Study texts for

# DIMATIA - DIMACS
# Research Experience for Undergraduates

July 15. – August 5. 1999

Prague

Supported by NSF - AV ČR grant
KONTAKT ME 337

and

GAČR 201/99/0242

# Bárány and Larman's combinatorial approach to the circle problem

Martin Klazar

June 1999

**Abstract**

We describe Bárány and Larman's combinatorial-geometric proof of the $O(N^{1/3})$ bound in the circle problem. Auxiliary results are presented without proofs. However, the proofs are indicated in a series of exercises equipped with hints.

**1. The circle problem and the content of this article.** The *circle problem* asks about the order of the error term in the asymptotics of the average number of representations of $n$ as a sum of two integral squares. More precisely, for $n \in \mathbf{N} = \{0, 1, 2, \ldots\}$ and $r(n)$ the number of solutions $x, y \in \mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$ of the equation $n = x^2 + y^2$, how big is the error in the asymptotic formula for $R(N) = \sum_{n=0}^{N} r(n)$. For example, $R(5) = 21$ because $r(0) = 1, r(1) = 4, r(2) = 4, r(3) = 0, r(4) = 4$ and $r(5) = 8$.

The main term of the asymptotics is determined readily by explaining the word *circle*: $R(N)$ counts the number of lattice points (i.e. points with both coordinates integral) inside or on the plane circle with radius $\sqrt{N}$ and centre in the origin (Exercise 1). Thus $R(N)$ is approximated by the area of the circle, which provides the main term $R(N) \sim \pi N$. By an easy geometric argument (Exercise 2), the error term can be bounded by the perimeter:

$$R(N) = \pi N + O(N^{1/2}).$$

Already C. F. Gauss knew this. One century later W. Sierpinski lowered the exponent from $1/2$ to $1/3$. Since then generations of researchers in analytic number theory tested their ingenuity on this problem to obtain the best bound on the exponent. The right value is almost surely $1/4$ (see section 5) but at present, it appears, we have still a long way to climb (or to descend?) to that distant shining summit. Or, maybe, sometime somebody in a giant leap ...

Let $\nu$ be the infimum of all exponents $\alpha > 0$ such that $R(N) = \pi N + O(N^\alpha)$. The altitudes reached in the ascent to the peak $1/4 = 0.25$ so far are:

| | | |
|---|---|---|
| $\nu \le \frac{1}{2}$ | $= 0.50000\ldots$ | C. F. Gauss |
| $\nu \le \frac{1}{3}$ | $= 0.33333\ldots$ | W. Sierpinski 1906 |
| $\nu < \frac{1}{3}$ | $= 0.33333\ldots$ | J. van der Corput 1923 |

$\nu \leq \frac{37}{112} = 0.33035\ldots$ E. Landau 1924, J. Littlewood and A. Walfisz 1924
$\nu \leq \frac{163}{494} = 0.32995\ldots$ A. Walfisz 1927
$\nu \leq \frac{27}{82} = 0.32926\ldots$ L. Nieland 1928
$\nu \leq \frac{15}{46} = 0.32608\ldots$ E. Titchmarsh 1934
$\nu \leq \frac{13}{40} = 0.32500\ldots$ Hua Loo-keng 1942
$\nu \leq \frac{12}{37} = 0.32432\ldots$ Chen Jing-run 1963
$\nu \leq \frac{35}{108} = 0.32407\ldots$ W.-G. Nowak 1984
$\nu \leq \frac{139}{429} = 0.32400\ldots$ G. Kolesnik 1985
$\nu \leq \frac{7}{22} = 0.31818\ldots$ H. Iwaniec and C. Mozzochi 1988 [10]
$\nu \leq \frac{23}{73} = 0.31506\ldots$ M. Huxley 1993 [8]

By M. Huxley [9], the potential of the Bombieri–Iwaniec–Mozzochi method is far from being exhausted and we can look forward to better bounds.

The methods for obtaining bounds below $1/3$ are delicate and use exponential sums. To get $\nu \leq 1/3$ is much easier but one still needed a nonnegligible dose of real analysis. Bárány and Larman eliminated the analysis almost completely (two simple integral estimates of sums, Propositions 4 and 5, suffice) and replaced it by fine geometric arguments. Their beauty is presented on the following pages.

We invite the reader[1] to take an active part in the advanture by proving auxiliary results independently. The corresponding exercises and hints are given in the sixth section. In the second section we review necessary definitions. Then we state all results needed. The proof follows in full in the fourth section. The fifth section contains remarks.

