Fuzzy Frames

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Introduction

Let $\mathbb{T}$ be a complete lattice. Recall that a (Chang-Goguen) $\mathbb{T}$-fuzzy space is a couple (X, L) such that L (the $\mathbb{T}$-fuzzy topology on the set X, or, the set of open fuzzy subsets of the space) is a subset of the set of mappings $\mathbb{T}^X$

(1) containing the constant zero and the constant one maps,

(2) closed under meets $\alpha \wedge \beta$, $(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x)$, and

(3) closed under general joins $\bigvee \alpha_i$, $(\bigvee \alpha_i)(x) = \bigvee \alpha_i(x)$.

Obviously, if $\mathbb{T}$ is a frame (see 1.3 below) then such an L, being a subframe of the frame $\mathbb{T}^X$, is a frame as well.

If $\mathbb{T}$ is spatial (1.3 ; for instance, if $\mathbb{T}$ is linearly ordered, which is the case we will restrict ourselves to in this article), also L is spatial; hence it is

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isomorphic to a (crisp) topology on another set Y. Thus, the lattice (frame) structure of L cannot contain, in a non-trivial case, enough information to reconstruct the $\mathbb{T}$-fuzzy space (X, L). The main purpose of this article is a study of an enrichment of the frame L (a point-free one, that is, defined in lattice theoretic terms) which would carry the information required. That is, we wish for an enrichment of L from which we would be able to determine

- what the underlying set is, and
- in what way the elements of our frame appear on this set as fuzzy subsets.

Or, let us put it as follows: we are heading for a structure, which we will call a $\mathbb{T}$-fuzzy frame, related to a frame in a way parallel to that in which a $\mathbb{T}$-fuzzy space is related to a space, and in the same time generalizing $\mathbb{T}$-fuzzy spaces in a way parallel to frames generalizing spaces. To visualize the situation see the diagram

$$\begin{array}{ccc} \text{fuzzy spaces} & \longrightarrow & \text{fuzzy frames} \\ \uparrow & & \uparrow \\ \text{spaces} & \longrightarrow & \text{frames} \end{array}$$

where the vertical arrows indicate the enrichment by fuzzification, and the horizontal ones indicate the extension by going point-free (it should be emphasized right away that the point is NOT in fuzzifying the frame as an algebra by mechanically considering the associated fuzzy algebra: that would not make sense in the upper horizontal arrow).

At this moment we should explain one thing: Strictly speaking, even a (crisp) space (X, τ) cannot be recovered from the lattice τ of open set in a quite general setting. The class of spaces in which one looks for the representation has to be restricted. For instance, one takes the restriction to *sober* spaces and then the (X, τ) can be reconstructed from τ by the well-known *spectrum* construction (see, e.g., [8]). There is a notion of $\mathbb{T}$-sobriety and a $\mathbb{T}$-spectrum construction (see [4]) reconstructing (exactly) the $\mathbb{T}$-fuzzy spaces that are $\mathbb{T}$-sober. Thus, at the first sight it may seem that the motive for the above mentioned enrichment of frames is not all that strong: One has to restrict the classes anyway, and hence, similarly as in the classical restriction to sober spaces, we can restrict ourselves to $\mathbb{T}$-sober $\mathbb{T}$-fuzzy ones, and everything is all right. But it is not. The point is that while the (crisp) sobriety is a property satisfied in a very large class (containing, e.g., all the Hausdorff spaces, and all the finite T_0-ones; moreover, the concept comes from elsewhere and is not specially tailored for the purposes of point-free topology), the $\mathbb{T}$-sobriety is an ad hoc defined property satisfied in very few $\mathbb{T}$-fuzzy spaces (even in very few finite ones).

Another restriction allows for a fairly satisfactory spectrum theory. B. Banaschewski has recently suggested ([1]) an elegant construction suitable for stratified $\mathbb{T}$-fuzzy spaces. This is a very important and useful class of fuzzy spaces. For our purposes, however, it still has the drawback that each of its members has to use all the values of $\mathbb{T}$ (so that, for instance, the "crisp ones in among the $\mathbb{T}$-fuzzy spaces", the $\mathbb{T}$-fuzzy spaces in which the incidence values of the (fuzzy) open sets just happen to be only 0 and 1, do not qualify).

The range of the $\mathbb{T}$-fuzzy spaces determined by he $\mathbb{T}$-fuzzy frames we are going to deal with in this article is much broader, containing all the $\mathbb{T}$-fuzzy spaces the natural crisp modification of which is sober in the classical sense.

There are two natural forgetful functors from the category of $\mathbb{T}$-fuzzy frames into the category of frames. One of them has a left adjoint determining for a (plain) frame L a universal fuzzy frame with the operating part L; in particular, it yields for a given fixed spatial frame L the system of all $\mathbb{T}$-fuzzy spaces (X, L') with L' isomorphic to L. The other one has a right adjoint yielding for a given frame L the universal fuzzy frame carried by L; in particular it yields for a fixed topological space (X, τ) the system of all the subframes L of $\mathbb{T}^X$ the crisp modification of which is τ.

Throughout this article, the value frame $\mathbb{T}$ is assumed to be linearly ordered. This restriction is done, partly, for technical reasons. It was rightly pointed out by U. Höhle that what we do is in fact working with the meet-irreducibles of $\mathbb{T}$, the structure of which is particularly simple in the linear case, and that one may use the meet-irreducibles also in more general cases (say, for spatial $\mathbb{T}$'s). Extensions of this sort are indeed possible, but, besides the technicalities, there are also some essential questions that deserve a carefull analysis and a thorough discussion. That will be done in a separate article. On the other hand, it is in particular the linear case in which we envisage a potential use of the notion of a fuzzy frame. One of the questions of interest in the fuzzy space theory is that of well-founded definitions of the structures of the uniformity type (see, e.g., [6], [11], [14]). The case of the linear value lattice $\mathbb{T}$, most important from the historical point of view, and still very important from the point of view of applications, does not allow the direct approach through uniformizing the fuzzy topology L as a frame: a uniformity on L induces a uniformity on the set of values (as observed by B. Banaschewski) and since the only linearly ordered frame admitting a uniformity is the two-point Boolean algebra $\{0, 1\}$ we would be left with the crisp case. One can, however, think of definitions based on the concept of fuzzy frame which would be more satisfactory.

Finally, let us note that an extra multiplication on the value lattice $\mathbb{T}$, or a quasi-complement (which appear in some modified definitions of a fuzzy-space) do not seem to present dificulties and could be built into the theory if we wished so. In order to preserve the transparency of the exposition, and of course also in order to keep the size of the article in reasonable limits, we have not pursued this line here.

The paper is organized as follows: Section 1 contains some general facts from category theory and a basic informatiom on frames. In Section 2 we discuss basics on fuzzy spaces, and present a technical lemma which is then repeatedly used throughout the text. In Section 3 we first discuss a point-free description of fuzzy spaces (in essence, equivalent with the usual one); justified by this, we present a definition of fuzzy frame and fuzzy frame homomorphism. Section 4 is devoted to the fuzzy spectrum and its properties. In Section 5, basic properties of the category of fuzzy frames are discussed. In Sections 6 and 7 the free and co-free functors are described; as applications we present the ensuing constructions of universal fuzzy spaces (X_L, L) into which all the (X, L') with L' isomorphic to L can be naturally embedded as subspaces, and, dually, universal fuzzy spaces (X, L_τ) from which one can derive all the fuzzy topologies L on X with crisp modification τ.

1. Preliminaries

1.1. Notation: A mapping associating elements a_i with elements i of an index set J will be sometimes called a *family* or a *system* and denoted by

$$(a_i \mid i \in J), \quad \text{or simply} \quad (a_i)_{i \in J}, \quad \text{or} \quad (a_i)_i$$

as opposed to $\{a_i \mid i \in J\}$, the set of the elements a_i where the order or repetition of the elements is irrelevant). Sometimes we will think of indexed sets as families even if the actual indexing is not important.

1.2. Recall that a family of morphisms $(\varphi_i : A \to B_i)_{i\in J}$ (resp. $(\varphi_i : A_i \to B)_{i\in J}$) in a category $\mathcal{C}$ is said to be *collectionwise monomorphic* – briefly, c.m. – (resp. *collectionwise epimorphic* – briefly, c.e. –) if

$$(\forall i \in J,\ \varphi_i\alpha = \varphi_i\beta \ \ (\text{resp, } \alpha\varphi_i = \beta\varphi_i)) \quad \Rightarrow \quad \alpha = \beta.$$

A c.m. (resp. c.e.) system is said to be collectionwise extremally monomorphic resp. collectionwise extremally epimorphic – briefly, c.e.m., resp. c.e.e. – if, moreover

$$\begin{array}{lll} & (\forall i \in J,\ \varphi_i = \varphi_i'\varepsilon)\ \&\ \varepsilon \text{ epimorphic} \quad \Rightarrow \quad \varepsilon \text{ is an isomorphism} \\ (\text{resp.} & (\forall i \in J,\ \varphi_i = \mu\varphi_i')\ \&\ \mu \text{ monomorphic} \quad \Rightarrow \quad \mu \text{ is an isomorphism }). \end{array}$$

Note that

(1.2.1) *limits constitute c.e.m. systems, and colimits constitute c.e.e. ones.*

We will use the following simple fact:

Lemma Let $\mathcal{C}$ have coequalizers. Then a system $(\varphi_i : A \to B_i)_{i\in J}$ is c.m. iff each $\gamma : A \to C$ such that $(\forall i\ \exists\psi_i : C \to B_i$ such that $\psi_i\gamma = \varphi_i)$ is a monomorphism.

Proof: Let $(\varphi_i)_i$ be c.m., let $\gamma\alpha = \gamma\beta$. Then $\psi_i\gamma\alpha = \psi_i\gamma\beta$ and hence $\alpha = \beta$. On the other hand, let the condition hold and let $\varphi_i\alpha = \varphi_i\beta$ for all i. Take $\gamma = \mathrm{Coequ}(\alpha, \beta)$ Then the ψ_i exist, γ is a monomorphism, hence an isomorphism, and hence $\alpha = \beta$.

□

1.3. Recall that a *frame* is a complete lattice L satisfying the distributivity law

$$(\bigvee S) \wedge b = \bigvee\{a \wedge b \mid a \in S\}$$

for any $b \in L$ and any subset $S \subseteq L$, and that a *frame homomorphism* $h : L \to M$ is a mapping preserving all joins (including the bottom 0) and finite meets (including the top 1). The category of frames and frame homomorphisms will be denoted by

Frm.

The one-element frame $\{0 = 1\}$ will be denoted by $\mathbf{1}$ and the two-element frame (Boolean algebra) $\{0 < 1\}$ will be denoted by $\mathbf{2}$. For details about frames see, e.g., [8], or [18].

