STRETCHABILITY OF JORDAN ARC CONTACT SYSTEMS

H. DE FRAYSSEIX AND P. OSSONA DE MENDEZ

ABSTRACT. We prove that a necessary and sufficient condition for a contact system of Jordan arcs to be stretchable is to be extendable into a weak arrangement of pseudo-lines.

1. INTRODUCTION

A *Jordan arc* is an arc homeomorphic to a straight line segment. A *pseudo-line* is an arc homeomorphic to a straight line.

A *contact system* is a set of Jordan arcs in the plane, such that two arcs intersect at most once at a point which is internal to at most one arc. A contact system is *stretchable* if there exits an homeomorphism which transforms it into a contact system whose arcs are straight line segments.

Definition 1.1. A set $\mathcal{L} = (L_1, \ldots, L_k)$ of pseudo-lines is a *weak arrangement of pseudo-lines* if it satisfies :

$$\forall i \neq j, \quad |L_i \cap L_j| \leq 1 \tag{1}$$

Definition 1.2. A contact system $\mathcal{A} = (A_1, \ldots, A_k)$ is *extendable into a weak arrangement of pseudo-lines* if there exists an arrangement of pseudo-lines (resp. a weak arrangement of pseudo-lines) $\mathcal{L} = (L_1, \ldots, L_k)$, such that $A_i \subseteq L_i$ for each $i \in \{1, \ldots, k\}$.

Remark 1.1. An arrangement of pseudolines is a weak arrangement of pseudolines with the additional axiom :

$$\forall i, j, k, \quad (L_i \cap L_j = \emptyset \text{ and } L_i \cap L_k \neq \emptyset) \Rightarrow L_j \cap L_k \neq \emptyset$$

It is easy to prove that a contact system is extendable into an arrangement of pseudo-lines if and only if it is extendable into a weak arrangement of pseudo-lines.

1991 *Mathematics Subject Classification.* Primary 05C10; Secondary 57M15.

This work was partially supported by the **Esprit LTR Project no 20244-ALCOM IT.**.

2. Extendibility and Extremal Points

Definition 2.1. An *extremal point* of a contact system is a point of the union of the arcs which is interior to no arc. We note $\delta(\mathcal{A})$ the set of the extremal points of the contact system $\mathcal{A}$.

Definition 2.2. A *maximal contact system* is a contact system whose extremal points belong to the unbounded region.

A contact system $\mathcal{A} = (A_1, \ldots, A_k)$ is *extendable into a maximal contact system* if there exists a maximal contact system $\mathcal{A}' = (A'_1, \ldots, A'_k)$, such that $A_i \subseteq A'_i$ for each $i \in \{1, \ldots, k\}$.

Remark 2.1. If a contact system is extendable into a weak arrangement of pseudo-lines, it is extendable into a maximal contact system, which in turn is extendable into a weak arrangement of pseudo-lines.

Let $\mathcal{A}$ be a maximal contact system extendable into a weak arrangement of pseudo-lines $\mathcal{L}$.

- Let p be an interior point of an arc A (but no other arc) on the unbounded region of $\mathcal{A}$.
- Let L be the pseudoline extending A in $\mathcal{L}$, and let L^- and L^+ be the two halves of L delimited by p.
- Let H be a half pseudo-line having its endpoint at p and having no other intersection in $\mathcal{A}$.

Then, L and H induce a "partition" of A into three new contact systems A^-, A^+ and A^0 (see Fig. 1 and 2) :

- H, L^- and L^+ define three unbounded regions $\mathcal{R}^-, \mathcal{R}^+$ and $\mathcal{R}^0$ having $H \cup L^-$, $H \cup L^+$ and L as respective frontiers.
- We define J^-, J^+ and J^0 as the sub-arcs of $H \cup L^-, H \cup L^+$ and L strictly including all the intersections of these later pseudolines with $\mathcal{A}$.

