Proceedings of

Fifth Prague Combinatorial Workshop

July 28. — August 1. 1997

(edited by Jiří Fiala)

Contents

Preface

The Fifth Prague Midsummer Combinatorial Workshop was held from July 28 to August 1, 1997 at Malostranské náměstí building of Charles University whose one side is depicted by a drawing on the cover of these KAM Series. The workshop was organized by Department of Applied Mathematics (KAM) of Charles University jointly with its DIMATIA centre.

Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Moshe Rosenfeld and Imre Barany among us. It was important that graduate students could take an active part. This included not only our graduate students but as well graduate students from Université de Paris and Mathematical Institute in Budapest. The workshop followed an informal daily routine which included morning and early afternoon discussions. This report reflects some of the discussions during the workshop. Perhaps you can digest from these proceedings some of the athmosphere at the workshop and you will also see that the fruitfull exchange of ideas led directly to some new results and papers.

This volume is edited by Jiří Fiala. Most of the problems were supplied by the authors as TeX files, the rest in print form or as handwritings. In any case we appologize for any inaccuracies which might occur in the editing process.

The conference photos were taken in Wallenstein Palace under Prague Castle on Monday, July 28. Wallenstein Palace currently houses the Senate (i.e. the Upper House) of the Czech Republic. We were given a professional tour (translated by Dr. Helena Nešetřilová free of charge) and the grand setting of this palace is illustrated by the conference photo. The conference programm included also an organ concert in St. Clement church whose programm is included in these proceedings.

This summer workshop was partly supported by our University grant GAUK 194, our Czech grant GAČR 0194 and Discrete Mathematik Network (DIMANET).

Based on our past experience and beeing encouraged by several participants we hope to organize the Sixth Prague Combinatorial Workshop in 1999. Yes, no workshop in 1998. This has a very simple reason: From July 6 – 11, 1998 DIMATIA and KAM organizes the Fifth Czech - Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applicatons in Prague. We hope to meet you all there!

(Write to cs98org@kam.ms.mff.cuni.cz or visit our www page: http://www.ms.mff.cuni.cz/acad/kam/conferences/c-s98.html.)

Jaroslav Nešetřil

Concert Programme

Programme of the organ concert performed by Irena Kosíková in Church of st. Kliment on Wednesday, July 30.

Johann Sebastian Bach (1685 – 1750)

Prelude and Fugue in B minor

Aus den Tiefe rufe ich

Toccata and Fugue in D minor — doric

Jehan Alain (1911 – 1940)

Intermezzo

Litanies

Robert Babilon

3–colorability of certain planar graphs

Definition. A curve φ is a pseudoline if φ is not selfintersecting and both its ends go to the infinity. A system S of pseudolines is a generalized arrangement of pseudolines (GAP) if every two pseudolines of S have at most one point in common and if no three pseudolines pass one point. For any GAP S we define a graph $G(S)$ in such a way that its vertices are all the intersection points of pseudolines of S and its edges connect adjacent vertices on pseudolines. We call such graphs GAP-graphs.

Definition. A system T (finite or infinite) of rhombs is a tiling if a intersection of any two rhombs is an empty set or one point or a whole common edge. For a tiling T we define a graph $G(T)$ in such a way that its vertices are the rhombs and its edges connect adjacent rhombs (i. e. with a common edge).

Theorem. Every GAP-graph is 3–colorable.

Theorem. For every tiling T a graph $G(T)$ is 3–colorable.

Theorem. For every compact tiling T (without holes inside) a graph $G(T)$ is GAP-graph.

Theorem. For every GAP-graph G there exists such a tiling T that $G = G(T)$.

Problem. If we remove the condition that ends of pseudolines go to the infinity, we obtain "pseudosegments". Is the associated graph (GAS-graph) still 3–colorable?

Krystyna T. Balińska and Louis V. Quintas

Distance Problems and a Random Graph Model

Problems concerning distances in graphs [1] whose vertices are graphs with bounded vertex degree are considered.

A graph with no vertex of degree greater than f is called an *f-graph.* Let $U(n,f)$ denote the graph whose vertex set consists of the unlabeled f-graphs of order n and such that $\{G,H\}$ is an edge in $U(n,f)$ if and only if H is obtainable from G by either the deletion or the insertion of a single edge.

The *eccentricity* $e(G) = \max\{d(G,X) \mid X \in V(U(n,f))\}$, where $d(G,X)$ is the distance between G and X, is defined for each vertex G of $U(n,f)$. As in any graph the *diameter* of $U(n,f)$ is the maximum eccentricity of its vertices and the *radius* the minimum eccentricity.

The diameter and radius for the unbounded case ($f = n-1$) are $\binom{n}{2}$ and $\lceil \frac{1}{2}\binom{n}{2} \rceil$, respectively [2]. The following three theorems state what is known about the diameter of $U(n,f)$ for fixed $f \geq 2$ (Problems 1 and 2 [3]).

Theorem 1 *([3][4]). If $f = 2$ or $(n-1)/2 \leq f \leq n-1$, then*

$$diam U(n,f) = \lfloor nf/2 \rfloor. \ \square$$

Theorem 2 *([4]). For fixed $f \geq 3$ and sufficiently large n,*

$$\frac{nf(f-1)}{f+1} - \frac{(f+1)^2}{8} \leq diam U(n,f) \leq \frac{nf^2}{f+1}. \ \square$$

Theorem 3 *([4]). If $\lim_{n\to\infty} (diam U(n,f))/n$ exists for fixed $f \geq 3$, then*

$$1 \leq \frac{2(f-1)}{f+1} \leq \lim_{n\to\infty} \frac{diam U(n,f)}{nf/2} \leq \frac{2f}{f+1} < 2. \ \square$$

Problem 1. What is the diameter of $U(n, f)$ for fixed f with $3 \leq f < (n-1)/2$?
Problem 2. For fixed $f \geq 3$, what is the least integer $w(f)$ for which

$$\frac{diam U(w(f), f)}{w(f)f/2} > 1 \text{ ?}$$

The best known bounds for $w(f)$ for up to $f = 7$ are given in the next theorem.

Theorem 4 *([4]).* $8 \leq w(3) \leq 12$*;* $10 \leq w(4) \leq 20$*;* $12 \leq w(5) \leq 26$*;* $14 \leq w(6) \leq 49$*; and* $16 \leq w(7) \leq 52$. $\square$

A theorem concerning the diameter of $U(n, f)$ when f is a function of n is given below (Problem 3 [3]).

Theorem 5 *([4]). If* $f = f(n)$ *tends to infinity with* n *and* $f < (n-1)/2$*, then* $diam U(n, f) = f(n-f) + o(nf)$ *as* n *goes to infinity.* $\square$

Problem 3. What is the radius of $U(n, f)$ for fixed f with $2 \leq f \leq n-2$?

The *center* and *periphery* are two subgraphs of $U(n, f)$ induced by the vertices G such that $e(G) = rad U(n, f)$ and $e(G) = diam U(n, f)$, respectively. In the unbounded case, a vertex G is in the center, if $d(G, G^c) \leq 1$ (G^c is the complement of G); the only vertices in the periphery are K_n and K_n^c (see [2]).
Problem 4. Determine the center of $U(n, f)$.
Problem 5. Determine the periphery of $U(n, f)$ for fixed f with $2 \leq f \leq n-2$.

$U(n, f)$ is the underlying graph of the transition digraph $D(n, f)$ for the probability model known as the Random f-Graph Process (see [5]). However, since $D(n, f)$ is not a symmetric digraph its distance properties are different from those of $U(n, f)$. Below we introduce a new model which is a generalization of this process. It has a symmetric transition digraph with $U(n, f)$ as its underlying graph.

The Reversible Random f-Graph Process

Let n, f and k be positive integers such that f is fixed with $2 \leq f \leq n-1$. Starting with the graph G_0 on n labeled vertices and no edges, obtain a sequence $G_1, G_2, \ldots, G_k$ of f-graphs as follows. At step $i < k$, let A_i be the set of edges of G_i together with those edges of G_i^c each of which when inserted in G_i results in an f-graph. To obtain G_{i+1}, first select an edge uniformly from A_i. Next, if this edge belongs to G_i, delete it; otherwise insert it in G_i.

Note that the Reversible Random f-Graph Process is an ergodic Markov chain with the sequence of graphs $G_1, G_2, \ldots, G_k$ being a random walk of length k. In this context a random walk need not start at the empty graph nor be restricted to considering one walk at a time.

A relation between the stationary distribution and the central and peripheral nodes of the transition digraph might yield an insight to the process. These comments suggest that both the graph theoretical properties of $U(n, f)$ (e.g., those concerning distance) and the probabilistic characterization of this new model are of interest.

Problem 6. For a given n and f, what is the stationary distribution for the Reversible Random f-Graph Process?

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Imre Bárány

The contribution of a single vertex to the volume

Given a convex polygon $P \subset R^2$ with vertex set $V = \{v_1, \ldots, v_n\}$ (the vertices are listed in clockwise order) the relative contribution of vertex v_i to the area of P is

$$\gamma(P, v_i) = \frac{\text{Area}\,\text{conv}(v_{i-1}, v_i, v_{i+1})}{\text{Area}\,P}.$$

Rényi and Sulanke proved in [RS] that

$$\prod_1^n \gamma(P, v_i) \leq \prod_1^n \gamma(Q_n, v_i) \tag{1}$$

where Q_n is the regular n–gon, and equality holds if and only if P is an affine regular n–gon. Set $\gamma(P) = \min_{i=1,\ldots,n} \gamma(P, v_i)$ Then (1) implies, after a short and simple calculation, that for every convex polygon P with n vertices

$$\gamma(P) \leq \frac{4}{n} \sin^2 \frac{\pi}{n} = \frac{4\pi^2}{n^3}(1 + o(1)).$$

So defining $\gamma(n) = \max\{\gamma(P)\}$ where max is taken over all convex polygons with n vertices, we have

$$\gamma(n) = \frac{4}{n} \sin^2 \frac{\pi}{n}.$$

What should be the corresponding statement in higher dimensions? In this short note I propose two possible extensions of the definition of γ, some results, and a conjecture.