If X is a topological space, the lattice $\mathfrak{O}(X)$ of its open sets is a frame, and if $f : X \to Y$ is a continuous map then $\mathfrak{O}(f) : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ defined by $\mathfrak{O}(f)(U) = f^{-1}(U)$ is a frame homomorphism. Thus we have a functor (**Top** is the category of spaces)

$$\mathfrak{O} : \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}.$$

The category $\mathbf{Frm}^{\mathrm{op}}$ is called the category of *locales* and denoted by **Loc**.

A frame L is said to be *spatial* if $L \cong \mathfrak{O}(X)$ for some space X; or, equivalently, if for any two distinct $a, b \in L$ there is a frame homomorphism $h : L \to \mathbf{2}$ such that $h(a) \neq h(b)$.

For convenience, we will sometimes write $\mathfrak{P}(X)$ for the set of all subsets of a set X (thus, $\mathfrak{P}(X) = \mathfrak{O}(X')$ where X' is X endowed with the discrete topology), and if $f : X \to Y$ is a mapping then $\mathfrak{P}(f) = (M \mapsto f^{-1}(M)) : \mathfrak{P}(Y) \to \mathfrak{P}(X)$.

1.4. If L is a frame and S any subset of L, the subframe generated by S will be denoted by

$$\langle S \rangle.$$

Here are some standard facts about the category **Frm**:

- **Frm** is complete and cocomplete;
- $(h_i : L \to M_i)_{i \in J}$ is c.m. in **Frm** iff for any two distinct $a, b \in L$ there is an $i \in J$ such that $h_i(a) \neq h_i(b)$; in particular, monomorphisms in **Frm** are exactly the one-one homomorphisms;
- $(h_i : L_i \to M)_{i \in J}$ is c.e.e. in **Frm** iff $\langle \bigcup_{i \in J} h_i[L_i] \rangle = M$; in particular, extremal epimorphisms in **Frm** are exactly the homomorphisms onto.

We immediately infer the following

Lemma Let $(\mu_i : L \to M_i)_i$ be c.m. in **Frm**.

1. Let $h : N \to M$ be a mapping such that each $\mu_i h : N \to M_i$ is a frame homomorphism. Then h is a frame homomorphism.

2. Let h, k be mappings such that $\mu_i h \leq \mu_i k$ for all i. Then $h \leq k$.

Let $(\varepsilon_i : L_i \to M)_i$ be c.e.e. in **Frm**.

3. Let h, k be frame homomorphisms such that $h\varepsilon_i \leq k\varepsilon_i$ for all i. Then $h \leq k$.

□

In **Frm** we have the obvious (extremal epi, mono)-factorization of morphisms:

$$(h : L \to M) = (L \xrightarrow{e=(a\mapsto h(a))} h[L] \xrightarrow{m=\subseteq} M).$$

Due to the contravariant relation of spaces to frames, an onto frame homomorphism (an extremal epimorphism in **Frm** and hence an extremal monomorphism in **Loc**) $h : L \to M$ represents an embedding of the generalized space represented by M into the generalized space represented by L; hence, they are referred to as *sublocales* (of course, not to be confused with subframes). Recall, further, that (natural equivalence classes of) subframes of a fixed frame constitute a complete lattice, and that the intersection of sublocales $(h_i : L \to M_i)_{i\in J}$ is obtained from the colimit of the naturally ensueing diagram.

1.5. Proposition Let $(h_i : L \to M)_{i\in J}$ be a system of frame homomorphisms linearly ordered by the natural order. Then $h = \bigvee_{i\in J} h_i = (a \mapsto \bigvee_{i\in J} h_i(a))$ is a frame homomorphism.

Proof: Obviously $h(0) = 0$, $h(1) = 1$, $h(\bigvee a_j) = \bigvee h(a_j)$ and $h(a \wedge b) \leq h(a) \wedge h(b)$. Finally,

$$h(a) \wedge h(b) = \bigvee h_i(a) \wedge \bigvee h_i(b) = \bigvee\{h_i(a) \wedge h_j(b) \mid i, j \in J\} = *$$

and denoting by h_k the larger of h_i, h_j we obtain

$$* \leq \bigvee\{h_k(a) \wedge h_k(b) \mid i, j \in J\} = \bigvee\{h_k(a \wedge b) \mid k \in J\} = h(a \wedge b).$$

□

1.6. An element a of a frame is *meet-irreducible* if $a \neq 1$ and if the inequality $a \geq a \wedge b$ implies that either $a \geq b$ or $a \geq c$ (equivalently, $a = b \wedge c \;\;\Rightarrow\;\; a = b$ or $a = c$). Obviously, in the frame $\mathfrak{O}(X)$, each $X \setminus \overline{\{x\}}$ is meet-irreducible. A space X is said to be *sober* if it is T_0 and if there are no other meet irreducibles but the $X \setminus \overline{\{x\}}$.

It is a standard fact that if Y is a sober space then $f \mapsto \mathfrak{O}(f)$ is an invertible correspondence between the continuous maps $f : X \to Y$ and the frame homomorphisms $h : \mathfrak{O}(Y) \to \mathfrak{O}(X)$. Thus, $\mathfrak{O}$ constitutes a full embedding

$$\mathbf{Sob} \to \mathbf{Loc}$$

(where **Sob** is the full subcategory of **Top** generated by the sober spaces).

1.7. Finally, recall the standard *spectrum* construction

$$\mathfrak{S}L = (\{\alpha \mid \alpha : L \to \mathbf{2} \text{ frame homomorphism}\}, \{\mathfrak{S}_a \mid a \in L\})$$

(where $\mathfrak{S}_a = \{\alpha \mid \alpha(a) = 1\}$) for frames L, and for frame homomorphisms $h : L \to M$

$$\mathfrak{S}h : \mathfrak{S}M \to \mathfrak{S}L \;\text{ defined by }\; \mathfrak{S}h(\alpha) = \alpha h.$$

This constitutes a functor

$$\mathfrak{S} : \mathbf{Loc}(= \mathbf{Frm}^{\mathrm{op}}) \to \mathbf{Top}.$$

The following are well-known facts:

- $\mathfrak{S}$ is a right adjoint for $\mathfrak{O}$,
- each $\mathfrak{S}L$ is sober,
- the unit map $X \to \mathfrak{S}\mathfrak{O}(X)$ is a homeomorphism iff X is sober, and
- the unit morphism $\mathfrak{O}\mathfrak{S}L \to L$ is an isomorphism iff L is spatial.

2. Fuzzy spaces

2.1. Let $\mathbb{T}$ be a frame. Recall that a (Chang-Goguen) *T-fuzzy space* is a couple (X, L) where L is a subframe of the frame $\mathbb{T}^X$ of all mappings $X \to \mathbb{T}$ ([2], [3]). A $\mathbb{T}$-*fuzzy continuous map* $(X, L) \to (Y, M)$ is a map $f : X \to Y$

such that the correspondence $(u \mapsto u \cdot f)$ maps M into L. Note that then, necessarily,

$$(u \mapsto u \cdot f) : M \to L \quad \textit{is a frame homomorphism.}$$

The resulting category will be denoted by

$$\mathbb{T}\text{–}\,\mathbf{FuzzTop}.$$

As the value frame $\mathbb{T}$ will be, typically, fixed, we will usually drop the prefix and speak of *fuzzy space, fuzzy continuous map* , and of the category **FuzzTop**.

Convention: The bottom resp. top of $\mathbb{T}^X$ (and hence of L) is the constant map sending all $x \in X$ to 0 resp. 1. Denoting these maps by 0 resp. 1 will hardly create confusion.

2.2. Throughout this article, $\mathbb{T}$ will be a *linearly ordered* complete lattice. We will often work with the poset

$$\mathbb{T}' = \mathbb{T} \setminus \{1\}.$$

(Note that, due to the linearity, $\mathbb{T}'$ is constituted by exactly the meet-irreducible elements of $\mathbb{T}$. The set of meet irreducibles of $\mathbb{T}$ is what one has to concentrate upon when generalizing the facts which will be presented in this article for more general $\mathbb{T}$; such generalizations need, however, much more than just changing the system $\mathbb{T}'$ and go beyond the scope of this article.

Also note that each linearly ordered complete lattice is a spatial frame.)

2.3. Remark: Whenever $\mathbb{T}$ is spatial (in particular, linearly ordered), each L in a $\mathbb{T}$-fuzzy space (X, L) is spatial as well. Indeed, let $u_0 \neq u_1$. Choose an $x \in X$ such that $u_0(x) \neq u_1(x)$ and a frame homomorphism $h : \mathbb{T} \to \mathbf{2}$ such that $h(u_0(x)) \neq h(u_1(x))$. Then $(u \mapsto h(u(x))) : L \to \mathbf{2}$ is a frame homomorphism separating u_0 and u_1.

2.4. Let (X, L) be a $\mathbb{T}$-fuzzy frame. For $u \in L$ and $t \in \mathbb{T}'$ set

$$\omega_t(u) = \{x \mid u(x) > t\} \subseteq X. \tag{2.4.1}$$

The topology on X with the subbase $\{\omega_t(u) = \{x \mid u(x) > t\} \mid t \in \mathbb{T}',\ u \in L\}$ (in other words, to use the notation of 1.4, the subframe $\langle\{\omega_t(u) = \{x \mid u(x) > t\} \mid t \in \mathbb{T}',\ u \in L\}\rangle$ of $\mathfrak{P}(X)$) will be denoted by

$$\tau L$$

and the space $(X, \tau L)$ (or the τL itself) will be sometimes referred to as the *crisp modification* of (X, L) resp. L.

2.5. Let $f : (X, L) \to (Y, M)$ be a fuzzy continuous map. Then for each $u \in M$ we have $uf \in L$ and

$$f^{-1}(\omega_t^M(u)) = \{x \mid f(x) \in \omega_t^M(u)\} = \{x \mid uf(x) > t\} = \omega_t^L(uf). \quad (2.5.1)$$

Thus,

$$f : (X, \tau L) \to (Y, \tau M) \textit{ is continuous.}$$

2.6. We say that a fuzzy space (X, L) is $\mathfrak{l}$-T_0 if for any two distinct $x, y \in X$ there is a $u \in L$ such that $u(x) \neq u(y)$. Obviously, this is equivalent to stating that $(X, \tau L)$ is T_0.

2.7. Recall from [4] the elegant extension of the spectrum adjunction:
The functor

$$\mathfrak{O}_{\mathbb{T}} : \mathbb{T}\text{--}\mathbf{FuzzTop} \to \mathbf{Loc}(= \mathbf{Frm}^{\mathrm{op}})$$

given by $\mathfrak{O}_{\mathbb{T}}(X, L) = L$, $\mathfrak{O}_{\mathbb{T}}(f)(u) = uf$, has a right adjoint

$$\mathfrak{S}_{\mathbb{T}} : \mathbf{Loc}(= \mathbf{Frm}^{\mathrm{op}}) \to \mathbb{T}\text{--}\mathbf{FuzzTop}$$

defined by

$$\mathfrak{S}_{\mathbb{T}}L = (\{\alpha \mid \alpha : L \to \mathbb{T} \text{ frame homomorphism}\}, \{\tilde{a} \mid a \in L\}), \quad \tilde{a}(\alpha) = \alpha(a)$$
$$\text{and} \quad \mathfrak{S}_{\mathbb{T}}h(\alpha) = \alpha \cdot h \quad \text{for} \quad h : L \to M.$$

2.8. We will repeatedly use the following simple fact:

Lemma Consider the spectrum $\mathfrak{S}M$ from 1.7 as a poset with the natural order. Let $\nu : \mathbb{T}' \to \mathfrak{S}M$ be an antitone map such that

for all non-void $S \subseteq \mathbb{T}'$, $\nu(\bigwedge S) = \bigvee\{\nu(s) \mid s \in S\}$.

