Using combinatorial maps in graph-topological computations

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Abstract

Possible use of combinatorial maps in graph-topological calculations are investigated continuing [10]. Some new permutational functions with interesting graph-topological interpretations are considered.

1 Introduction

We continue to investigate the combinatorial maps, see [1, 3, 4, 6]. We apply the idea used in [8] considering the corners between the edges in the embedding of the graph on the surface to be the elements on which the permutations act. This work is a continuation of [8, 9, 10].

We use permutation calculus as a tool for describing graphs on surfaces.In particular we use the fact that there can be established a one-one map between permutations and graphs on surfaces. A possible way how to do it is shown in [8]. A way, how to exploit this fact, is to find for some chosen feature of the graphs on surfaces the corresponding characteristics in permutations. Thus, every permutational formula has some equivalent operation on graphs. For a chosen map that has a graph on the surface in correspondence, any computable in permutations partial map has an object in this graph in correspondence. It is interesting to find such partial maps that has such graphic objects in correspondence that has some graphic or graph-topological interpretation. The other way round, if we can find for some operations on graphs corresponding operations on permutations, then we can hope, that some nontrivial manipulations on graphs can be done using some sufficiently simple operations on permutations, i.e. multiplication of permutations, selection of submaps of maps etc. In this work we prove some simple conjectures on permutations with less trivial graph-topological conjectures in correspondence.

We use computer program of permutation calculus checking our ideas on different series of [manually entered or randomly generated] maps. In future we hope to find sufficiently large set of permutational formulas with graph-topological operations in correspondence to use them independently for topological calculus without any other algorithmic support. From computational point of view it is sufficiently important that permutational calculus with simplest operations mostly can be done linearly with respect to the order of permutations or updated parts of them.

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2 Permutations, combinatorial maps and partial combinatorial maps

Permutations act on a universal set C the elements of which we call corners because of their geometrical interpretation. For a permutation P and $c \in C$ c^P denotes the element of C in which P maps c. Permutations we multiply from left to right.

In general we use the same terminology for combinatorial maps as in [8].

A pair of permutations (P, Q) we call *combinatorial map* whenever $P^{-1} \cdot Q$ is a matching, i.e. involution without fixed elements. The main characteristics of combinatorial map is its edge-rotation ϱ equal to $Q \cdot P^{-1}$ and next-edge-rotation π equal to $P^{-1} \cdot Q$.

Usually we are working with classes of maps with fixed π, that are closed under multiplication of maps from left.

Partial combinatorial maps (shorter *p-maps*) are pairs of permutation without any restriction on their multiplication, i.e. their edge rotation can be arbitrary permutation.

We use apparatus of permutations calculation developed in [9]. If C is divided into C_1 and C_2, then $p = \begin{cases} C_1 : & A \\ C_2 : & B \end{cases}$ denotes a permutation that is calculated using permutations A and B that act from C_1 into C [in case of A] and from C_2 into C [in case of B].

3 Cycles and cycle covers in combinatorial maps

For a combinatorial map (P, Q), a cyclic sequence of elements $c_1, \ldots, c_n$ $(n > 0)$ is called *cycle*, if for c_i, $(0 < i < n)$, next element c_{i+1} in the sequence is equal to c_i^P or c_i^Q. *Simple* cycle is a cycle without repetition of elements. For a combinatorial map simple cycles are transitive permutations on some subsets of C.

A *cycle cover* of the combinatorial map (P, Q) is a permutation acting on C, where each of its orbits is a cycle in (P, Q). Cycles in cycle covers are always simple cycles. *Trivial* cycle covers of (P, Q) are P and Q themselves.

Multiplications of cycle cover of some combinatorial map with submaps [8] of its next-edge-rotation π gives all possibly cycle covers of this combinatorial map.

Two cycles ζ_1 and ζ_2 in (P, Q) *touch* each other if e_1 belongs to ζ_1 and e_1^P or e_1^Q or e_1^{-P} or e_1^{-Q} belongs to cycle ζ_2 too.