Assume P is a convex polytope in R^d with positive volume and vertex set $V = \{v_1, \ldots, v_n\}$. Define the relative contribution of vertex v_i to the volume as

$$\gamma(P, v_i) = \frac{\operatorname{Vol}(P \setminus \operatorname{conv}(V \setminus \{v_i\}))}{\operatorname{Vol} P}.$$

Set $\gamma(P) = \min_{i=1,\ldots,n} \gamma(P, v_i)$ and $\gamma(n, d) = \max \gamma(P, v_i)$ where max is taken over all convex polytopes in R^d with n vertices (and positive volume).

Preposition 1 *For large enough n,*

$$\gamma(n, d) \geq c_1 n^{-\frac{d+1}{d-1}},$$

where c_1 is a constant depending only on d.

Proof (Sketch) We use a result from approximation theory [Gr]. Let P_n be a best approximating polytope with n vertices inscribed in the Euclidean unit ball B^d of R^d. Best approximation is meant here in the sense that $\operatorname{Vol}(B^d \setminus P_n)$ is minimal. Then the limit of $n^{\frac{2}{d-1}} \operatorname{Vol}(B^d \setminus P_n)$, as $n \to \infty$, exists and is a positive number $c(d)$ depending only on d. Now to prove the Proposition, take P_n and delete m vertices $v_{i_1}, \ldots, v_{i_m}$ so that the stars of these v_{i_j}s are pairwise disjoint. Here, as one can show, m can be chosen to be at least $b(d)n$ (where $b(d)$ is a small but positive constant). Now $Q = \operatorname{conv}(V \setminus \{v_{i_1}, \ldots, v_{i_m}\})$ is a polytope on $n - m$ vertices, inscribed in B^d, so the approximation result implies $\operatorname{Vol} B^d - \operatorname{Vol} Q \geq c(d)(n - m)^{-\frac{2}{d-1}}$. Using that P_n is a best approximating polytope we get

$$\operatorname{Vol} P_n - \operatorname{Vol} Q \geq c(d) \left[(n - m)^{-\frac{2}{d-1}} - n^{-\frac{2}{d-1}} \right].$$

Then, as a simple computation reveals, the average of the $\gamma(P, v_{i_j})$ is at least $\frac{1}{m} c(d) \left[(n - m)^{-\frac{2}{d-1}} - n^{-\frac{2}{d-1}} \right] = \operatorname{const}(d) n^{-\frac{d+1}{d-1}}$. $\square$

I think Proposition 1 is best possible so I propose the following

Conjecture *For large enough n,*

$$\gamma(n,d) \le c_2 n^{-\frac{d+1}{d-1}},$$

where c_2 is a constant depending only on d.

If true this would imply a theorem of G. E. Andrews [An] which says that for a lattice polytope $P \subset R^d$ with n vertices and positive volume

$$n^{\frac{d+1}{d-1}} \le \mathrm{const}(d) \,\mathrm{Vol}\, P.$$

The implication is based on the fact that for a lattice polytope $\mathrm{Vol}(P \setminus \mathrm{conv}(V \setminus \{v_i\})) \ge \frac{1}{d!}$.

A suitable version of the Rényi–Sulanke inequality (1) may hold true in higher dimensions and would, of course, imply the conjecture. Maybe one should use a different, not the geometric, mean. One difficulty is the lack of regular polytopes (with many vertices) in higher dimensions.

Another way to extend the definition of $\gamma(P, v_i)$ is the following. Set

$$\beta(P) = \frac{\min\{\mathrm{Vol}(P \cap H)\ H \text{ is a halfspace and } |H \cap V| \ge d+1\}}{\mathrm{Vol}\, P}.$$

In the plane this is the same as $\gamma(P)$. Set $\beta(n,d) = \max \beta(P)$ with max taken over all polytopes with n vertices. Using the economic cap covering theorem from [BL] one can show

Preposition 2 *For large enough n,*

$$c_3 n^{-\frac{d+1}{d-1}} \ge \beta(n,d) \ge c_4 n^{-\frac{d+1}{d-1}},$$

with constants $c_3, c_4 > 0$ depending only on d.

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Jiří Fiala

On representing directed graphs by linear orderings

Let $N = \{c_1, ..., c_n\}$ be a set of candidates and $M = \{v_1, ..., v_m\}$ be a set of voters. Each voter v_i has a preference relation (i.e. linear ordering) of candidates denoted by $L(v_i)$ [1]. For a pair of cadidates $c_i, c_j, i \neq j$, let $M_{ij} \subset M$ be the set of voters which prefer candidate c_i to c_j, formally $M_{ij} = \{v : [i,j] \in L(v)\}$.

A directed graph T based on this data set is defined as follows: $V(T) = N$ and $\vec{e} = [i,j] \in E(T)$ if and only if more voters prefer i to j i.e. $|M_{ij}| > |M_{ji}|$. If m is odd then T is a toutnament (since $|M_{ij}| > |M_{ji}|$ or $|M_{ij}| < |M_{ji}|$ for every $i \neq j$).

Problem: What is the minimum $m = r_d(n)$ such that every directed graph with n vertices can be represented by a set of linear orderings of m voters?

Problem: How is the minimum $m = r_t(n)$ — similarly defined number for tournaments — related to $r_d(n)$?

We know the bounds $\frac{n}{2 \cdot \log n} \leq r_d(n) \leq 2\lfloor \frac{n+3}{2} \rfloor$

The lower bound is due to work of Jan Vondrák, student of Charles University.

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Yoav Kirsch

Bounding the Independence Number via Graph Homomorphisms

Finding the independence number of a graph is one of the significant problem in graph theory. We give here a technique that enable us to bound this number using graph homomorphism. A homomorphism of a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ for any $uv \in E(G)$. If there is a homomorphism from G to H and H is vertex transite graph then $\alpha(H) \leq \frac{|V(H)|\alpha(G)}{|V(G)|}$.

We use this technique to give simple proof of M. A. Perles to the Erdös-Ko-Rado theorem

Definition: A collection of sets $A_1, ..., A_m$ is called an *intersecting family* if $A_i \cap A_j \neq \emptyset$ for any i, j.

Theorem 1 *Erdös-Ko-Rado theorem let $\{A_1, A_2, ..., A_m\}$ be intersecting family of k-subsets of an n-set S, $k \leq \frac{n}{2}$, then $m \leq \binom{n-1}{k-1}$.*

Proof. The n, k Kneser graph denote by $KG_{n,k}$ is the graph whose vertex are the k-subset of $\{1, 2, ..., n\}$,$k \leq \frac{n}{2}$, two vertices are adjoin if their intersection is empty. Notice that any independence set in $KG_{n,k}$ is intersecting family so $m \leq \alpha(KG_{n,k})$. Let $G_{n,k}$ be the graph whose vertex are the numbers $[1, 2, ..., n]$ i, j are adjoin if $|i - j| \geq k$ all calculation are done mode n. $G_{n,k} \to KG_{n,k}$ by $f(i) = \{i, i+1, ..., i+k-1\}$. Since $\alpha(G_{n,k}) = k$ we get $\frac{|V(KG_{n,k})|}{\alpha(KG_{n,k})} \geq \frac{n}{k} \Rightarrow \alpha(KG_{n,k}) \leq \binom{n-1}{d-1}$.

Problem: can you find another families of graph that admit homomorphism into Kneser graphs?

This might be helpfull because many intersecting problems are problems on Kneser graphs.

Problem: Can you prove the genaral Erdös-Ko-Rado theorm if $A_1, ..., A_m$ is intersecting family of k-sets of n-set such that $|A_i \cap A_j| \geq t$. and $n >> k$ then $m \leq \binom{n-t}{k-t}$ using the graph homomorphism technique ?

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Martin Klazar

On up-down and sum-closed permutations

A permutation $a_1 a_2 \ldots a_n$ of $\{1, 2, \ldots, n\}$ is called an *up-down permutation* if $a_1 < a_2 > a_3 < a_4 > a_5 < \cdots$ Counting up-down permutations is a little gem of enumerative combinatorics — see the book [1]. It seems that not much is known about *sum-closed permutations* which I introduced [2] in connection with finite Sidon sets. Those are permutations satisfying relations (n is odd)

$$a_{2i-1} + a_{2i+1} = a_{2i} \text{ for } i = 1, 2, \ldots, (n-1)/2.$$

Clearly, any sum-closed permutation is up-down but the additive condition is much more restrictive. For example, there are only six sum-closed permutations for $n = 9$, namely

$$143968275, 572869341, 154739682, 286937451,$$

$$594623187, \text{ and } 781326495,$$

while the number of up-down permutations of length nine is 7936.

Let $Y(n)$ be the number of sum-closed permutations of $\{1, 2, \ldots, n\}$. Obviously, $Y(n)$ is even for any odd $n > 1$. The following three problems are listed in the order of presumed difficulty. I cannot solve any of them.

Three problems.

1. Show that $Y(n) > 0$ for any odd n.

2. Prove or disprove that

$$Y(n) < c^n$$

for a constant $c > 1$.

3. Find a precise asymptotics of $Y(n)$.

The first few values of $Y(n)$ are

$$(Y(2n-1))_{n \geq 1} = (1, 2, 4, 4, 6, 42, 126, 396, 1786, 7748, 37962, \ldots).$$

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Jan Kratochvíl

Choosability with separation

Let p, r be natural numbers, and $G = (V, E)$ a graph with vertex set V and edge set E. A (p, r)-*assignment* of G is a collection $\mathcal{L} = \{L(v) \mid v \in V\}$ of lists (sets) of "colors" assigned to the vertices such that

$|L(v)| = p$ for all $v \in V$, and

$|L(u) \cap L(v)| \leq p - r$ for all $uv \in E$.