This article is a compilation based on [1], [2] and [3]; I added also some effort of my own. I would like to thank Prof. Imre Bárány for answering patiently my questions and Martin Zeman for obtaining for me copies of some articles. I thank warmly Katharina Langkau and Andrew Rechnitzer for their valuable comments.

**2. Notation.** $\mathbf{R}$ and $\mathbf{Z}^2$ denote real numbers and lattice points. In sequel everything takes place in the plane $\mathbf{R}^2$. A set is *convex* if it contains with any two points the connecting segment. Smallest convex set containing $X \subset \mathbf{R}^2$ is said to be the *convex hull* of $X$. A *halfplane* $H$ consists of all points lying on the same side of or on a line $L$. A *strip* consists of all points between or on two paralell lines. A *halfstrip* is the intersection of a strip with a halfplane perpendicular to it. By a *polygon* we mean any bounded intersection of finitely many halfplanes. Or, equivalently, it is the convex hull of finitely many points. *Lattice polygon* is a polygon whose vertices are lattice points. $\partial X$ denotes the *boundary* of $X$ and $\mathrm{int}(X) = X \backslash \partial X$ the *interior* of $X$. Symbol $\mathrm{per}(X)$ denotes the length of the perimeter of $X$, where $X$ will be a circle or a polygon. Similarly

[1]This text was written for the student participants of the International Spring School in Combinatorics that took place in Šumava (Böhmerwald) mountains in April 1999.

$\mathrm{Vol}(X)$ denotes the volume of $X$ (in our plane case simply the area) and $\mathrm{ver}(M)$ the number of vertices of a polygon $M$. Points inside or on a circle form a *disc*.

*D-segments* $U = D \cap H$, where $D$ is a disc and $H$ a halfplane, will play an important role. *Base* of $U$ is the segment $b = \partial H \cap D$, the *length* of $U$ is the length of $b$ and the *width* of $U$ is measured in the direction perpendicular to the base. The *radius* of $U$ is the radius of $D$.

For $v = (v_1, v_2) \in \mathbf{R}^2$ the norm $(v_1^2 + v_2^2)^{1/2}$ is denoted $\|v\|$. Symbol $v \cdot w$ means, for $v, w \in \mathbf{R}^2$, the scalar product $v_1 w_1 + v_2 w_2$. Lattice point is *primitive* if it has coprime coordinates. For $v \in \mathbf{R}^2$ we set $v^\perp = (-v_2, v_1)$; thus $v \cdot v^\perp = 0$. For $X \subset \mathbf{R}^2$ and $v \in \mathbf{R}^2, \|v\| > 0$, we set

$$t(v, X) = \sup_{x \in X} v \cdot x - \inf_{x \in X} v \cdot x.$$

It is simply the thickness of $X$ in the direction of $v$, unit length $= 1/\|v\|$. *Distance* of two subsets $A, B$ of $\mathbf{R}^2$ is defined as

$$\mathrm{dist}(A, B) = \inf_{x \in A, y \in B} \|x - y\|.$$

For a statement $\varphi$, the indicator function $\langle \varphi \rangle$ is defined as 1 if $\varphi$ holds and 0 else. Cardinality of a set is indicated by $|\cdots|$. In paralell to the notation $f = O(g)$ we use the relation symbol $f \ll g$ with the same meaning; $f \ll_c g$ means that the implicit constant depends only on the parameter(s) mentioned.

## 3. Auxiliary combinatorial-geometric results.

**Proposition 1 (Pick's theorem)** *For every lattice polygon $M$ it holds*

$$\mathrm{Vol}(M) = |\mathbf{Z}^2 \cap M| - \tfrac{1}{2}|\mathbf{Z}^2 \cap \partial M| - 1.$$

Proof — Exercises 3–7.

**Proposition 2** *For a polygon $M$ inside a circle $C$ we have always the inequality* $\mathrm{per}(M) < \mathrm{per}(C)$.

Proof — Exercise 8.

**Proposition 3** *Every lattice polygon $M$ satisfies* $\mathrm{ver}(M) \ll \mathrm{per}(M)^{2/3}$.

Proof — Exercises 9–11.

**Proposition 4** *We have, for each fixed $c > 0$ and $r \to \infty$, the estimate*

$$\sum_{v \in \mathbf{Z}^2} \langle 0 < \|v\| < cr^{1/3} \rangle \cdot \|v\|^{-3/2} \ll_c r^{1/6}.$$

Proof — Exercise 12.

