Disjoint and fold domination in graphs

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Abstract

We consider the problem of finding k disjoint dominating sets with a minimum size of their union, in a given network. We show that this problem can be solved in polynomial time for interval graphs and odd-sun-free graphs. We also relate this question to the so called k-fold domination in graphs.

1 Problem definition and motivation

We consider finite undirected graphs without loops or multiple edges. A set of vertices of a graph is called *dominating* if every vertex not in the set is adjacent (i.e., dominated) by at least one vertex from the set. Finding a dominating set of minimum cardinality is one of the basic optimization problems in computational graph theory. Many variants of this problem are studied in what is now commonly called domination theory in graphs [7, 8]. We propose to study the following problem

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k **Disjoint Domination** (abreviated k-**DD**)
Instance: A graph G and a number m.
Question: Does G contain k disjoint dominating sets such that their union has at most m vertices?

One can be also interested in a variant of this problem in which k is not a fixed parameter of the problem but rather a part of the input. As one can expect, k Disjoint Domination is NP-complete for very narrow classes of graphs, as the ordinary domination problem is.

We motivate the concept of k-fold domination by considering needs for private communication over public channels. If two parties wish to communicate in secret or confidence over an insecure communication channel they can do so using encryption. Typically the encryption algorithm will be DES or something similar, that is, an encryption algorithm whose security relies on a secret key and on the fact that without the key it is "difficult enough" to break the code. The key cannot be sent over an insecure channel in the clear form - it must either be sent enciphered (possibly using a public key encryption) or it must be sent via a secure channel (e.g., a personal courrier). Most computers do not have a direct line of communication between them; usually the message has to be relayed through a number of computer sites. To provide a link between any two nodes of a computer network, the network must contain one or more dominating sets of nodes. To provide some degree of fault tolerance, there are usually several such dominating sets, not necessarily disjoint. However, in case it becomes known that there has been a breach of security in the network, it would be advantageous to have the option of switching over to an alternative route. In the case when it becomes known that one or more nodes in a particular route has been compromised without knowing exactly which node(s) it is, any further secret communication should be done using a new route, totally disjoint from the compromised one. That is, in such a case we require that there be two or more disjoint dominating sets of nodes. It would then be possible to continue "business as usual" using the alternative channel while at the same time investigating where exactly the breach has occurred and whether it was the key or the encryption algorithm itself that has been exposed, possibly by sending false encrypted messages over the compromised route and analysing their effect. Such a procedure could be also carried out in the case of a mere suspicion of a security breach of a channel - or even as a routine preventive check.

Our paper is structured as follows. In section 3, we show that k Disjoint Domination is polynomially solvable for several classes of graphs for which also the ordinary domination is polynomially solvable. The classes for which we will develop efficient solution algorithms are the interval graphs, strongly chordal graphs and odd-sun-free graphs. Our arguments employ a new notion of unfolding of k-fold dominating sets: A set is k-fold dominating if every vertex of the graph is dominated by at least k vertices. We also use results of Berge [1] and Brouwer *et al.* [2] on balanced matrices. In Section 3 we investigate in detail the concept of unfolding of k-fold dominating sets. Section 4 is devoted to the complete discussion of the unfolding status of cycles, while computational complexity aspects of unfolding are considered in Section 5. We discuss a generalization of this concept in Section 6.

2 Fold and unfold – a two-step way to the solution

In this section we design a strategy for solving the k-DD problem for some special classes of graphs including interval graphs, strongly chordal graphs and odd-sun-free graphs. This strategy is the reverse of the well known "divide and conquer" method, but instead of "conquer and divide" we prefer the more descriptive "fold and unfold". The idea is that the union of the disjoint dominating sets we are looking for has to be a minimum k-fold dominating set in the given graph (see the definitions below). The task thus reduces to finding a minimum k-fold dominating set, provided we would be able to further partition such a set into k dominating sets (i.e., unfold it). It turns out that for most of the classes of graphs for which k-DD is known to be solvable in polynomial time, every k-fold dominating set indeed can be unfolded in polynomial time. In practice, this two step approach leads to a much more transparent algorithm than any direct algorithm known so far.

