# On the linear and hereditary discrepancies

Jiří Matoušek

Department of Applied Mathematics
Charles University
Malostranské nám. 25, 118 00 Praha 1
Czech Republic

**Abstract**

We exhibit a set system $\mathcal{S}_0$ such that any set system containing it has linear discrepancy at least 2.

Let $V$ be a finite set and $\mathcal{S} \subseteq 2^V$ a system of subsets of $V$. The *incidence matrix* $A$ of $(V, \mathcal{S})$ has rows indexed by the sets of $\mathcal{S}$ and columns indexed by the points of $V$. The column corresponding to a point $v \in V$ is denoted by $a_v$.

A *coloring* of $V$ is a mapping $\chi: V \to \{-1, +1\}$. The *discrepancy* of $\mathcal{S}$, denoted by $\mathrm{disc}(\mathcal{S})$, is given by

$$\mathrm{disc}(\mathcal{S}) = \min_{\chi} \mathrm{disc}(\chi, \mathcal{S}),$$

where the minimum taken is over all colorings $\chi$ of $V$ and $\mathrm{disc}(\chi, \mathcal{S}) = \max_{S \in \mathcal{S}} |\chi(S)|$, with $\chi(S) = \sum_{v \in S} \chi(v)$. The *hereditary discrepancy* of $\mathcal{S}$, denoted by $\mathrm{herdisc}(\mathcal{S})$, is

$$\mathrm{herdisc}(\mathcal{S}) = \max_{U \subseteq V} \mathrm{disc}(\mathcal{S}|_U),$$

where $\mathcal{S}|_U = \{S \cap U : S \in \mathcal{S}\}$. (In the sequel, by saying that a set system $\mathcal{S}$ *contains* a set system $\mathcal{S}'$ we mean that $\mathcal{S}' \subseteq \mathcal{S}|_U$ for some subset $U$ of the ground set of $\mathcal{S}$.)

The *linear discrepancy* arises in the following "rounding" problem. Each point $v \in V$ is assigned a weight $w_v \in [-1, 1]$. We want a coloring of $V$ for which the sum of the colors in each set $S \in \mathcal{S}$ is close to the total weight of its points. The discrepancy of $\mathcal{S}$ with respect to the given weights is thus

$$\min_{\chi: V \to \{-1,1\}} \max_{S \in \mathcal{S}} |\chi(S) - w(S)|,$$

and the linear discrepancy of $\mathcal{S}$ is the supremum of this quantity over all choices of the weight vector $w \in [-1,1]^V$.

These definitions can be expressed in terms of the incidence matrix $A$ of $\mathcal{S}$, and in that form, they can be used to define the various discrepancies for arbitrary real matrices $A$. Namely, for an $m \times n$ matrix $A$ we have

$$\mathrm{disc}(A) = \min_{x \in \{-1,1\}^n} \|Ax\|_\infty,$$

$$\mathrm{herdisc}(A) = \max_B \mathrm{disc}(B)$$

where $B$ ranges over all submatrices of $A$, and

$$\mathrm{lindisc}(A) = \max_{w \in [-1,1]^n} \min_{x \in \{-1,1\}^n} \|A(x-w)\|_\infty.$$

The linear and hereditary discrepancies were considered by Lovász, Spencer, and Vesztergombi [3] (also see Spencer [4] or [1] for background on combinatorial discrepancy theory). We should remark that they use $0/1$ colorings instead of $-1/+1$ colorings, and thus their discrepancy quantities differ from ours: to translate their notions of discrepancy to our setting, all discrepancies must be multiplied by 2.

They proved that

$$\mathrm{lindisc}(A) \leq 2 \cdot \mathrm{herdisc}(A)$$

for any matrix $A$ (and consequently for any set system). (Recently, Doerr [2] improved this slightly, to the near-tight estimate $\mathrm{lindisc}(A) \leq 2(1 - \frac{1}{2m}) \,\mathrm{herdisc}(A)$, where $m$ is the number of rows of $A$.)

