EQUATIONS OF TOURNAMENTS ARE NOT FINITELY BASED

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Abstract

The aim of this paper is to prove that there is no finite basis for the equations satisfied by tournaments. This solves a problem posed in Müller, Nešetřil and Pelant [10].

1 Introduction

By a tournament we mean a directed graph with loops, such that for any two distinct vertices a and b exactly one of the two cases, either $a \to b$ or $b \to a$, takes place.

For any tournament T we can define multiplication on T by setting $ab = ba = a$ whenever $a \to b$. With respect to this multiplication, T becomes a groupoid (a universal algebra with one binary operation). Moreover, T is uniquely determined by this multiplication. It is easy to see that the class of groupoids obtained from tournaments in this natural way is just the class of commutative groupoids satisfying $ab \in \{a, b\}$ for all a and b. (One could equivalently say: commutative groupoids, every subset of which is a subgroupoid.) Because of the one-to-one correspondence, we will identify tournaments with their corresponding groupoids. So, a tournament is a commutative groupoid satisfying $ab \in \{a, b\}$ for all a and b. For a tournament T, we have $a \to b$ if and only if $ab = a$.

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A complete bibliography on algebraic representations of tournaments would include the papers [2], [3], [4], [5], [6], [9] and [10].

One can easily check that tournaments satisfy, for example, the following equations:

(1) $xx = x$
(2) $xy = yx$
(3) $x(xy) = xy$
(4) $x((xy)(xz)) = (xy)(xz)$
(5) $((xy)z)y = ((xz)y)z$
(6) $((xy)(xz))((xy)(yz)) = (xy)z$

(On the other hand, the associative law is not satisfied.) It is natural to ask whether a list of equations like this one is complete, in the sense that any equation satisfied in all tournaments would be derivable. Our main result, Theorem 3, states that not only the six-item list is not complete, but there is no finite complete list of equations for tournaments at all. That question has been first formulated in Müller, Nešetřil and Pelant [10].

2 Universal algebraic background

For the basics of universal algebra, the reader is referred to either [8] or [1]. We are going to recall here only a few facts that are essential for the proof of our main result.

A variety is a class of (general) algebras of the same similarity type that can be defined by a set of equations. A variety is called finitely based if there is a finite set B of equations satisfied in V, such that every equation satisfied in V is a (logical) consequence of B. An arbitrary class of algebras (such as the class of tournaments) is called finitely based if it generates a finitely based variety.

A variety is said to be locally finite if any finitely generated algebra in V is finite. A variety is locally finite if and only if its free algebras on n generators, for any positive integer n, are all finite.

For a variety V and a positive integer n, we denote by V^n the variety of algebras determined by the equations in at most n variables that are satisfied in V. In this way we obtain a chain of varieties for any given variety V:

$$V^1 \supseteq V^2 \supseteq V^3 \supseteq \cdots \supseteq V.$$

It is not difficult to see that an algebra belongs to V^n if and only if all its subalgebras generated by at most n elements belong to V.

One can easily prove that a locally finite variety V of algebras of a finite similarity type is finitely based if and only if $V = V^n$ for at least one positive integer n.

In order to be able to apply this characterization to the variety of groupoids generated by tournaments, we need to know that the variety is locally finite. This will follow from the following observation.

Lemma 1 *Let K be a class of finite algebras of a finite similarity type, closed under forming of subalgebras. The variety V generated by K is locally finite if and only if for every positive integer n there are, up to isomorphism, only finitely many n-generated algebras in K.*

Proof. If V is locally finite, then (for any n) the free n-generated algebra in V is finite, so it has (up to isomorphism) only finitely many homomorphic images; these include all the n-generated algebras in V.

In order to prove the converse, denote by F_n the algebra of terms over a set of n variables, and by E the set of all the ordered pairs (t, s) of elements of F_n that represent an equation $t = s$ satisfied in V. Then E is a congruence of F_n and F_n/E is the free n-generated algebra in V. An equation in n variables belongs to E if and only if it is satisfied in all algebras in K, but we only need to check the at most n-generated ones. If the set S of the n-generated algebras in K is finite (it is sufficient to consider just the nonisomorphic ones), then E has only finitely many blocks, since every block is uniquely determined by a function, assigning to any algebra $A \in S$ and any interpretation of the n variables in A an element of A; consequently, the free algebra F_n/E is finite. $\square$

The variety generated by tournaments will be denoted by $\mathbf{T}$.

Corrollary 2 (Crvenković and Marković [2]) *The variety $\mathbf{T}$ is locally finite.* $\square$

Because of the lack of associativity, we need to distinguish between expressions like $(xy)z$ and $x(yz)$. In order to avoid using too many parentheses, let us make the following convention: if parentheses are missing, they are always assumed to be grouped to the left. So, for example, $xy(z(uv)t)$ stands for $(xy)((z(uv))t)$.