These regions define three contact systems of Jordan arcs :

$$\mathcal{A}^- = \{J^-\} \cup \{A \cap \mathcal{R}^-, \quad A \in \mathcal{A} \text{ and } A \cap \mathcal{R}^- \text{ is a nonempty arc}\}$$
$$\mathcal{A}^+ = \{J^+\} \cup \{A \cap \mathcal{R}^+, \quad A \in \mathcal{A} \text{ and } A \cap \mathcal{R}^+ \text{ is a nonempty arc}\}$$
$$\mathcal{A}^0 = \{J^0\} \cup \{A \cap \mathcal{R}^0, \quad A \in \mathcal{A} \text{ and } A \cap \mathcal{R}^0 \text{ is a nonempty arc}\}$$

These three contact systems have some nice properties :

Lemma 2.2. *The contact systems $\mathcal{A}^-, \mathcal{A}^+$ and $\mathcal{A}^0$ are maximal and extendable into weak arrangements of pseudo-lines.*

Proof. The maximality is straightforward, as the extremal points of these contact systems are either extremal points of $\mathcal{A}$ or the extremities of J^-, J^+ or J^0.

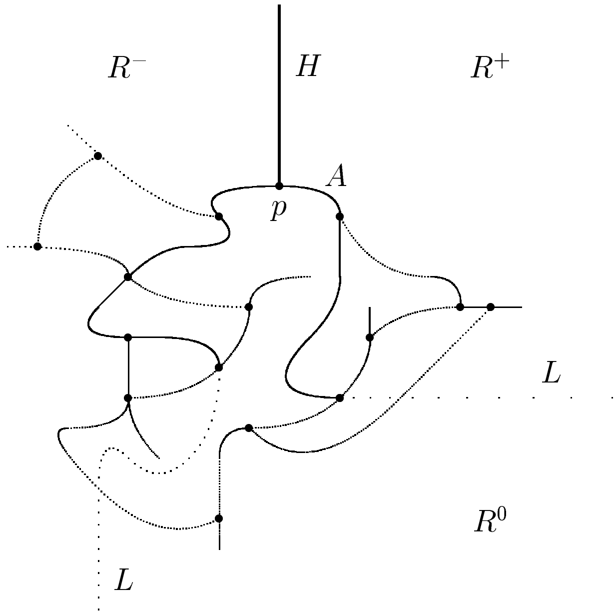

FIGURE 1. A Contact System

The contact systems $\mathcal{A}^-$ and $\mathcal{A}^+$ are homeomorph to the contact systems where J^- (resp. J^+) is replaced by J^0. Then, the weak arrangement of pseudo-lines which extends $\mathcal{A}$ is an extension of all of the contact systems. □

Lemma 2.3. *If the arc A includes no extremal point of $\mathcal{A}$, we have :*

$$2 \leq |\mathcal{A}^-| < |\mathcal{A}| \tag{2}$$

$$2 \leq |\mathcal{A}^+| < |\mathcal{A}| \tag{3}$$

$$3 \leq |\mathcal{A}^0| \leq |\mathcal{A}| \tag{4}$$

and each of $\mathcal{A}^-, \mathcal{A}^+$ and $\mathcal{A}^0$ has two extremal points belonging to a same arc.

Proof. As A includes no extremal point of $\mathcal{A}$ the extremity of A in A^- (resp. A^+) is interior to some arc B^- (resp. B^+). As A may not intersect an arc twice, we have $B^- \neq B^+$. Moreover, each of these arcs intersects at most once the extension L of A, and hence the arc B^- (resp. B^+) does not meet the region $\mathcal{R}^+$ (resp. $\mathcal{R}^-$). Thus, $|\mathcal{A}^+| \leq |\mathcal{A}| - 1$ and the same holds for $|\mathcal{A}^-|$. Notice also that $\mathcal{A}^-$ and $\mathcal{A}^+$ include at least two arcs : J^- (resp. J^+) with two extremal points on it and the non empty portion of B^- (resp. B^+) in $\mathcal{R}^-$ (resp. $\mathcal{R}^+$). Moreover, $\mathcal{A}^0$ includes at least three arcs : J^0 (with two extremal points on it) and the portions of B^- and B^+ in $\mathcal{R}^0$. □

