Properties of the Descartes' construction of triangle-free graphs with high chromatic number

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Abstract

We present some nice properties of the classical construction of triangle-free graphs with high chromatic number given by Blanche Descartes and its modifications. In particular, we construct colour-critical graphs and hypergraphs of high girth with moderate average degree.

1 Introduction

We cite the description of the famous construction by Blanche Descartes in [8]:

The graph G_3 is a circuit of just 7 edges. (Any larger odd number would do.) When G_i is defined, with m_i nodes say, we construct G_{i+1} as follows. We take $\binom{im_i-i+1}{m_i}$ disjoint copies of G_i. We adjoin $im_i - i + 1$ extra nodes, which we call "central". We set up a 1-1 correspondence between the copies of G_i and the sets of m_i central nodes. We join each copy of G_i to the members of the corresponding set of central nodes by m_i new edges of which no two have a common end. The resulting graph is G_{i+1}.

The key properties of this beautiful construction (observed in [8]) are the following.

Property 1 *For each $i \geq 3$, the chromatic number $\chi(G_i)$ of the graph G_i is at least i.*

Property 2 *For each $i \geq 3$, the girth $g(G_i)$ of the graph G_i is at least* 6.

The aim of this note is to list some other properties of the construction and its modifications. This is mostly done in the next section. We show that this classical construction gives several statements which were proved by an ad hoc argument and, in fact, its easy generalization yields a strengthening of these results (to graphs

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with arbitrary girth). In Section 3, we extend the construction to hypergraphs. In the last section, we use this construction to prove the existence, for each $k \geq 3$, of k-(chromatically)-critical graphs of arbitrary girth with average degree less than $2(k-2)$ and k-(chromatically)-critical r-uniform hypergraphs of arbitrary girth with average degree less than $2(k-2)+\epsilon$ for any $\epsilon > 0$. In [11] it is shown that any such graph must have the average degree at least $2k - o(k)$.

2 Degree properties

A graph is called *k-degenerate* if each its non-empty subgraph has a vertex of degree at most k. Clearly, the chromatic number of any k-degenerate graph is at most $k+1$. In a sense, any k-degenerate graph can be "efficiently" coloured with at most $k+1$ colours.

Property 3 *For each $i \geq 3$, the graph G_i is $(i-1)$-degenerate.*

PROOF. Trivially, the graph G_3 is 2-degenerate. Assume that the property holds for G_i, and consider an arbitrary subset W of $V(G_{i+1})$. If W contains only "central" vertices, then each $w \in W$ has degree 0 in $G_{i+1}(W)$. Otherwise, let H be a copy of G_i with $V(H) \cap W \neq \emptyset$. By the induction assumption, there is $w \in V(H) \cap W$ of degree in $H(W \cap V(H))$ at most $i-1$. Then the degree of w in $G_{i+1}(W)$ is at most i. □

Star forest is a forest with each component being a star. I. Křiž [12] constructed for each n and k a k-chromatic graph $G(n,k)$ of girth n such that its edge-set is the union of $k-1$ star forests.

Property 4 *For each $i \geq 3$, the graph G_i is the union of a set of vertex-disjoint cycles and $i-3$ star forests.*

PROOF. For $i=3$, the statement is trivial. Assume that G_i is the union of a set of vertex-disjoint cycles and $i-3$ star forests, and consider G_{i+1}. By the assumption, the subgraph of G_{i+1} induced by all copies of G_i also is the union of a set of vertex-disjoint cycles and $i-3$ star forests, and the edges incident to "central" vertices form one more star forest. □

Property 4 is stronger than the property of being the union of $k-1$ star forests. On the other hand, Křiž's construction gives graphs of arbitrary girth. To avoid this disadvantage, observe (as it was made in [10, 14]) the following. We can view the Descartes' construction of G_{i+1} as replacing each hyperedge in the complete m_i-uniform hypergraph on $im_i - i + 1$ vertices by a copy of G_i joined by a matching with vertices of this hyperedge. If, instead of the complete m_i-uniform hypergraph on $im_i - i + 1$ vertices, we take any m_i-uniform $(i+1)$-chromatic hypergraph then the chromatic number of the resulting G_{i+1} also can not be less than $i+1$ (for the same reasons). Moreover, the properties 3 and 4 hold as well (the proofs are the same). In addition, if we choose as G_3 an odd cycle of length at least g, and on every step the auxiliary m_i-uniform $(i+1)$-chromatic hypergraph also is of girth g (which exists by the classical result of P. Erdős and A. Hajnal [9]), then the girth of the resulting G_{i+1} also is at least g.‘

Remark. Actually, girth $\lceil g/3 \rceil$ of the auxiliary m_i-uniform $(i+1)$-chromatic hypergraph would be enough for G_{i+1} to have girth at least g.

For a (hyper)graph G, let *the density* $den(G)$ be the maximum of the ratio $\frac{|E(H)|}{|V(H)|}$ over all sub(hyper)graphs H of G.