Conversely, one can ask whether the hereditary discrepancy can be bounded by a function of the linear discrepancy. For real matrices, this is not the case: Lovász et al. [3] exhibited matrices $A$ for which $\mathrm{lindisc}(A) \leq 1$ while $\mathrm{herdisc}(A)$ is as large as desired; their example is the single-row matrix $(1, 2, 2^2, \ldots, 2^{n-1})$.

Here we consider this question for set systems. That is, do there exist set systems with linear discrepancy bounded by some constant $M$ and with arbitrarily large hereditary discrepancy? In other words, are there set systems with arbitrarily large discrepancy that can be exended, by adding new points, to systems with linear discrepancy at most $M$? If yes, one can further ask whether *every* set system can be extended to a set system with linear discrepancy at most $M$, or whether there are set systems that force high linear discrepancy of any system containing them.

It seems tempting to believe that any set system can be extended, by adding enough new points to each of its sets, to a set system of linear discrepancy at most 1. Here we show that this is not the case, by exhibiting a set system forcing linear discrepancy at least 2 for any system containing it. In fact, we show it for the "restricted linear discrepancy" $\mathrm{lindisc}_1$, for which the weight of each point is restricted to be $-1$, 0, or $+1$. By the method of Lovász et al. [3] (also see Spencer [4]), it is easy to check that $\mathrm{lindisc}_1(\mathcal{S}) \leq \mathrm{lindisc}(\mathcal{S}) \leq 2\,\mathrm{lindisc}_1(\mathcal{S})$.

**Theorem.** *There exists a set system $\mathcal{S}_0$ such that* $\mathrm{lindisc}(\mathcal{S}) \geq \mathrm{lindisc}_1(\mathcal{S}) \geq 2$ *for any $\mathcal{S}$ containing $\mathcal{S}_0$.*

**Proof.** As we will see below, any $\mathcal{S}_0$ with the following properties will do:

(i) Each set of $\mathcal{S}_0$ has an even size.

(ii) The number of sets is strictly larger than the number of points.

(iii) The discrepancy of $\mathcal{S}_0$ is nonzero (and thus at least 2), and this holds not only for colorings of points by $\pm 1$, but also for colorings by arbitrary odd integers.

(iv) The system $\mathcal{S}_0$ is *closed on real linear combinations of points*; that is, the real vector space generated by the columns of the incidence matrix $A_0$ of $\mathcal{S}_0$ contains no 0/1 vector distinct from the columns of $A_0$.

A simple example of such an $\mathcal{S}_0$ is the system of all nonempty subsets of $\{1, 2, \ldots, 5\}$ of even cardinality. Checking (iii) is straightforward. As for (iv), let $a_1, \ldots, a_5$ be the column vectors of the incidence matrix, and suppose that $\sum_{i=1}^{5} \alpha_i a_i$ is a 0/1 vector. For any $i < j \le 5$ there is a row with 1s in columns $i$ and $j$ and 0s elsewhere, and so $\alpha_i + \alpha_j \in \{0, 1\}$. Similarly we get that $\alpha_i + \alpha_j + \alpha_k + \alpha_\ell \in \{0, 1\}$ for any four distinct indices $i, j, k, \ell$. Hence if, say, $\alpha_1 + \alpha_2 = 1$ then $\alpha_3 = \alpha_4 = \alpha_5 = 0$ and either $\alpha_1 = 0$ or $\alpha_2 = 0$. This shows that $\mathcal{S}_0$ is closed on real linear combinations of points.

It remains to prove that $\mathrm{lindisc}_1(\mathcal{S}) \ge 2$ for any $\mathcal{S}$ containing an $\mathcal{S}_0$ satisfying (i) through (iv). Let $V$ be the ground set of $\mathcal{S}$ and $V_0 \subseteq V$ the ground set of $\mathcal{S}_0$. We suppose that $|\mathcal{S}| = |\mathcal{S}_0| = m > |V_0|$.

