v_n(X)$ *are* n *distinct points of* X

(ii) $e_1(X), e_2(X), \ldots, e_m(X)$ *are* m *open arcs in* X, *disjoint in pairs*

(iii) $v_1(X) \cap e_j(X) = \emptyset$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$

(iv) if $e_j = \{v_i, v_{i'}\}$ *then the open arc* $e_j(X)$ *has* $v_i(X)$ *and* $v_{i'}(X)$ *as endpoints,* $j = 1, 2, \ldots, m$

Any surface is also a topological space so the embedding of graphs on surfaces is defined by Definition 3.4. The fact that there exists an embedding of an arbitrary graph on a surface is easy. We can simply get an embedding of G on a surface of genus $|E(G)|$. But adding a 'handle' for each edge causes that we have a surface with higher genus than is necessary.

Definition 3.5 *Let G be a graph. Then an embedding $G(S)$ on a surface S is called a minimal embedding of G if the genus of S is the minimal possible genus of surfaces on which G can be embedded. The genus of the graph G, denoted $\gamma(G)$, is the genus $\gamma(S)$ of the surface S on which the embedding $G(S)$ is the minimal embedding of G.*

Definition 3.6 *Let $G(S)$ be an embedding of a graph G on a surface S. The $G(S)$ is called a 2-cell embedding if each face of $G(S)$ is a 2-cell. It means that each face is a homeomorphic image of an open unit disc.*

The following relation between minimal and 2-cell embeddings is proved in Youngs [7].

Theorem 3.7 *Let $G(S)$ be any minimal embedding of a connected graph G. Then $G(S)$ is a 2-cell embedding.*

Now, we present important result of [1] which enables us to prove our main theorem later.

Theorem 3.8 *Let G be a graph, connected or not, and let $B_1, B_2, \ldots, B_k$ be all blocks of G. Then $\gamma(G) = \sum_{i=1}^{k} \gamma(B_i)$.*

4 Main Theorem

Now we are ready to answer affirmatively the question whether two graphs with isomorphic graphic matroids have the same genus.

Theorem 4.1 *Let G and H be 2-isomorphic graphs. Then both graphs have the same genus.*

Since isolated vertices do not effect graphic matroid nor the genus of a graph, the Theorem 4.1 immediately implies following.

Corollary 4.2 *Let G and H be two graphs with isomorphic graphic matroids. Then G and H have the same genus.*

The same corollary holds for cographic matroids. Using the fact that two matroids are isomorphic if and only if their dual are also isomorphic we can easy see the following.

Corollary 4.3 *Let G and H be two graphs with isomorphic cographic matroids. Then both graphs have the same genus and their graphic matroids are also isomorphic.*

In order to prove Theorem 4.1 we will show that operations of vertex identification, vertex cleaving and twisting, described in Definition 2.1, do not change the genus of a graph. Hence we prove the following.

Theorem 4.4 *Let G be 2-connected graph. Let G' denote twisted G. Then $\gamma(G) = \gamma(G')$.*

This theorem and the results of previous chapter give us the way how to prove Theorem 4.1.

Proof of Theorem 4.1. From the definition of 2-isomorphism it suffices to prove that vertex identification, vertex cleaving and twisting do not effect the genus of a graph. The required result for vertex identification and vertex cleaving follows immediately from the Theorem 3.8.

If a graph is 2-connected then twisting does not effect its genus by the Theorem 4.4. Assume now a twisting of a graph, which is not 2-connected. If twisting effects only one of its blocks then the genus of this block is not changed and also the genus of the whole graph remains the same. The only remaining case is that both vertices of twisting are cut-vertices and then an operation of twisting can be replaced by a sequence of vertex cleaving and vertex identification operations. Hence twisting does not change the genus of a graph.

□

To prove Theorem 4.4 we will consider 2-connected graph G and we will usually use the notation u and v for twisting vertices and G_1 and G_2 for pieces of twisting. Also some structure will be needed which enables a change of some minimal embedding of a graph G into an embedding of G' of the same genus. This structure will be called *dividing systems*.

5 Dividing Systems

Definition 5.1 *Let G be a 2-connected graph with defined twisting operation. Let G_1 and G_2 be pieces of twisting and u, v be twisting vertices.*

Let $G(S)$ be a 2-cell embedding of G on a surface S. Then any system $D = \{\phi_1, \phi_2, \dots, \phi_k\}$ of finite number of curves on S is called a dividing system of $G(S)$ if the following conditions hold:

(i) if $\phi_i \in D$ then ϕ_i is an arc or a loop with endpoints in $\{u(S), v(S)\}$
($\phi_i(0), \phi_i(1) \in \{u(S), v(S)\}$, $i = 1, 2, \dots, k$)

(ii) the interiors of elements of D are pairwise disjoint
($\forall i \neq j$, $\forall s, t \in (0; 1)$: $\phi_i(s) \neq \phi_j(t)$)

(iii) interiors of curves of D have no common point with $G(S)$
($\forall i$, $\forall s \in (0; 1)$: $\phi_i(s) \notin G(S)$)

(iv) curves of D divide G_1 and G_2 on S
($\forall x_j \in G_j(S)$, $j = 1, 2$, $\forall$ arc ψ such that $\psi(0) = x_1$ and $\psi(1) = x_2$,
$\exists i \; \exists s, t \in [0; 1]$: $\psi(s) = \phi_i(t)$)