Property 5 *For each $i \geq 3$,*

$$den(G_{i+1}) < 1 + den(G_i).$$

PROOF. Consider an arbitrary (induced) subgraph H of G_{i+1}. Let V' be the set of "central" vertices of G_{i+1} which belong to $V(H)$ and $V'' = V(H) \setminus V'$. Let $H' = H(V')$ and $H'' = H(V'')$. Clearly,

$$\frac{|E(H'')|}{|V(H'')|} \leq den(G_i).$$

By the construction, $|E(H)| \leq |V(H'')| + |E(H'')|$ and for the equality, we need $V' \neq \emptyset$. It follows that

$$\frac{|E(H)|}{|V(H)|} \leq \frac{|V(H'')|}{|V(H)|} + \frac{|E(H'')|}{|V(H'')|} < den(G_i).$$

This proves the proposition.

Since k-degenerate graphs can be coloured with $k+1$ colours so easily and have average degree less than $2k$, it might be expected that the vertex-set of any such graph (or any such graph with large girth) can be partitioned into k subsets each of which induces a subgraph with maximum degree at most 100. But this is not true. O. V. Borodin [4] gave for each integer d an example of a 2-degenerate graph $G(d)$ such that in each bipartition $\{V_1, V_2\}$ of $V(G(d))$, either V_1 or V_2 induces a subgraph with maximum degree at least d. The idea of Descartes' construction lets to sharpen this result as follows.

Property 6 *For each positive integers k, g, and d, there exists a k-degenerate graph $G(k, g, d)$ of girth at least g such that in each partition $\{V_1, \ldots, V_k\}$ of $V(G(k, g, d))$, at least one of V_i induces a subgraph with maximum degree at least d.*

PROOF. We fix positive integers g and d, and construct the graph $G_k = G(k, g, d)$ inductively. The graph G_1 is the star $K_{1,d}$. Suppose that G_k is constructed and has m_k vertices. By [9], there exists an m_k-uniform $(d \cdot m_k)$-chromatic hypergraph of girth g. Such a hypergraph contains a $(d \cdot m_k)$-critical subhypergraph $H(m_k)$ whose minimum degree must be at least $d \cdot m_k - 1$. Let $H(m_k)$ have s_k vertices. Again, by [9], there exists an s_k-uniform $(k+2)$-chromatic hypergraph F of girth g. Replacing each hyperedge in F by a copy of $H(m_k)$, we obtain an m_k-uniform hypergraph F' with the property that in each partition of $V(F')$ into $k+1$ subsets, at least one of them induces a subhypergraph containing a copy of $H(m_k)$. The vertices of F' will be "central" in G_{k+1}. Then we adjoin $|E(F')|$ disjoint copies of G_k to F' and set up a 1-1 correspondence between the copies of G_k and the hyperedges of F'. Finally, we join the vertices of every hyperedge in F' by a matching with the corresponding copy of G_k and delete the hyperedges. This is G_{k+1}.

Exactly as above, G_{k+1} is $(k+1)$-degenerate and has girth at least g. Consider an arbitrary partition $\{V_1, \ldots, V_{k+1}\}$ of $V(G_{k+1})$. By above, at least one of V_i induces on $V(F')$ a subhypergraph containing a copy of $H(m_k)$. Hence, we may assume that some $W \subseteq V_1 \cap V(F')$ with $|W| = s_k$ induces a copy of $H(m_k)$. By construction, W is joined in G_{k+1} by matchings with at least $d \cdot s_k$ copies of G_k. If each of these copies contains a vertex in V_1, then W is joined in G_{k+1} at least $d \cdot s_k$ vertices in V_1, and some vertex in W has degree at least d in the subgraph of G_{k+1} induced by V_1. Otherwise, some copy C of G_k is disjoint from V_1, and by the induction assumption, some of the sets $V_2 \cap V(C), \ldots, V_{k+1} \cap V(C)$ induces a subgraph with maximum degree at least d. $\square$

3 A hypergraph construction

A version of Descartes' construction for r-uniform hypergraphs can be stated as follows.

The hypergraph $G_2 = G_2(r, g)$ is a hyperedge on just r vertices. When $G_i = G_i(r, g)$ is defined, with m_i nodes say, we construct G_{i+1} as follows. Let $F = F(r, g, i)$ be an $(r-1)m_i$-uniform $(i+1)$-chromatic hypergraph of girth g (which exists by [9]) with, say, l_i hyperedges. We take l_i disjoint copies of G_i. We adjoin a ("central") copy of F. We set up a 1-1 correspondence between the copies of G_i and the hyperedges of F. For each copy C of G_i, we delete the corresponding hyperedge e_C of F and join vertices of C to the set e_C by m_i new edges of size r of which no two have a common end and each has exactly one vertex in common with C. The resulting hypergraph is G_{i+1}.