We define suitable weights of the points of $V$. We first choose a vector $y \in \mathbf{R}^m$ with the following property: for any vector $a \in \{0, 1\}^m$, $y^T a = 0$ holds if and only if $a$ is one of the columns of the incidence matrix $A_0$ of $\mathcal{S}_0$. Such an $y$ must be perpendicular to the subspace of $\mathbf{R}^m$ generated by the columns of $A_0$, which has dimension smaller than $m$ by (ii). Since this subspace contains no other 0/1 vector by (iv), $y$ can be chosen in such a way that it is perpendicular only to the columns of $A_0$.

Having chosen $y$, we define the weight vector $w = (w_v : v \in V)$. For $v \in V_0$, we put $w_v = 0$, and for $v \in V \setminus V_0$, we let

$$w_v = \begin{cases} 1 & \text{if } y^T a_v \ge 0 \\ -1 & \text{if } y^T a_v < 0, \end{cases}$$

where $a_v$ is the column of $v$ in the incidence matrix of $\mathcal{S}$.

Let $x \in \{-1, 1\}^V$ be a vector corresponding to a coloring of $V$. Since the components of $x - w$ corresponding to points in $V \setminus V_0$ are even integers and since each set of $\mathcal{S}_0$ has even size, the entries of $A(x - w)$ are even integers. Hence it suffices to show that $A(x - w) \ne 0$.

Supposing for contradiction that $A(x - w) = 0$, we calculate

$$0 = y^T A(w - x) = \sum_{v \in V} y^T a_v (x - w)_v .$$

By the definition of $w$, one can check that all the addends on the right-hand side are $\le 0$, and therefore all of them are 0.

Consequently, $(x-w)_v = 0$ for all $v \in V \setminus \overline{V}_0$, where $\overline{V}_0 = \{v \in V : a_v = a_u \text{ for some } u \in V_0\}$. Thus, the vector $x - w$ can be viewed as a coloring of $V_0$ by odd integers, where $v \in V_0$ receives the color

$$x_v + \sum_{u \in \overline{V}_0 \setminus V_0 :\, a_v = a_u} (x_u - 1).$$

By the property (iii) of $\mathcal{S}_0$, some set has nonzero discrepancy under this odd-integer coloring, and so $A(x-w) \neq 0$. This finishes the proof. $\Box$

**Remarks.** The behavior of the linear discrepancy exhibited by the Theorem doesn't look very nice. Maybe a more natural notion of "rounding" discrepancy would be the analogoue of the linear discrepancy with the weights restricted to the interval $[-\frac{1}{2}, \frac{1}{2}]$ instead of $[-1, 1]$. In this way, there is always room for changing the value in both directions by rouding. It is easy to show that for this modified linear discrepancy, any set system can be extended to a system with modified linear discrepancy at most 1 by adding enough new points.

Can the constant 2 in the Theorem be increased? I believe that the following question is crucial for understanding this problem: What is the linear discrepancy of the set system dual to $(X, 2^X)$, i.e. of the "full Venn diagram" of $n$ sets? Does it become arbitrarily large as $|X| \to \infty$?

# References


[1] J. Beck and V. Sós. Discrepancy theory. In *Handbook of Combinatorics*, pages 1405–1446. North-Holland, Amsterdam, 1995.

[2] B. Doerr. Linear and hereditary discrepancy. Manuscript, Tech. Univ. Kiel, Germany, 1998.

[3] L. Lovász, J. Spencer, and K. Vesztergombi. Discrepancy of set-systems and matrices. *European J. Combin.*, 7:151–160, 1986.

[4] J. Spencer. *Ten lectures on the probabilistic method.* CBMS-NSF. SIAM, Philadelphia, PA, 1987.