3 The variety $\mathbf{T}$ is not finitely based

Theorem 3 *For every $n \geq 3$ there exists a groupoid $\mathbf{M}_n$ with $n+2$ elements such that $\mathbf{M}_n$ belongs to $\mathbf{T}^n$ but not to $\mathbf{T}^{n+1}$. Consequently, the variety $\mathbf{T}$ (the variety generated by tournaments) is not finitely based.*

Proof. Put $\mathbf{M}_n = \{a, b_0, \ldots, b_n\}$ and define commutative and idempotent multiplication on $\mathbf{M}_n$ by

$$\begin{aligned}
&ab_1 = b_0,\\
&ab_i = b_i \text{ for } i \leq n-1 \text{ and } i \neq 1,\\
&ab_n = a,\\
&b_i b_{i+1} = b_i \text{ for } i < n-1,\\
&b_n b_{n-1} = b_n,\\
&b_i b_j = b_{\max(i,j)} \text{ for } |i-j| \geq 2 \text{ and } i, j < n,\\
&b_n b_i = b_i \text{ for } i < n-1;
\end{aligned}$$

the multiplication in the other cases is given by commutativity and idempotency (see also Fig. 1).

Define terms $t_1, s_1, t_2, s_2, \ldots, t_n, s_n$ in $n+1$ variables $x, y_1, \ldots, y_n$ as follows:

$$\begin{aligned}
&t_1 = y_1 \text{ and } s_1 = xy_1;\\
&t_i = s_{i-1}y_i \text{ and } s_i = t_{i-1}y_i \text{ for } 2 \leq i \leq n-1;\\
&t_n = t_{n-1}y_{n-3}y_n t_{n-1} \text{ and } s_n = s_{n-1}y_{n-3}y_n t_{n-1} \text{ if } n \geq 4,\\
&\qquad \text{while } t_3 = t_2 s_1 y_3 t_2 \text{ and } s_3 = s_2 s_1 y_3 t_2 \text{ if } n = 3.
\end{aligned}$$

Finally, put $t = s_1 t_n s_n t_n (x t_n)$ and $s = t(s_1 t_n)$.

We are going to prove that the equation $t = s$ is satisfied in any tournament. There will be no confusion if we do not distinguish between a term and its value in a tournament under an interpretation. We distinguish two cases:

If $s_1 = x$, then

$$t = x t_n s_n t_n (x t_n)$$

and

$$s = x t_n s_n t_n (x t_n)(x t_n) = x t_n s_n t_n (x t_n) = t.$$

The other case is $s_1 = y_1$. Then we have $t_1 = s_1, t_2 = s_2, \ldots, t_n = s_n$. Consequently,

$$t = y_1 t_n (x t_n) \quad \text{and} \quad s = y_1 t_n (x t_n)(y_1 t_n); \tag{$*$}$$

clearly, these two values are equal. (In these arguments we have repeatedly used equation (3) from the list in Introduction.)

So, $t = s$ in every tournament under any interpretation.

This means that the equation $t = s$ is satisfied in $\mathbf{T}$. On the other hand, we are going to show that the equation is not satisfied in the groupoid $\mathbf{M}_n$. Consider the interpretation $x \mapsto a$, $y_i \mapsto b_i$. By induction on $i = 1, \dots, n$ we can see that $t_i \mapsto b_i$ and $s_i \mapsto b_{i-1}$. So, $t \mapsto a$ and $s \mapsto b_0$. Since $a \neq b_0$, the equation $t = s$ is not satisfied in $\mathbf{M}_n$.

We have proved that the groupoid $\mathbf{M}_n$ does not belong to $\mathbf{T}$. Since it is generated by $n + 1$ elements, it follows that it does not belong to $\mathbf{T}^{n+1}$. In order to prove that it belongs to $\mathbf{T}^n$, it is sufficient to show that every subgroupoid of $\mathbf{M}_n$ generated by at most n elements belongs to $\mathbf{T}$.

If we remove either a or b_1 from $\mathbf{M}_n$, we obtain a subtournament. If we remove b_0, we must remove either a or b_1 in order to obtain a subgroupoid. So, it is sufficient to prove that, for any $i = 2, \dots, n$, $\mathbf{M}_n - \{b_i\}$ is a subgroupoid belonging to $\mathbf{T}$. One can easily check that there are two congruences C_1 and C_2 of $\mathbf{M}_n - \{b_i\}$ with trivial intersection, such that both factors $(\mathbf{M}_n - \{b_i\})/C_1$ and $(\mathbf{M}_n - \{b_i\})/C_2$ are tournaments: C_1 is the congruence generated by (a, b_0) and C_2 is the congruence generated by (b_1, b_0). (It is easy to see that $\{a, b_0\}$ and $\{b_{i-1}, \dots, b_0\}$ are the only non-singleton blocks of C_1 and C_2, respectively.) Consequently, $\mathbf{M}_n - \{b_i\}$ is a subdirect product of two tournaments (its factor groupoids by C_1 and C_2) and hence belongs to $\mathbf{T}$. $\square$

4 Generalization to directed graphs

The one-to-one correspondence between tournaments and commutative groupoids satisfying $ab \in \{a, b\}$ can be naturally extended to a one-to-one correspondence between arbitrary directed graphs with loops (i.e., reflexive binary relations) and arbitrary groupoids satisfying $ab \in \{a, b\}$ for all a and b: we set $ab = a$ if $a \to b$, and $ab = b$ in the other case. These groupoids are called quasitrivial in some papers, e.g., in [7]. Here we will call them digraphs and identify them with directed graphs with loops. Digraphs are precisely the groupoids such that every subset is a subgroupoid. For two elements of a digraph, $a \to b$ if and only if $ab = a$.