In the sequel, we use the following standard notations. The closed neighborhood of a vertex v is denoted by $N[v]$, i.e., $N[v] = \{u : uv \in E(G)\} \cup \{v\}$. The neighborhood matrix of a graph G is denoted by $N(G)$ (i.e., $N(G)$ has rows and columns indexed by the vertices of G and $N(G)_{uv} = 1$ iff $u = v$ or $uv \in E(G)$).

A 0-1 matrix is called *balanced* if it does not contain an edge-vertex incidence matrix of an odd cycle as a submatrix. Vectors are considered columnwise, $\mathbf{1}$ stands for the all-one vector. The set of nonnegative integers is denoted by Z_0^+.

The domatic number of a graph G, $dom(G)$, is the largest number of disjoint dominating sets that G can be partitioned into. A graph G is called *domatically full* when $dom(G) = \delta(G)+1$ (where $\delta(G)$ is the minimum degree of a vertex in G). The maximum degree of a vertex in G is denoted by $\Delta(G)$.

Definition 2.1 *A set C of vertices of a graph G is called* k-fold dominating *if every vertex in G is adjacent to at least k vertices in C (i.e., every vertex in C has at least $k-1$ neighbors in C and every vertex not in C has at least k neighbors in C).*

Obviously, the union of k disjoint dominating sets in a graph is a k-fold dominating set, but the converse is not always true. For example, the vertices of a 4–cycle form a 3-fold dominating set, but a 4–cycle cannot be split into 3 disjoint dominating sets. Therefore we introduce the following notions:

Definition 2.2 *We call a graph k-*unfolding *if every k-fold dominating set can be partitioned into k sets, each of them dominating the original graph. We say that a graph is* unfolding *if it is k-unfolding for every k.*

The dominating number of a graph G is usually denoted by $\gamma(G)$. In this sense, let us denote by $\gamma_D(G,k)$ ($\gamma_F(G,k)$, resp.) the minimum size of the union of k disjoint dominating sets in G (the minimum size of a k-fold dominating set in G, respectively). If G does not contain k disjoint dominating sets or if it does not contain a k-fold dominating set, we define $\gamma_D(G,k) = \infty$ (resp. $\gamma_F(G,k) = \infty$). The following facts follow directly from the definitions:

Observation 2.3 *For any graph G and every positive integer k,*

$$\gamma_F(G,k) \leq \gamma_D(G,k)$$

and $\gamma_F(G,k) = \gamma_D(G,k)$ provided G is unfolding.

It is then clear that the following two problems are of particular interest.

k-fold Domination (abreviated **k-FD**)
Instance: A graph G and an integer m.
Question: Does G contain a k-fold dominating set of size at most m?

Proposition 2.4 *For every $k \geq 1$, the problems k-DD and k-FD are NP-complete even when restricted to chordal graphs.*

Proof: It is well known that the dominating set problem (i.e., 1-DD) is NP-complete for chordal graphs. For a given chordal graph G, consider $G' = G + K_{k-1}$ as the graph obtained from G by adding $k-1$ mutually adjacent extra vertices adjacent to all vertices of G. Obviously, $\gamma_D(G', k) = \gamma_F(G', k) = \gamma(G) + k - 1$ and the statement follows (membership in NP being straightforward). $\square$

k**-unfold**
Instance: A graph G and a k-fold dominating set C in G.
Question: Can C be partitioned into k dominating sets? If yes, find a partition.

We will prove in Section 3 that 2-**unfold** is also NP-hard even when restricted to chordal graphs. In this section, we will relate the unfolding property to domatic fullness of graphs and the concept of balanced matrices.