Definition 5.2 *The size $|D|$ of a dividing system D is the number of different curves in D .*

Proposition 5.3 *Let D be any dividing system of a 2-cell embedding $G(S)$. Then $|D| \geq 2$.*

Proof. Since G is 2-connected there exist $e_i \in G_i$, $i = 1, 2$, both incident with u. We may assume that $e_i(0) = u(S)$, $i = 1, 2$. The arc $\psi = e_1([0; 0.5]) \cup e_2([0; 0.5])$ connects points of $G_1(S)$ and $G_2(S)$. Hence it has to intersect a curve ϕ of D. Since $\psi \subset G(S)$, the only point of intersection can be the point $u(S)$. Hence there is a curve in D with endpoint $u(S)$, and using the same argument, also a curve in D with endpoint $v(S)$.

If $D = \{\phi\}$ where ϕ is an arc with endpoints $u(S)$ and $v(S)$, then there exists a loop in small neighbourhood of $u(S)$ intersecting ϕ in exactly one point. This loop can be drawn around the point $u(S)$ and if it is chosen small enough, it also intersects each edge incident with u in one point. There are again edges $e_i \in G_i$ incident with u, $i = 1, 2$. They divide the loop into two arcs. Exactly one of them intersects ϕ so the other one contradicts the condition (iv) of dividing systems.

□

Let us denote by $S - D$ the union of open 2-manifolds with boundary $S - (\bigcup_{\phi \in D} \phi)$.

Proposition 5.4 *Let $G(S)$ be a 2-cell embedding of G. Let D be a system satisfying conditions (i), (ii) and (iii) from definition of dividing system. Then the condition (iv) is equivalent to the following:*

(iv)' there is no component of $S - D$ containing points of both $G_1(S)$ and $G_2(S)$.

Proof. Suppose (iv) holds and (iv)' does not. Then there exist points X_1, X_2 in the same component of $S - D$ such that $X_i \in G_i(S)$, $i = 1, 2$. Hence there exists an arc ψ with endpoints X_1 and X_2 in that component. Hence ψ has no common point with the boundary of the component of $S - D$. So ψ has no common point with any $\phi \in D$ which contradicts (iv).

Now assume (iv)' holds but (iv) does not. There exists an arc ψ with endpoints from different $G_i(S)$, $i = 1, 2$, which does not intersect any curve of D. Since endpoints of ψ are from different $G_i(S)$, they are in different components of $S - D$. Hence ψ has to intersect the boundary of some components of $S - D$. This means ψ intersects a curve ϕ from D and (iv) holds a contradiction.

$\square$

With the knowledge of these properties of dividing systems it is now easy to prove that for any 2-cell embedding of a graph G on a surface S there exists a dividing system of $G(S)$.

Proposition 5.5 *Let $G(S)$ be a 2-cell embedding of G on a surface S. Then there exists a dividing system D of $G(S)$.*

Proof. As declared in previous section if $G(S)$ is a 2-cell embedding, each face of $G(S)$ is homeomorphic to an open disc.

Let Φ be a face of $G(S)$ which boundary contains edges of both G_1 and G_2. Consider the circuit boundering the image of a homeomorphic mapping of Φ to an open disc. We label the points of this circuit, corresponding to vertices incident with edges from different G_i, by $x_1, x_2, \dots, x_k$ in clockwise order. Each x_i is a copy of the vertex u or v. Then vertices labeled by x_i and x_{i+1} (indices computed *mod k*), $i = 1, 2, \dots, k$, will be connected by a straight line l_i in the image of Φ. Let $\phi_1^\Phi, \phi_2^\Phi, \dots, \phi_k^\Phi$ be curves on S such that homeomorphic image of ϕ_i^Φ in the open disc is the line l_i.

We define the system of curves D by $D = \bigcup_\Phi \{\phi_1^\Phi, \phi_2^\Phi, \dots, \phi_{k_\Phi}^\Phi\}$ over all faces Φ of $G(S)$ boundered by edges of both G_i. If $k = 2$ then $\phi_1^\Phi = \phi_2^\Phi$

and so only one of them is used. Clearly D is finite set of curves satisfying conditions (i), (ii) and (iii) of dividing system.

To prove that D satisfies the condition (iv)' we proceed by a contradiction. Suppose there is a component of $S-D$ containing points of both $G_i(S)$. Then this component contains a face of $G(S)$ boundered by edges of both G_i. But there have to be some curves of D constructed in this face. Their interiors are subsets of this face so they are subsets of the component of $S-D$. This is impossible and (iv)' holds.

$\square$

In the next proposition we will prove that any dividing system is close to the one constructed in the previous proof.

Proposition 5.6 *Let D be a dividing system of a 2-cell embedding $G(S)$ of G on a surface S. Let $\phi \in D$, then there exists unique face $\Phi = \Phi(\phi)$ of $G(S)$ such that $\phi((0;1)) \subset \Phi$.*

Proof. Let $s \in (0;1)$. Then $\phi(s) \notin G(S)$ so there exist face Φ^s such that $\phi(s) \in \Phi^s$. Suppose that there are $s, t \in (0;1)$ such that $\Phi^s \neq \Phi^t$. We may assume without loss of generality that $s < t$. Hence $\phi([s;t])$ is an arc on S which has to intersect the boundery of Φ^s in point $\phi(s')$, $s' \in (s;t)$. Since faces of $G(S)$ are components of $S - G(S)$, $\phi(s') \in G(S)$. Hence ϕ has common point with $G(S)$ and cannot be a curve of a dividing system.