An $\mathcal{L}$-*admissible coloring* of G is a function $F : V \to \bigcup_{v \in V} L(v)$ such that

$F(v) \in L(v)$ for all $v \in V$, and

$F(u) \neq F(v)$ for all $uv \in E$.

The graph G is said to be $(p, 1, r)$-*choosable* if it admits an $\mathcal{L}$-admissible coloring for every (p, r)-assignment $\mathcal{L}$.

This is a special case of (p, q, r)-choosability (for $q = 1$), which was introduced in [2], where mainly the computational complexity issues were considered. The particular case of $r = 0$ (p-choosability) was first considered by Vizing [6] and independently by Erdős, Rubin and Taylor [1], and has been studied extensively since then. We refer to the recent survey [5] for a detailed account with a large number of bibliographical items. The role of the second parameter, r, is restricting the attention to list assignments such that lists assigned to adjacent vertices have bounded intersections (i.e., are 'separated'). This constraint shows suprising effect on the choosability. E.g., the complete graph on n vertices is not $(n-1)$-choosable, but a simple observation shows that it is $(n-2, 1, 1)$-choosable, i.e., requiring that adjacent vertices are assigned distinct lists guarantees the existence of a feasible coloring even if the lists are smaller by 2 than the ordinary choice number.

In [3], we introduce the following notation. For a graph G and a nonnegative integer c, we denote by $\zeta(G, c)$ the minimum integer k such that G is $(k, 1, k - c)$-choosable. I.e., $\zeta(G, c)$ is the minimum integer k such that an $\mathcal{L}$-admissible coloring exists for every assignment of lists of size k with intersections of lists assigned to adjacent vertices being of size at most c. The following problems and results appear in [3].

Problem 1. Does there exist a limit $L = \lim_{n,c\to\infty} \frac{\zeta(K_n,c)}{\sqrt{nc}}$? If yes, determine the limit.

Remark For $c = 1$, $lim_{n\to\infty}\frac{\zeta(K_n,1)}{\sqrt{n}} = 1$. (More precisely, we know that $\zeta(K_n, 1) \leq \lfloor\sqrt{n - \frac{11}{4}} + \frac{3}{2}\rfloor$, and $\zeta(K_{q^2+1}, 1) = q + 1$ for any prime power q.)

For arbitrary c and a graph G of maximum degree $\Delta(G)$, $\zeta(G, c) \leq \lceil\sqrt{2\mathrm{e}c(\Delta(G) - 1)}\rceil$, where $\mathrm{e} = 2.718\ldots$ is the Euler constant. For complete graphs, we have the lower bound

$$\liminf_{n\to\infty} \frac{\zeta(K_n, c)}{\sqrt{n}} \geq \sqrt{\lfloor\frac{c}{2}\rfloor}.$$

Thus

$$\sqrt{\lfloor\frac{c}{2}\rfloor} \leq \liminf_{n\to\infty} \frac{\zeta(K_n, c)}{\sqrt{n}} \leq \limsup_{n\to\infty} \frac{\zeta(K_n, c)}{\sqrt{n}} \leq \sqrt{2\mathrm{e}c}$$

and if L exists, we have $\sqrt{\frac{1}{2}} \leq L \leq \sqrt{2\mathrm{e}}$.

Problem 2. Is every planar graph $(4, 1, 2)$-choosable?

Remark This problem seems to be one of the last open problems for choosability of planar graphs. Every planar graph is 5-choosable [4], and there are planar graphs which are not 4-choosable [7]. Examples of planar non-4-choosable graphs are known such that the infeasible list assignments assign distinct lists to adjacent vertices, i.e., not all planar graphs are (4,1,1)-choosable. On the other hand, we have proved that every planar graph is $(4, 1, 3)$-choosable. (This proof is constructive, i.e., given list assignment such that list assigned to adjacent vertices are almost disjoint and every list has size 4, an admissible coloring can be found in polynomial time.)

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Peter Mihók

Unique factorization of graph properties

Let us denote by $\mathcal{I}$ the class of all finite simple graphs. If $\emptyset \neq \mathcal{P} \subset \mathcal{I}$ be a nonempty isomorphism-closed subset of $\mathcal{I}$, then $\mathcal{P}$ denote the graph property. A property $\mathcal{P}$ is said to be *monotone* whenever $H \subseteq G$ and $G \in \mathcal{P}$ implies $H \in \mathcal{P}$, *induced hereditary* if it is closed under induced subgraphs and *additive* if it is closed under the disjoint union of graphs. (For more details see [1] and [2]).

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties, a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ - partition of a graph G is a partition $(V_1, V_2, \ldots, V_n)$ of the vertex set $V(G)$ such that the induced subgraph $G[V_i] \in \mathcal{P}_i$ for $i = 1, 2, \ldots, n$.

Let us denote by $U(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ the set of all graphs which posses unique vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ - partition and $\mathcal{R} = \mathcal{P}_1.\mathcal{P}_2.\ldots.\mathcal{P}_n$ the set of all vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ - partitionable graphs. It is easy to see, that if $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are additive and monotone (induced hereditary) graph properties, then $\mathcal{R} = \mathcal{P}_1.\mathcal{P}_2.\ldots.\mathcal{P}_n$ is monotone (induced hereditary) and additive, too.

A property $\mathcal{R} = \mathcal{P}_1.\mathcal{P}_2.\ldots.\mathcal{P}_n$, $n \geq 2$ is said to be reducible.

A graph G is said to be $\mathcal{P}$ -maximal if $G \in \mathcal{P}$ but for any edge e of its complement $G + e \notin \mathcal{P}$.

To characterize the reducible properties of graphs the following problems have been investigated.

Problem 1 *Unique Factorization Problem [2]*
Let $\mathcal{R}$ be a reducible property of graphs and $\mathcal{R} = \mathcal{P}_1.\mathcal{P}_2.\ldots.\mathcal{P}_n$, $n \geq 2$ be a factorization of $\mathcal{R}$ into irreducible factors. Is the factorization unique (apart from the order of factors)?

This question is related to the following problems:
Let $\mathcal{R} = \mathcal{P}_1.\mathcal{P}_2.\ldots.\mathcal{P}_n$, $n \geq 2$ be a factorization of the property R into irreducible factors. Do the following statements hold?

Problem 2 *The Structure of Maximal Graphs*
If the property $\mathcal{R}$ is reducible, then all $\mathcal{R}$-maximal graphs are joins of at least n graphs. Is this necessary condition also sufficient?

Problem 3 *The Existence of uniquely Partitionable Graphs*
$U(\mathcal{P}_1.\mathcal{P}_2.\ldots.\mathcal{P}_n) \neq \emptyset$, moreover $U(\mathcal{P}.\mathcal{P}.\ldots.\mathcal{P}) \neq \emptyset$ if and only if $\mathcal{P}$ is irreducible.

For additive and monotone graph properties we proved that all problems have an affirmative solution (see [3]).
However we found counterexamples in all cases if the additivity of

graph properties is not required.
At the present the problems are open for additive and induced hereditary graph properties.

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Jana Maxová, Jaroslav Nešetřil

Oriented Perfect Path Double Cover

Let $G = (V, E)$ be an undirected graph, let $\vec{G} = (V, \vec{E})$ be the symmetric orientation of G (i.e. $\vec{E} = \{(x, y); \{x, y\} \in E\}$).

A family $\mathcal{P} = \{P_1, ..., P_t\}$ of oriented paths in $\vec{G}$ is called Oriented Perfect Path Double Cover (OPPDC) of G if the following holds:

1. every arc of $\vec{G}$ belongs to exactly one of the paths P_i
2. every vertex $v \in V$ is the starting point of exactly one of the paths P_i

One can check easily that K_3 fails to have OPPDC and by a little more elaborate argument also K_5 fails to have OPPDC (this has been shown by J. Turzíková [2]). Presently these are the only known examples of connected graphs without OPPDC.

Problem: Does OPPDC exist for all but finitely many connected graphs?

Trees K_{2n}, $K_{m,n}$, C_n, $(n \geq 4)$, odd graphs (i.e. each vertex is of odd degree) and 2-connected graphs (except of K_3) with $|E(G) \leq 2n-1$ $(n = |V(G)|)$ are known to have OPPDC. One can also show that G admits OPPDC if each block of G admits OPPDC. Also the minimal counterexample must be a 2-connected graph with all degrees ≥ 4 (except of K_3) [2].

Note: This s oriented version of the notion of Perfect Path Double Cover (PPDC). It has been shown [1] that PPDC exists for any finite graph, see also [3].

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Jaroslav Nešetřil

The Critical Asymmetric Oriented Graph

An oriented graph $G = (V, E)$ is any orientation of undirected graph (i.e. we assume $(x, y) \in E \Rightarrow (y, x) \notin E$).

G is said to be an asymmetric graph G if the identity is the only automorphism of G. G is said to be Critical Asymmetric graph if G is asymmetric and any vertex deleted graph $G - x$ fals to be asymmetric.

Conjecture: K_1 (i.e. the single vertex graph) is the only critical asymmetric oriented graph.

I proposed this conjecture about 15 years ago and it has been stated at some the previous workshops. The conjecture can be verified for orientation of trees and, more generally, for acyclic graphs (Wojcik, Comment. Math. Univ. Carol. 1996).