Let us suppose, that for some combinatorial map (P, Q) with a fixed cycle cover the elements of C are colored in such a way, that 1)elements of cycles of the cycle cover are colored with the same color, and 2) cycles with equal element coloring do not touch. Such a coloring of elements of combinatorial map we call *a cycle cover coloring.*

4 Two-colorable cycle covers in combinatorial maps

We consider two-colorable cycle covers.

Let for a combinatorial map (P, Q) with cycle cover ζ elements be colored in two colors in the way that this is also a cycle cover coloring. Let an arbitrary edge be with its (possibly not all distinct) corners c_1, c_2, c_3, c_4, such that $c_2 = c_1^{-P}$, $c_3 = c_1^{\pi}$ and $c_4 = c_1^{-Q}$. There are three possibilities:

1) c_1 and c_2 have the same color and c_3 and c_4 have the same other color , [we call such

an edge **cut-edge**];
2) c_1 and c_4 have the same color and c_2 and c_3 have the same other color, [we call such an edge **cycle-edge**];
3) all corners of the edge are of the same color,[we call such an edge **inner edge**].

Let for a combinatorial map (P,Q) with cycle cover ζ elements are colored with two colors, green and red, in such a way that this is also a cycle cover coloring and there are not inner edges. Then $\mid C_{green} \mid = \mid C_{red} \mid$, where $C_{green} \cup C_{red} = C$, and π and ϱ are one-one matches between C_{green} and C_{red}.

We call a zigzag walk cover *a knot* in the combinatorial map [1] and [8]. Zigzag walk always has orbits of even degree, thus it is naturally to connect with a zigzag walk a cycle cover coloring [theorem 3 [10]]. Inversely, each cycle cover without inner edges fixes some knot.

Let for a chosen knot μ of (P,Q) the corresponding cycle cover be ζ and let us express it as $\zeta_{green} \cdot \zeta_{red}$, where ζ_{green} contains cycles of green elements, and ζ_{red} contains cycles of red elements.

It holds [theorem 4 [10]]
1) $\zeta \cdot \pi = \zeta_{altern}$, where ζ_{altern} is a cycle cover with alternating coloring of its elements;
2) $\zeta^{-1} \cdot P = \zeta_{altern}^{-1} \cdot Q = \pi_{cycle}$, where π_{cycle} have all possible cycle-edges;
3) $\zeta_{altern}^{-1} \cdot P = \zeta^{-1} \cdot Q = \pi_{cut}$, where π_{cut} have all possible cut-edges;
4) π_{cycle} and π_{cut} are complementary involutions: $\pi_{cycle} \cdot \pi_{cut} = \pi$.

5 Graphs on surfaces

In [10] we considered a map $[P \cdot P_{cycle}^{-1}, Q]$. It deserves an interest because it may be interpreted as map $[P,Q]$ with cut some cycles. Let us define some new permutational functions that may have interesting graph-topological interpretations. Let us define four following multiplications $A = P \cdot P_{cycle}^{-1}$, $B = Q \cdot Q_{cut}^{-1}$, $C = P \cdot P_{cut}^{-1}$ and $D = Q \cdot Q_{cycle}^{-1}$.

Next theorem shows that these permutations have similar properties:

Theorem 1.

$$A = P \cdot P_{cycle}^{-1} = \zeta \cdot \zeta_{cycle}^{-1},$$
$$B = Q \cdot Q_{cut}^{-1} = \zeta \cdot \zeta_{cut}^{-1},$$
$$C = P \cdot P_{cut}^{-1} = \zeta_{altern} \cdot \zeta_{altern_{cut}}^{-1}$$

and

$$D = Q \cdot Q_{cycle}^{-1} = \zeta_{altern} \cdot \zeta_{altern_{cycle}}^{-1}.$$

Proof. □

Theorem 2. *The restrictions of the partial maps*

$$[\zeta, Q]_{cut},$$
$$[\zeta, P]_{cycle},$$
$$[\zeta_{altern}, Q]_{cycle}$$

and

$$[\zeta_{altern}, P]_{cut}$$

are geometric maps.