$\square$

Having these properties of dividing systems it is possible to prove some important properties of twisting of G.

6 Minimal Dividing System

In this section we consider only minimal embeddings of graphs. By the Theorem 3.7 we know that minimal embeddings of connected graphs are 2-cell embeddings.

To prove Theorem 4.4 dividing systems of minimal size will be used because these dividing systems have some useful properties.

Definition 6.1 *Let D be a dividing system of any minimal embedding $G(S)$ of G on surface S. Then D is called a minimal dividing system if its size $|D|$ is minimal over all dividing systems of minimal embeddings of G.*

Lemma 6.2 *Let D be a dividing system of a 2-cell embedding $G(S)$. Let C_1, C_2 be two different components of $S - D$ such that points of at most one $G_i(S)$ are embedded on $C_1 \cup C_2$. Let x be a common point of boundaries of C_1 and C_2 distinct from $u(S)$ and $v(S)$. Then there $\exists \phi \in D$ and $\exists s \in (0;1)$ such that $\phi(s) = x$ and $\phi([0;1]) \subset \partial C_1 \cap \partial C_2$.*

Proof. Since x is a point of the boundery of any component of $S - D$ there exists $\phi \in D$ and $s \in [0;1]$ such that $x = \phi(s)$. Moreover, x distinct from $u(S)$ and $v(S)$ gives $s \neq 0$ and $s \neq 1$. the boundery of C_i is the union of some curves from D. These curves have disjoint interiors and $x \in \partial C_i$, hence $\phi([0;1]) \subset \partial C_i$.

□

Proposition 6.3 *Let $G(S)$ be a minimal embedding of G such that there is a minimal dividing system D of $G(S)$. Then each component of $S - D$ contains points of exactly one of $G_i(S)$, $i = 1, 2$, and the boundery of two components containing points of the same G_i have common points only from $\{u(S), v(S)\}$.*

Proof. Each component of $S - D$ contains points of at most one $G_i(S)$, $i = 1, 2$. Let C be a component of $S - D$ containing no point of $G(S)$. Let $\phi \in D$ be one from bounder curves of C. Then it is clear that the system $D - \{\phi\}$ satisfies conditions (i), (ii) and (iii). Since C does not contain any point of $G(S)$ there has to be curves of D different from ϕ incident with $u(S)$ and $v(S)$. Moreover, there is C', the component of $S - D$, such that $C \cup \phi((0;1)) \cup C'$ is a component of $S - (D - \{\phi\})$ and any other component of $S - (D - \{\phi\})$ is also a component of $S - D$. Since $C \cup \phi((0;1))$ does not contain any point of $G(S)$, $D - \{\phi\}$ also satisfies(iv)' and hence $D - \{\phi\}$ is a dividing system of $G(S)$. So D cannot be minimal.

If C_1 and C_2 are two different components of $S - D$ containing points of the same $G_i(S)$ and having common point distinct from $u(S)$ and $v(S)$ then by Lemma 6.2, $\phi \in D$ can be found such that $\phi([0;1]) \subset \partial C_i$, $i = 1, 2$. Then system $D - \{\phi\}$ is also a dividing system of $G(S)$ and D is not minimal.

□

The proof of Theorem 4.4 will be done by changing a minimal embedding of G into an embedding of twisted G of the same genus. The change will use a minimal dividing system D of $G(S)$. If D does not contain a loop, an embedding of twisted G can be drawn on the same surface as the given minimal embedding of G. But if D contains a loop we will have to use some properties of minimal dividing systems.

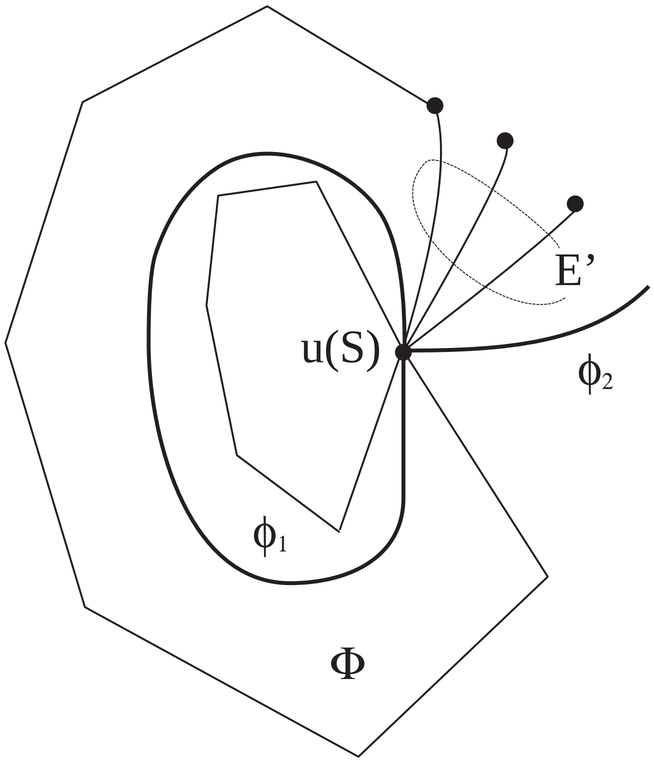
Lemma 6.4 *Let D be a minimal dividing system containing a loop with endpoint $u(S)$. Then there is no other curve in D incident with $u(S)$. The same holds for $v(S)$.*