Klaus Reuter

The Graph of Linear Extensions

1 Introduction

Let $P = (X, \leq)$ be a finite poset. The *graph of linear extensions* $G(P)$ has the linear extensions of P as vertices with two being connected if they differ by an adjacent transposition. For example, the $\mathbf{N}$-shaped poset $(\{1,2,3,4\}; 1 < 3, 2 < 3, 2 < 4\})$ has five linear extensions $\alpha = 2413$, $\beta = 2143$, $\gamma = 1243$, $\delta = 2134$, $\epsilon = 1234$ and the edges of $G(\mathbf{N})$ are $\{\alpha, \beta\}$, $\{\beta, \gamma\}$ $\{\beta, \delta\}$, $\{\gamma, \epsilon\}$, $\{\delta, \epsilon\}$. The author has become interested in these graphs of linear extensions ever since he (re)discovered that several order theoretic concepts are related to convexity concepts in $G(P)$. A subset of a graph is called *convex*, if with two points all points on shortest paths between them are contained in this subset. The purpose of this abstract is to present some of the relationships to convexity and to state several challenging open problems.

2 Results and Problems

Continuing with the example, observe that $G(\mathbf{N})$ is a convex subgraph of the permutahedron graph with 4! vertices, which is just G(4-antichain). More general it turns out that $G(P)$ is a convex subgraph of $G(Q)$ if and only if P is an extension of Q. There exists even a lattice antiisomophism [1, 5]:

Theorem 1 *Let $Ext(P)$ be the lattice of extensions of P and $Conv(P)$ the lattice of convex subgraphs of $G(P)$. Then $Ext(P)$ is antiisomorphic to $Conv(P)$.*

A non-empty subset X of $G(P)$ is called *biconvex* (or sometimes called a *halfspace*) if X as well as $\bar{X} = G(P) - X$ are convex. The splits of $G(P)$ into biconvex pairs $(X, \bar{X})$ correspond to the incomparable pairs of P. This fact is the starting point for proving that the comparability graph $Comp(P)$ can be reconstructed from the (unlabeled !) graph of linear extensions of P [6]:

Theorem 2 *$G(P) \cong G(Q)$ implies $Comp(P) \cong Comp(Q)$*

By a theorem of Djoković [2] the graph of linear extensions is an isometric subgraph of a hypercube. But not every such isometric subgraph is isomorphic to some $G(P)$.

Problem 1 *Characterize the graphs of linear extensions. What is the complexity to recognize them ?*

Let $led(P)$ denote $diam(G(P))$, the diameter of $G(P)$. In [3] it is shown:

Theorem 3

$$led(P) = \max\{inc(Q) | Q \text{ is a 2-dimensional extension of } P\},$$

where $inc(Q)$ is the number of incomparable pairs of Q.

In case of the Boolean Poset B_n it is not hard to find two linear extensions with $led(B_n) \geq 2^{2n-2} - (n+1)2^{n-2}$. We conjecture that this is best possible [3]:

Problem 2 *Is $led(B_n) = 2^{2n-2} - (n+1)2^{n-2}$?*

A *critical pair* (x, y) of P is an incomparable pair with the property that $a < x$ implies $a < y$ and $b > y$ implies $b > x$ for all $a, b \in P - \{x, y\}$. A critical pair of P corresponds to a minimal (wrt inclusion) biconvex set of $G(P)$.

Problem 3 *Let the linear extensions L_1 and L_2 realize the diameter of $G(P)$. Is L_1 or L_2 (or both) always contained in a minimal biconvex set of $G(P)$? Equivalently: Does L_1 or L_2 (or both) reverse a critical pair of P?*

An affirmative answer to Problem 3 would in particular imply that the formula of Problem 2 holds. The following is a well known conjecture on linear extensions [4] which in our context reads as follows:

Problem 4 *Is there always a biconvex subset X of $G(P)$ such that $1/3 \leq |X|/|G(P)| \leq 2/3$?*

The Carathéodory, Helly and Radon numbers of $G(P)$ are defined similar as in classical convexity theory [5]. We denote them by $c(P), h(P)$ and $r(P)$. We only recall the definition of $r(P)$, since we follow [7] in defining it diminished by 1 against the classical one: $r(P)$ is the least r, such that for every $X \subseteq G(P)$ with $|X| > r$ there exists a partition $X = X_1 \uplus X_2$ with $conv(X_1) \cap conv(X_2) \neq \emptyset$.

Theorem 4 *For $n = |P|$ it holds:*
(i) $c(P) \leq \max\{\lfloor \frac{1}{4} n^2 \rfloor, n\}$,
(ii) $h(P) = width(P)$,
(iii) $r(P) \leq n$.

The proof of (iii) uses a transformation to the R^n-convexity [5]. It would be interesting to have a direct combinatorial proof.

Problem 5 *Find a combinatorial proof showing that the Radon number $r(P)$ is at most $|P|$.*

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Moshe Rosenfeld

We say that a 3-CNF instance $f(x_1 \dots x_n)$ is *critical* if it is not satisfiable but the removal of any clause makes the resulting instance satisfiable. For instance, if we take $f(x_1, x_2, x_3)$ to include all 8 distinct clauses on 3 variables then clearly, it is critical.

Problem 1 What is the maximum number of clauses in a 3-CNF critical instance with n variables?

The following construction yields critical instances with n variables and about $(1+\frac{1}{logn})n$ clauses. Let $k \geq 4$ be an integer. Consider first the instance $g(x_1 \dots x_k)$ that contains all 2^k possible distinct clauses $(y_1, \dots, y_k)$ on the k variables $x_1, \dots, x_k$. Clearly, g is not satisfiable but if we delete any clause from g the resulting instance is satisfiable (if the clause $(y_1, \dots, y_k)$ is deleted, then the assignment

x_i = true if $y_i = \bar{x}_i$ and false otherwise will make $g \setminus (y_1, \ldots, y_k)$ satisfiable).

We now replace each clause $(y_1, \ldots, y_k)$ by the following $k-2$ clauses

$$(y_1 + y_2 + z_1)(\bar{z}_1 + z_2 + y_3) + \cdots + (\bar{z}_{k-3} + y_{k-1} + y_k)$$

where the new $k-3$ variables $z_1, \ldots, z_{k-3}$ are specific to each clause $(y_1, \ldots, y_k)$. Each such set of $k-2$ clauses has the property that if any clause is deleted, the remaining $k-3$ clauses can be satisfied by assigning appropriate values to the new variables z_i (regardless of the values assigned to the variables y_i). It is a simple matter to verify that the resulting instance is a critical 3-CNF instance with $(k-2)2^k$ clauses and $k + (k-3)2^k$ variables.

It is known that if one chooses randomly about $(1.462...)n$ 3-clauses then almost surely the resulting instance is not satisfiable, so a natural question would be whether there are critical instances with more than $1.5n$ clauses.

Problem 2 Is it true that if the smallest eigenvalue of a graph G of order n has multiplicity $> (\frac{k-1}{k})n$ then G contains a K_{k+1} as a subgraph?

I proved that this is true for $k = 2$ that is if the multiplicity of the smallest eigenvalue of G is $> \frac{n}{2}$ then G contains a triangle. Furtheremore, the extremal graphs are unique. That is if the smallest eigenvalue of G has multiplicity $\frac{n}{2}$ and G does not contain a triangle then G is the disjoint union of edges (the complement of the Turán graph $K_{2,n}$). Barry Guiduli pointed out that by A. Hoffman's bound on the chromatic number,

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$$

(λ_1 is the largest eigenvalue and λ_n the smallest) one can easily show that the chromatic number of the graph refered to in the problem is $\geq k+1$ which supports our conjecture. Again, the complements of the Turán graphs $K_{k,n}$ provide examples of graphs with smallest eigenvalue of multiplicity $= (\frac{k-1}{k})n$ that do not contain a K_{k+1} as a subgraph. It will be interesting to know whether these are the only extremal graphs. We also note that the multiplicity assumption must be associated with the smallest eigenvalue. The complete bipartite graph has $2n$ vertices and the eigenvalue 0 has multiplicity $2n-2$.

Miklós Ruszinkó

Intersecting Systems

An *intersecting system of type* $(\exists, \forall, k, n)$ is a collection $\mathbb{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$ of pairwise disjoint families of k-subsets of an n-element set satisfying the following condition. For every ordered pair $\mathcal{F}_i$ and $\mathcal{F}_j$ of distinct members of $\mathbb{F}$ there exists an $A \in \mathcal{F}_i$ that intersects every $B \in \mathcal{F}_j$. Let $I_n(\exists, \forall, k)$ denote the maximum possible cardinality of an intersecting system of type $(\exists, \forall, k, n)$. Ahlswede, Cai and Zhang conjectured that for every $k \geq 1$, there exists an $n_0(k)$ so that $I_n(\exists, \forall, k) = \binom{n-1}{k-1}$ for all $n > n_0(k)$. R. Ahlswede, N. Alon, P.L. Erdős, M. Ruszinkó and L.A. Székely showed [1] that this is true for $k \leq 3$, but false if k is large:

$$I_n(\exists, \forall, k) \geq c\binom{n-1}{k-1}$$

for $n > n_0(k)$ and $k \geq 8$ ($c > 1$ is an absolute constant). On the other hand, if $k \geq 8$ we can prove only an

$$I_n(\exists, \forall, k) \leq k\binom{n-1}{k-1}$$

upper bound. (Observe that this bound is the Erdős-Ko-Rado one multiplied by k.) Therefore, we would like to ask if at least in magnitude the Erdős-Ko-Rado bound $\binom{n-1}{k-1}$ for intersecting systems does hold, i.e. is it true that for all $n > k$

$$I_n(\exists, \forall, k) = \Theta\left(\binom{n-1}{k-1}\right)?$$

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József Solymosi

Integral distances in the plane

1 Point sets

The set of interpoint distances realised by a set of points in the plane is a fairly investigated topic. The present note deals with the set of integral distances. The first result in this field was the following theorem of A.Anning and P.Erdős in 1945:

Theorem 1 *[1] Let H be an infinite set of points in the plane, such that all the interpoint distances are integer. Then the points lie on one line.*

Few months later Erdős found a simpler proof of this theorem [2]. Using the basic idea of his proof we can prove a quantitative version of Theorem 1.