Proof. Firstly, ζ restricted on C_{cycle} (C_{cut}) multiplied by π or π_{cycle} (π_{cut}) give the same result. Thus, the dual of the restriction coincide with the restriction of its dual. Certainly, the same is true for ζ_{altern}, P and Q too.

Further, $\zeta_{cut} \cdot \pi$ maps into $P \cdot \pi$ and $Q_{cut} \cdot \pi$ maps into $\zeta_{altern} \cdot \pi$, thus the map $[\zeta, Q]_{cut}$ is geometric.

Similarly we conclude for other restrictions.

$\zeta_{cycle} \cdot \pi$ maps into $Q \cdot \pi$ and $P_{cycle} \cdot \pi$ maps into $\zeta_{altern} \cdot \pi$, thus the map $[\zeta, P]_{cycle}$ is geometric.

$\zeta_{altern_{cycle}} \cdot \pi$ maps into $P \cdot \pi$ and $Q_{cycle} \cdot \pi$ maps into $\zeta \cdot \pi$, thus the map $[\zeta_{altern}, Q]_{cycle}$ is geometric.

$\zeta_{altern_{cut}} \cdot \pi$ maps into $Q \cdot \pi$ and $Q_{cut} \cdot \pi$ maps into $\zeta \cdot \pi$, thus the map $[\zeta_{altern}, P]_{cut}$ is geometric. □

Theorem 3.

$$\gamma[\zeta, P] = \gamma[\zeta, P]_{cycle};$$

$$\gamma[\zeta_{altern}, Q] = \gamma[\zeta_{altern}, Q]_{cycle};$$

$$\gamma[\zeta_{altern}, P] = \gamma[\zeta_{altern}, P]_{cut};$$

$$\gamma[\zeta, Q] = \gamma[\zeta, Q]_{cut}.$$

Proof. Let us prove that $\gamma[\zeta, P] = \gamma[\zeta, P]_{cycle}$. By previous theorem the restriction $\gamma[\zeta, P]_{cycle}$ is a geometric map. As follows, the next edge rotation $\pi_{[\zeta,P]}$ is equal to π_{cycle}. It means, that the edge rotation of $[\zeta, P]$ has "normal edges" from π_{cycle} and "half edges"corresponding to cut edges from π_{cut}. Eliminating an edge (a, b) from P that belongs to π_{cut} and choosing a new knot that has the same coloring of elements [that is always possible only by eliminating cut edges] the new value of P has the same π_{cycle} and π_{cut}. Thus, the new value of ζ should be $\zeta \mid_{a,b}$, i.e. with eliminated elements corresponding to the eliminated edge. It means that genus of $[P, \zeta]$ does not change by eliminating a cut edges.

We have proved that $\gamma[\zeta, P] = \gamma[\zeta, P]_{cycle}$. Duality gives us that $\gamma[\zeta, Q] = \gamma[\zeta, Q]_{cut}$. To prove that $\gamma[\zeta_{altern}, Q] = \gamma[\zeta_{altern}, Q]_{cycle}$ we use conjecture $\zeta_{altern}^{-1} \cdot Q = \pi_{cycle}$ and similarly as previously considerations. Applying duality we get $\gamma[\zeta_{altern}, P] = \gamma[\zeta_{altern}, P]_{cut}$. □

To the previous theorem we add some new conjectures.

Theorem 4.

$$\gamma[\zeta, P] = \gamma[\zeta, P]_{cycle} = \gamma[B, P];$$

$$\gamma[\zeta_{altern}, Q] = \gamma[\zeta_{altern}, Q]_{cycle} = \gamma[C, Q];$$

$$\gamma[\zeta_{altern}, P] = \gamma[\zeta_{altern}, P]_{cut} = \gamma[D, P];$$

$$\gamma[\zeta, Q] = \gamma[\zeta, Q]_{cut} = \gamma[A, Q].$$

Before to prove this we need a simple theorem.