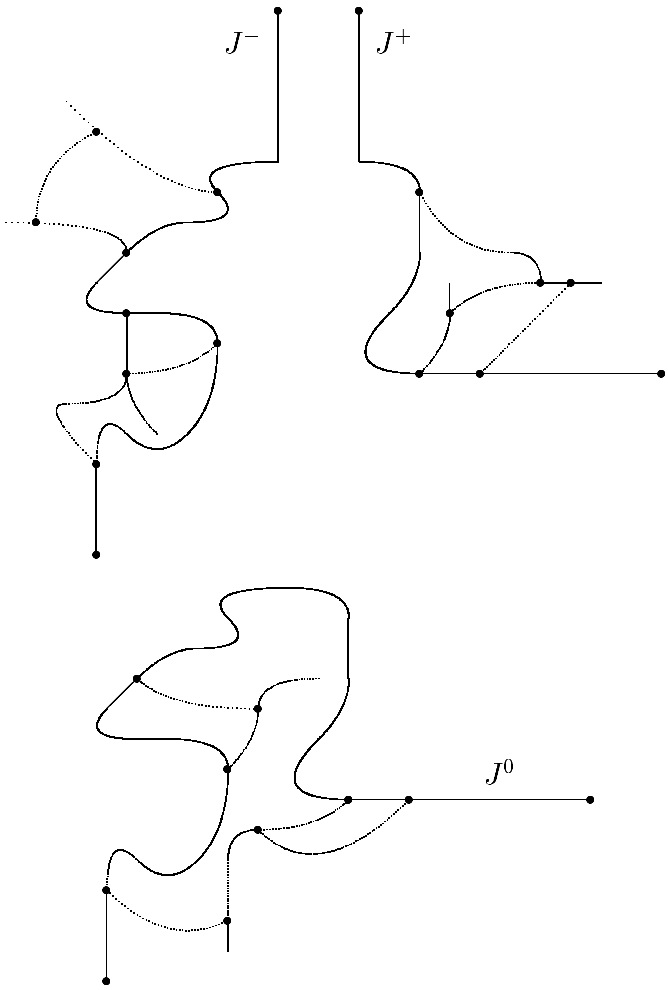

FIGURE 2. Partitioning a Contact System

Lemma 2.4. *Any contact system extendable into an arrangement of pseudo-lines has at least 3 extremal points, unless it has cardinality at most one.*

Proof. First notice that we may prove the lemma for maximal contact systems only, according Remark 2.1 and the fact that the extended contact system may not have more extremal points than the original one.

The lemma is straightforward for contact systems of cardinality at most two. So, assume that for any $2 \leq i < k$, the lemma holds for maximal contact systems of cardinality i and assume there exists a maximal contact system $\mathcal{A}$ of cardinality k having at most 2 extremal points.

Notice that at least 3 arcs of $\mathcal{A}$ meet the unbounded region in a sub-arc. Otherwise, the contact system would be bounded by a single arc or by two arcs intersecting each other twice, and the system could not be extended into a weak arrangement of pseudo-lines.