Observation 2.5 *Unfolding implies domatically full.*

Proof: The entire vertex set of a graph G is itself a $(\delta(G) + 1)$-fold dominating set. $\square$

Observation 2.6 *Domatically full does not imply unfolding.*

Proof: The 3-sun (cf. Figure 2) is domatically full but not unfolding. Indeed, $\{1, 1'\}, \{2, 2'\}, \{3, 3'\}$ are 3 disjoint dominating sets (the minimum degree of the 3-sun is 2), but the 2-fold dominating set $\{1, 2, 3\}$ does not unfold into 2 disjoint dominating sets. $\square$

Observation 2.7 *A graph G is k-unfolding if and only if for every 0-1 vector x,*

$$N(G)x \geq k\mathbf{1}$$

implies the existence of k 0-1 vectors $x_1, x_2, \ldots, x_k$ such that

$$x = \sum_{i=1}^{k} x_i \text{ and } N(G)x_i \geq \mathbf{1}, i = 1, 2, \ldots, k.$$

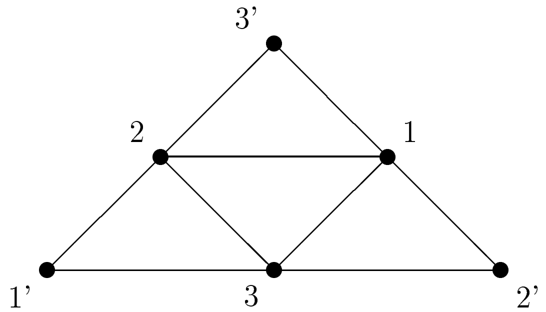

Figure 1: The 3-sun.

In other words, G is unfolding iff for every choice of a subset C of columns of the neighborhood matrix $N(G)$, one can color the columns of C using k colors (where k is the minimum row sum in $N(G)[C]$) so that every row of $N(G)[C]$ has at least one non-zero entry of every color. Berge [1] proved that the columns of a balanced matrix can be colored by k colors, where k is the minimum row sum, in such a way that for every color c, every row contains a 1 in a column of color c. As noted in [3], his method can be easily extended to show that graphs with balanced neighborhood matrices are unfolding:

Proposition 2.8 *If $N(G)$ is balanced then G is unfolding and any k-fold dominating set can be unfolded in polynomial time.*

It also follows from Berge's theory of balanced matrices that k-FD is polynomially solvable on graphs with balanced neighborhood matrices. We may thus conclude:

Corollary 2.9 *The problem k-DD is polynomially solvable on graphs with balanced neighborhood matrices.*

Chordal graphs (i.e., graphs without induced cycles of length greater than 3) form an important class of graphs, however, it is known that the domination problem itself is NP-complete for these graphs [6]. Further well studied subclasses of chordal graphs are interval graphs (intersection graphs of intervals on a line) and the so called strongly chordal graphs (graphs with no induced trampoline subgraph, cf. [5]). Interval and strongly chordal graphs have balanced neighborhood matrices [5], and according to Brouwer *et al.* [2], a chordal graph has balanced neighborhood matrix if and only if it does not contain odd suns as induced subgraphs. Therefore we have:

Theorem 2.10 *The k-DD problem can be solved in polynomial time for odd-sun-free chordal graphs (and therefore also for strongly chordal and interval graphs).*

The algorithm for finding minimum-size k-fold dominating set in a graph with balanced neighborhood matrix is algebraical (i.e., using the linear programming method). Therefore we find it useful to include a combinatorial algorithm for the narrower class of strongly chordal graphs (which includes interval graphs as well). Recall that the algorithm for the k-unfold problem presented in [3] (called tranversal partitioning in there) is combinatorial.