Theorem 5. *There holds:*

$$\zeta = \begin{cases} C_{cut}: & A \\ C_{cycle}: & B \end{cases}$$

and

$$\zeta_{altern} = \begin{cases} C_{cycle}: & C \\ C_{cut}: & D \end{cases}.$$

Proof. For the first conjecture:

$$\zeta = \begin{cases} C_{cut}: & A \\ C_{cycle}: & B \end{cases} = \begin{cases} C_{cut}: & \zeta \cdot \zeta_{cycle}^{-1} \\ C_{cycle}: & \zeta \cdot \zeta_{cut}^{-1} \end{cases} = \zeta.$$

For the second conjecture:

$$\zeta_{altern} = \begin{cases} C_{cycle}: & C \\ C_{cut}: & D \end{cases}$$

$$= \begin{cases} C_{cycle}: & \zeta_{altern} \cdot \zeta_{altern_{cut}}^{-1} \\ C_{cut}: & \zeta_{altern} \cdot \zeta_{altern_{cycle}}^{-1} \end{cases} = \zeta_{altern}.$$

□

Now we return to the proof of theorem [4]. First we prove fourth right equation:

Theorem 6. $\gamma[\zeta, Q] = \gamma[A, Q]$.

To prove this conjecture we need additional lemma.

Lemma 7. *If* $\gamma[Y, Z] = 0$ *and* $X_{C_\beta} = i$ *then*

$$\gamma[\begin{cases} C_\alpha: & X \\ C_\beta: & Y \end{cases}, Z] = \gamma[X, Z].$$

According theorem 5 $\zeta = \begin{cases} C_{cut}: & A \\ C_{cycle}: & B \end{cases}$. We must prove that $\gamma[B, Q] = 0$ and $A_{C_{cycle}} = i$.

The first equation is proven by the following lemma.

Lemma 8. $\gamma[B, Q] = \gamma[Q \cdot Q_{cut}^{-1}, Q] = 0$.

Proof. □

The second equation is proven by the following trivial lemma.

Lemma 9. $\begin{cases} C_\alpha: & i \\ C_\beta: & X \cdot X_{alpha}^{-1} \end{cases} = i.$

Indeed, $A_{C_{cycle}} = \begin{cases} C_{cut}: & i \\ C_{cycle}: & P \cdot P_{cycle}^{-1} \end{cases} = i$. This completes the proof of the theorem.

Now the first right equation from theorem 4:

Theorem 10. $\gamma[\zeta, P] = \gamma[B, P]$.

Proof. From theorem 5

$$\zeta = \left\{ \begin{array}{ll} C_{cycle}: & B \\ C_{cut}: & A \end{array} \right. .$$

$$\gamma[A, P] = \gamma[Q \cdot Q_{cut}^{-1}, Q] = 0$$

by lemma 8 applied dually.

$$B_{C_{cut}} = \left\{ \begin{array}{ll} C_{cycle}: & i \\ C_{cut}: & Q \cdot Q_{cut}^{-1} \end{array} \right. = i$$

by lemma 9. □

Now the second right equation from theorem 4:

Theorem 11. $\gamma[\zeta_{altern}, Q] = \gamma[C, Q]$.

Proof. From theorem 5

$$\zeta_{altern} = \left\{ \begin{array}{ll} C_{cycle}: & C \\ C_{cut}: & D \end{array} \right. .$$

$$\gamma[D, Q] = \gamma[Q \cdot Q_{cycle}^{-1}, Q] = 0$$

by lemma 8 applied dually.

$$C_{C_{cycle}} = \left\{ \begin{array}{ll} C_{cut}: & i \\ C_{cycle}: & P \cdot P_{cycle}^{-1} \end{array} \right. = i$$

by lemma 9. □

Now the third right equation from theorem 4:

Theorem 12. $\gamma[\zeta_{altern}, P] = \gamma[D, P]$.