Theorem 2 *There exists a natural number $N(d)$ for every natural number d with the following property: if H is a finite set of points in the plane such that*

(i) all the interpoint distances realized by H are integer,

(ii) $|H| \geq N(d)$

(iii) the minimum distance in H is at most d

then at least $|H| - 2d$ points of H are collinear.

Problem 1 *Let H be a set of points in the plane such that all the interpoint distances are integer and no 3 of them be collinear. Then what can we say about the minimum distance $d = d(|H|)$?*

2 Graphs

Let M be a countable infinite set of points in the plane, no 3 of them be collinear. The *integer graph* of M is defined as follows: the vertex set is M, and if two vertces are adjacent then their distance is integer. Erdős proved, that an integer graph cannot contain a complete bipartite graph $K_{3,\aleph_0}$ and asked, whether this condition characterizes the integer graphs [3]. Our answer is negative. There exists an infinite, bipartite graph with finite degrees which is not an integer graph. The characterization of integer graphs must be a hard task in view of this example and the following theorem:

Theorem 3 *Any subgraph induced by the common neighbours of any two verices of an integer graph has finite chromatic number.*

Problem 2 *Is there exists a C_4-free graph which is not an integer graph? (I think, the ansver is no)*

Problem 3 *Is there exists a graph with bounded degrees which is not an integer graph?*

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Petra Špičková

Homomorphism duality for rooted oriented paths

Let (P, r) be a fixed rooted oriented path. The (P, r)-*colouring problem* is the problem of deciding which rooted digraphs (G, s) admit a rooted homomorphism to (P, r), i. e. for which rooted digraphs (G, s) there is a homomorphism $f : G \to P$ which maps the vertex s to the vertex r. I characterized the nonexistence of such a homomorphism by the existence of a rooted oriented cycle (C, t), which is homomorphic to (G, s) but not homomorphic to (P, r). Such a property of the graph is called *rooted cycle duality.* This extends the analogical result for unrooted oriented paths given in [1]: for an oriented path P and an arbitrary digraph G the nonexistence of a homomorphism $G \to P$ is guaranteed by the existence of an oriented path W homomorphic to G but not homomorphic to P. My result makes it possible to modify the polynomial time algorithm given in [2] for deciding the P-colouring problem to the algorithm for deciding the (P, r)-colouring problem. I also introduce the notion of *comprimed tree duality.* I can show that comprimed tree duality of a rooted digraph (H, r) implies a polynomial algorithm for the (H, r)-colouring problem.

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Claude Tardif

Homomorphisms of vertex-transitive graphs

1 Introduction

The problem presented here is inspired by the odd cycles: It is well known that these are the only 3-critical graphs. They also happen to be vertex-transitive, and the same holds for K_1 and K_2, which are the critical graphs with smaller chromatic number. This suggests the following:

Problem 6 For any graph G, there exists a vertex-transitive graph H such that H admits a homomorphism into G and $\chi(H) = \chi(G)$.

Here, a homomorphism is a map $\phi : H \mapsto G$ from the vertex set of H to the vertex set of G such that $[\phi(u), \phi(v)]$ is an edge of G whenever $[u, v]$ is an edge of G. We write $H \to G$ if there exists a homomorphism from H to G. This definition clearly implies that if $H \to G$, then $\chi(H) \leq \chi(G)$. However note that H need not be a subgraph of G. Hence, minimality does not play an essential role as it does in the definition of χ-critical graphs. In particular, G need not contain a vertex-transitive $\chi(G)$-critical subgraph, but note that there exists infinitely many $\chi(G)$-critical graphs that admit homomorphisms into G, while only a finite number of them are subgraphs of G.

At the present, it is not even known that there exists a number m such that for any graph G with chromatic number at least m, there exists a 4-chromatic graph admitting a homomorphism into G. Nonetheless, there are some grounds for believing that Conjecture 6 is true, based on the construction presented in the next section.

2 Vertex-transitive graphs admitting a homomorphism into G

Let $O_1, \ldots, O_n$ denote the orbits of a graph G under the automorphism group of G. Then for any nonnegative integers $r_1, \ldots, r_n$, we define the graph $G(r_1, \ldots, r_n)$ as follows: Put $s = r_1 + \ldots + r_n$; the vertices of $G(r_1, \ldots, r_n)$ are all functions $f : \{1, \ldots, s\} \mapsto V(G)$ such that

$$|\{i : f(i) \in O_k\}| = r_k \text{ for all } k = 1, \ldots, n,$$

and two functions f, g are joined by an edge in $G(r_1, \ldots, r_n)$ if and only if $[f(i), g(i)] \in E(G)$ for all $i = 1, \ldots s$.

We then have $G(r_1, \ldots, r_n) \to G$, since the map $\phi : G(r_1, \ldots, r_n) \mapsto G$ where $\phi(f) = f(1)$ is a homomorphism from $G(r_1, \ldots, r_n)$ to G. Furthermore, $G(r_1, \ldots, r_n)$ is vertex-transitive: Let f, g be two vertices of $G(r_1, \ldots, r_n)$. Then, there exists a permutation ψ of $\{1, \ldots, s\}$ such that $f(\psi(i)) \in O_j$ if and only if $g(i) \in O_j$. We can then find automorphisms $\alpha_1, \ldots, \alpha_s$ of G such that $\alpha_i(f(\psi(i))) = g(i), i = 1, \ldots, s$. Define $\alpha : G(r_1, \ldots, r_n) \mapsto G(r_1, \ldots, r_n)$ by $\alpha(h) = h'$, where $h'(i) = \alpha_i(h(\psi(i)))$. Then α is an automorphism of $G(r_1, \ldots, r_n)$ such that $\alpha(f) = g$. Thus, $G(r_1, \ldots, r_n)$ is vertex-transitive.

This construction provides infinitely many vertex-transitive graphs admitting homomorphisms to a given graph G. If $\chi(G(r_1, \ldots, r_n)) = \chi(G)$ for a suitable choice of integers $r_1, \ldots, r_n$, then Conjecture 6 is true. Thus, this construction looks like a good starting point for testing the conjecture. The following result shows that in fact, it is the only construction that needs to be considered.

Proposition 1 *Let G be a graph with orbits $O_1, \ldots, O_n$ and K a graph such that $K \to G$. If there exists a vertex-transitive graph H such that $K \to H \to G$, then $K \to G(r_1, \ldots, r_n) \to G$ for some integers $r_1, \ldots, r_n$.*

Proof. Suppose that H is a vertex-transitive graph such that there exist homomorphisms $\phi : K \mapsto H$ and $\psi : H \mapsto G$, and let $\{\alpha_1, \ldots, \alpha_s\}$ be the set of all automorphisms of H. For $i = 1, \ldots, n$, put $r_i = |\psi^{-1}| \cdot m$, where m is the cardinality of the stabilizer of any vertex of H in the group of automorphisms of H. For any vertex u of K, we then have $|\{j \in \{1,\ldots,s\} : \psi(\alpha_j(\phi(u))) \in O_i\}| = r_i$. Thus, we can define a homomorphism $\hat{\phi} : K \mapsto G(r_1,\ldots,r_n)$ by putting $\hat{\phi}(u) = f_u$, where $f_u(j) = \psi(\alpha_j(\phi(u))), j = 1,\ldots,s$.

■

In [1], Albertson and Booth have shown that for any graph K, we can find a graph G such that $K \to G$ and the only possible vertex-transitive graphs satisfying $K \to H \to G$ are those that also satisfy $H \to K$, if any such graphs exist. A careful examination of the proof of Proposition 1 shows that such a vertex-transitive graph exists only if there are 'enough' homomorphisms between K and G. More precisely, if $\{\phi_1,\ldots,\phi_k\}$ is the set of homomorphisms from K to G, then there exists a vertex-transitive graph H such that $K \to H \to G$ if and only if there exists nonnegative integers $a_1,\ldots,a_k$ such that for any two vertices u, v of K, we have $\sum\{a_j : \phi_j(u) \in O_i\} = \sum\{a_j : \phi_j(v) \in O_i\}, i = 1,\ldots,n$.

Therefore, the validity of Conjecture 6 depends on the existence of $\chi(G)$-chromatic graphs endowed with a rich class of homomorphisms into G. There are some grounds for believing that such graphs indeed exist, based on the results of El-Zahar and Sauer [3] on Hedetniemi's conjecture. These results show that the critical subgraphs in a categorical product of graphs tend to be a lot smaller than the product itself. In particular, El-Zahar and Sauer outline a construction for some 4-chromatic subgraphs of a product of 4-chromatic graphs. It is possible that this construction, properly applied to categorical powers of G, may produce a graph with the desired structure.

3 Odd wheels

In this section, we verify Conjecture 6 for the simplest class of 4-chromatic graphs that do not contain K_4, that is, the odd wheels. For any $k \geq 4$, the wheel W_k is the graph obtained from the cycle C_k by adding a new vertex u that is adjacent to all vertices of C_k. Thus, the chromatic number of W_k is 3 or 4, depending on whether k is even or odd. The orbits of W_k are $O_1 = \{u\}$, the universal vertex, and $O_2 = \{0,\ldots,k-1\}$, the vertex-set of C_k. We prove the following:

Proposition 2 *For any odd number $k \geq 5$ and any $n \geq 2$, $\chi(W_k(n, 2n)) = 4$.*

Proof. Obviously, $\chi(W_k(n, 2n)) \leq 4$ and it suffices to show that $W_k(n, 2n)$ is not 3-colourable. We will use the following result of Albertson and Collins [2], where $i(G)$ denotes the independence ratio of a graph G, that is, $i(G) = \alpha(G)/|V(G)|$, where $\alpha(G)$ is the independence number of G.