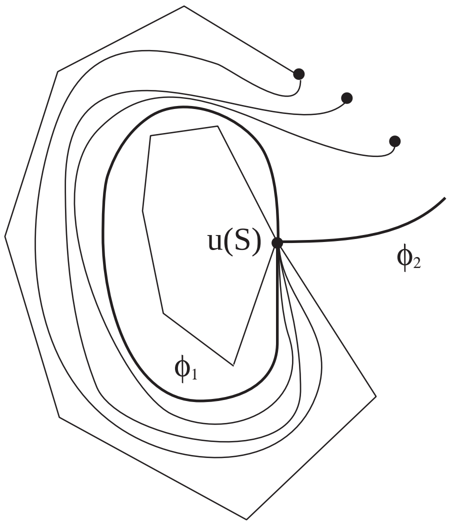
Proof. Suppose $\phi_1 \in D$ is a loop with endpoint $u(S)$ and there is at least one other curve in D incident with $u(S)$. A small neighbourhood N of $u(S)$ on S can be treated as a plane and curves representing edges incident with u and curves of D incident with $u(S)$ can be considered as lines in N. Two of these lines, say l_1 and l_2, are parts of ϕ_1. There also exists a curve ϕ_2 incident with $u(S)$ such that the angle A in N, boundered by a line corresponding to ϕ_2 and l_1 or l_2, contains only lines corresponding to curves representing edges of G. Denote the set of edges whose lines belong to A by E'.

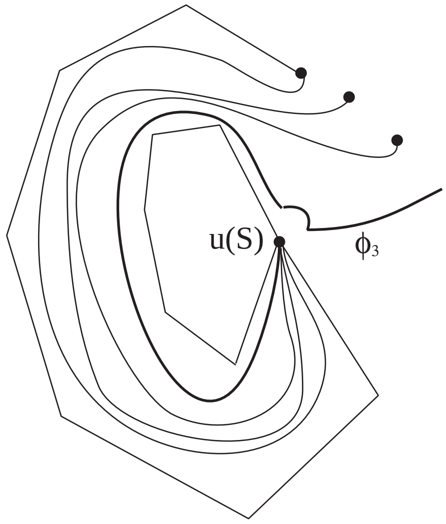
Let Φ be a face of $G(S)$ such that $\phi_1((0;1)) \subset \Phi$. Now delete parts of curves representing edges of E' in N and draw them along ϕ_1 into $u(S)$ 'from another side'. This gives a new embedding of G on S such that in the angle A there is no other curve. Let $D' = D - \{\phi_1, \phi_2\} \cup \{\phi_3\}$ where ϕ_3 is the curve obtained from $\phi_1 \cup \phi_2$ so that the ends of ϕ_1 and ϕ_2 corresponding to the lines boundering the angle A are deleted in N and replaced by an arc in A connecting the remaining parts of ϕ_1 and ϕ_2. More precisely, let ψ be a loop boundering N. Let $\phi_1(s_1) = \psi(t_1)$ and $\phi_2(s_2) = \psi(t_2)$ be common points of ψ and ϕ_1, resp. ϕ_2 such that they divide ψ into two arcs and the interior of one of them (which belongs to the angle A) is not intersected by any curve of D or any curve representing an edge of G. Call this arc ψ_1. Also ϕ_1 and ϕ_2 are divided into two arcs and one of them is contained in the closure of N boundered by ψ. We may assume these parts are $\phi_1([s_1; 1])$ and $\phi_2([0; s_2])$. Also without lost of generality $0 \leq t_1 < t_2 \leq 1$ and $\psi_1 = \psi([t_1; t_2])$. Then we let $\phi_3 = \phi_1([0; s_1]) \cup \psi([t_1; t_2]) \cup ([s_2; 1])$.

It is clear that each curve of $D' = D - \{\phi_1, \phi_2\} \cup \{\phi_3\}$ is a loop or an arc with endpoints in $\{u(S), v(S)\}$. Their interiors do not contain points of $G(S)$ since the curves of D hold this condition and ψ_1 also does. Let C be a component of $S - D$ containing the angle A. Let C_1 and C_2 be components of $S - D$ such that they are neighbours of C in N (maybe $C_1 = C_2$). Then components of $S - D'$ are the same as the components of $S - D$ except of the following case. There are two new components, one obtained from C as $C_1' = C - \overline{N}$ and another obtained from C_1 and C_2 as $C_2' = C_1 \cup C_2 \cup ((N \cap A) \cup \phi_1((s_1; 1)) \cup \phi((0; s_2)))$. Without loss of generality we can assume that C contains points of $G_1(S)$ and hence C_1' also contains points of $G_1(S)$ and no point of $G_2(S)$. Since $(N \cup A)$ does not contain any point of $G(S)$, and C_1 and C_2 are both neighbours of C, C_2' contains points of $G_2(S)$ and no point of $G_1(S)$. Hence condition (iv)' of dividing systems holds and D' is the dividing system of size $|D| - 1$. Hence D cannot be the minimal dividing system.