Then

$$\nu(t)(a) = 1 \quad \text{iff} \quad t < \bigvee\{s \in \mathbb{T}' \mid \forall r < s,\ \nu(r)(a) = 1\} \ (\in \mathbb{T}).$$

Proof: If $t < \bigvee\{s \in \mathbb{T}' \mid \forall r < s, \cdots\}$ then there is an $s \in \mathbb{T}'$ such that $t < s$ and for all $r < s$, $\nu(r)(a) = 1$. Hence $\nu(t)(a) = 1$.

Now let $\nu(t)(a) = 1$. Set $S = \{s \in \mathbb{T}' \mid t < s\}$, $s_0 = \bigwedge S$. Then either $t = s_0$ and hence $S \neq \emptyset$, and $\nu(t)(a) = \bigvee_{s \in S} \nu(s)(a) = 1$; then there is an $s > t$ such that $\nu(s)(a) = 1$, and consequently $\nu(r)(a) = 1$ for all $r \leq s$ and $t < s \leq \bigvee\{s \in \mathbb{T}' \mid \forall r < s, \cdots\}$. Or $t < s_0$ and then for all $r < s_0$ we have $r \leq t$ and hence $\nu(r)(a) = 1$ and $t < s_0 \leq \bigvee\{s \in \mathbb{T}' \mid \forall r < s, \cdots\}$. $\square$

3. Fuzzy frames

3.1. Let (X, L) be a $\mathbb{T}$-fuzzy space. The formula (2.4.1) defines mappings

$$\omega_t : L \to \tau L, \quad t \in \mathbb{T}'.$$

Using the linearity of $\mathbb{T}$ we easily check that

each ω_t is a frame homomorphism.

Also, it is easy to check that

$$\text{for each non-void } S \subseteq \mathbb{T}',\ \omega_{\wedge S} = \bigvee_{s \in S} \omega_s \tag{3.1.1}$$

(hence, in particular, $s \leq t \ \Rightarrow\ \omega_t \leq \omega_s$), and that $(\omega_t)_{t \in \mathbb{T}'}$ is a collectionwise monomorphic system. By definition of τL it is also collectionwise extremally epimorphic. We will show that it contains all the information necessary to recover the structure of (X, L). More precisely, we have

Proposition Let (X, τ) be a topological space. Let $(\varphi_t : M \to \tau)_{t \in \mathbb{T}'}$ be a c.m. and c.e.e. system of frame homomorphisms satisfying (3.1.1). Then there is an isomorphism $\kappa : M \to L$ such that

(1) L is a subframe of $\mathbb{T}^X$,

(2) $\tau = \tau L$, and

(3) for each $t \in \mathbb{T}'$, $\omega_t \cdot \kappa = \varphi_t$.

Proof: For $a \in M$ define $\kappa(a) : X \to \mathbb{T}$ by setting

$$\kappa(a)(x) = \bigvee\{t \mid \forall s < t,\ x \in \varphi_s(a)\}\ (\in \mathbb{T})$$

and put

$$L = \{\kappa(a) \mid a \in M\}.$$

For $t \in \mathbb{T}'$ define $\nu(t) : M \to \mathbf{2}$ by setting $\nu(t)(a) = 1$ iff $x \in \varphi_t(a)$. We easily check that each $\nu(t)$ is a frame homomorphism. For non-void $S \subseteq \mathbb{T}'$ we have

$$\nu(\bigwedge S)(a) = 1 \quad \text{iff} \quad x \in \varphi_{\wedge S}(a) = \bigcup \varphi_s(a) \quad \text{iff} \quad \bigvee_{s \in S} \nu(s)(a) = 1.$$

Thus, we can apply Lemma 2.8 to obtain that

$$t < \kappa(a)(x) \quad \text{iff} \quad x \in \varphi_t(a). \tag{$*$}$$

From this formula (using the linearity of $\mathbb{T}$) we immediately infer that $\kappa(0) = 0$, $\kappa(1) = 1$, $\kappa(a \wedge b) = \kappa(a) \wedge \kappa(b)$, and $\kappa(\bigvee a_i) = \bigvee \kappa(a_i)$. Thus

- L is a subframe of $\mathbb{T}^X$, and
- $\kappa : M \to L$ is a frame homomorphism.

Furthermore, $(*)$ says that $x \in \varphi_t(a)$ iff $x \in \omega_t(\kappa(a))$ so that

$$\varphi_t = \omega_t \cdot \kappa \quad \text{and} \quad \tau = \tau L$$

(for the second equality we have used the fact that $(\varphi_t)_t$ is c.e.e.). Finally, by the Lemma in 1.2, κ is one-one, and as it is onto by definition, it is an isomorphism.

□

3.2. Recall 2.5. If $f : (X, L) \to (Y, M)$ is a continuous map, we have $\mathfrak{O}(f) : \tau M \to \tau L$, and a frame homomorphism $h = (u \mapsto uf) : M \to L$. By 2.5.1 we obtain that

$$\forall t \in \mathbb{T}', \quad \mathfrak{O}(f) \cdot \omega_t^M = \omega_t^L \cdot h.$$

3.3. The facts from 3.1 and the observation in 3.2 justify the following definition:

A $\mathbb{T}$-*fuzzy frame* (in the sequel, the prefix $\mathbb{T}$ will be, again, dropped) is a system of frame homomorphisms

$$L = (\varphi_t^L : L^{\mathfrak{u}} \to L^{\mathfrak{l}} \mid t \in \mathbb{T}')$$

such that

(F0) for each non-void $S \subseteq \mathbb{T}'$, $\varphi^L_{\wedge S} = \bigvee_{s\in S} \varphi^L_s$,

(F1) $(\varphi^L_t)_t$ is c.e.e. in **Frm**, that is, $L^{\mathfrak{l}} = \langle \bigcup_t \varphi^L_t[L^{\mathfrak{u}}]\rangle$,

(F2) $(\varphi^L_t)_t$ is c.m. in **Frm**.

A *fuzzy homomorphism* $h : L \to M$ is a couple

$$(h^{\mathfrak{u}} : L^{\mathfrak{u}} \to M^{\mathfrak{u}}, h^{\mathfrak{l}} : L^{\mathfrak{l}} \to M^{\mathfrak{l}})$$

satisfying

$$h^{\mathfrak{l}} \cdot \varphi^L_t = \varphi^M_t \cdot h^{\mathfrak{u}} \quad \text{for all } t \in \mathbb{T}'.$$

The resulting category will be denoted by

FuzzFrm.

3.3.1. Note: From (F1) and (F2) we immediately see that in a fuzzy homomorphism $h = (h^{\mathfrak{u}}, h^{\mathfrak{l}})$, each of the frame homomorphisms $h^{\mathfrak{u}}$, $h^{\mathfrak{l}}$ determines the other one.

3.4. Remark: In [16], S. Rodabaugh points out that several notions in fuzzy topology yield categories that can be viewed, in a natural way, as functor categories. This concerns, e.g., the Hutton approach ([5], [7]) which can be based on the category with two objects and a single non-trivial morphism between them, topologies with degrees of stratification, and others ([13], [16]).

Also the notion of a fuzzy frame can be viewed in a similar way. Only, we have to consider enriched categories (see, e.g., [12]), that is, roughly speaking, categories with structured morphism sets:

Consider **Frm** with the morphism sets endowed by the natural order. The base category **T** has two objects $\mathfrak{u}$, $\mathfrak{l}$ and, besides the units, morphisms $t : \mathfrak{u} \to \mathfrak{l}$, $t \in \mathbb{T}'$ with $\mathbf{T}(\mathfrak{u}, \mathfrak{l})$ ordered by the inverse order of $\mathbb{T}$. Trivially, $\mathcal{T} = (t)_{t\in\mathbb{T}'}$ is both c.m. and c.e.e. in **T**. Now, fuzzy frames can be viewed as functors L :**T**$\to$ **Frm** that

(a) preserve the c.m. and c.e.e. property of $\mathcal{T}$, and

(b) preserve non-void joins.

Fuzzy morphisms then coincide with natural transformations between such functors.

3.5. Recalling 3.1 and 3.2 we see that we have a functor

$$\Omega : \mathbf{FuzzTop} \to \mathbf{FuzzFrm}^{\mathrm{op}}$$

defined by

$$\begin{aligned}
&\Omega(X, L) = (\omega_t^L : L \to \tau L \mid t \in \mathbb{T}'), \\
&\Omega(f) = (\Omega^{\mathfrak{u}}(f), \Omega^{\mathfrak{l}}(f)), \quad \Omega^{\mathfrak{u}}(f)(u) = uf,\ \Omega^{\mathfrak{l}}(f)(U) = f^{-1}(U).
\end{aligned}$$

We say that a fuzzy frame is *spatial* if it is isomorphic to an $\Omega(X, L)$.

Let us say that a fuzzy space (X, L) is $\mathfrak{l}$-*sober* if the space $(X, \tau L)$ is sober. Denote by **FuzzLSob** the full subcategory of **FuzzTop** generated by $\mathfrak{l}$-sober fuzzy spaces.

In analogy with the situation in spaces we have the following statement:

Proposition The restriction of the functor Ω to

$$\Omega : \mathbf{FuzzLSob} \to \mathbf{FuzzFrm}^{\mathrm{op}}$$

is a full embedding.

Proof: The functor is obviously faithful (for this, already the T_0 property of the spaces suffice). Now let $(h^{\mathfrak{u}}, h^{\mathfrak{l}}) : \Omega(X, L) \to \Omega(Y, M)$ be a morphism in **FuzzFrm**. Then, first, $h^{\mathfrak{l}} : \tau M \to \tau L$ is a frame homomorphism and hence, by sobriety, there is an $f : X \to Y$ such that $h^{\mathfrak{l}}(U) = f^{-1}(U)$ for all U. Consequently,

$$\omega_t(h^{\mathfrak{u}}(u)) = h^{\mathfrak{l}}(\omega_t(u)) = f^{-1}(\omega_t(u)).$$

Thus,

$$\{x \mid h^{\mathfrak{u}}(u)(x) > t\} = \{x \mid f(x) \in \omega_t(u)\} = \{x \mid uf(x) > t\},$$

and hence, for all t,

$$h^{\mathfrak{u}}(u)(x) > t \quad \text{iff} \quad uf(x) > t.$$

Thus, $h^{\mathfrak{u}}(u) = uf$, f is a fuzzy continuous map, and $(h^{\mathfrak{u}}, h^{\mathfrak{l}}) = \Omega(f)$.