Lemma 3 (No-Homomorphism lemma [2]) *Let G, H be graphs such that H is vertex-transitive and $G \to H$. Then,*

$$i(G) \geq i(H).$$

Also, if $i(G) = i(H)$, then for any homomorphism $\phi : G \mapsto H$ and for any independent set I of maximal cardinality in H, we have

$$|\phi^{-1}(I)| = \alpha(G).$$

Note that $W_k(n, 2n)$ contains many triangles, which can all be described as follows: Let $\{1, \ldots, 3n\}$ be partitionned into sets A, B, C such that $|A| = |B| = |C| = n$, and select vertices x_i, y_i of C_k such that $[x_i, y_i] \in E(C_k), i = 1, \ldots, 3n$. Then the functions f, g, h induce a triangle in $W_k(n, 2n)$, where $f(i) = u, g(i) = x_i, h(i) = y_i$ if $i \in A$, $f(i) = x_i, g(i) = y_i, h(i) = u$ if $i \in B$ and $f(i) = y_i, g(i) = u, h(i) = x_i$ if $i \in C$. This shows that $K_3 \to W_k(n, 2n)$, and by Lemma 3, we have $1/3 = i(K_3) \leq i(W_k(n, 2n))$, that is, every independent set of $W_k(n, 2n)$ contains at most one third of the vertices of $W_k(n, 2n)$. In fact, this bound is achieved by each of the sets

$$I_i = \{f \in V(W_k(n, 2n)) : f(i) = u\}, i = 1, \ldots, 3n.$$

Thus, $i(W_k(n, 2n)) = 1/3$.

Now suppose that $W_k(n, 2n)$ is 3-colourable. Then it can be partitionned into three independent sets, each containing exactly one third of the vertices. In order to show that this is impossible, it suffices to show that the only independent sets of maximal cardinality in $W_k(n, 2n)$ are the sets $I_1, \ldots, I_{3n}$ defined above. Indeed, any two of them share common vertices, so it is impossible to partition $W_k(n, 2n)$ using three of these sets.

We first need to consider smaller independent sets in $W_k(n, 2n)$. For any n-subset A of $\{1, \ldots, 3n\}$, put

$$J_A = \{f \in V(W_k(n, 2n)) : f^{-1}(u) = A\}.$$

Then, using the fact that k is odd and the description of the triangles of $W_k(n, 2n)$ given above, it can be seen that for every $f, g \in J_A$, there exists a sequence $f = f_0, f_1, \ldots, f_m = g$ of elements of J_A such that f_{i-1} and f_i have two common neighbours for all $i = 1, \ldots, m$.

Let I be an independent set of maximal cardinality in $W_k(n, 2n)$. Since $i(W_k(n, 2n)) = i(K_3)$, the second part of Lemma 3 states that I contain a vertex of every triangle of $W_k(n, 2n)$. Therefore I intersects a set J_A if and only if J_A is contained in I, and

$$I = \bigcup\{A : J_A \subseteq I\}.$$

Note that whenever A is disjoint from B, J_A contains a vertex f that is adjacent to some vertex g of J_B. Therefore, $\{A : J_A \subseteq I\}$ is a family of pairwise intersecting subsets of $\{1, \ldots, 3n\}$ which has maximum cardinality. Hence by Hilton and Milner's theorem [4], there exists an integer i such that

$$\{A : J_A \subseteq I\} = \{A : i \in A\}.$$

We then have $I = I_i$, which concludes the proof.

■

Note that since W_k has only two orbits, one of which containing no edges, there is a ressemblance between the construction of $W_k(a, b)$ and that of the well-known Kneser graphs. For integers m, n the Kneser graph $K(m, n)$ is the graph whose vertices are the n-subsets of $\{1, \ldots, m\}$, two vertices being joined by an edge if and only if they are disjoint. Thus, if we allow loops in our construction, $K(m, n)$ can be described as the graph $G(n, m-n)$, where G is the graph obtained from K_2 by adding a loop to one vertex.

The object of Kneser's conjecture was to show that the chromatic number of $K(m, n)$ is $m - 2n + 2$, which was achieved by Lovász [5]. Note that there exists a homomorphism $\phi : W_k(a, b) \mapsto K(a + b, a)$ defined by $\phi(f) = f^{-1}(u)$, so that $\chi(W_k(a, b)) \leq \chi(K(a + b, a))$.

However, it is not known whether $\chi(W_k(a,b)) = 4$ whenever $\chi(K(a+b,a)) \geq 4$. An investigation of the large independent sets similar to the argument of the proof of Proposition 2 was insufficient to settle the case of all Kneser graphs. Lovász's method relies on some topological properties of Kneser graphs. It would be interesting to know whether these methods can be adapted to some cases of Conjecture 6.

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Eberhard Triesch

Problems and Results in Combinatorial Group Testing

Abstract

We survey some of the most fundamental problems in Combinatorial Group Testing. Using some results about the equidistribution of sequences, we prove that it is NP-hard to decide whether a graph has an induced subgraph with exactly half the number of edges.

1 Introduction

Group testing problems were first studied during World War II in a paper by Dorfman (see [8]). In mathematical language, the problem is as follows: Assume that some hypergraph $G = (V, E)$ is given, i.e., some finite set V (the vertex set) and a set $E \subset 2^V$ of (hyper-)edges. We want to find some (unknown) edge e^* (usually called the set of defective elements) by successively choosing subsets $W \subset V$ and testing whether $e^* \cap W$ is empty or not. The worst case complexity of this search problem is defined as the minimum number of tests needed in the worst case to find e^*. We denote it by $L(G)$. Since each test has at most two possible answers, it is clear that

$$L(G) \geq \lceil \log_2 |E| \rceil.$$

This inequality is called the information theoretic bound.

Similarly, if we have some probability distribution on the edges, i.e. some function $p : E \to (0, \infty)$ with $\sum_{e \in E} p(e) = 1$, we may ask for the expected length $L(G, p)$ of an optimal search algorithm. The information theoretic bound in this case is given by the entropy of the distribution:

$$L(G,p) \geq H(p) = -\sum_{e\in E} p(e)\log_2 p(e).$$

The application studied originally by Dorfman is as follows: Suppose you want to find and isolate all people infected with some contagious desease in some given population. After drawing blood samples from everybody you might mix the blood of a group of people and test whether at least one person of that group is infected. Since you can discard the whole group if your test is negative, this approach should be superior to testing each probe separately if the probability of being infected is low.

As introductions to search problems, see [2] and [1]. The book [9] gives the most detailed account on group testing problems including applications and some remarks about their history.

We begin with a natural question: Under which conditions is it optimal to test each element of V individually? Suppose first that each element of V has some fixed probability p of being defective, more precise: $E = 2^V$ and $p(e) = p^{|e|}(1-p)^{|V|-|e|}$. For this model, Ungar proved in 1960 ([14]) that individual tests are optimal if and only if $p \geq \frac{1}{2}(3-\sqrt{5})$. One of the most interesting problems in this area is the following conjecture, due to Du and Hwang (see [9]):

Denote by $L(n,d)$ the complexity $L(G)$ with $|V| = n$ and $|E| = \binom{V}{d}$. (As usual, $\binom{V}{d}$ denotes the set of d-element subsets of V.)

Conjecture 1 *$L(n,d) = n$ if and only if $d \geq n/3$*

It is easy to see that $L(n,d) < n$ for $d < n/3$. Du and Hwang proved that $L(n,d) = n$ for $d \geq (8/21)n$ (see [10]). This seems to be the only significant lower bound on $L(n,d)$ except the information theoretic bound.

Concerning upper bounds, we have the following general result (see [13])

Theorem 4 *Suppose $G = (V,E)$ is a hypergraph and $p(e), e \in E$ are (positive) probabilities on its edges. Then there is a search algorithm finding the unknown edge e^* by at most $\lceil -\log p(e^*)\rceil + |e^*|$ tests.*

If all edges in G have cardinality d or less, the number of tests can actually bounded by $\lceil -\log p(e^*)\rceil + (d-1)$.

Suppose now that G is a graph, i.e., all edges have cardinality 2. Choosing p as the equidistribution on E, we get the upper bound $L(G) \leq \lceil \log|E|\rceil + 1$ which differs from the information theoretic bound by exactly 1. This result for graphs was first proved by Damaschke (see [7]). An investigation by Chang, Hwang and Lin (see [6]) shows that both equalities $L(G) = lc\log|E|\rceil$ and $L(G) = lc\log|E|\rceil + 1$ are satisfied by infinitely many complete graphs.

Let us call a graph G optimal if $L(G) = lc\log|E|\rceil$. Is it difficult to recognize whether a graph is optimal or not? We have the following conjecture:

Conjecture 2 *It is an NP-complete problem to recognize optimal graphs.*

It is not difficult to see that the above problem is in NP. A nondeterministic Turing machine might guess the tests for an algorithm proving optimality. On the other hand, in what follows we want to give some evidence that the construction of good test sets $W \subset V$ is not an easy problem. If we want to come close to the information theoretic bound, in each step we have to construct a test which approximately halves the candidate edges and we are now going to prove that halving a graph exactly is an NP–hard problem. We use some well–known facts from the theory of NP–completeness as developed in [11]. For graph–theoretical notions, see, e.g., [4].

2 GRAPH HALVING is NP–complete

Consider the following decision problem:

GRAPH HALVING (GH)

Input: A graph $G = (V, E)$.

Question: Does there exist a subset $W \subset V$ such that $m(W) := |\{f : f \in E, f \cap W \neq \emptyset\}|$, the number of edges of G which are incident to some point in W, is exactly $|E|/2$?