One of the characterizations of strongly chordal graphs states that these graphs allow *strong elimination schemes*, i.e., linear ordering of the vertices $v_1, v_2, \ldots, v_n$ so that for every $h < i < j$ such that $v_h v_i, v_h v_j \in E(G)$, $v_i v_j \in E(G)$ and $N[v_i] \cap \{v_x : x \geq h\} \subseteq N[v_j] \cap \{v_x : x \geq h\}$ [5].

Algorithm k-fold domination
Input: A graph $G = (V, E)$ and its strong elimination scheme $v_1, \ldots, v_n$.
Constant: k.
Variable: X (the constructed k-fold dominating set).
if $\delta(G) < k - 1$ **then ouput**('No k-fold dominating set') and **halt**;
for $i := 1$ **to** n **do**
  begin
  $m := |N[v_i] \cap X|$;
  if $m < k$ **then**
    begin
    let X_i be the set containing the $k - m$ elements
      of $N[v_i] - X$ with the largest indices;
    $X := X \cup X_i$
    endif
  enddo;
output(X).

Proof of correctness of the Algorithm. If $\delta(G) < k - 1$ then G obviously has no k-fold dominating set. If the minimum degree is at least $k-1$, the entire vertex set is a k-fold dominating set. We will prove that in this case, the outcome of the algorithm is a minimum k-fold dominating set. Let A be a minimum k-fold dominating set such that the sum of the indices of its vertices is as large as possible (i.e., A maximizes the sum $\sum_{v_i \in A} i$). We will prove by induction on i that $X_i \subset A$ for every i. It will follow that $X = \bigcup_{i=1}^{n} X_i \subseteq A$ and hence $X = A$ due to the minimality of A.

Trivially, $i = 0$, $X_i = \emptyset \subset A$.

Consider $i \geq 1$ and suppose $X_i - A \neq \emptyset$. Take an $x = v_a \in X_i - A$. Since v_i forced $x \in X_i$, v_i is not adjacent to enough (i.e., k) vertices of $A \cap \bigcup_{j \leq i} X_j$. Hence there is $y = v_b \in A - (\bigcup_{j \leq i} X_j)$ such that $yv_i \in E$. Since X_i took the neighbors of v_i with the largest possible indices, we conclude that $b < a$. We claim that $A' = (A - \{y\}) \cup \{x\}$ is also a k-fold dominating set. By the induction hypothesis, every v_j with $j < i$ is adjacent to k vertices in $A \cap \bigcup_{j<i} X_j$. Since $b < a$, $N[y] \cap \{v_j : j \geq i\} \subseteq N[x] \cap \{v_j : j \geq i\}$ and x dominates all vertices $v_j, j \geq i$ that are dominated by y. However, $|A'| = |A|$ and $\sum_{v_j \in A'} j > \sum_{v_j \in A} j$, contradicting the choice of A. $\square$

3 More about unfolding

In this section, we pay closer attention to the question of characterization of unfolding graphs. It will be shown in Section 5 that this problem is NP-hard and therefore a nice characterization leading to a polynomial recognition algorithm is most probably (unless P=NP) hopeless to look for. With the hope that narrower classes of graphs might actually be easier to recognize, we define two stronger notions, hereditary unfoldingness and strong unfoldingness. These are natural and theoretically interesting generalizations of the concept of unfolding graphs. Recall that a graph property is called *hereditary* if it is closed under taking induced subgraphs.

Observation 3.1 *Being unfolding is not a hereditary property.*

Proof: Consider the 3-dimensional cube with vertices $1, 2, 3, 4, 1', 2', 3', 4'$ and edges $12, 23, 34, 41, 1'2', 2'3', 3'4', 4'1', 13', 24', 31', 42'$. This cube is unfoldin:g Every pair $\{1, 1'\}, \{2, 2'\}, \{3, 3'\}, \{4, 4'\}$ dominates every vertex exactly once (i.e., each of these pairs is a perfect code in the cube). Therefore if a k-fold dominating set C contains such a pair of antipodal vertices, that pair can be separated from C as a dominating set, and the remaining C' is a $(k-1)$-fold dominating set. Thus we may restrict our attention only to sets C that do not contain pairs of antipodal vertices. It is, however, easy to see that such a C cannot be 2-fold dominating.