$\square$

4. Fuzzy spectrum

4.1. Denote by $|\mathfrak{S}(L)|$ the set $\{\alpha : L \to \mathbf{2} \mid \alpha \text{ frame homomorphism }\}$ (that is, $\mathfrak{S}(L)$ without the space structure).

Let $L = (\varphi_t^L : L^{\mathfrak{u}} \to L^{\mathfrak{l}} \mid t \in \mathbb{T}')$ be a fuzzy frame, let a be in $L^{\mathfrak{u}}$. Define

$$\Sigma_a : |\mathfrak{S}(L^{\mathfrak{l}})| \to \mathbb{T}$$

by putting

$$\Sigma_a(\alpha) = \bigvee\{t \mid \forall s < t, \quad \alpha(\varphi_s^L(a)) = 1\}$$

Setting $M = L^{\mathfrak{u}}$ and $\phi(t) = \alpha\varphi_t^L : L^{\mathfrak{u}} \to \mathbf{2}$ we obtain from Lemma 2.8 that

$$s < \Sigma_a(\alpha) \quad \text{iff} \quad \alpha(\varphi_s^L(a)) = 1 \tag{4.1.1}$$

and from this formula we easily infer that

$$\Sigma_0 = 0, \ \Sigma_1 = 1, \ \Sigma_{a\wedge b} = \Sigma_a \wedge \Sigma_b \quad \text{and} \quad \Sigma_{\vee S} = \bigvee\{\Sigma_a \mid a \in S\}. \tag{4.1.2}$$

Consequently,

$\{\Sigma_a \mid a \in L^{\mathfrak{u}}\}$ *is a* $\mathbb{T}$*-fuzzy topology on* $|\mathfrak{S}(L^{\mathfrak{l}})|$.

Now, define

$$\Sigma L = (\mathfrak{S}L^{\mathfrak{l}}, \{\Sigma_a \mid a \in L^{\mathfrak{u}}\}) \ \ (\in \mathbf{FuzzTop}).$$

4.2. For a fuzzy homomorphism $h = (h^{\mathfrak{u}}, h^{\mathfrak{l}}) : L \to M$ define a mapping $\Sigma h : |\mathfrak{S}M^{\mathfrak{l}}| \to L^{\mathfrak{l}}$ by setting

$$\Sigma h(\alpha) = \alpha \cdot h^{\mathfrak{l}}.$$

Lemma We have

$$\Sigma_a \cdot \Sigma h = \Sigma_{h^{\mathfrak{u}}(a)}.$$

Consequently, Σh is a fuzzy continuous mapping $\Sigma M \to \Sigma L$.

Proof: We have

$$\begin{aligned}\Sigma_a(\Sigma h(\alpha)) &= \Sigma_a(\alpha h^{\mathfrak{l}}) = \bigvee\{t \mid \forall s < t, \quad \alpha(h^{\mathfrak{l}}\varphi^L_s(a)) = 1\} = \\ &= \bigvee\{t \mid \forall s < t, \quad \alpha(\varphi^L_s(h^{\mathfrak{u}}(a))) = 1\} = \Sigma_{h^{\mathfrak{u}}(a)}(\alpha).\end{aligned}$$

$\square$

4.3. Thus we have obtained a functor

$$\Sigma : (\mathbf{FuzzFrm})^{\mathrm{op}} \to \mathbf{FuzzTop}.$$

We will call it the *fuzzy spectrum* functor.

4.4. The transformation σ : By (4.1.2) the mapping

$$\sigma^{\mathfrak{u}}_L : L^{\mathfrak{u}} \to \Omega\Sigma L^{\mathfrak{u}}$$

defined by putting $\sigma^{\mathfrak{u}}(a) = \Sigma_a$ is a frame homomorphism.

Further, for $b \in L^{\mathfrak{l}}$ set

$$\sigma^{\mathfrak{l}}_L(b) = \{\alpha \in |\mathfrak{S}(L^{\mathfrak{l}})| \mid \alpha(b) = 1\}.$$

Obviously,

$$\sigma^{\mathfrak{l}}(b \wedge c) = \sigma^{\mathfrak{l}}(b) \cap \sigma^{\mathfrak{l}}(c) \quad \text{and} \quad \sigma^{\mathfrak{l}}(\bigvee b_i) = \bigcup \sigma^{\mathfrak{l}}(b_i).$$

By (4.1.1) we obtain

$$\sigma^{\mathfrak{l}}_L(\varphi^L_t(a)) = \{\alpha \mid \alpha\varphi^L_t(a) = 1\} = \{a \mid \Sigma_a(\alpha) > t\} = \omega_t(\Sigma_a) = \omega_t(\sigma^{\mathfrak{u}}(a)).$$

Thus, $\sigma^{\mathfrak{l}}_L(b) \in (\Omega\Sigma L)^{\mathfrak{l}}$ and we have a homomorphism $\sigma^{\mathfrak{l}}_L : L^{\mathfrak{l}} \to (\Omega\Sigma L)^{\mathfrak{l}}$ constituting with $\sigma^{\mathfrak{u}}_L$ a fuzzy morphism

$$\sigma_L = (\sigma^{\mathfrak{u}}_L, \sigma^{\mathfrak{l}}_L) : L \to \Omega\Sigma L.$$

Using Lemma 4.2 we easily check that we have obtained a natural transformation

$$\sigma : \mathrm{Id} \to \Omega\Sigma.$$

4.5. The transformation ρ : Let (X,L) be a fuzzy space. The formula

$$(\rho_{(X,L)}(x))(U) = 1 \quad \text{iff} \quad x \in U$$

obviously defines a homomorphism $\Omega(X,L)^{\mathfrak{l}} \to \mathbf{2}$. Thus, $\rho_{(X,L)}$ is a mapping sending (X,L) into $\mathfrak{S}(\Omega(X,L))^{\mathfrak{l}}$. It satisfies the equality

$$\Sigma_u \cdot \rho_{(X,L)} = u. \tag{4.5.1}$$

(Indeed, by (4.1.1) we have

$$s < \Sigma_u(\rho(x)) \quad \text{iff} \quad \rho(x)(\omega_s(u)) = 1 \quad \text{iff} \quad x \in \omega_s(u) \quad \text{iff} \quad s < u(x).\ )$$

Thus,

$$\rho_{(X,L)} : (X,L) \to \Sigma\Omega(X,L)$$

are fuzzy continuous maps and it is easy to check that they constitute a natural transformation

$$\rho : \mathrm{Id} \to \Sigma\Omega.$$

4.6. Proposition The spectrum Σ is a right adjoint for the functor Ω. The transformations σ and ρ are units of the adjunction.

Proof: Consider the composition

$$\Omega(X,L) \xrightarrow{\sigma_{\Omega(X,L)}} \Omega\Sigma\Omega(X,L) \xrightarrow{\Omega(\rho_{(X,L)})} \Omega(X,L).$$

On the upper level we have, by (4.5.1), for $u \in \Omega(X,L)^{\mathfrak{u}} = L$

$$(\Omega\rho_{(X,L)})^{\mathfrak{u}}(\sigma^{\mathfrak{u}}_{\Omega(X,L)}(u)) = (\Omega\rho_{(X,L)})^{\mathfrak{u}}(\Sigma_u) = \Sigma_u \cdot \rho_{(X,L)} = u.$$

On the lower level we have for $U \in \Omega(X,L)^{\mathfrak{l}}$ $(\subseteq \mathfrak{P}(X))$

$$\begin{aligned}(\Omega\rho_{(X,L)})^{\mathfrak{l}}(\sigma^{\mathfrak{l}}_{\Omega(X,L)}(U)) &= \rho^{-1}(\{\alpha \in |\mathfrak{S}(\Omega(X,L)^{\mathfrak{l}})| \mid \alpha(U) = 1\}) = \\ &= \{x \mid \rho(x)(U) = 1\} = \{x \mid x \in U\} = U.\end{aligned}$$

As for the composition

$$\Sigma L \xrightarrow{\rho_{\Sigma L}} \Sigma\Omega\Sigma L \xrightarrow{\Sigma\sigma_L} \Sigma L,$$

take an $\alpha \in |\mathfrak{S}L^{\mathfrak{l}}|$, that is, $\alpha : L^{\mathfrak{l}} \to \mathbf{2}$ and a $b \in L^{\mathfrak{l}}$. We have $(\Sigma\sigma(\rho_{\Sigma L}(\alpha)))(b) = \rho_{\Sigma L}(\alpha)(\sigma^{\mathfrak{l}}(b)) = 1$ iff $\alpha \in \sigma^{\mathfrak{l}}(b)$, that is, iff $\alpha(b) = 1$, and hence $\Sigma\sigma(\rho_{\Sigma L}(\alpha)) = \alpha$.

$\square$

4.7. Since $\sigma_L^{\mathfrak{u}}$ is, by definition, always onto we see from 3.3.1 that σ_L is always an epimorphism in **FuzzFrm**. Consequently, the equality $\Omega(\rho_{(X,L)}) \cdot \sigma_{\Omega(X,L)} = \mathrm{id}$ implies that

both $\Omega(\rho_{(X,L)})$ *and* $\sigma_{\Omega(X,L)}$ *are isomorphisms.* (4.7.1)

In particular, we obtain, similarly as in spaces,

Corollary A fuzzy frame L is spatial if and only if

$$\sigma_L : L \to \Omega\Sigma L$$

is an isomorphism.

4.8. Lemma Let L be a fuzzy frame. Then $(|\mathfrak{S}L^{\mathfrak{l}}|, \Omega^{\mathfrak{l}}(X, M)) = \mathfrak{S}L^{\mathfrak{l}}$.

Proof: Since $\mathfrak{S}_a \cap \mathfrak{S}_b = \mathfrak{S}_{a\wedge b}$ and $\mathfrak{S}_{\vee a_i} = \bigcup \mathfrak{S}_{a_i}$, it suffices to show that $\bigcup_t \omega_t(L^{\mathfrak{u}})$ consists of the $\mathfrak{S}_a$'s with a in a set generating $L^{\mathfrak{l}}$.

By (4.1.1) we have

$$\omega_t(\Sigma_a) = \{\alpha \mid \Sigma_a(\alpha) > t\} = \{\alpha \mid \alpha(\varphi_t^L(a)) = 1\} = \mathfrak{S}_b$$

where $b = \varphi_t^L(a)$, and $\bigcup_t \varphi_t^L[L^{\mathfrak{u}}]$ generates $L^{\mathfrak{l}}$.