Proof. From theorem 5

$$\zeta_{altern} = \left\{ \begin{array}{ll} C_{cut}: & D \\ C_{cycle}: & C \end{array} \right. .$$

$$\gamma[C, P] = \gamma[P \cdot P_{cut}^{-1}, P] = 0$$

by lemma 8.

$$D_{C_{cycle}} = \left\{ \begin{array}{ll} C_{cut}: & i \\ C_{cycle}: & Q \cdot Q_{cycle}^{-1} \end{array} \right. = i$$

by lemma 9. □

We have ended the proof of the theorem 4.

In [10] in theorem 6 (Main theorem) inequality is too strong.There are maps that reach equality. We give this theorem in a new appearance.

Theorem 13.

$$\gamma_{(P,Q)} \geq \gamma_{(P,\zeta)} + \gamma_{(Q,\zeta)}$$

and

$$\gamma_{(P,Q)} >= \gamma_{(P,\zeta_{altern})} + \gamma_{(Q,\zeta_{altern})}$$

Proof. $\square$

To prove this theorem we must prove a lemma.

Lemma 14.

$$\|\zeta\| \geq c_{[P,\zeta]} + c_{[Q,\zeta]} - c_{[P,Q]},$$

$$\|\zeta_{altern}\| \geq c_{[P,\zeta_{altern}]} + c_{[Q,\zeta_{altern}]} - c_{[P,Q]}.$$

Let us add a theorem without proof.

Theorem 15. *For cubic maps $[P,Q]$ there hold:*

$$\gamma[P,\zeta] = \gamma[P,\zeta]_{cycle} = \gamma[P,\zeta_{altern}] = \gamma[P,\zeta_{altern}]_{cut} = 0;$$

$$\gamma[B,P] = \gamma[D,P] = \gamma[A,D] = 0;$$

$$\gamma[A,D] = \gamma[B,C] = 0.$$

Partial maps $[A,Q]$ and $[B,C]$ are of particular interest because the first contains all cut-edges and the second - all cycle-edges and they are two cut parts of $[P,Q]$ together comprising all $[P,Q]$.

In order to illustrate all characteristics that we have considered we give an example of a combinatorial map corresponding to K_5.

$P =$ (1 7 5 3)(2 10 18 16)(4 12 20 17)(6 14 15 19)(8 13 11 9)

$Q =$(1 8 14 16)(2 9 7 6 13 12 19 5 4 11 10 17 3)(15 20 18)

$\varrho =$ (1 9)(2 11)(3 16)(4 13)(5 17)(6 8)(7 19)(10 20)(12 15)(14 18)

$\mu=$ (1 2 11 12 15 16 3 4 13 14 18 17 5 6 8 7 19 20 10 9)

$\alpha =$ (1 8 14 11 10 17 16 19 5 4)(2 7 13 12 9 18 15 20 6 3)

$\beta =$ (1 7)(2 3)(4 5)(6 19)(8 14)(9 11)(10 17)(12 13)(15 20)(16 18)

$\pi_{cut} =$ (1 2)(3 4)(11 12)(13 14)(17 18)(19 20)

$\pi_{cycle} =$ (5 6)(7 8)(9 10)(15 16)

$A :$ (1 5 3)(2 16)(4 12 20 17)(6 14)(8 13 11)(10 18)(15 19)

$B :$ (1 8)(2 9 7 6)(5 19)(10 11)(14 16)(15 18)

$C :$ (1 7 5)(2 10)(6 19)(8 11 9)(14 15)(16 18)

$D :$ (1 16)(2 10 17 3)(4 11 5)(6 13 12 19)(8 14)(15 20 18)