The variety generated by digraphs will be denoted by $\mathbf{D}$. This variety is again locally finite, according to Lemma 1. The proof of Theorem 3 works, with the same algebras $\mathbf{M}_n$, even for this non-commutative case. There are only two changes to be made:

First of all, replace the definition of s_1 with $s_1 = xy_1x$. The second change is that the equality of t and s in $(*)$ is not as clear as in the commutative case. But it is again true. It follows from the fact that if $s_1 = y_1$, then $xy_1 = y_1x = y_1$ and the implication

$$xy = yx = y \Longrightarrow (yz)(xz)(yz) = (yz)(xz)$$

is true in any digraph, which can be easily verified.

So, we know that also digraphs have no finite basis for their equations. In fact, we have proved more:

Theorem 4 *Let V be any variety contained in $\mathbf{D}$ and containing $\mathbf{T}$. Then V is not finitely based.* $\square$

5 Quasiequations

In the proof of Theorem 3 we have shown that every n-generated subalgebra of $\mathbf{M}_n$ is a subdirect product of tournaments. We have concluded that $\mathbf{M}_n \in \mathbf{T}^n$. But there is more in it: it follows that $\mathbf{M}_n$ satisfies not only all the at most n-variable equations, but also all the at most n-variable quasiequations that are satisfied in all tournaments. Since $\mathbf{M}_n \notin \mathbf{T}^{n+1}$, the algebras $\mathbf{M}_n$ do not satisfy all the at most n-variable quasiequations that are satisfied in all tournaments. Consequently, the quasiequations of tournaments are not finitely based.

The same argument can be applied in the non-commutative case. We have proved:

Theorem 5 *The quasivariety generated by tournaments is not finitely based. Moreover, there is no finitely based quasivariety containing the quasivariety generated by tournaments and contained in the quasivariety generated by digraphs.* $\square$

We were not able to determine, however, whether the quasivariety generated by tournaments is properly contained in the variety generated by tournaments. It may be that the two classes of groupoids are the same.

References

[1] S. Burris and H.P. Sankappanavar, *A course in universal algebra,* Graduate Texts in Mathematics, Springer-Verlag, New York, 1981.

[2] S. Crvenković and P. Marković, *Decidability of tournaments* (to appear).

[3] P. Erdös, E. Fried, A. Hajnal and E.C. Milner, *Some remarks on simple tournaments,* Algebra Universalis **2** (1972), 238–245.

[4] P. Erdös, A. Hajnal and E. C. Milner, *Simple one-point extensions of tournaments,* Mathematika **19** (1972), 57–62.

[5] E. Fried and G. Grätzer, *A nonassociative extension of the class of distributive lattices,* Pacific Journal of Mathematics **49** (1973), 59–78.

[6] G. Grätzer, A. Kisielewicz and B. Wolk, *An equational basis in four variables for the three-element tournament,* Colloquium Mathematicum **63** (1992), 41–44.

[7] J. Ježek and T. Kepka, *Quasitrivial and nearly quasitrivial distributive groupoids and semigroups,* Acta Univ. Carolinae **19** (1978), 25–44.

[8] R. McKenzie, G. McNulty and W. Taylor, *Algebras, Lattices, Varieties, Volume I,* Wadsworth & Brooks/Cole, Monterey, CA, 1987.

[9] J.W. Moon, *Embedding tournaments in simple tournaments,* Discrete Mathematics **2** (1972), 389–395.

[10] Vl. Müller, J. Nešetřil and J. Pelant, *Either tournaments or algebras?* Discrete Mathematics **11** (1975), 37–66.

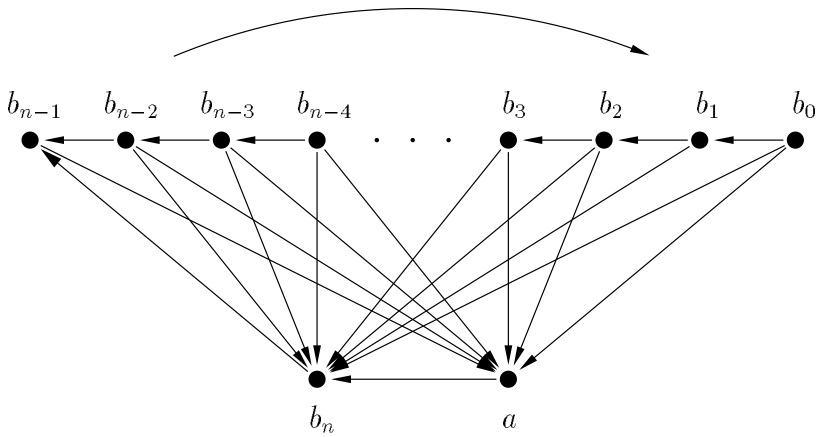

Fig. 1