$\square$

4.9. Recall the definition of $\mathfrak{l}$-sobriety from 3.5. Similarly as in spaces we have

Proposition Each ΣL is $\mathfrak{l}$-sober, and a fuzzy space (X, M) is $\mathfrak{l}$-sober if and only if $\rho_{(X,M)} : (X, M) \to \Sigma\Omega(X, M)$ is an isomorphism.

Proof: As each $\mathfrak{S}M$ is sober we obtain the $\mathfrak{l}$-sobriety of ΣL from 4.8; thus, if $\rho_{(X,M)}$ is an isomorphism, (X, M) is $\mathfrak{l}$-sober. On the other hand, if (X, M) is $\mathfrak{l}$-sober, since $\Omega(\rho_{(X,M)})$ has an inverse $\alpha (= \sigma_{\Omega(X,M)})$ by (4.7.1), we have by the Proposition in 3.5 an inverse f to $\rho_{(X,M)}$, namely the fuzzy continuous map for which $\alpha = \Omega(f)$.

$\square$

4.10. A fuzzy continuous map $f : (X, L) \to (Y, M)$ is an *embedding* if it is one-one and if for each $u \in L$ there is a $v \in M$ such that $u = vf$ (that is, if the frame homomorphism $(v \mapsto vf)$ is a sublocale homomorphism). Recall 2.6. In analogy with spaces we have

Proposition The following statements on a fuzzy space (X, L) are equivalent:

(1) (X, L) is $\mathfrak{l}$-T_0,

(2) $\rho_{(X,L)}$ is one-one,

(3) $\rho_{(X,L)}$ is an embedding.

Proof: (1)$\Rightarrow$(2): Let $x \neq y$. Take a $u \in L$ such that, say $u(x) > u(t) = t$. Then $y \neq \omega_t(u) \ni x$ and $\rho(x)(\omega_t(u)) = 1 \neq \rho(y)(\omega_t(u))$.

(2)$\Rightarrow$(3): By (4.5.1) we have, for each $u \in L$, $u = \Sigma_u \rho$.

(3)$\Rightarrow$(1): By 4.9, $\Sigma\Omega(X, L)$ is $\mathfrak{l}$-T_0. As obviously embeddings preserve the $\mathfrak{l}$-T_0-property, (X, L) is $\mathfrak{l}$-T_0.

□

5. The category FuzzFrm

5.1. In this section we will discuss the basic properties of the category **FuzzFrm** : completeness, cocompleteness, and factorization of morphisms.

We will denote by

$$\mathcal{U}^{\mathfrak{u}} \text{ resp. } \mathcal{U}^{\mathfrak{l}} : \mathbf{FuzzFrm} \to \mathbf{Frm}$$

the natural forgetful functors defined by $\mathcal{U}^{\mathfrak{x}}(L) = L^{\mathfrak{x}}$, $\mathcal{U}^{\mathfrak{x}}(h) = h^{\mathfrak{x}}$, $\mathfrak{x} = \mathfrak{u}$ resp. $\mathfrak{l}$. They will be referred to as the *upper* resp. *lower* forgetful functor.

For technical reasons, we will introduce two larger categories

$$\mathfrak{F}_0 \quad \text{and} \quad \mathfrak{F}_1.$$

The objects of $\mathfrak{F}_0$ are the systems $L = (\varphi_t^L : L^{\mathfrak{u}} \to L^{\mathfrak{l}} \mid t \in \mathbb{T}')$ in **Frm** satisfying just (F0), and

the objects of $\mathfrak{F}_1$ are the systems $L = (\varphi_t^L : L^{\mathfrak{u}} \to L^{\mathfrak{l}} \mid t \in \mathbb{T}')$ in **Frm** satisfying (F0) and (F1).

The morphisms are defined as in 3.3 and are, again, called fuzzy homomorphisms, in both extensions. The objects of $\mathfrak{F}_i$ are referred to as $\mathfrak{F}_i$-objects.

By abuse of notation, we will use the symbols $\mathcal{U}^{\mathfrak{x}}$, $\mathfrak{x} = \mathfrak{u}$ resp. $\mathfrak{l}$, also for the obvious extensions of the upper and lower forgetful functors.

5.2. Factorization systems : Let $\mathcal{M}$ be the system of all fuzzy homomorphisms h with $h^{\mathfrak{x}}$ one-one, $\mathfrak{x} = \mathfrak{u}$ resp. $\mathfrak{l}$, and let $\mathcal{E}$ be the system of all the h with $h^{\mathfrak{x}}$ onto, $\mathfrak{x} = \mathfrak{u}$ resp. $\mathfrak{l}$. We will use this notation in any of the categories $\mathfrak{F}_0$, $\mathfrak{F}_1$ and **FuzzFrm**.

Proposition $(\mathcal{M}, \mathcal{E})$ is a factorization system in any of $\mathfrak{F}_0$, $\mathfrak{F}_1$ and **FuzzFrm**.

Proof: Consider an $h : L \to M$ in $\mathfrak{F}_0$. Factor $h^{\mathfrak{u}} = m^{\mathfrak{u}} e^{\mathfrak{u}}$, $h^{\mathfrak{l}} = m^{\mathfrak{l}} e^{\mathfrak{l}}$ in **Frm** with $m^{\mathfrak{u}}, m^{\mathfrak{l}}$ one-one and $e^{\mathfrak{u}}, e^{\mathfrak{l}}$ onto (recall

$$\begin{tikzcd}
L^{\mathfrak{u}} \arrow[r, "e^{\mathfrak{u}}"] \arrow[d, "\varphi_t^L"] & K^{\mathfrak{u}} \arrow[r, "m^{\mathfrak{u}}"] \arrow[d, "\varphi_t^K"] & M^{\mathfrak{u}} \arrow[d, "\varphi_t^M"] \\
L^{\mathfrak{l}} \arrow[r, "e^{\mathfrak{l}}"] & K^{\mathfrak{l}} \arrow[r, "m^{\mathfrak{l}}"] & M^{\mathfrak{l}}
\end{tikzcd}$$

define φ_t^K as the mappings satisfying

$$\varphi_t^K(b) = e^{\mathfrak{l}}(\varphi_t^L(a)) \text{ for some } a \text{ such that } b = e^{\mathfrak{u}}(a) \tag{5.2.1}$$

which is easily seen to determine φ_t^K uniquely. From the unicity of the formula we readily infer also that the φ_t^K are frame homomorphisms and that $\varphi_{\wedge S}^K(b) = \bigvee_{s \in S} \varphi_s^K(b)$. Immediately, $\varphi_t^K e^{\mathfrak{u}} = e^{\mathfrak{l}} \varphi_t^L$ and also

$$m^{\mathfrak{l}} \varphi_t^K(b) = m^{\mathfrak{l}} e^{\mathfrak{l}}(\varphi_t^L(a)) = \varphi_t^M(m^{\mathfrak{u}} l^{\mathfrak{u}}(a)) = \varphi_t^M m^{\mathfrak{u}}(b).$$

If we are in $\mathfrak{F}_1$ set $S = \bigcup \varphi_t^K[K^{\mathfrak{u}}]$, $S' = \bigcup \varphi_t^L[L^{\mathfrak{u}}]$. We have $e^{\mathfrak{l}}[S'] \subseteq S$ and $e^{\mathfrak{l}}[S']$, as an image of a generating set under an onto homomorphism is a generating set. Thus $K = (\varphi_t^K : K^{\mathfrak{u}} \to K^{\mathfrak{l}} \mid t \in \mathbb{T}')$ is in $\mathfrak{F}_1$.

Finally, if we are in **FuzzFrm** we infer that $(\varphi_t^K)_t$ is c.m. in the obvious way from the right hand square in the diagram.

$\square$

5.3. Proposition $\mathfrak{F}_0$ is complete and cocomplete and the functors

$$\mathcal{U}^{\mathfrak{x}} : \mathfrak{F}_0 \to \mathbf{Frm}, \quad \mathfrak{x} = \mathfrak{u} \text{ resp. } \mathfrak{l},$$

preserve all limits and colimits.

Proof: Let $D : C \to \mathfrak{F}_0$ be a diagram in $\mathfrak{F}_0$, $D(i) = L_i = (\varphi_t^i : L_i^{\mathfrak{u}} \to L_i^{\mathfrak{l}} \mid t \in \mathbb{T}')$; let

$$(\gamma_i^{\mathfrak{x}} : L_i^{\mathfrak{x}} \to L^{\mathfrak{x}} \mid i \in |C|)$$

($\mathfrak{x} = \mathfrak{u}$ resp. $\mathfrak{l}$) be the colimits of $\mathcal{U}^{\mathfrak{x}}D$ in **Frm**. Then for $\alpha = D(\alpha')$, $\alpha' : i \to j$ in C, we have

$$\gamma_j^{\mathfrak{l}}\varphi_t^i\alpha^{\mathfrak{u}} = \gamma_j^{\mathfrak{l}}\alpha^{\mathfrak{l}}\varphi_t^i = \gamma_i^{\mathfrak{l}}\varphi_t^i$$

and hence we have for each t exactly one $\varphi_t : L^{\mathfrak{u}} \to L^{\mathfrak{l}}$ such that

$$\varphi_t\gamma_i^{\mathfrak{u}} = \gamma_i^{\mathfrak{l}}\varphi_t^i \quad \text{for all } i. \tag{$*$}$$

By 1.4, $\varphi_t \leq \varphi_s$ for $s \leq t$ and hence, by 1.5, $\bigvee_{s\in S}\varphi_s$ is a frame homomorphism and as

$$\varphi_{\wedge S}\cdot\gamma_i^{\mathfrak{u}} = \gamma_i^{\mathfrak{l}}\varphi_{\wedge S}^i = \gamma_i^{\mathfrak{l}}\cdot\bigvee\varphi_s^i = \bigvee(\gamma_i^{\mathfrak{l}}\varphi_s^i) = \bigvee(\varphi_s\gamma_i^{\mathfrak{u}}) = (\bigvee\varphi_s)\gamma_i^{\mathfrak{u}}$$

we have $\varphi_{\wedge S} = \bigvee_{s\in S}\varphi_s$ and $L = (\varphi_t : L^{\mathfrak{u}} \to L^{\mathfrak{l}} \mid t \in \mathbb{T}')$ is an $\mathfrak{F}_0$-object. If $(h_i : L_i \to M)_{i\in|C|}$ is an upper bound of D then $(h_i^{\mathfrak{x}} : L_i^{\mathfrak{x}} \to M^{\mathfrak{x}})_{i\in|C|}$, $\mathfrak{x} = \mathfrak{u}$ resp. $\mathfrak{l}$, are upper bounds of $\mathcal{U}^{\mathfrak{x}}D$ and hence there are unique $g^{\mathfrak{x}} : L^{\mathfrak{x}} \to M^{\mathfrak{x}}$ such that $g^{\mathfrak{x}}\gamma_i^{\mathfrak{x}} = h_i^{\mathfrak{x}}$. We have

$$\varphi_t^M g^{\mathfrak{u}}\gamma_i^{\mathfrak{u}} = \varphi_t^M h_i^{\mathfrak{u}} = h_i^{\mathfrak{l}}\varphi_t^i = g^{\mathfrak{l}}\gamma_i^{\mathfrak{l}}\varphi_t^i = g^{\mathfrak{l}}\varphi_t\gamma_i^{\mathfrak{u}}$$

and hence $\varphi_t^M g^{\mathfrak{u}} = g^{\mathfrak{l}}\varphi_t$.