**Proposition 5** *Suppose* $(y_i, x_i)$, *where* $0 \le y_i < x_i$ *and* $i = 1, \dots, m$, *are disjoint real intervals such that* $x_i - y_i < c$ *for all* $i$. *Then*

$$\sum_{i=1}^{m} \frac{(x_i - y_i)^3}{x_i^{3/2}} \ll_c 1.$$

Proof — Exercise 13.

**Proposition 6 (plane Flatness theorem)** *There is an absolute constant* $c > 0$ *such that* $t(q, X) < c$ *holds for any convex set* $X$, $\mathrm{int}(X) \neq \emptyset$ *and* $\mathrm{int}(X) \cap \mathbf{Z}^2 = \emptyset$, *and appropriate* $q \in \mathbf{Z}^2$.

Proof — Exercises 14–23. See also Exercise 24.

**Proposition 7** *Let* $U$ *be a* $D$*-segment,* $\mathrm{int}(U) \cap \mathbf{Z}^2 = \emptyset$, *whose base contains* $\ge 3$ *lattice points* $v_0, v_1, \dots$ *so that* $v_0$ *and* $v_1$ *are neighbours. Then*

$$t((v_1 - v_0)^{\perp}, U) < 2.$$

Proof — Exercise 25.

**Proposition 8** *The length* $2\rho$ *and width* $t$ *of a* $D$*-segment of radius* $r$ *satisfy the formula*

$$\rho = \sqrt{(2r - t)t}.$$

Proof — Exercise 26.

**4. Bárány and Larman's proof of $\nu \le 1/3$.** Set $B_r = \{x \in \mathbf{R}^2 : \|x\| \le r\}$ and $M_r =$ the convex hull of $\mathbf{Z}^2 \cap B_r$. We need to prove that $|\mathbf{Z}^2 \cap M_r| - \mathrm{Vol}(B_r) \ll r^{2/3}$. Proposition 1 gives us a geometric representation of the error:

$$\begin{aligned} |\mathbf{Z}^2 \cap M_r| - \mathrm{Vol}(B_r) &= \tfrac{1}{2}|\mathbf{Z}^2 \cap \partial M_r| - \mathrm{Vol}(B_r \backslash M_r) + 1 \\ &= S_1 - S_2 + 1. \end{aligned}$$

We show that both $S_1$ and $S_2$ are $\ll r^{2/3}$.

On $\partial M_r$ we count clockwise. For each side $s = vw$ of $M_r$ containing $l(s) + 1$ lattice points, $s \cap \mathbf{Z}^2 = \{v_0 = v, v_1, \dots, v_{l(s)} = w\}$, we define $p_s = v_1 - v_0$ and $U_s = B_r \cap H$, where $H$ is the halfplane given by $s \subset \partial H$ & $M_r \not\subset H$. Notice that distinct $s$ have distinct vectors $p_s$. Let $t_s$ be the width of $U_s$.

Let us bound $S_1$. We have

$$\begin{aligned} 2S_1 = \sum_s l(s) &= \sum_s \langle l(s) = 1 \rangle + \sum_s \langle l(s) > 1 \rangle \cdot l(s) \\ &= S_3 + S_4. \end{aligned}$$

By Propositions 2 and 3, $S_3 \ll r^{2/3}$. By Proposition 7, $l(s) > 1$ implies $t_s < 2/\|p_s\|$. Using this and Proposition 8, we obtain

$$\|p_s\| \le \rho < \sqrt{2rt_s} \le \sqrt{\frac{4r}{\|p_s\|}}.$$

Thus $\|p_s\| < (4r)^{1/3}$ and $l(s) \le 2\rho/\|p_s\| \le 4(r/\|p_s\|^3)^{1/2}$. By Proposition 4,

$$S_4 < 4\sqrt{r} \sum_{p \in \mathbf{Z}^2} \langle 0 < \|p\| < (4r)^{1/3} \rangle \cdot \|p\|^{-3/2} \ll \sqrt{r} \cdot r^{1/6} = r^{2/3}.$$

So, $S_1 = (S_3 + S_4)/2 \ll r^{2/3}$.