The process of loop reduction can be seen on following figures :







□

Proposition 6.5 *Let D be a minimal dividing system containing a loop. Then $|D| = 2$ and both curves of D are loops, one with endpoint $u(S)$ and the second with endpoint $v(S)$.*

Proof. Let $\phi_1 \in D$ be a loop with endpoint $u(S)$. Then by Lemma 6.4 there is no other curve from D incident with $u(S)$. Since $|D| \geq 2$, there have to be a curve $\phi_2 \in D$. Hence ϕ_2 is a loop with endpoint $v(S)$. By Lemma 6.4 there is no other curve of D incident with $v(S)$. Hence $D = \{\phi_1, \phi_2\}$.

□

7 The Proof of Theorem 4.4

In previous sections we proved some properties of graph embeddings which enable us now to present the proof of Theorem 4.4.

Proof of Theorem 4.4. Let G be a 2-connected graph. Let G_1 and G_2 be pieces of twisting and let u and v be twisting vertices. Denote twisted G by G'. Suppose $G(S)$ is a minimal embedding of G and the dividing system D of $G(S)$ is a minimal dividing system of G. We will construct an embedding of G' on S or on a surface of the same genus. It will imply that $\gamma(G) \geq \gamma(G')$.

Using the fact that double twisting gives the same graph (twisted G' is graph isomorphic to G), it will follow that $\gamma(G) = \gamma(G')$.

According to Proposition 6.5 D consists of two loops with different endpoints or D contains no loop.

Two loops case. Let $D = \{\phi_u, \phi_v\}$ such that ϕ_u is a loop with endpoint $u(S)$ and ϕ_v is a loop with endpoint $v(S)$. Let Φ_u be a face of $G(S)$ containing points of $\phi_u((0;1))$ and Φ_v the face containing points of $\phi_v((0;1))$.

The graph G is 2-connected so G_1 and G_2 are connected graphs. Hence the union of components of $S - D$ containing G_i including their boundaries is a connected 2-manifold, $i = 1, 2$.

Since D consists of two loops, $S - D$ can have at most three components. If $S - D$ would have exactly three components then two of them contain points of the same G_i, assume G_1. Since ϕ_u and ϕ_v have no common point, the union of those components containing points of G_1 is not connected. Hence the number of components of $S - D$ is exactly two. Call them C_1 and C_2 such that C_i contains points of G_i, $i = 1, 2$.

The boundary curves of each C_i are some loops. Since S is "cutted" only along loops ϕ_u and ϕ_v, the boundary of each C_i can be formed by these loops only. If C_1 has a boundary consisting from only one loop, say ϕ_u, then $v(S)$ have to be an inner point of C_1 because $v(S) \notin \phi_u$ but G_1 contains edges incident with the vertex v. This is a contradiction. Hence the boundery of C_1 is formed by both ϕ_u and ϕ_v. The same property holds for C_2.

Let $C_i' = C_i \cup \phi_u \cup \phi_v$. Then $G_i(S) \subset C_i'$. Denote $u_i(S)$ and $v_i(S)$ the points $u(S)$ and $v(S)$ of C_i', $i = 1, 2$.

Now let a new surface S' be created from C_1' and C_2' by identifying the boundary curve ϕ_u of C_1' with the boundary curve ϕ_v of C_2' such that the points $u_1(S)$ and $v_2(S)$ are identified, and identifying the boundery curve ϕ_u of C_2' with the boundery curve ϕ_v of C_1' such that the points $u_2(S)$ and $v_1(S)$ are identified.

This operation is very close to the twisting operation. It is clear that the embedding of G_1 on C_1' and the embedding of G_2 on C_2' create the embedding of G' on S'. The only remaining thing to prove is that $\gamma(S) = \gamma(S')$. Since G and G' have the same number of vertices and edges it is enough to compute the number of faces of $G(S)$ and $G'(S')$. If these numbers are the same and if $G'(S')$ is 2-cell embedding, the proof is completed because both surfaces have the same euler characteristic.

The faces of $G(S)$ different from Φ_u and Φ_v are not effected by the operation described above. If $\Phi_u \neq \Phi_v$ then they are both divided by D into two parts. Since Φ_u can be map by a homeomorphism to the open unit disc, image of $\phi_u((0;1))$ is an open arc in this mapping. So we can assume the image of $\phi_u((0;1))$ is a line such that it is a diameter of the unit disc. So $\Phi_u \cap C_i$ is a homeomorphic image of the open half-disc for both $i = 1, 2$. The same thing holds for the face Φ_v.

Combining C_1' and C_2' gives one new face created from one part of Φ_u and one part of Φ_v and the second new face from the remaining parts of Φ_u and Φ_v. Each of these new faces is a 2-cell because is combined from two parts each homeomorphic to an open half-disc. Hence the number of faces remains the same and both new faces are open 2-cells.