The proof for the limits is quite analogous.

$\square$

Note : In particular, both $\mathcal{U}^{\mathfrak{x}} : \mathfrak{F}_0 \to$ **Frm** preserve monomorphisms so that the monomorphisms in $\mathfrak{F}_0$ are exactly the h with both $h^{\mathfrak{u}}$, $h^{\mathfrak{l}}$ one-one. Thus, the factorization from 5.2 is in $\mathfrak{F}_0$ the (mono,extremal-epi) one.

Not so in **FuzzFrm**. There, as we will see shortly (in 5.6), the monomorphisms are exactly the h with $h^{\mathfrak{u}}$ one-one, which does not imply $h^{\mathfrak{l}}$ being one-one (see 6.4 below). Thus, in **FuzzFrm**, $\mathcal{M}$ is smaller than the class of all monomorphisms, and $\mathcal{E}$ is larger than the class of all extremal epimorphisms.

5.4. Proposition $\mathfrak{F}_1$ is mono-coreflective in $\mathfrak{F}_0$. Consequently, it is complete and cocomplete.

Proof: Set $\widetilde{L}^{\mathfrak{u}} = L^{\mathfrak{u}}$, $\widetilde{L}^{\mathfrak{l}} = \langle \bigcup_t \varphi_t^L(L^{\mathfrak{u}})\rangle$ and define $\varphi_t^{\widetilde{L}} : \widetilde{L}^{\mathfrak{u}} \to \widetilde{L}^{\mathfrak{l}}$ by $\varphi_t^{\widetilde{L}}(a) = \varphi_t^L(a)$. Then obviously $\widetilde{L}$ is an $\mathfrak{F}_1$-object and $(\mathrm{id}, \subseteq) : \widetilde{L} \to L$ a monomorphism in $\mathfrak{F}_0$. These morphisms $(\mathrm{id}, \subseteq)$ constitute a coreflection: Let M be an $\mathfrak{F}_1$-object and $h : M \to L$ a fuzzy homomorphism. Let $a \in \bigcup \varphi_t^M[M^{\mathfrak{u}}]$, say $a = \varphi_t^M(b)$. Then $h^{\mathfrak{l}}(a) = h^{\mathfrak{l}}\varphi_t^M(b) = \varphi_t^L(h^{\mathfrak{u}}(b)) \in \widetilde{L}^{\mathfrak{l}}$. Thus, $h^{\mathfrak{l}}(\bigcup \varphi_t^M[M^{\mathfrak{u}}]) \subseteq \widetilde{L}^{\mathfrak{l}}$ and $\bigcup \varphi_t^M[M^{\mathfrak{u}}] \subseteq (h^{\mathfrak{l}})^{-1}(\widetilde{L}^{\mathfrak{l}})$ which is a subframe and hence $M^{\mathfrak{l}} = \langle \bigcup \varphi_t^M[M^{\mathfrak{u}}]\rangle \subseteq (h^{\mathfrak{l}})^{-1}(\widetilde{L}^{\mathfrak{l}})$ and finally $h^{\mathfrak{l}}[M^{\mathfrak{l}}] \subseteq \widetilde{L}^{\mathfrak{l}}$ so that $h^{\mathfrak{l}}$ factorizes through $\widetilde{L}^{\mathfrak{l}}$. $\square$

5.5. Proposition **FuzzFrm** is epireflective in $\mathfrak{F}_1$. Consequently, it is complete and cocomplete.

Proof: For an $\mathfrak{F}_1$-object L choose a representing system $\mathcal{E}(L)$ of all the fuzzy homomorphisms $e : L \to L_e$ such that $e = (e^{\mathfrak{u}}, \mathrm{id})$ with $e^{\mathfrak{u}}$ onto; suppose that $(\mathrm{id}, \mathrm{id}) \in \mathcal{E}(L)$. Take the colimit

$$(\gamma_e : L_e \to \widetilde{L} \mid e \in \mathcal{E}(L))$$

of the diagram

$$e : L \to L_e, \quad e \in \mathcal{E}(L).$$

Set $\gamma_L = \gamma_{(\mathrm{id},\mathrm{id})}$; hence $\gamma_e \cdot e = \gamma_L$ for all e. By 1.4, $\gamma_L^{\mathfrak{u}}$ is the intersection of the sublocales $\gamma_e^{\mathfrak{u}}$ with $e \in \mathcal{E}(L)$, and we can have $\gamma_e^{\mathfrak{l}} = \mathrm{id}$ for all $e \in \mathcal{E}(L)$.

If $h : \widetilde{L} \to M$ is a fuzzy morphism, we have by 5.2 a factorization $h = \mu \cdot \varepsilon$ with $\varepsilon^{\mathfrak{u}}$ onto, $\mu^{\mathfrak{u}}$ one-one and $\mu^{\mathfrak{l}} = \varepsilon^{\mathfrak{l}} = \mathrm{id}$. Then $\varepsilon\gamma$ has a representative $e \in \mathcal{E}(L)$, that is, $\varepsilon\gamma = \varepsilon\gamma_e e = \alpha e$ with an isomorphism α, hence $\varepsilon\gamma_L = \alpha$ is an isomorphism, and as γ_L is onto, ε is an isomorphism. Thus $\widetilde{L}$ is a fuzzy frame.

If L was a fuzzy frame we have $\mathcal{E}(L) = \{\mathrm{id}\}$ and hence $\gamma_L = \mathrm{id}$.

Finally, let $h : L \to M$ be a fuzzy morphism. Consider the pushout

$$\begin{tikzcd}
L \arrow[r, "h"] \arrow[d, "\gamma_L"'] & M \arrow[d, "e"] \\
\widetilde{L} \arrow[r, "g"] & N
\end{tikzcd}$$

By 5.3 and 5.4 we have the pushout in **Frm**

$$
\begin{array}{ccc}
L^{\mathfrak{u}} & \xrightarrow{h^{\mathfrak{u}}} & M^{\mathfrak{u}} \\
{\scriptstyle \gamma_L^{\mathfrak{u}}}\downarrow & & \downarrow{\scriptstyle e^{\mathfrak{u}}} \\
\widetilde{L}^{\mathfrak{u}} & \xrightarrow{g^{\mathfrak{u}}} & N^{\mathfrak{u}}
\end{array}
$$

and $e^{\mathfrak{l}} = \mathrm{id}$. As pushouts send extremal epimorphisms to extremal epimorphims, $e^{\mathfrak{u}}$ is onto and we can assume that $e \in \mathcal{E}(M)$. Set $\widetilde{h} = \gamma_e g$ which gives

$$\widetilde{h} \cdot \gamma_L = \gamma_M \cdot h$$

and as γ_L is onto, $\widetilde{h}$ is the unique frame homomorphism satisfying this equation. Thus, the correspondence

$$L \mapsto \widetilde{L}, \quad h \mapsto \widetilde{h}$$

is functorial. $\square$

5.6. As the coreflection in 5.4 is identical on the upper part, and the reflection in 5.5 is identical on the lower part,

$$
\begin{aligned}
&\mathcal{U}^{\mathfrak{u}} : \mathbf{FuzzFrm} \to \mathbf{Frm} \quad \text{preserves limits, and} \\
&\mathcal{U}^{\mathfrak{l}} : \mathbf{FuzzFrm} \to \mathbf{Frm} \quad \text{preserves colimits.}
\end{aligned}
$$

In particular,

the monomorphisms in **FuzzFrm** *are precisely the h with $h^{\mathfrak{u}}$ one-one.*

(See also 6.4 below.)

6. The free functor

6.1. Since $\mathcal{U}^{\mathfrak{u}} : \mathbf{FuzzFrm} \to \mathbf{Frm}$ preserves limits it is to be expected it has a left adjoint. The construction is straightforward, but it may be of some use to recall the following standard fact on quotients of frames (see, e.g., [8], [9]):

Let L be a frame and let $R \subseteq L \times L$ be an arbitrary binary relation on the set L. Then there is a sublocale $\kappa : L \to L/R$ such that

(1) $(a,b) \in R \;\Rightarrow\; \kappa(a) = \kappa(b)$*, and*

(2) *for each frame homomorphism* $h : L \to M$ *such that* $(a,b) \in R \;\Rightarrow\; h(a) = h(b)$ *there is a frame homomorpism* $\overline{h} : L/R \to M$ *such that* $\overline{h} \cdot \kappa = h$.

For a frame L set $\mathcal{L}(L)^{\mathfrak{u}} = L$. Take the copower $(\iota_t : L \to {}^{\mathbb{T}'}\!L \mid t \in \mathbb{T}')$ and consider the relation

$$R = \{(\iota_{\wedge S}(a), \bigvee\{\iota_s(a) \mid s \in S\}) \mid \emptyset \neq S \subseteq \mathbb{T}',\; a \in L\}$$

on ${}^{\mathbb{T}'}\!L$. Set

$$\mathcal{L}(L)^{\mathfrak{l}} = {}^{\mathbb{T}'}\!L/R, \quad \mu_t = \kappa \cdot \iota_t,$$

where $\kappa : {}^{\mathbb{T}'}\!L \to {}^{\mathbb{T}'}\!L/R$ is the quotient homomorphism from above.

6.1.1. Lemma For each $\mathfrak{F}_0$-object M and each frame homomorphism $h : L \to M^{\mathfrak{u}}$ there is (exactly one) frame homomorphism $h^{\mathfrak{l}} : \mathcal{L}(L)^{\mathfrak{l}} \to M^{\mathfrak{l}}$ such that $(h, h^{\mathfrak{l}}) : \mathcal{L}(L) \to M$ is a fuzzy homomorphism.