By the definition, $G_i(r, g)$ is an r-uniform hypergraph. Analogously to the above, the following properties hold for G_i.

Property 1′ *For each $i \geq 2$, the chromatic number $\chi(G_i)$ of the hypergraph G_i is at least i.*

Property 2′ *For each $i \geq 2$, $g \geq 3$, the girth $g(G_i(r,g))$ of the hypergraph $G_i(r,g)$ is at least g.*

Property 3′ *For each $i \geq 2$, the hypergraph G_i is $(i-1)$-degenerate.*

Define *a hypergraph star forest* to be a hypergraph whose every edge contains a vertex of degree 1. Then, similarly to Property 4, the following holds.

Property 4′ *For each $i \geq 2$, the hypergraph G_i is the union of a matching and $i-2$ star forests.*

Property 5′ *For each $i \geq 2$,*

$$den(G_{i+1}) < 1 + den(G_i).$$

M. I. Burstein [5], L. Lovász [13], P. D. Seymour [15] and D. R. Woodall [16] proved independently that the density of any 3-chromatic hypergraph is at least 1. M. I. Burstein [5] has constructed for every $r \geq 2$ an example of a 3-chromatic (in particular, 3-critical) r-uniform hypergraph with the density 1. It looks that the examples of 3-critical r-uniform hypergraphs of girth 4, 5 and 6 given by H. L. Abbott, D. R. Hare, and B. Zhou [2, 3] can have the density arbitrarily close to 1. Descartes' construction provides such examples for arbitrary girth.

Property 7 *For each positive integers $g \geq 3$, $r \geq 2$ and m, there exists a 3-chromatic r-uniform hypergraph $G_3(r,g,m)$ of girth at least g such that*

$$den(G_3(r,g,m)) < 1 + 1/m.$$

PROOF. We prove the property by induction on m. Since $den(G_2(r,g)) = 1/r$, we conclude from Property 5′ that $den(G_3(r,g)) \leq 1 + 1/r$ and so we can take $G_3(r,g,2) = G_3(r,g)$.

Now, let the property be true for all $m \leq m_0$ and any r and g. In order to construct $G_3(r,g,2m_0)$, we apply the Construction to the hyperedge $G_2 = G_2(r,g)$ with $F = G_3(r(r-1),g,m_0)$. We claim that the resulting hypergraph G possesses the required property.

Indeed, assume that $den(G) > 1 + 1/2m_0$. Let a subhypergraph G' of G be the smallest (with respect to the number of vertices) subhypergraph with $\frac{|E(G')|}{|V(G')|} > 1 + 1/2m_0$. Then it cannot have vertices of degree 0 and 1. It follows that together with any "non-central" vertex v of G, G' contains the "new" edge including v and all edges connecting this edge with central vertices. In other words, the edges of G' can be partitioned into groups of cardinality $r+1$ such that each group was created in the course of constructing G by replacing an edge of F by the edges of this group. Thus, if we have x such groups and G' contains y central vertices, then

$$\frac{|E(G')|}{|V(G')|} = \frac{x(r+1)}{rx+y} = \frac{r+1}{r+y/x}.$$

Since $den(F) \leq 1 + 1/m_0$, we have $x/y \leq 1 + 1/m_0$, and so

$$\frac{|E(G')|}{|V(G')|} \leq \frac{r+1}{r + \frac{m_0}{1+m_0}} = \frac{r+1}{r+1-\frac{1}{1+m_0}} = 1 + \frac{1/(1+m_0)}{r+1-\frac{1}{1+m_0}} < 1 + \frac{1}{rm_0}.$$

This contradicts the choice of G'.

4 Sparse colour-critical graphs and hypergraphs

In [11] it is shown that any k-critical hypergraph without graph edges on n vertices has at least $kn(1-3k^{-1/3})$ edges. Constructions by H. L. Abbott, D. R. Hare, and B. Zhou [1, 2, 3] show that this bound is asymptotically (in k) tight even for k-critical r-uniform hypergraphs of girth g for $g \leq 5$ and arbitrary r. Namely, they construct for every $\epsilon > 0$, $k \geq 3$ and r an example of a k-critical r-uniform hypergraph G of girth 5 with $|E(G)|/|V(G)| \leq k - 2 + \epsilon$. Descartes' construction allows us to extend the lower bound to any girth.

Theorem 1 *For each $\epsilon > 0$, $k \geq 3$, $g \geq 3$ and $r \geq 2$, there exists a k-critical r-uniform hypergraph $G = G(k, g, r, \epsilon)$ of girth g with $|E(G)|/|V(G)| \leq k - 2 + \epsilon$.*

PROOF. By Properties 7 and $5'$, there exists a k-chromatic r-uniform hypergraph $H = H(k, g, r, \epsilon)$ of girth g with $den(H) \leq k - 2 + \epsilon$. This H contains a k-critical subhypergraph $G = G(k, g, r, \epsilon)$. By the definition of density this G is what we need.

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