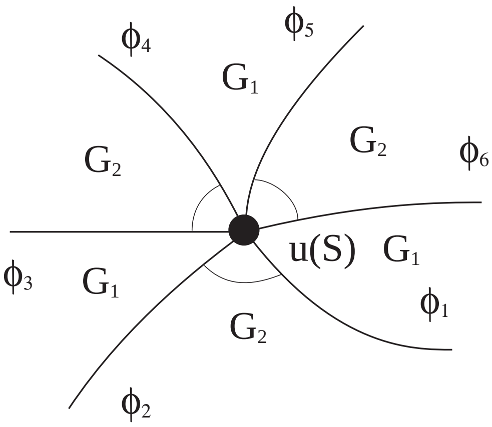
If $\Phi_u = \Phi_v = \Phi$, D divides this face into three parts. the construction of S' combines these parts of Φ into one new face of $G'(S')$. Hence the number of faces of $G(S)$ and $G'(S')$ is the same and this face is also an open 2-cell. Hence $\gamma(G) \geq \gamma(G')$.

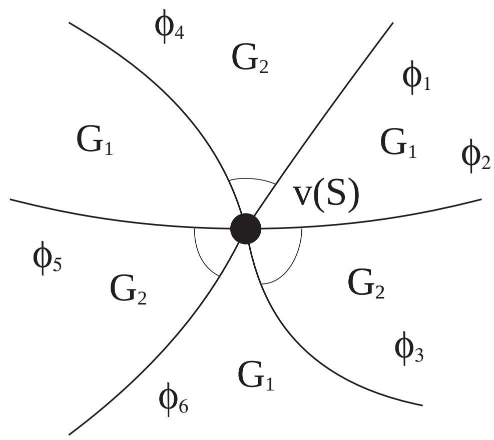
No loop case. Let $D = \{\phi_1, \phi_2, \ldots, \phi_k\}$ be a minimal dividing system which elements are only arcs. The arc ϕ_i will be called *oriented from u to v* if $\phi_i(0) = u(S)$ and $\phi_i(1) = v(S)$ and will be called *oriented from v to u* otherwise. Replacing any curve ϕ_i of D by a curve ${\phi_i}'(t) = \phi_i(1-t)$ does not effect the properties of D but only the orientation on ϕ_i. Curves of D can be considered as lines in a neighbourhood of the point $u(S)$. We may assume that they are indexed in the clock-wise order in this neighbourhood. The angles between two neighbouring lines representing curves of D can be labeled by G_1 or G_2 due to the pertinence to the component of $S - D$ and due to the incidency of points of $G(S)$ in this component. Then two angles having the common boundary line are labeled by different G_i from Lemma 6.2. In the neighbourhood of $v(S)$ the lines corresponding to the curves of D appear in an unknown order but again, two angles having common boundary belong to components of $S - D$ labeled by different G_i.

Note that G' is the same graph as G, only the edges of G_2 incident with u are drawn to v and the edges incident with v are drawn to the vertex u. Hence the embedding of $G(S)$ in which edges of G_2 incident with u become edges incident with v and edges incident with v become incident with u gives an embedding of G'.

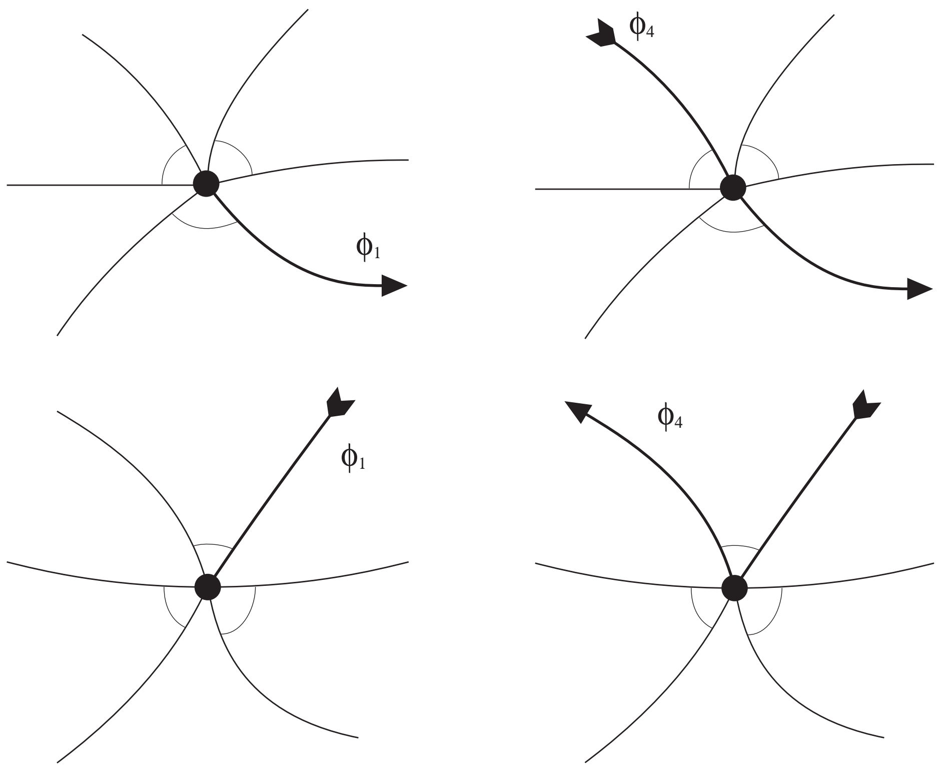
The orientation of ϕ_i (from u to v or conversally) does not effect the embedding of ϕ_i. Hence we will choose the orientation of curves of D such that each angle in the neighbourhood of $u(S)$ and $v(S)$ labeled by G_2 is boundered by one line corresponding to the arc oriented from u to v and another one oriented from v to u.