Now we bound $S_2$ that is the area of $B_r$ missed by $M_r$. Consider the annulus $A_r = \{x \in \mathbf{R}^2 : \ r - r^{-1/3} \le \|x\| \le r\}$. Clearly,

$$\begin{aligned} S_2 &\le \mathrm{Vol}(A_r) + \mathrm{Vol}(\bigcup\{U_s : \ t_s \ge r^{-1/3}\}) \\ &= \mathrm{Vol}(A_r) + S_5. \end{aligned}$$

Obviously, $\mathrm{Vol}(A_r) \ll r^{2/3}$. It remains to bound $S_5$.

We employ Proposition 6 and assign to each $U_s$ a $q \in \mathbf{Z}^2$ so that $t(q, U_s) < c$; $c > 0$ is an absolute constant. We can assume that the accute angle between $q$ and the base of $U_s$ is $\ge \pi/4$; if not $q$ is replaced by $q^\perp$. For a $q \in \mathbf{Z}^2$ we denote $\mathcal{M}(q)$ the set of all $U_s$ of $S_5$ corresponding to $q$ and estimate $\mathrm{Vol}(\bigcup \mathcal{M}(q))$. Summing the estimates over all $q$, we obtain the bound on $S_5$. Since $r^{-1/3} \le t_s < c/\|q\|$, we have $\|q\| \ll r^{1/3}$ for all (nonempty) $\mathcal{M}(q)$.

Let $L_1$ and $L_2$ be two perpendicular lines going through the origin, $L_1$ in direction $q$, and let $L_1 \cap \partial B_r = \{u^*, v^*\}$. The lines split $B_r$ into four quadrants $Q_1, \ldots, Q_4$. We have the partition $\mathcal{M}(q) = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_6$, where $\mathcal{M}_1 = \{U \in \mathcal{M}(q) : \ u^* \in U\}$, $\mathcal{M}_2 = \{U \in \mathcal{M}(q) : \ v^* \in U\}$, and $\mathcal{M}_3, \ldots, \mathcal{M}_6$ contain the $U \in \mathcal{M}(q)$ lying in $Q_1, \ldots, Q_4$, respectively. Notice that no $U$ can cross $L_2$.

Clearly, $\bigcup \mathcal{M}_1 \subset V$ for a $B_r$-segment $V$ that is perpendicular to $L_1$, has width $t < c/\|q\|$ and $u^* \in \partial V$. Therefore the length $b$ of $V$ satisfies $b/2 = \sqrt{(2r-t)t} \ll \sqrt{r/\|q\|}$ and we have

$$\mathrm{Vol}(\bigcup \mathcal{M}_1) \le \mathrm{Vol}(V) < bt \ll \sqrt{r}\|q\|^{-3/2}.$$

The same bound holds for $\mathrm{Vol}(\bigcup \mathcal{M}_2)$.

Let $\mathcal{M}_3 = \{U_1, \ldots, U_m\}$ be, clockwise, all $U \in \mathcal{M}(q)$ lying above $L_1$ and to the left of $L_2$. We can assume that also $u^* \in Q_1$. Let $b_i$ be the base or the length of $U_i$, $t_i$ its width, and $s_i = M_r \cap U_i$ the corresponding side. Let $I_i$ and $J_i$ be the projections of $b_i$ and $s_i$ on $L_1$, respectively. Let $f_i$ and $g_i$, $f_i > g_i$, be the distances of the endpoints of $I_i$ from $u^*$ and $x_i'$ and $y_i'$ be given by $f_i = x_i'/\|q\|$ and $g_i = y_i'/\|q\|$. Let $x_i$ and $y_i$ be defined analogously for the interval $J_i$. Let $d_i$ be the length of the projection of $b_i$ on $L_2$.

Notice that $x_i - y_i \le x_i' - y_i' < c$, $x_i' - y_i' \le 3(x_i - y_i)$ (because $\mathbf{Z}^2 \cap (B_r \backslash M_r) = \emptyset$), $x_i \le x_i' \le r\|q\|$, $b_i \le \sqrt{2} d_i$, and the intervals $(y_i, x_i)$ are disjoint. With the help of Proposition 8,

$$\begin{aligned} d_i &= \sqrt{(2r - f_i)f_i} - \sqrt{(2r - g_i)g_i} \\ &= \frac{2r(f_i - g_i) - f_i^2 + g_i^2}{\sqrt{(2r - f_i)f_i} + \sqrt{(2r - g_i)g_i}} \\ &\ll \frac{r(x_i' - y_i')/\|q\|}{\sqrt{r x_i'/\|q\|}} \ll (x_i - y_i)\sqrt{\frac{r}{x_i\|q\|}}. \end{aligned}$$