Proof: Define, first, $\overline{h} : {}^{\mathbb{T}'}\!L \to M^{\mathfrak{l}}$ by requiring that

$$\overline{h}\iota_t = \varphi_t^M h.$$

Then

$$\overline{h}(\iota_{\wedge S}(a)) = \varphi_{\wedge S}^M(h(a)) = \bigvee_{s \in S} \varphi_s^M(h(a)) = \bigvee_{s \in S} \overline{h}\iota_s(a) = \overline{h}(\bigvee \iota_s(a))$$

and hence $\overline{h}$ equalizes the relation R. Thus, there is (exactly one) $h^{\mathfrak{l}} : {}^{\mathbb{T}'}\!L/R \to M^{\mathfrak{l}}$ such that $h^{\mathfrak{l}}\kappa = \overline{h}$ and we obtain that

$$h^{\mathfrak{l}}\mu_t = h^{\mathfrak{l}}\kappa\iota_t = \overline{h}\iota_t = \varphi_t^M h$$

and see that

$$(h, h^{\mathfrak{l}}) : \mathcal{L}(L) \to M$$

is a fuzzy homomorphism, obviously unique such that the upper coordinate is $h : L \to M^{\mathfrak{u}}$. $\square$

6.1.2. Lemma Each μ_t^L is a coretraction. Moreover, all the μ_t^L have a common left inverse.

Proof: Consider the constant fuzzy frame $(\mathrm{id}_L : L \to L \mid t \in \mathbb{T})$. By 6.1.1 there is an α (the $h^{\mathfrak{l}}$ for $h = \mathrm{id}_L$) such that

$$\forall t, \quad \alpha \cdot \mu_t^L = \mathrm{id}_L.$$

$\square$

6.2. From the definition of the relation R we see that $\mathcal{L}(L)$ satisfies (F0). Obviously, as κ is onto (recall also (1.2.1) and 1.4), it satisfies (F1), and by 6.1.2 it satisfies (much more than) (F2). Thus,

> $\mathcal{L}(L)$ *is a fuzzy frame.*

For a frame homomorphism $h : L \to M$ set

$$\mathcal{L}(h) = (h, h^{\mathfrak{l}}) : \mathcal{L}(L) \to \mathcal{L}(M)$$

where $h^{\mathfrak{l}}$ is the homomorphism associated by 6.1.1 with $h : L \to M = \mathcal{L}(M)^{\mathfrak{u}}$. Obviously this defines a functor

$$\mathcal{L} : \mathbf{Frm} \to \mathbf{FuzzFrm}.$$

Proposition $\mathcal{L}$ is a left adjoint for $\mathcal{U}^{\mathfrak{u}} : \mathbf{FuzzFrm} \to \mathbf{Frm}$.

Proof: The fact from 6.1.1 establishes an invertible correspondence between the frame homomorphisms $h : L \to \mathcal{U}^{\mathfrak{u}}(M) = M^{\mathfrak{u}}$ and the fuzzy homomorphisms $(h, h^{\mathfrak{l}}) : \mathcal{L}(L) \to M$. Checking that it is natural in L, M is straightforward. $\square$

6.3. The left unit : The right unit $L \to \mathcal{U}^{\mathfrak{u}}\mathcal{L}(L)$ is the identity. The left one,

$$\lambda_M = (\mathrm{id}_M, \lambda_M^{\mathfrak{l}}) : \mathcal{L}\mathcal{U}^{\mathfrak{u}}(M) \to M$$

is more interesting. First, from the formula

$$\lambda_M^{\mathfrak{l}} \cdot \mu_t = \varphi_t^M$$

we immediately see that

> $\lambda_M^{\mathfrak{l}}$ *is onto (that is, a sublocale).*

Thus,

for a frame L we have a "free fuzzy frame" $(\mu_t : L \to \widetilde{L} \mid t \in \mathbb{T}')$ such that each fuzzy frame $(\varphi_t : L \to L_0 \mid t \in \mathbb{T}')$ with the upper part L is a composition of $(\mu_t)_{t\in\mathbb{T}'}$ with a sublocale $\lambda : \widetilde{L} \to L_0$.

6.4. Monomorphisms and extremal epimorphisms in FuzzFrm: We already know that the monomorphisms in **FuzzFrm** are precisely the morphisms h with $h^{\mathfrak{u}}$ one-one. Consequently, as $\lambda_M^{\mathfrak{u}} = \mathrm{id}$,

each λ_M is a monomorphism in **FuzzFrm**.

Now take a fuzzy frame L such that $L^{\mathfrak{u}} = M$ that is not isomorphic to $\mathcal{L}(M)$ (such an L exist since else all L with $L^{\mathfrak{u}} = M$ would be isomorphic, which is obviously not true). Since $\lambda_M^{\mathfrak{l}}$ is onto,

- *λ_M is a monomorphism although $\lambda_M^{\mathfrak{l}}$ is not one-one, and*

- *λ_M is not an extremal epimorphism in* **FuzzFrm** *although both $\lambda_M^{\mathfrak{u}}$ and $\lambda_M^{\mathfrak{l}}$ are extremal epimorphisms in* **Frm**.

6.5. The free functor $\mathcal{L}$ can be used for a construction of a universal fuzzy space on which a given frame L operates.

A *universal fuzzy space* for a (spatial) frame L is a fuzzy space $(X_L, \overline{L})$ together with an isomorphism $\sigma_L : L \to \overline{L}$ such that for each $\mathfrak{l}$-T_0 fuzzy space (X, L') admitting an isomorphism $\iota : L \to L'$ there is an embedding

$$j : (X, L') \to (X_L, \overline{L})$$

such that

$$\forall a \in L, \quad \sigma_L(a) \cdot j = \iota(a) \tag{$*$}$$

(obviously, any one-one maping satisfying $(*)$ is an embedding).

Observation A universal fuzzy space for a given L is uniquely determined up to isomorphism.

Proof: Let us have, besides the $(X, \overline{L})$, $\sigma : L \to \overline{L}$, another universal fuzzy space (Y, M) with an isomorphism $\theta : L \to M$. Then we have embeddings

$$j : (Y, M) \to (X_L, \overline{L}), \quad k : (X_L, \overline{M}) \to (Y, M)$$

such that, for all $a \in L$,

$$\sigma(a) \cdot j = \theta(a), \quad \theta(a) \cdot k = \sigma(a).$$

Thus, for each for each $u \in \overline{L}$, $u(jk(x)) = u(x)$ and for each $u \in M$, $u(kj(x)) = u(x)$, and hence (recall 2.6), j and k are mutually inverse.

$\square$

Proposition Let L be a spatial frame. Set $(X_L, \overline{L}) = \Sigma\mathcal{L}(L)$ and define $\sigma : L \to (\Sigma\mathcal{L}(L))^{\mathfrak{u}}$ by $\sigma(a) = \Sigma_a$. Then $(X_L, \overline{L})$ is a universal fuzzy space for L.

Proof: By the Corollary in 4.7, σ is an isomorphism. Now let (X, L') be an $\mathfrak{l}$-T_0 fuzzy space and $\iota : L \to L'$ an isomorphism. By 4.10 we have the embedding $\rho : (X, L') \to \Sigma\Omega(X, L')$. Further consider the morphism

$$\lambda = \lambda_{\Omega(X,L')} : \mathcal{L}(L') = \mathcal{L}(\mathcal{U}^{\mathfrak{u}}\Omega(X, L')) \to \Omega(X, L').$$

As $\lambda^{\mathfrak{l}}$ is onto, the mapping

$$\Sigma\lambda : \Sigma\Omega(X, L') \to \Sigma\mathcal{L}(L')$$

is one-one (recall the definition of Σh in 4.2). Define j as the composition

$$(X, L') \xrightarrow{\rho} \Sigma\Omega(X, L') \xrightarrow{\Sigma\lambda} \Sigma\mathcal{L}(L') \xrightarrow{\Sigma\mathcal{L}(\iota)} \Sigma\mathcal{L}(L).$$

We have

$$\sigma(a) \cdot j = \Sigma_a \cdot \Sigma\mathcal{L}(i) \cdot \Sigma\lambda \cdot \rho = *.$$

As $\lambda^{\mathfrak{u}} = \mathrm{id}$ and $\mathcal{L}(\iota)^{\mathfrak{u}} = \iota$ we conclude by Lemma in 4.2 and by (4.5.1) that

$$* = \Sigma_{\iota(a)}\rho = \iota(a).$$

$\square$

6.6. Remark : The functor Σ, as a right adjoint, preserves limits, that is, in frame reasoning, sends colimits to limits. Thus, we can give a more explicit description of the universal fuzzy frame $(X_L, \overline{L})$, namely as a subset

$$X_L = \{(\alpha_t)_t \mid S \neq \emptyset, S \subseteq \mathbb{T}' \;\Rightarrow\; \alpha_{\wedge S} = \bigvee_{s \in S} \alpha_s\} \quad \text{of} \quad \mathfrak{S}L^{\mathbb{T}'}$$

on which the L operates by the formula

$$\sigma(u)((\alpha_t)_t) = \bigvee\{s \mid \forall r < s,\ \alpha_r(u) = 1\}.$$

The embedding

$$j : (X, L) \to (X_L, \overline{L})$$

is then defined by setting

$$j(x) = (j(x)_t)_t \quad \text{where} \quad j(x)_t(a) = 1 \quad \text{iff} \quad \iota(a)(x) > t.$$

6.7. Remark : Combine the spectrum adjunction $\Omega \dashv \Sigma$ with the adjunction $\mathcal{U}^{\mathfrak{u}} \dashv \mathcal{L}$ considered as of functors between $\mathbf{FuzzFrm}^{\mathrm{op}}$ and $\mathbf{Frm}^{\mathrm{op}}$ (hence, $\mathcal{U}^{\mathfrak{u}}$ is now to the left and $\mathcal{L}$ to the right. As $\mathcal{U}^{\mathfrak{u}}\Omega$ is the $\mathfrak{O}_{\mathbb{T}}$ from 2.7, $\Sigma\mathcal{L}$ is naturally equivalent with the spectrum from [4] (in the present notation, $\mathfrak{S}_{\mathbb{T}}$). To be more specific, the natural isomorphism associates the $\alpha : \mathcal{L}(L)^{\mathfrak{l}} \to \mathbf{2}$ with the $\overline{\alpha} : L \to \mathbb{T}$ defined by $\overline{\alpha}(a) = \Sigma_a(\alpha)$.

7. The co-free functor

In this section we will briefly discuss the right adjoint to $\mathcal{U}^{\mathfrak{l}} : \mathbf{FuzzFrm} \to \mathbf{Frm}$. It will be constructed in an obvious dual analogy with that of the free functor from the previous section.

7.1. For a frame M consider the power $M^{\mathbb{T}'}$ and the subset

$$\mathcal{R}(M)^{\mathfrak{u}} = \{(a_t)_t \mid \text{ for all non-void } S \subseteq \mathbb{T}',\ a_{\wedge S} = \bigvee_{s\in S} a_s\} \ \subseteq \ M^{\mathbb{T}'}.$$

Observation $\mathcal{R}(M)^{\mathfrak{u}}$ is a subframe of $M^{\mathbb{T}'}$.