On the other hand, the cycle of length 4 is an induced subgraph of the cube and we have observed before that C_4 is not unfolding. $\square$

This leads us to the following definition.

Definition 3.2 *A graph is* hereditary unfolding *if every induced subgraph of G is unfolding.*

We observe that, as a hereditary class, the class of all hereditary unfolding graphs is already hereditary, and as such it can be described by forbidden induced subgraphs (namely by minimal not hereditary unfolding graphs). It is not clear whether the number of these forbidden subgraphs is finite or infinite. We include this problem among open problems at the end of this section.

Observation 2.7 leads to the following generalization of the concept of unfolding sets.

Definition 3.3 *A graph G is* strong unfolding *if for every k and every vector $x \in (Z_0^+)^n$,*

$$N(G)x \geq k\mathbf{1}$$

implies the existence of k 0-1 vectors $w_1, w_2, \ldots, w_k$ such that

$$x \geq \sum_{i=1}^{k} w_i \text{ and } N(G)w_i \geq \mathbf{1}, i = 1, 2, \ldots, k.$$

Obviously, strong unfoldingness implies unfoldingness. Every complete graph is strong unfolding, and as an example of a nontrivial strong unfolding graph we mention the 3-dimensional cube from the proof of Observation 3.1 (the proof of strong unfoldingness is left to the reader). The following problems might seem too daring to ask, but we do not know any counterexample that would deny an affirmative answer:

Problem 1. Does unfolding imply strong unfolding?

Similarly to the hereditary unfolding graphs, we define a graph to be *hereditary strong unfolding* if each of its induced subgraphs is strong unfolding. Then of course hereditary strong unfoldingness implies hereditary unfoldingness, and if the answer to Problem 1 is negative, we can still ask

Problem 1'. Does hereditary unfolding imply hereditary strong unfolding?

We should note that these questions apply to graphs with induced cycles, since for chordal graphs the questions are settled by the result of Brouwer *et al.* [2] that the neighborhood matrix of a chordal graph is balanced if and only if the graph is odd-sun-free. It implies the following result.

Theorem 3.4 *If G is chordal, then G is hereditary strong unfolding iff it is hereditary unfolding iff $N(G)$ is balanced iff G is odd-sun-free. In particular, interval graphs and strongly chordal graphs are hereditary strong unfolding.*

We will continue the investigation of properties related to unfolding of dominating sets.

Observation 3.5 *Hereditary strong unfolding does not imply that the neighborhood matrix is balanced.*

Proof: The cycle C_6 is hereditary strong unfolding, but its neighborhood matrix is not balanced. □

The proof of the following theorem is presented in Section 4.

Theorem 3.6 *The only cycles that are unfolding are C_3, C_6 and C_9. These particular cycles are hereditary strong unfolding.*

We conclude this section by listing some open problems:

Problem 2. Characterize the classes of unfolding (strong unfolding, hereditary unfolding, hereditary strong unfolding) graphs.

Problem 3. Is there a nice description for the class of forbidden induced subgraphs for hereditary unfolding (hereditary strong unfolding) graphs?

Problem 4. How difficult is it to recognize unfolding (strong unfolding, hereditary unfolding, hereditary strong unfolding) graphs?

Note that a reasonable answer to Problem 3 would place recognition of hereditary unfolding and hereditary strong unfolding graphs in co-NP. At this point, membership in either NP or co-NP is unclear.