From $b_i/2 = \sqrt{(2r - t_i)t_i} \geq \sqrt{rt_i}$ we get $t_i \ll b_i^2/r \leq (\sqrt{2}d_i)^2/r$. Hence

$$\mathrm{Vol}(U_i) < b_i t_i \leq \sqrt{2} d_i t_i \ll d_i^3/r \ll \frac{\sqrt{r}(x_i - y_i)^3}{(x_i\|q\|)^{3/2}}.$$

By Proposition 5,

$$\mathrm{Vol}(\bigcup \mathcal{M}_3) \leq \sum_{i=1}^{m} \mathrm{Vol}(U_i) \ll \sqrt{r}\|q\|^{-3/2}.$$

By symmetry, the same bound holds for $\mathcal{M}_4, \mathcal{M}_5$, and $\mathcal{M}_6$.

Alltogether we have

$$\mathrm{Vol}(\bigcup \mathcal{M}(q)) \leq \sum_{i=1}^{6} \mathrm{Vol}(\bigcup \mathcal{M}_i) \ll \sqrt{r}\|q\|^{-3/2}.$$

Recall, $\|q\| \ll r^{1/3}$. Summing over $q$ and using Proposition 4, we have again

$$S_5 \leq \sum_{q \in \mathbf{Z}^2} \mathrm{Vol}(\bigcup \mathcal{M}(q)) \ll \sqrt{r} \sum_{q \in \mathbf{Z}^2} \langle 0 < \|q\| \ll r^{1/3} \rangle \cdot \|q\|^{-3/2} \ll \sqrt{r} \cdot r^{1/6} = r^{2/3}.$$

So, $S_5 \ll r^{2/3}$ and $S_2 \ll r^{2/3}$. Sierpinski's bound is proven.

**5. Remarks.** The circle problem has a big brother with a similar history and bounds, the *divisor problem*; see [13] and [16]. In 1915 E. Landau and also G. H. Hardy proved that $\nu \geq 1/4$. This lower bound has not yet been improved. Let $A = 0 \leq a_1 \leq a_2 \leq \ldots$ be a sequence of integers and $r_A(n) = \sum_{i,j} \langle n = a_i + a_j \rangle$. P. Erdős and W. Fuchs proved [5] (see also [6] or [15]) that if

$$\sum_{n=0}^{N} r_A(n) = cN + O(N^\alpha)$$

with $c > 0$ a constant, then $\alpha \geq 1/4$. This greatly generalizes Landau's and Hardy's result corresponding to the special case $a_i = i^2$.

Proposition 1 is due to Georg Pick (1859–1942) who was born in Vienna, spent most of his career as a professor of mathematics at the German University in Prague (in the school year 1900/01 he was a dean of the Faculty of Arts) and perished in the Theresienstadt (Terezín) ghetto ([4, 11]). Proposition 6 is an important result and holds in general in $\mathbf{R}^n$, see [12] for more information. Three other proofs of $\nu \leq 1/3$ can be given: by means of Bessel functions (E. Landau's method, see [13] or [14]) or by trigonometric sums (J. van der Corput's method, see [16]) or the perhaps most elegant proof based on simple real analysis and Dirichlet's approximation theorem (I. M. Vinogradov's method, see [7]). For strengthenings and generalizations of the bounds on $S_1$ and $S_2$ see [3].

**6. Exercises and hints.**

Exercises.

1. Prove that $R(N)$ is the cardinality of $\mathbf{Z}^2 \cap \{v \in \mathbf{R}^2 : \ \|v\| \leq \sqrt{N}\}$.

2. Prove geometricly that $R(N) = \pi N + O(N^{1/2})$.

3. A *base* is a set $B = \{v, w\} \subset \mathbf{R}^2$ consisting of linearly independent vectors. *Parallelogram* $P(B)$ spanned by $B$ is $P(B) = \{\alpha v + \beta w : \ \alpha, \beta \in [0,1)\}$. Show that
$$\mathrm{Vol}(P(B)) = \pm \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}.$$

4. A *lattice with base* $B = \{v, w\}$ is the set $\Lambda = \Lambda(B) = \{av + bw : \ a, b \in \mathbf{Z}\}$. Prove with the help of the previous exercise that
$$\Lambda(B_1) = \Lambda(B_2) \Longrightarrow \mathrm{Vol}(P(B_1)) = \mathrm{Vol}(P(B_2)),$$
i.e. the area of the *elementary lattice parallelogram* $P(B)$ depends only on $\Lambda$ and not on the base of $\Lambda$.