Theorem 5 *The decision problem GH is NP–complete*

Proof. Membership in NP is trivial. Our aim is to prove that GH is NP–complete by constructing a transformation from

VERTEX COVER (VC)

Input: A graph $G = (V, E)$ and some number $k < n := |V|$.

Question: Does there exist a subset $W \subset V$, $|W| = k$, such that $m(W) = |E|$?

So suppose that an input G, k for VC is given. The graph H we are going to construct will consist of two components where one component is a complete graph K_N and the other one, which we call F, has vertex set $V \cup U \cup X \cup Y$ where the sets V, U, X and Y are pairwise disjoint with $|U| = |E| + 1, |X| = (k+1)|U|$ and $|Y| \geq |X|$ will be defined later. The induced subgraph $F[V]$ is equal to G. Furthermore, F contains the complete bipartite graphs $K_{V,U}, K_{U,X}, K_{X,Y}$ and the induced graphs $F[U]$ and $F[X]$ are empty. The induced graph $F[Y]$ will be specified later. Note that F contains a vertex set W with $m(W) = |E| + k|U|$ if and only if G has a vertex cover of cardinality k.

Suppose first that i vertices are chosen in K_N. Obviously, they are incident to exactly $f(i) := \binom{i}{2} + i(N - i) = i(N - \frac{i+1}{2})$ edges. We are going to investigate the solution $z = z(N)$ of the equation $f(z) = \frac{1}{2}\binom{N}{2}$ in the interval $(0, N)$. By elementary algebra we find that

$$z(N) = N - \frac{1}{2} - \frac{1}{2}\sqrt{2N^2 - 2N + 1}.$$

Let $\gamma := 1 - 1/\sqrt{2}$. We need:

Lemma 1

$$z(N) = \gamma N - \frac{\gamma}{2} + O(\frac{1}{N}) \qquad (N \to \infty).$$

Proof of Lemma 1. By elementary calculations,

$$\left| z(N) - \gamma N + \frac{\gamma}{2} \right| = \left| \frac{\sqrt{2}}{2} \frac{N - \frac{1}{2}}{\sqrt{N^2 - N + \frac{1}{2}} + N} - \frac{1}{2\sqrt{2}} \right|$$

$$= \frac{\sqrt{2}}{2}\left|\frac{N - \frac{1}{2} - \frac{1}{2}N - \frac{1}{2}\sqrt{N^2 - N + \frac{1}{2}}}{\sqrt{N^2 - N + \frac{1}{2}} + N}\right|$$

$$\leq \frac{\sqrt{2}}{2}\left|\frac{\frac{1}{2}(N-1) - \frac{1}{2}N}{N - \frac{1}{2} + N}\right| = \frac{\sqrt{2}}{4}\left|\frac{1}{2N - \frac{1}{2}}\right| = O(\frac{1}{N}).$$

□

Consider the sequence $\xi = (\xi_N)_{N=1}^{\infty}$ where $\xi_N := \gamma N - \lfloor \gamma N \rfloor$. The *discrepancy* $D_N = D_N(\xi)$ of ξ is defined as

$$D_N(\xi) := \sup_{J \subset [0,1)} \left| \frac{1}{N} |\{j \leq N : \xi_j \in J\}| - \ell(J) \right|$$

where the supremum extends over all intervals $J \subset [0,1)$ and $\ell(J)$ is the length of J. Since γ is an irrational solution of the equation $2\gamma^2 - 4\gamma + 1 = 0$, a well–known theorem from the theory of equidistribution (see, e.g.,[12], p.97) implies that $D_N(\xi) = O(\frac{\log N}{N})$. It is clear that the discrepancy of the sequence $\zeta = (\zeta_N)_{N=1}^{\infty}$ with $\zeta_N := \lceil \gamma N - \gamma/2 \rceil - (\gamma N - \gamma/2)$ has the same property:

Lemma 2 *There is some constant $C > 0$ such that, for all N, the discrepancy $D_N(\zeta)$ satisfies the inequality*

$$D_N(\zeta) \leq C \frac{\log N}{N}.$$

Now suppose that $J \subset (0,1)$ is some interval and $M \in \mathbf{N}$. We want to make sure that at least one of the numbers $\zeta_{M+1}, \zeta_{M+2}, \ldots, \zeta_{M+K}$ is in J by choosing K as small as possible. The results on the discrepancy imply that $K = O(\log M)$ suffices:

Lemma 3 *There is some constant $D > 0$ (depending on J) such that for all $M \in \mathbf{N}$ and $K > D \log M$ at least one of the numbers $\zeta_{M+1}, \zeta_{M+2}, \ldots, \zeta_{M+K}$ is contained in J.*

Proof of Lemma 3. Let $A(i,j) := A_J(i,j) := |\{l : \zeta_l \in J, i \leq l \leq j\}|$. By the preceding lemma, we have the inequalities:

$$\frac{A(1,M)}{M} \leq \ell(J) + C \frac{\log M}{M}$$

and

$$\frac{A(1,M+K)}{M+K} \geq \ell(J) - C\frac{\log(M+K)}{M+K}$$

whence

$$\begin{array}{rcl} A(M+1,M+K) & = & A(1,M+K) - A(1,M) \\ & \geq & k\ell(J) - C(\log M + \log(M+K)) \end{array}$$

The last expression is positive if

$$K > \frac{C}{\ell(J)}\log M + \frac{C}{\ell(J)}\log(M+K)$$

from which the result follows.

□

Before we go on, note that the following estimates hold for the difference $f(z+\alpha) - f(z)$:

$$f(z+\alpha)-f(z) = \alpha(N-z-\frac{\alpha}{2}-\frac{1}{2})\begin{cases} = N-z-1 & \text{if } \alpha = 1 \\ < \alpha(N-z) & \text{if } 0 < \alpha < 1 \\ > \alpha(N-z-1) & \text{if } 0 < \alpha < 1. \end{cases}$$

Denoting by $e(H), e(F)$ and $e(G)$ the number of edges in H, F and G, respectively, we have $e(H)/2 = (\binom{N}{2}+e(F))/2 = f(z(N))+e(F)/2 = f(\lceil z\rceil)+e(F)/2-(f(\lceil z\rceil)-f(z))$. N and F should be chosen such that $e(H)$ is bounded by a polynomial in the size of G and the following three conditions hold:

(i) $e(F) = 2(f(\lceil z\rceil) - f(z) + e(G) + k|U|)$.

(ii) $f(\lceil z\rceil + 1) > e(H)/2$.

(iii) $f(\lceil z\rceil - 1) + e(F) < e(H)/2$.

It is clear that (i)–(iii) imply that the only way to halve H is to choose $\lceil z\rceil$ points in K_N and some subset W of F such that $e(W) = e(G) + k|U|$. Let $\alpha := \alpha_N := \lceil z\rceil - z$. By Lemmas 1 and 3 we can choose some N such that

$$1/5 < \alpha_N < 2/5 \tag{2}$$

and

$$|X| = 1 + e(G) + k|U| < 1/5(N - \lceil z \rceil). \tag{3}$$

We also want N to be so large that there is some set Y such that

$$\begin{aligned} e(F) - e(Y) &= e(G) + |V||U| + |U||X| + |X||Y| \\ &< 2\left(e(G) + k|U| + \frac{1}{5}(N - z - 1)\right) \end{aligned} \tag{4}$$

and

$$e(F) - e(Y) + \binom{|Y|}{2} > 2\left(e(G) + k|U| + \frac{2}{5}(N - z)\right). \tag{5}$$

It is straightforward from Lemmas 1 and 3 that N and Y satisfying (1)–(4) can be found such that N and $|Y|$ are bounded by a polynomial in $\max(|V|, e(G))$. It remains to define $F[Y]$ and to check conditions (i)–(iii): First of all, we note that (1) implies

$$\frac{1}{5}(N - z - 1) < \alpha(N - z - 1) < f(z + \alpha) - f(z) =$$
$$= f(\lceil z \rceil) - f(z) < \alpha(N - z) < \frac{2}{5}(N - z)$$

and thus by (3) and (4) we can choose $F[Y]$ such that (i) is fulfilled. Since (i) implies $e(H)/2 = f(\lceil z \rceil) + e(G) + k|U|$, condition (ii) holds if $f(\lceil z \rceil + 1) = f(\lceil z \rceil) + N - z - 1 > f(\lceil z \rceil) + e(G) + k|U|$, i.e. if $N - z - 1 > e(G) + k|U|$ which follows from (2).

Condition (iii) holds if and only if $f(\lfloor z \rfloor) + e(F) < e(F)/2 + f(z)$ which is equivalent to

$$e(F)/2 < f(z) - f(\lfloor z \rfloor) = f(\lfloor z \rfloor + 1 - \alpha) - f(\lfloor z \rfloor). \tag{6}$$

This last expression is bounded from below by $(1-\alpha)(N - \lfloor z \rfloor - 1) = (1-\alpha)(N - \lceil z \rceil) > \frac{3}{5}(N - \lceil z \rceil)$, hence (5) is implied by (3) if $\frac{2}{5}(N - z) + e(G) + k|U| \le \frac{3}{5}(N - \lceil z \rceil)$, i.e. if $\frac{2}{5}\alpha + e(G) + k|U| \le \frac{1}{5}(N - \lceil z \rceil)$ which follows from (2).

The proof is complete.