- Assume $\mathcal{A}$ two extremal points belonging to a same arc.
  From the previous construction and according to Lemma 2.2, $\mathcal{A}^-$ and $\mathcal{A}^+$ are extendable into weak arrangements of pseudo-line and, according to Lemma 2.3, have a smaller cardinality than $\mathcal{A}$ but include at least two arcs. By induction, they have at least 3 extremal points. Among these, there is one which is an extremal point of $\mathcal{A}$ belonging to the interior of $\mathcal{R}^-$ (resp. $\mathcal{R}^+$). Notice that these extremal points do not belong to a same arc : otherwise, this arc would intersect A twice. Hence, as the contact system $\mathcal{A}$ has an arc with two extremal points on it, it has at least 3 extremal points, what leads to a contradiction.
  Hence, the lemma holds for any maximal contact system with at most k arcs having two extremal points belonging to a same arc.
- Assume $\mathcal{A}$ does not include two extremal points belonging to a same arc.
  From the same construction and according to Lemma 2.2, the contact system $\mathcal{A}^0$ is maximal and, according to Lemma 2.3, it has cardinality at most k and has two extremal points belonging to a same arc. According to the previous case, it has at least 3 extremal points. Among these, one is an extremal point of $\mathcal{A}$ belonging to the interior of $\mathcal{R}^0$. Together with the 2 extremal points of $\mathcal{A}$ belonging to the interior of $\mathcal{R}^-$ and $\mathcal{R}^+$ (as in the previous case), we get 3 extremal points, which lead to a contradiction.

□

Lemma 2.5. *Any contact system $\mathcal{A}$ extendable into a weak arrangement of pseudo-lines is extendable into a maximal contact system, each subsystem of which has at least 3 extremal points unless it has cardinality at most one.*

Proof. As any subsystem $\mathcal{A}'$ of the extension of $\mathcal{A}$ into a maximal contact system is extendable into a weak arrangement of pseudo-lines, the result follows from Lemma 2.4. □

3. Barycentric Representation of Graphs

Consider the plane graph G whose vertices are the points which are extremity of a least an arc of $\mathcal{A}$ and whose edges are the sub-arcs between such points. Consider the α function from $V(G) \times V(G)$ to $\mathbb{R}$, such that $\alpha(x,y) = \frac{1}{2}$ if y is interior to some arc $A \in \mathcal{A}$ and x is a neighbor of y in A, and $\alpha(x,y) = 0$, otherwise. This function induces a directed graph D_α such that (x,y) is a directed edge of D_α if and only if $\alpha(x,y) \neq 0$. Let $S(G)$ be the vertices of G corresponding to the sources of D_α.

Definition 3.1. An α-barycentric representation of G is a straight-line representation of G, such that each vertex v that does not belong to $S(G)$ is the barycenter of its neighbors w with weights $\alpha(w,v)$.

Definition 3.2. Let H be a connected subgraph of G. The *α-frontier* $\partial_\alpha(H)$ is the set of the vertices of H which either belong to $S(G)$ or have in D_α an incoming arc from a vertex that do not belong to H.

Notation 3.3. Given an arc $A \in \mathcal{A}$, the path $P(A)$ is the path of G, whose vertices and edges are induced by A. The set $\mathcal{P}$ of these paths induces a covering of G by edge disjoint paths.

By extension, given a contact system $\mathcal{A}'$, $P(\mathcal{A}')$ denotes the union of the paths $P(A)$ $(A \in \mathcal{A}')$.

Notation 3.4. Let H be a connected subgraph of G with at least one edge. We note $\mathcal{A}(H)$ the set of the arcs of $\mathcal{A}$ having at least two points corresponding to vertices of H.

Theorem 3.1 ([1])**.** *Let G be a plane graph, such that the vertices of $S(G)$ belong to a same face.*

Then, G has a plane α-barycentric representation (with the same embedding as G) if and only there exists no connected subgraph H of G (with at least one edge) such that :

$$\Big(|\partial_\alpha(H)| = 1\Big) \text{ or } \Big(|\partial_\alpha(H)| = 2 \text{ and } H \text{ is not a path}\Big) \tag{5}$$

We now may link this result with the stretching problem, using the next notations and lemmas.

Lemma 3.2. *Let $\mathcal{A}'$ be a subsystem of $\mathcal{A}$. Then,*

$$\partial_\alpha(P(\mathcal{A}')) = \delta(\mathcal{A}') \tag{6}$$

Proof. A vertex of $H = P(\mathcal{A}')$ belongs to the α-frontier of H if and only if it has no incoming edge in D_α (and thus is an extremal point of $\mathcal{A}$) or if its has an incoming edge in D_α from a vertex which does not belong to $V(H)$ (and thus is internal to some arc in $\mathcal{A} \setminus \mathcal{A}'$ or, equivalently, is an extremal point of $\mathcal{A}'$ but not an extremal point of $\mathcal{A}$).