5. Show that, for a lattice $\Lambda(B)$, $\{P(B) + v : \ v \in \Lambda(B)\}$ is a partition of $\mathbf{R}^2$.

6. Suppose that $\Delta$ is a closed lattice triangle that contains no lattice points except the vertices. Using the previous two exercises, show that $\mathrm{Vol}(\Delta) = 1/2$.

7. From the previous exercise derive Pick's theorem.

8. Prove Proposition 2.

9. For any triangle $\Delta$ with sides $a$ and $b$ and the angle $\alpha$ between them we have $\mathrm{Vol}(\Delta) = \frac{1}{2} ab \sin \alpha$.

10. We call a side $s$ of a polygon $M$ *short* if $|s| < 10\mathrm{per}(M)/\mathrm{ver}(M)$. Similarly, an inner angle $\alpha$ of $M$ is *big* if $\pi - \alpha < 20\pi/\mathrm{ver}(M)$. Show that $M$ must have two consecutive short sides which determine a big inner angle.

11. Use Exercises 6, 9, and 10 to prove Proposition 3.

12. Prove Proposition 4.

13. Prove Proposition 5.

14. Proposition 6 says that there is an absolute constant $c > 0$ such that for every $X$ as stated a lattice direction $v$ exists so that $X$ can be covered by a strip with direction perpendicular to $v$ and width $< c/\|v\|$.

15. There exists a set $Z$ such that (i) $Z$ is convex, (ii) $Z \supset X$, (iii) $\mathrm{int}(Z) \cap \mathbf{Z}^2 = \emptyset$ and (iv) $Z$ is maximal to $\supset$ with respect to the properties (i)–(iii). Clearly it suffices to prove Proposition 6 for $Z$ in place of $X$.

16. First handle the (easier) case of unbounded $Z$. Show that any unbounded convex set $A$ with nonempty interior contains a halfstrip with a positive width.

17. Show that for every irrational $\alpha \in \mathbf{R}$ the set $\{\{n\alpha\} : n \in \mathbf{N}\}$, where $\{n\alpha\}$ is the fractional part of $n\alpha$, is dense in $[0,1]$. Infer from this that the halfstrip contained in $Z$ has a lattice direction.

18. Using the previous exercise together with Exercises 14 and 6, show that in the case of unbounded $Z$ we have $t(v, Z) = 1$ for some $v \in \mathbf{Z}^2$.

19. Proceed to the case of bounded $Z$. Show that for any convex set $A$ and any point $x \notin \operatorname{int}(A)$ there is a halfplane $H$ such that (i) $A \subset H$, (ii) $x \in \partial H$ and (iii) $\operatorname{dist}(\partial H, A) = \operatorname{dist}(x, A)$. $H$ is called the *supporting halfplane* of $x$ (with respect to $A$).

20. Thus, $Z$ is a polygon, each of which sides contains a lattice point different from the vertices.

21. In fact, $Z$ is a triangle or a quadrilateral.

22. Moreover, the lattice points inside the edges of $Z$ can be assumed to be $(0,0)$, $(0,1)$, and $(1,0)$ in the former case and $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$ in the latter case.

23. For such restricted $Z$, finish the proof of Proposition 6 by showing that one of the vectors $v = (1,0), (0,1)$, and $(1,1)$ works in the former case and $v = (1,0)$ or $v = (0,1)$ works in the latter case.

24. Does Proposition 6 hold for the family of all convex sets $X$ disjoint to $\mathbf{Z}^2$?

25. With the help of Exercises 6 and 14, prove Proposition 7.

26. Prove Proposition 8.

Hints.

1. What is the geometric meaning of $r(n)$?

2. Wrap each lattice point in the disc by a unit square and bound the error term by the number of squares intersecting the boundary circle.

3. Start from the formula $\operatorname{Vol}(P(B)) = \|v\| \cdot \|w\| \cdot |\sin\alpha|$; $\alpha$ is the angle between $v$ and $w$.

4. Express the vectors of $B_1$ as integral combinations of vectors in $B_2$ and vice versa. Use integrality of the transfer matrices.