This is possible for example by the following construction. We start with D containing *"non-oriented"* curves.

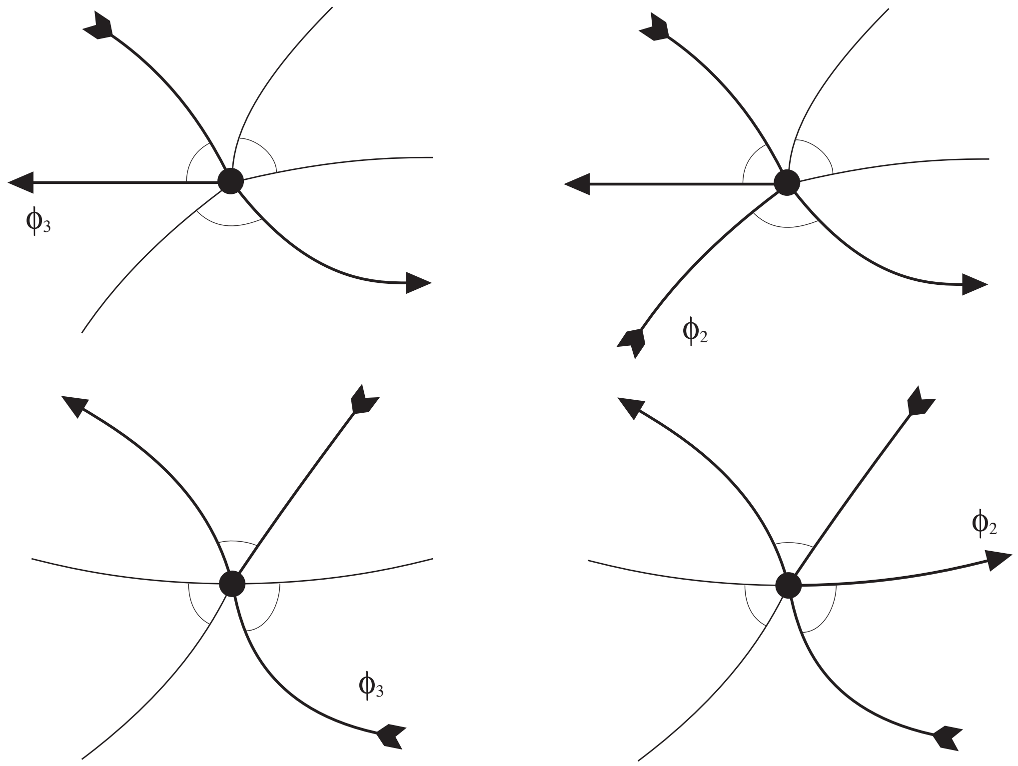




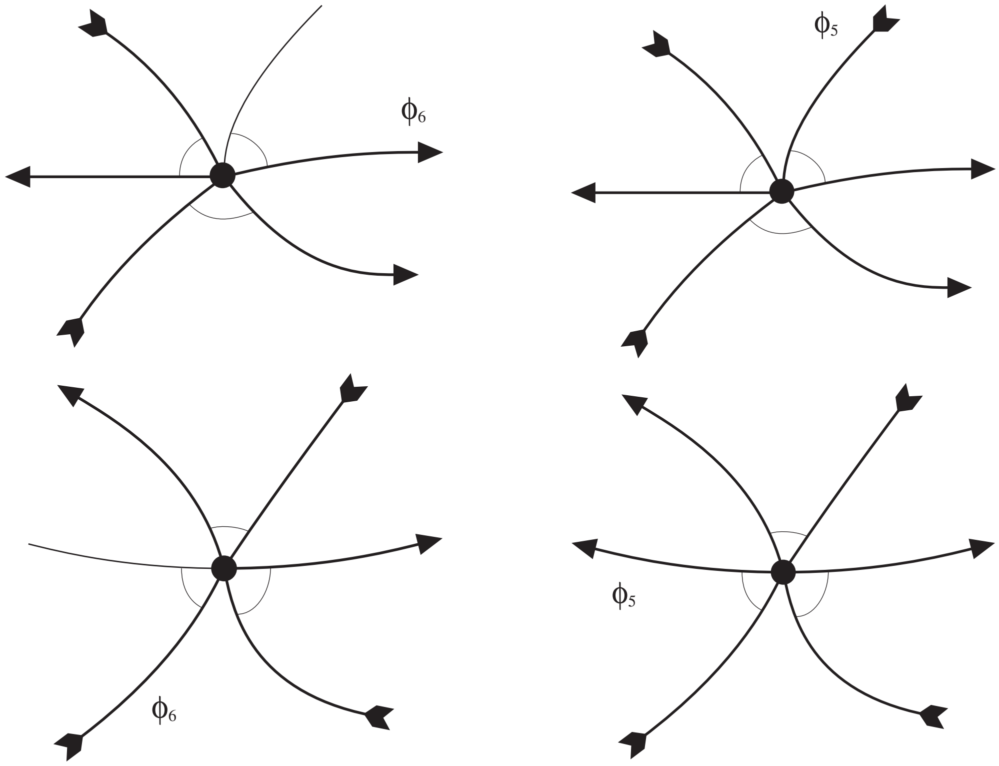
In each step we orient one of these curves. Let ϕ_1 be oriented from u to v. Let ϕ_{i_2} be the second boundary line of the angle boundered by ϕ_1 and labeled by G_2, in the neighbourhood of $v(S)$. We will orient this arc from v to u.