All together, we get that the α-frontier of H is exactly the set of the extremal vertices of $\mathcal{A}'$. □

Lemma 3.3. *Let H be a connected subgraph of G with at least one edge. Then,*

$$H \subseteq P(\mathcal{A}(H)) \tag{7}$$

$$|\partial_\alpha(H)| \geq |\partial_\alpha(P(\mathcal{A}(H)))| \tag{8}$$

Proof. Inclusion (7) is straightforward as each edge of H is included in the path corresponding to some arc of $\mathcal{A}(H)$.

Consider the connected graph H' obtained from H by adding $k \geq 0$ paths corresponding to arcs in $\mathcal{A}(H)$ and assume that $|\partial_\alpha(H')| = |\partial_\alpha(H)|$ holds. Let A be some arc of $\mathcal{A}(H)$, such that $P(A)$ does not belong to H'.

- Assume H' include two vertices v and w of $P(A)$ but none of the edges of P linking them.
  Then, by adding those missing edges and vertices, no vertex is added to the α-frontier of H'.
  After this first completion, the internal vertices of $P(A)$ have at most one incoming edge from the complement of H'.
- Otherwise, H' includes an interior vertex v of $P(A)$, but none of the vertices between v and an extremity w of P.
  Then, v belongs to the α-frontier of H'. By adding the sub-path from v to w, exactly one vertex is added to the α-frontier of H', namely w. As $P(A)$ has at least two vertices in common with H', v is an interior vertex of $P(A)$. Moreover, as the previous case does not apply, v had initially one incoming edge (exactly) from the complement of H'. Thus, after the addition of the sub-path from v to w, the vertex v will no longer belong to the α-frontier of H'.

Thus, we may add a new path to H' without increasing $|\partial_\alpha(H')|$, which completes the induction. □

Theorem 3.4. *Let $\mathcal{A}$ be a contact system of Jordan arcs. Then, the following conditions are equivalent :*

(C_1) *$\mathcal{A}$ is stretchable,*
(C_2) *$\mathcal{A}$ is extendable into a weak arrangement of pseudo-lines,*

Proof. It is straightforward that Condition (C_1) implies condition (C_2). According to Lemma 2.5, this second condition implies that $\mathcal{A}$ is extendable into a maximal contact system, each subsystem of which has at least 3 extremal points or has cardinality at most one. Consider any connected subgraph H of G with at least one edge. Then, according to Lemma 3.3 and Lemma 3.2, $|\partial_\alpha(H)| \geq |\partial_\alpha(P(\mathcal{A}(H)))| = |\delta(\mathcal{A}(H))|$. But, either $\mathcal{A}(H)$ includes a single arc and hence H is a path and $|\partial_\alpha(H)| \geq 2$ or $\mathcal{A}(H)$ includes at least two arcs and then $|\partial_\alpha(H)| \geq 3$, according to the hypothesis. The result now follows from the application of Theorem 3.1. $\square$

4. Conclusion

The equivalence between stretchability and extendibility into an arrangement of pseudo-lines for contact systems of Jordan arcs does not extend to intersection systems. With more work, another equivalent condition may be given : A contact system is stretchable if and only if any subsystem has at least 3 extremal points on its unbounded region, unless it has at most an arc.

References

[1] M. Las Vergnas and P. Ossona de Mendez, *On barycentric representation of graphs*, in preparation.

Hubert de Fraysseix, CNRS UMR 0017, E.H.E.S.S., 54 Bd Raspail, 75006 Paris, France
E-mail address: hf@ehess.fr

Patrice Ossona de Mendez, CNRS UMR 0017, E.H.E.S.S., 54 Bd Raspail, 75006 Paris, France
E-mail address: pom@ehess.fr