5. Express $x \in \mathbf{R}^2$ as a real combination of vectors in $B$.

6. Move $\Delta$ so that one vertex coincides with the origin and extend $\Delta$ to an elementary lattice parallelogram. Show that the lattice is, in fact, $\mathbf{Z}^2$.
7. Prove (by induction, say) that $M$ can be split in the right number of triangles of the right type.
8. Reduce this to the situation when $C$ goes through two neighbouring vertices of $M$ and we need a stronger inequality. Prove it by induction.
9. What is the high school formula for the area of a triangle?
10. Show that $M$ has many short sides and many big inner angles.
11. Bound from below and above the area of that small flat triangle.
12. Evaluate the double integral $\int\int \|x\|^{-3/2} dx_1 dx_2$ over the disc (use polar coordinates).
13. Notice that $(x_i - y_i)^3 \leq x_i^2(x_i - y_i), c^2(x_i - y_i)$ and apply integral estimates.
14. This is really a mere reformulation.
15. Use (calmly!) Zorn maximality principle.
16. Let $x \in \operatorname{int}(A)$. Consider an accumulation point of the directions $(y - x)/\|y - x\|$ for $y \in A, y \to \infty$.
17. First show by the pigeon hole principle that 0 is an accumulation point.
18. Let $v \in \mathbf{Z}^2$ be the primitive direction of the halfstrip. Consider all lines going through lattice points in direction $v$.
19. For $x \in \partial A$ rotate line going through $x$. For $\operatorname{dist}(x, A) > 0$ take the line perpendicular to the distance.
20. Consider the supporting halfplanes for all $v \in \mathbf{Z}^2$. Also, $Z$ is maximal.
21. Because among each five lattice points two determine segment that has lattice middle point.
22. Move one of the lattice points to the origin. The lattice in question is, in fact, $\mathbf{Z}^2$. Thus ...
23. Triangle $Z$ detemined by three lines going through $(0, 0)$, $(0, 1)$, and $(1, 0)$, respectively, contains no lattice point inside. The square case is easy.
24. No, consider an appropriate line.
25. Again, consider all lines going through lattice points in direction $v_1 - v_0$.
26. Use well known results of Pythagoras, Euclid, and Thales.

## References

[1] I. Bárány, lecture on the 5th Prague Combinatorial Workshop, July 1997.

[2] I. Bárány, an e-mail message, October 1998.

[3] I. Bárány and D. G. Larman, The convex hull of the integer points in a large ball, *Math. Ann.* **312** (1998), 167–181.

[4] M. Bruckheimer and A. Arcavi, Farey series and Pick's area theorem, *Math. Intell.* **17** (1995), 64–67.

[5] P. Erdős and W. H. J. Fuchs, On a problem of additive number theory, *J. London Math. Soc.* **31** (1956), 67-73.

[6] H. Halberstam and K. F. Roth, *Sequences*, Clarendom Press, Oxford, 1966.

[7] Hua Loo Keng, *Introduction to number theory*, Springer, Berlin, 1982.

[8] M. Huxley, Exponential sums and lattice points II, *Proc. London Math. Soc.* **66** (1993), 279–301.

[9] M. Huxley, personal communication, February 1999.

[10] H. Iwaniec and C. Mozzochi, On the divisor and circle problems, *J. Number Theory* **29** (1988), 60–93.

[11] Fr. Kafka a J. Petráň (redaktoři), *Dějiny Univerzity Karlovy*, svazek III (1802–1918) a IV (1918–1990), Karolinum, Praha, 1997 a 1998. [Fr. Kafka and J. Petráň (editors), *History of Charles University*, volume III (1802–1918) and IV (1918–1990), Karolinum, Praha, 1997 and 1998.]

[12] R. Kannan and L. Lovász, Covering minima and lattice-point-free convex bodies, *Ann. of Math.* **128** (1988), 577–602.

[13] E. Krätzel, *Lattice points*, Kluwer Academic Publishers, Dordrecht, 1988.

[14] E. Landau, *Vorlesungen über Zahlentheorie* (3 volumes), Hirzel, Leipzig, 1927; reprinted by Chelsea, New York, 1969.

[15] D. J. Newman, *Analytic number theory*, Springer, New York, 1998.

[16] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge University Press, Cambridge UK, 1995.

*Department of Applied Mathematics, Charles University, Malostranské náměstí 25, 118 00 Praha, Czech Republic;* `klazar@kam.ms.mff.cuni.cz`