4 Unfolding cycles

Theorem 4.1 *The cycle* $C_n, n \geq 3$ *is*

a) 3-unfolding *if and only if* n *is divisible by 3;*

b) 2-unfolding *if and only if* $n = 3, 5, 6, 8, 9, 11, 14$ *or* 17.

c) unfolding *if and only if* $n = 3, 6$ *or* 9*;*

Proof: Let the vertex set of C_n be denoted $V = \{v_1, v_2, \ldots, v_n\}$ in this order. Every dominating set C in C_n has at least $\lceil \frac{n}{3} \rceil$ vertices. Since $C = V$ is a 3-fold dominating set, C_n is not 3-unfolding, unless n is divisible by 3. On the other hand, $C = V$ is the only 3-fold dominating set, and it can be partitioned into 3 disjoint dominating sets $C_i = \{v_k | k \bmod 3 = i\} 1 \leq k \leq n, 0 \leq i \leq 2$. This proves a).

Since a cycle is unfolding if and only if it is 2-unfolding and 3-unfolding, c) will follow from a) and b). The proof of b) is provided by the following lemmas:

Lemma 1. For $k \geq 1$, C_{3k+1} is not 2-unfolding.
Lemma 2. For $k \geq 4$, C_{3k} is not 2-unfolding.
Lemma 3. For $k \geq 7$, C_{3k-1} is not 2-unfolding.
Lemma 4. $C_3, C_5, C_6, C_8, C_9, C_{11}, C_{14}$ and C_{17} are 2-unfolding.

Proof of Lemma 1. Set $C = \{v_{3i+1}, v_{3i+2} | i = 0, 1, \ldots, k-1\} \cup \{v_{3k}\}$. This is a 2-fold dominating set of cardinality $2k + 1$. Since every dominating set contains at least $k+1$ vertices, C cannot be split into two disjoint dominating sets.

Proof of Lemma 2. Set $C = \{v_2, v_3, v_4, v_6, v_7, v_8, v_{10}, v_{11}, v_{12}\} \cup \{v_{3i+2}, v_{3i+3} | i = 4, 5, \ldots, k-1\}$. Clearly, C is a 2-fold dominating set. Suppose $C = A \cup B$, where A and B are disjoint dominating sets. Suppose, without loss of generality, that $v_2 \in A$. Since $v_1 \notin C$, v_2 can only be dominated by B if $v_3 \in B$. Similarly, as $v_5 \notin C$, v_4 can only be dominated by A if $v_4 \in A$. Similar reasoning yields $\{v_7, v_{10}, v_{12}\} \subset A$, $\{v_6, v_8, v_{11}\} \subset B$ and then $v_{3i+2} \in B$, $v_{3i+3} \in A$, $i = 4, 5, \ldots, k-1$. Since $v_2, v_{3k} \in A$, v_1 is not dominated by B.

Proof of Lemma 3. Set $C = \{v_2, v_3, v_4, v_6, v_7, v_8, v_{10}, v_{11}, v_{12}, v_{14}, v_{15}, v_{16}, v_{18}, v_{19}, v_{20}\} \cup \{v_{3i-2}, v_{3i-1} | i = 8, 9, \ldots, k\}$. This is a 2-fold dominating set. If

$C = A \cup B$, where A and B are disjoint dominating sets and $v_2 \in A$, a reasoning similar to above yields $A = \{v_2, v_4, v_7, v_{10}, v_{12}, v_{15}, v_{18}, v_{20}\} \cup \{v_{3i-1} | i = 8, 9, \ldots, k\}$ and the vertex v_1 is not dominated by B.

Proof of Lemma 4. Call a set $C \subset V(C_n)$ *critical 2-fold dominating* if it is 2-fold dominating but no proper subset of it is. Obviously, if C_n is not 2-unfolding then it contains a critical 2-fold dominating set that does not unfold into two disjoint dominating sets. Without loss of generality we assume in the sequel that C is a critical 2-fold dominating set and $v_1 \notin C$. We call a set of indices $\{i, i+1, \ldots, i+l\}$ an *interval* of C if $v_{i-1} \notin C$, $v_j \in C$ for $j = i, i+1, \ldots, i+l$ and $i+l = n$ or $v_{i+l+1} \notin C$. In such a case the length of the interval is $l+1$.