□

3 Concluding Remarks.

The estimates of $f(z)$ in the preceding section show that there is some positive constant $c > 0$ and infinitely many complete graphs K_n such that

$$\min\{|m(W) - \frac{1}{2}\binom{n}{2}| : W \subset V(K_n)\} \geq cn.$$

For bipartite graphs, however, the situation might be different: We have the following conjecture by Chang and Hwang (see [5]):

Conjecture 3 *Every bipartite graph is optimal.*

An interesting reformulation of the conjecture is the following:

Conjecture 4 *Every bipartite graph whose number of edges is a power of two has an induced subgraph with exactly half the number of edges.*

In [3], a sequence of (complete) bipartite graphs $G_n$is constructed where the difference $\min\{|m(W) - \frac{1}{2}e(G_n)| : W \subset V(G_n)\}$ goes to infinity (essentially as $\log e(G_n)/\log\log e(G_n)$). It remains to improve this construction.

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Paola Vocca

Lower and Upper Bounds on Lattices Representation

1 Introduction.

The study of partial orders has been extensively tackled in many fields of computer science. Due to their relevance, posets computational complexity is an interesting problem. In particular, given a class $\mathcal{P}$ of finite partially ordered sets (*posets*) $\mathcal{P} = (N, \prec_P)$, one counting problem is automatically raised. How many bits of information are needed to represent posets in $\mathcal{P}$?

A trivial upper bound is $O(n^2 \log n)$[1], where $|N| = n$. Although, the complexity of some classes is much lesser than this. By example, 2-dimensional posets need $O(n \log n)$ bits of information. More generally, k-dimensional posets need $O(kn \log n)$ bits. Thus, in some cases, the order dimension [2] represents an useful tool for evaluating the computational complexity of a poset. Even more, the order dimension, evaluating the degree of non-linearity of a poset, has been often related to the complexity of the poset itself. Unfortunately, this approach is not valid for all posets. In fact, there are classes having $O(n)$ order dimension and strictly less than $n^2 \log n$ representation complexity. For example, interval orders and distributive lattices can be represented by subsets of n set, using $O(n \log n)$ bits of information [7, 1]; partial lattices can be represented by more complex structures using $O(n\sqrt{n} \log n)$ bits [8, 10, 9]. Hence, there is no general approach for evaluating the representation complexity of a poset.

[1] Unless stated otherwise, all logarithms are to the base 2.

Due to results otained in [8, 10, 9], we are specifically interested in evaluating tight lower and upper bounds on general lattices representation.

In literature there are some previous results on the complexity of bipartite lattices, expressed in terms of the maximum number of edges. One way for evaluating the complexity of a lattice is the following. Given a finite bipartite lattice, i.e. a lattice whose maximal chains contain at most two elements other than the greatest and the least elements of the lattice, the corresponding Hasse Diagram contains no $K_{2,2}$ as vertex induced subgraph. Based on this property and on a result in [11], it is possible to evaluate the number of edges of a finite bipartite lattice

Unfortunately, the result in [11] can not be applied to finite general lattices as the above property does not hold for this class of posets [4], as shown in the following section.

2 Previous Results.

Some known results on the complexity of lattices representation derive from enumeration of labeled posets [3].

Let $L(n)$ denote the number of lattices containing $n+2$ elements where the elements other than 0 and 1 are labeled. The best known lower bound for $L(n)$ is the following:

Fact 2.1 ([6])

$$\alpha^{n\sqrt{n}+o(n\sqrt{n})} < L(n), \qquad \textit{with} \quad \alpha = 2^{\sqrt{2}/4} \approx 1.2777.$$

The proof relies on estimating the number $B(n)$ of bipartite lattices. Observing that $B(n)$ is also the number of n–element bipartite graphs containing no $K_{2,2}$ as a vertex induced sub-graphs, W. Klotz and L. Lucht enumerate the comparabilities of $B(n)$, so deriving the desired bound.

The importance of the above result is substantiated by the following theorem which can be derived from the proof.

Theorem 2.1 *There exists a lattice containing* $n+2$ *elements whose Hasse Diagram has* $o(n\sqrt{n})$ *edges.*

Moreover, for $B(n)$ a tight upper bound has been proved [5].

Theorem 2.2

$$B(n) < \gamma^{n\sqrt{n}+o(n\sqrt{n})}, \qquad \textit{with} \quad \gamma \approx 1.6994.$$

If we denote $E(\mathcal{L})$ the number of edges of the Hasse Diagram of a finite lattice then for bipartite lattices we can derive $E(\mathcal{L})$ starting from $B(n)$. In fact, deleting any edge, the resulting bipartite is still a lattice. Hence, we trivially have:

Corollary 2.3

$$E(\mathcal{L}) = \log B(n) < \delta n\sqrt{n} + o(n\sqrt{n}) \qquad \textit{with} \quad \delta = \log \gamma.$$

Another approach for deriving the number of edges of a bipartite lattice is to apply bipartite Ramsey theorems [4]. In particular, Zarankiewicz [11] proved an interesting relationship between the number of edges of a bipartite graph and the presence of $K_{a,a}$. If $k_a(n)$ denotes the minimal number of edges so that if G is a subgraph of $K_{n,n}$ and contains $e \geq k_a(n)$ edges then G contains a $K_{a,a}$ then the following theorem holds:

Theorem 2.4 *If* $n \binom{e/n}{a} > (a-1)\binom{n}{a}$ *then* $k_a(n) \leq e$

From the above theorem it follows:

Corollary 2.5 *Let* $H_{n,m}$ *be a bipartite graph , with* $e = |E(h)|$. *If* $H_{n,m}$ *contains no* $K_{2,2}$ *then* $e \leq \frac{2}{\sqrt{3}}(n+m)\sqrt{n+m}$

Unfortunately, we can not use the extension of the Ramsey theorem to k–partite graphs for tightly evaluating the complexity of general lattices, since, in this case the forbidden sub-structure is not the k–partite analogue of a $K_{2,2}$.

It also worth noting the following upper bound [5].

Theorem 2.6

$$L(n) < \beta^{n\sqrt{n}+o(n\sqrt{n})}, \qquad \textit{with} \qquad \beta \approx 6.11343.$$

Unfortunately, theorem 2.6 can not help for deriving an upper bound on the complexity of general finite lattices as it was possible for bipartite lattices (see theorem 2.3). In fact, the lattice property is not *monotone*, in the sense that a subset of a lattice can not be a lattice. Hence, deleting an edge from the Hasse Diagram the resulting partial order not always is a lattice as shown in figure 1.

Figure 1: Non-monotonicity.

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List of participants

Robert Babilon, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
babilon@kam.ms.mff.cuni.cz

Krystyna T. Balińska, Computer Science Center, The Technical University of Poznan, Pl. M. Sklodowskiej-Curie 5, 60-965 Poznan, Poland;
balinska@ruby.poz.edu.pl

Imre Bárány, Math. Institute, Hungarian Academy of Sciences, Reáltanoda u. 13–15, H 1364 Budapest, Hungary;
barany@math-inst.hu

Karel Čulík Department of Computer Science, University of South Carolina, Columbia, SC 29208, U.S.A.;
culik@cs.sc.edu

Christian Delhomme Universität Bielefeld, SFB 343, Postfach 100131, D-33501 Bielefeld, Germany;
delhomme@mathematik.uni-bielefeld.de

Jiří Fiala, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
fiala@kam.ms.mff.cuni.cz

Ivan Havel, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha, Czech Republic;
haveli@earn.cvut.cz

Petr Hliněný, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
hlineny@kam.ms.mff.cuni.cz

Tomáš Kaiser, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
kaiser@kam.ms.mff.cuni.cz

Yoav Kirsch, Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, Jerusalem, Israel;
kirsch@math.huji.ac.il

Martin Klazar, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
klazar@kam.ms.mff.cuni.cz

Jan Kratochvíl, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
honza@kam.ms.mff.cuni.cz

Martin Loebl, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
loebl@kam.ms.mff.cuni.cz

Jiří Matoušek, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
matousek@kam.ms.mff.cuni.cz

Peter Mihók, Mathematical Institute, Slovak Akademy of Sciences, Grešákova 6, 040 01 Košice, Slovak Republic;
mihok@duro.upjs.sk

Mirka Miller, University of Newcastle, Australia;
mirka@cs.newcastle.edu.au

Jaroslav Nešetřil, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
nesetril@kam.ms.mff.cuni.cz

Patrice Osona de Mendez, EHESS-CNRS UMR 17, 54 Bd Raspail, 75006 Paris, France;
pom@msh-paris.fr

Andrzej Proskurowski, University of Oregon, Eugene, U.S.A.;
andrzej@cs.uoregon.edu

Louis V. Quintas, Mathematics Department, Pace University, New York, NY 10038, U.S.A.;
quintas@pacevm.dac.pace.edu

Klaus Reuter, Mathematisches Seminar, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany;
reuter@math.uni-hamburg.de

Moshe Rosenfeld, Dept. of Computer Science Pacific Lutheran University & Dept. of Mathematics University of Washington, Seattle, WA. 98195, U.S.A.;
moishe@math.washington.edu

Miklós Ruszinkó, Computer and Automation Research Institute of the Hungarian Academy of Sciences, Budapest, P.O.B. 63, Hungary 1518;
ruszinko@lutra.sztaki.hu

Jiří Sgall, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
sgall@kam.ms.mff.cuni.cz

József Solymosi, Mathematical Institute of the Hungarian Academy of Sciences, Budapest, P.O.B. 127, Hungary 1364;
solymosi@cs.elte.hu

Petra Špičková, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
spickova@kam.ms.mff.cuni.cz

Claude Tardif, Universität Bielefeld, SFB 343, Postfach 100131, D-33501 Bielefeld, Germany;
tardif@mathematik.uni-bielefeld.de

Eberhard Triesch, Lehrstuhl II für Mathematik, RWTH Aachen, Templergraben 55, D-52062 Aachen, Germany triesch@math2.rwth-aachen.de

Pavel Valtr, Department of Applied Mathematics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic;
valtr@kam.ms.mff.cuni.cz

Paola Vocca, c/o Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, I-00173 Rome, Italy;
vocca@mat.utovrm.it