Also ϕ_{i_2} is one boundary of an angle in the neighbourhood of $u(S)$. the second boundary of this angle call ϕ_{i_3}. If $\phi_{i_3} \neq \phi_1$ we will orient ϕ_{i_3} from u to v. Then we will continue by induction alternately in the neighbourhoods of $u(S)$ and $v(S)$.

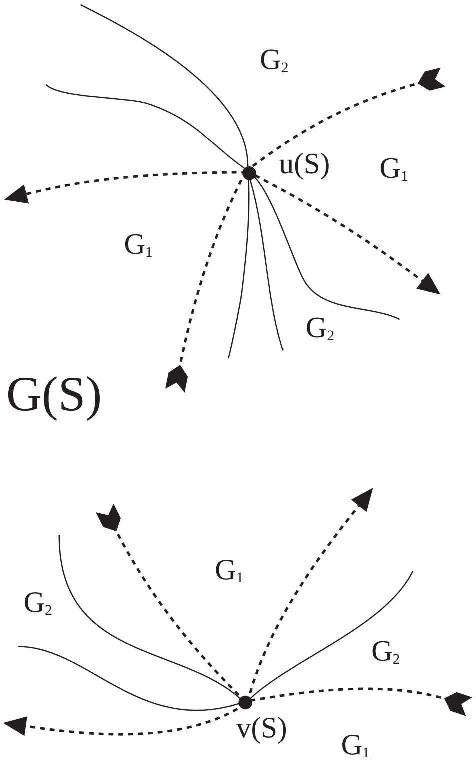


If we stop before orienting all curves of D, since the second boundary in an angle boundered by last oriented curve of D (this second boundary curve have to be ϕ_1), we will choose an arbitrary non-oriented curve of D and start with this curve as with ϕ_1 in the begining of the construction.

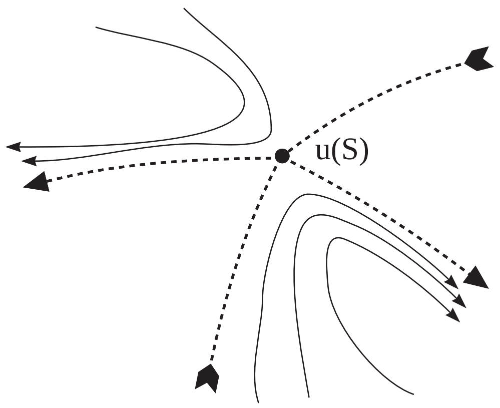


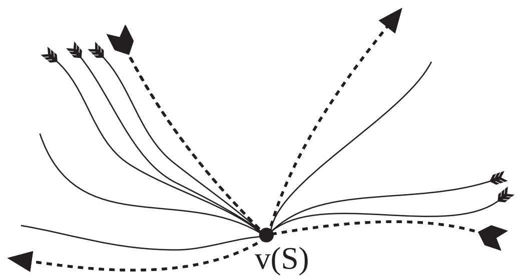
The construction will be finished when all curves of D are oriented. It is clear that the orientation of the curves of D have the property described above.

Now an embedding of $G'(S)$ derived from $G(S)$ will be constructed.

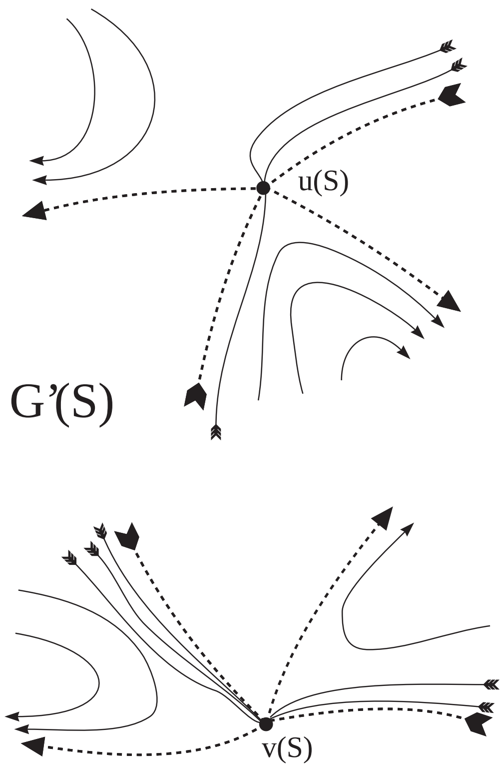


Let E' be the set of edges of G_2 in one angle in the neighbourhood of $u(S)$. They will be deleted in this neighbourhood and will be extended to point $v(S)$ along the curve of D corresponding to that line boundering that angle, which is oriented from u to v. It can be done such that the new edges do not intersect the arcs representing edges of G, arcs from D and also do not intersect themselves.





If it is done for all angles labeled by G_2 in the neighbourhood of $u(S)$ and then also in the neighbourhood of $v(S)$, all edges of G_2 incident with u become incident with v and conversally. Hence we obtain an embedding of G' on S.



Hence $\gamma(G) \geq \gamma(G')$.

$\square$

8 Acknowledgement

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