Proof: Let $(a^i_t)_t$, $i \in J$, be in $\mathcal{R}(M)^{\mathfrak{u}}$, $a_t = \bigvee_i a^i_t$. Then obviously

$$a_{\wedge S} = \bigvee_{s\in S} a_s. \tag{7.1.1}$$

Let $a = (a_t)_t,\ b = (b_t)_t \in N$. We have (since $\mathbb{T}$ is linear an hence $s \wedge t$ is always one of the s, t, and since we know by (7.1.1) that $s \le t \ \Rightarrow \ a_s \ge a_t$)

$$\begin{aligned}(a \wedge b)_{\wedge S} \ &= \ a_{\wedge S} \wedge b_{\wedge S} = \bigvee\{a_s \wedge b_t \mid s, t \in S\} \le \bigvee\{a_{s\wedge t} \wedge b_{s\wedge t} \mid s, t \in S\} \le \\ &\le \ \bigvee\{a_s \wedge b_s \mid s \in S\} \le \bigvee\{(a \wedge b)_s \mid s \in S\} \le (a \wedge b)_{\wedge S}.\end{aligned}$$

□

7.2. Now set $\mathcal{R}(M)^{\mathfrak{l}} = M$ and define $\varepsilon_t : \mathcal{R}(M)^{\mathfrak{u}} \to \mathcal{R}(M)^{\mathfrak{l}} = M$ by restricting the projections $\pi_t : M^{\mathbb{T}'} \to M$ to $\mathcal{R}(M)^{\mathfrak{u}}$. From (7.1.1) we immediately deduce that

$$\mathcal{R}(M) = (\varepsilon_t : \mathcal{R}(M)^{\mathfrak{u}} \to \mathcal{R}(M)^{\mathfrak{l}} \mid t \in \mathbb{T}')$$

is an $\mathfrak{F}_0$-object.

Lemma For each $\mathfrak{F}_0$-object L and each frame homomorphism $h : L^{\mathfrak{l}} \to M$ there is (exactly one) frame homomorphism $h^{\mathfrak{u}} : L^{\mathfrak{u}} \to \mathcal{R}(M)^{\mathfrak{u}}$ such that $(h^{\mathfrak{u}}, h) : L \to \mathcal{R}(M)$ is a fuzzy homomorphism.

Proof: Consider, first, the $\overline{h} : L^{\mathfrak{u}} \to M^{\mathbb{T}'}$ determined by

$$\pi_t \overline{h} = h\varphi_t^L.$$

Let $a \in L$, $b = (b_t)_t = \overline{h}(a)$. We have, for $\emptyset \neq S \subseteq \mathbb{T}'$,

$$b_{\wedge S} = \pi_{\wedge S}\overline{h}(a) = h(\varphi^L_{\wedge S}(a)) = h(\bigvee_{s\in S} \varphi^L_s(a)) = \bigvee_{s\in S} h\varphi^L_s(a) = \bigvee_{s\in S} \pi_s\overline{h}(a) = \bigvee_{s\in S} b_s.$$

Thus, $\overline{h}$ factorizes through $\mathcal{R}(M)^{\mathfrak{l}}$. Denote by

$$h^{\mathfrak{u}} : L^{\mathfrak{u}} \to \mathcal{R}(M)^{\mathfrak{u}}$$

the homomorphism defined by $h^{\mathfrak{u}}(a) = \overline{h}(a)$. Now we have a fuzzy morphism

$$(h^{\mathfrak{u}}, h) : L \to \mathcal{R}(M).$$

□

7.3. Lemma $\mathcal{R}(M)$ is a fuzzy frame. Moreover, each ε_t^M is a retraction and they all have a common right inverse.

Proof: We have already observed that $\mathcal{R}(L)$ satisfies (F0).

Now consider the constant fuzzy frame $(\mathrm{id}_M : M \to M \mid t \in \mathbb{T}')$. By 7.2 there is an α (the $h^{\mathfrak{u}}$ for $h = \mathrm{id}_M$) such that

$$\forall t, \quad \varepsilon_t^M \cdot \alpha = \mathrm{id}_M.$$

Thus, in particular, $\mathcal{R}(L)$ satisfies (F1).

(F2) follows from (1.2.1) since the limit is combined with an embedding. $\square$

Now setting for a frame homomorphism $h : L \to M$

$$\mathcal{R}(h) = (h^{\mathfrak{u}}, h) : \mathcal{R}(L) \to \mathcal{R}(M)$$

where $h^{\mathfrak{u}}$ is the homomorphism associated by 7.2 with $h : L = \mathcal{R}(L)^{\mathfrak{l}} \to M$ we immediately see that we have obtained a functor

$$\mathcal{R} : \mathbf{Frm} \to \mathbf{FuzzFrm}.$$

Proposition $\mathcal{R}$ is a right adjoint for $\mathcal{U}^{\mathfrak{l}} : \mathbf{FuzzFrm} \to \mathbf{Frm}$.

Proof: The fact from 7.2 establishes an invertible correspondence between the frame homomorphisms $h : \mathcal{U}^{\mathfrak{l}}(L) \to M$ and the fuzzy homomorphisms $(h^{\mathfrak{u}}, h) : L \to \mathcal{R}(M)$. Checking that it is natural in L, M is straightforward. $\square$

7.4. The right unit : The left unit $\mathcal{U}^{\mathfrak{l}}\mathcal{L}(M) \to M$ is the identity. Again, the other one,

$$\rho_L = (\rho_L^{\mathfrak{u}}, \mathrm{id}_L) : L \to \mathcal{R}\mathcal{U}^{\mathfrak{l}}(L)$$

is more interesting. First, from the formula

$$\varepsilon_t \cdot \rho_L^{\mathfrak{u}} = \varphi_t^M$$

we see that

> $\rho_L^{\mathfrak{u}}$ *is one-one.*

Thus,

> *for a frame M we have a "co-free fuzzy frame" $(\varepsilon_t : \overline{M} \to M \mid t \in \mathbb{T}')$ such that each fuzzy frame $(\varphi_t : M_1 \to M \mid t \in \mathbb{T}')$ is a composition of $(\varepsilon_t)_{t \in \mathbb{T}'}$ with an embedding $\rho : M_1 \to \overline{M}$.*

7.5. Similarly as in 6.5 we can obtain a universal construction of a fuzzy space with a given underlying topological space. Here the fact is more immediate and will be presented in the vein analogous to 6.6.

Let (X, θ) be a T_0-topological space. A *universal fuzzy space for* (X, θ) is a fuzzy space (X, L_θ) such that $\tau L_\theta = \theta$ and that for each fuzzy space (X, L) with $\tau L = \theta$ there is a one-one frame homomorphism

$$h : L \to L_\tau$$

such that for all $u \in L$ and $x \in X$,

$$h(u)(x) = u(x).$$

Obviously again, it is uniquely determined up to isomorphism.

Based on 7.3 and 7.4 we can construct a universal fuzzy frame for (X, θ) as follows: Set

$$L_\theta = \{\alpha((U_t)_{t \in \mathbb{T}'}) \mid U_t \in \theta, \text{ and } (\forall S \neq \emptyset, S \subseteq \mathbb{T}', \ U_{\wedge S} = \bigcup_{s \in S} U_s)\}$$

where the mappings $\alpha((U_t)_{t \in \mathbb{T}'}) : X \to \mathbb{T}$ are defined by setting

$$\alpha((U_t)_t)(x) = \bigvee \{s \mid \forall r < s, \ x \in U_r\}.$$

Using 2.8 one shows that thus defined L_θ is a subframe of $\mathbb{T}^X$ (one shows that $\bigvee_{i \in J} \alpha((U_t^i)_t) = \alpha((\bigcup_{i \in J} U_t^i)_t)$ and that

$$\alpha((U_t^1)_t) \wedge \alpha((U_t^2)_t) = \alpha((U_t^1 \cap U_t^2)_t))$$

and that $\tau L_\theta = \theta$ (one shows that $\omega_s(\alpha((U_t)_t)) = U_s$).

Now if (X, L) is such that $\tau L = \theta$, define $h : L \to L_\theta$ by setting

$$h(u) = (\omega_t(u))_t.$$

Again by 2.8 we obtain that

$$h(u)(x) > s \quad \text{iff} \quad x \in \omega_s(u) \quad \text{iff} \quad u(x) > s$$

and hence

$$h(u)(x) = u(x)$$

(which also implies that h is one-one).

7.6. Note : The functors $\mathcal{U}^{\mathfrak{u}}, \mathcal{U}^{\mathfrak{l}}$ viewed as functors $\mathfrak{F}_0 \to \mathbf{Frm}$ preserve all limits and colimits. Here we have, of course, also the trivial right resp. left adjoint $\mathcal{R}'$ resp. $\mathcal{L}' : \mathbf{Frm} \to \mathfrak{F}_0$ to $\mathcal{U}^{\mathfrak{u}}$ resp. $\mathcal{U}^{\mathfrak{l}}$ given by $\mathcal{R}'(L) = (L \to \mathbf{1} \mid t \in \mathbb{T}')$ and $\mathcal{L}'(L) = (\mathbf{2} \to L \mid t \in \mathbb{T}')$.

References

[1] B. Banaschewski, preliminary communication

[2] C.L. Chang: *Fuzzy topological spaces,* J. Math. Anal. Appl. **24** (1968), pp. 182-190

[3] J.A. Goguen: *The fuzzy Tychonoff theorem,* J. Math. Anal. Appl. **43** (1973), pp. 734-742

[4] U. Höhle: *Locales and L-Topologies,* Mathematik-Arbeitspapiere (Universität Bremen) **48** (1997) pp. 223-250

[5] B. Hutton: *Products of fuzzy topological spaces,* Topology Appl. **11** (1980), pp. 59-67

[6] B. Hutton: *Uniformities on fuzzy topological spaces,* J. Math. Anal. Appl. **58** (1977), pp. 559-571

[7] B. Hutton and I. Reilly: *Separation axioms in fuzzy topological spaces,* Fuzzy Sets and Systems **3** (1980), pp. 93-104

[8] P.T. Johnstone: *"Stone Spaces"*, Cambridge University Press, Cambridge, 1982

[9] P.T. Johnstone: *Tychonoff's theorem without the axiom of choice,* Fund. Math. **113** (1981), pp. 31-35

[10] R. Lowen: *Fuzzy topological spaces and fuzzy compactness,* J. Math. Anal. and Appl. **56** (1976), pp. 621-633

[11] R. Lowen: *Fuzzy uniform spaces,* J. Math. Anal. Appl. **82** (1981) pp. 370-385

[12] S. Mac Lane: *"Categories for the Working Mathematician"*, Springer-Verlag, Berlin, 1971

[13] S.E. Rodabaugh : *A categorical accomodation of various notions to fuzzy topology,* Fuzzy Sets and Systems **9** (1983), pp. 241-265

[14] S.E. Rodabaugh: *A theory of fuzzy uniformities with applications to the fuzzy real lines,* J. Math. Anal. Appl. **129** (1988), pp. 37-70

[15] S.E. Rodabaugh: *Categorical Frameworks for Stone Representation Theories,* in [17]

[16] S.E. Rodabaugh: *Point-set lattice-theoretic topology,* Fuzzy Sets and Systems **40** (1991) pp. 297- 345

[17] S.E. Rodabaugh, E.P. Klement, U. Höhle (editors): *"Applications of Category Theory to Fuzzy Subsets"*, Kluwer Academic Publ., 1991

[18] S. Vickers : *"Topology via Logic"*, Cambrige Tracts in Theor. Comp. Sci., Number 5, Cambridge University Press, Cambridge, 1985