$\pi_{[A,D]}$: (1 2)(3 4)(5 16 10 15 6 8)(11 12)(13 14)(17 18)(19 20)

$\pi_{[C,B]}$: (1 19 2 11)(5 6)(7 8)(9 10)(14 18)(15 16)

$[\zeta, P]_{cycle}$=(5 7)(6 15)(8 9)(10 16)
=(5 8 10 15)(6 16 9 7)

$[\zeta_{altern}, P]_{cut}$=(1 3)(2 18)(4 12 20 17)(11 13)(14 19)
=(1 4 11 14 20 18)(2 17 3)(12 19 13)

$[\zeta, P]_{cycle}$=(1 14)(2 13 12 19 4 11 17 3)(18 20)
=(1 13 11 18 19 3)(2 14)(4 12 20 17)

$[\zeta_{altern}, Q]_{cycle}$=(5 10 9 7 6)(8 16)
=(5 9 8 15 16 7)

ζ =(1 8 13 11 10 18 15 19 5 3)(2 9 7 6 14 16)(4 12 20 17)

ζ_{altern} =(1 7 5 4 11 9 8 14 15 20 18 16)(2 10 17 3)(6 13 12 19).

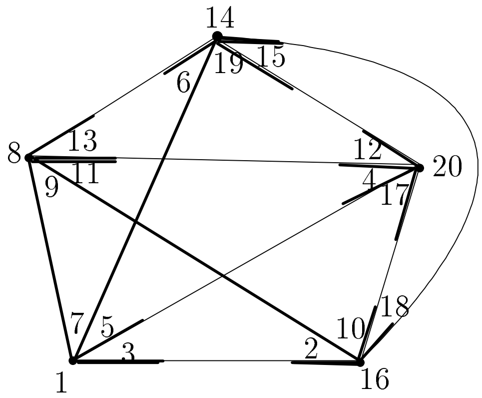

Figure 1: Combinatorial map $[P, Q]$ corresponding to K_5. $[P, \zeta]$ is drawn bold. $[P, \zeta_{altern}]$ can be seen changing cut-edges to cycle-edges and cycle-edges to cut-edges

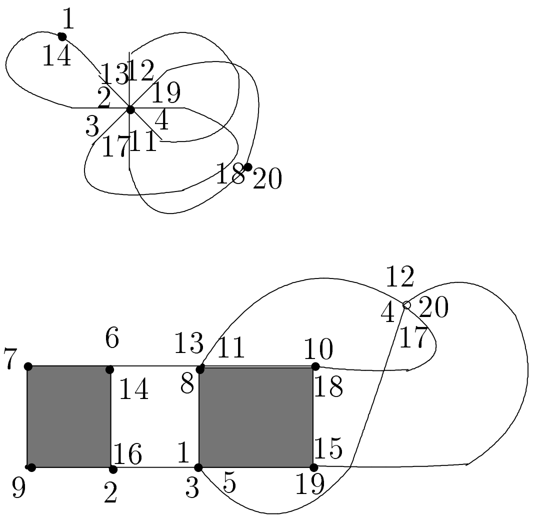

Figure 2: Cycle dual submap $[\zeta, Q]_{cut}$ and partial map $[A, Q]$. Orbits of ζ are easy seen as cycles in $[A, Q]$.

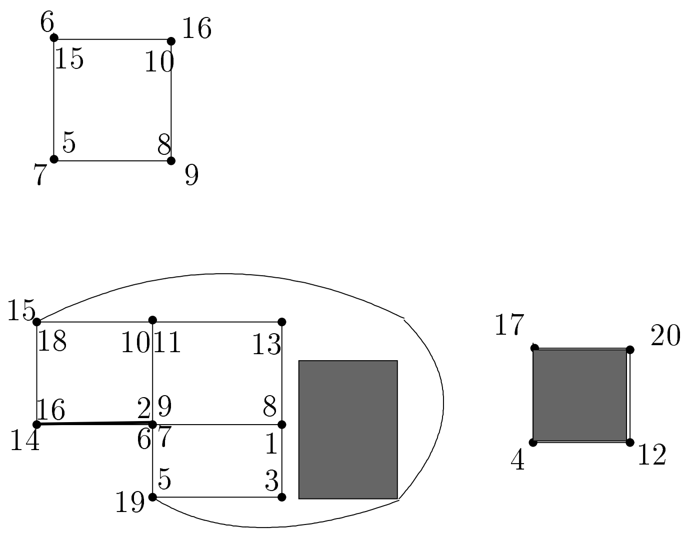

Figure 3: Cycle submap $[\zeta, P]_{cycle}$ and partial map $[B, P]$. Orbits of ζ are easy seen as cycles in $[B, P]$.

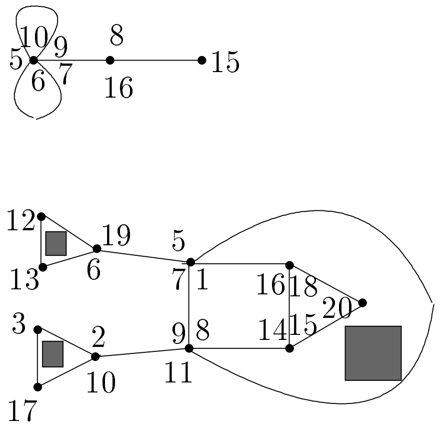

Figure 4: Cut dual submap $[\zeta_{altern}, Q]_{cycle}$ and partial map $[C, Q]$. Orbits of ζ_{altern} are easy seen as cycles in $[C, Q]$.

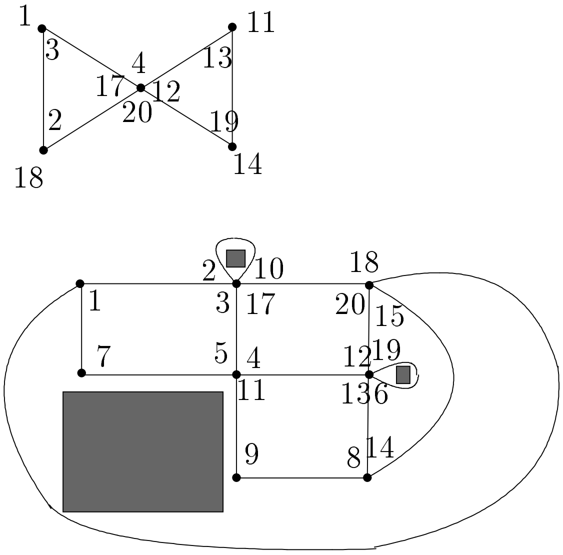

Figure 5: Cut submap $[\zeta_{altern}, P]_{cut}$ and partial map $[D, P]$. Orbits of ζ_{altern} are easy seen as cycles in $[D, P]$.

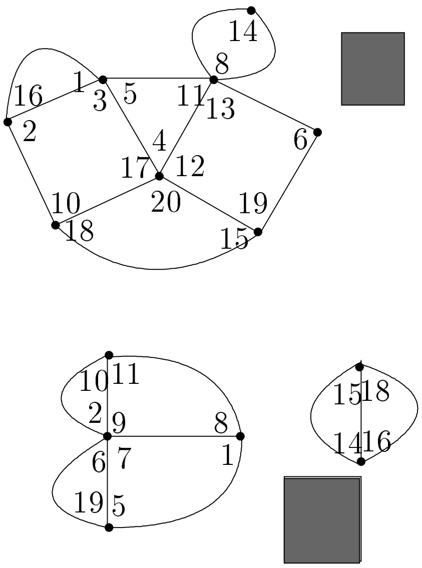

Figure 6: Partial maps $[A, D]$ and $[B, C]$. $[A, Q]$ have all cut-edges and $[B, C]$ have all cycle-edge.

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