Claim 1. A 2-fold dominating set C is critical if and only if every interval of C has length 2, 3 or 4.

Proof: Intervals in a 2-fold dominating set must have length greater than 1. If some interval had length greater than 4, deleting one of its middle vertices would result in a 2-fold dominating set that would be a proper subset of C.

On the other hand, no proper subset is 2-fold dominating when all intervals have lengths smaller than 5.

Let us proceed with the proof of Lemma 4. Suppose that its statement is not true and let n be the minimum of $\{3, 5, 6, 8, 9, 11, 14, 17\}$ such that C_n is not 2-unfolding. Let then C be a critical 2-fold dominating set C_n that does not unfold (still keeping the assumption $v_1 \notin C$).

Claim 2. a) C does not contain an interval of length 2.
b) C does not contain two consecutive intervals of length 3.
c) C may contain an interval of length 4 only if $n \in \{9, 17\}$.

Proof: a) Suppose C contains an interval of length 2. Without loss of generality we may suppose that this is the last interval, i.e., $v_{n-2} \notin C, v_{n-1}, v_n \in C$. If $n = 3$ then obviously C does unfold. Note that $n \neq 5$, since the only critical 2-fold dominating set in C_5 consists of one interval of length 4. If $n > 5$, then $C' = C - \{v_{n-1}, v_n\}$ is a 2-fold dominating set in C_{n-3}. Since $n - 3 \in \{3, 5, 6, 8, 9, 11, 14\}$, C_{n-3} is 2-unfolding due to the choice of n. That means that $C' = A' \cup B'$, where A', B' are disjoint dominating sets in C_{n-3}, say $v_2 \in A'$ and $v_{n-3} \in B'$. Then $A = A' \cup \{v_{n-1}\}$ and $B = B' \cup \{v_n\}$ are disjoint dominating sets in C_n and C does unfold.

b) For C to contain two intervals of length 3, n has to be greater than or equal to 8 and $n \neq 9$. In the case of $n = 8$, C unfolds into $A = \{v_2, v_4, v_7\}$ and $B = \{v_3, v_6, v_8\}$. If $n \in \{11, 14, 17\}$, $n - 8 \in \{3, 6, 9\}$ and we proceed as in the proof of case a).

c) If C contains an interval of length 4, n has to be greater than 4 and $n \neq 6$. In the case of $n = 5$, $C = \{v_2, v_3, v_4, v_5\}$ unfolds into $A = \{v_2, v_4\}$ and $\{v_3, v_5\}$. If $n \in \{8, 11, 14\}$, $n - 5 \in \{3, 6, 9\}$ and the proof proceeds as in case a).

Let C have x intervals of length 3 and y intervals of length 4. Then $4x + 5y = n$, which leaves the following possibilities:
(i) $y = 0, x = 2, n = 8$,
(ii) $y = 1, x = 1, n = 9$,
(iii) $y = 1, x = 3, n = 17$,
(iv) $y = 2, x = 1, n = 14$.
Case (iv) is ruled out by Claim 2c), cases (i) and (iii) are ruled out by Claim 2b). In the last case, (ii), C is (up to a cyclic permutation) unique and unfolds easily: $C = \{v_2, v_3, v_4, v_6, v_7, v_8, v_9\}, A = \{v_2, v_4, v_7, v_8\}, B = \{v_3, v_6, v_9\}$. $\square$

Theorem 4.2 *The cycles C_3, C_6 and C_9 are strong unfolding.*

Proof: The cycle C_3 can be viewed as the complete graph on 3 vertices, and as such it is strong unfolding.

Consider C_6 with vertices $v_1, v_2, \ldots, v_6$ and edges $v_1v_2, v_2v_3, \ldots, v_6v_1$. Let $x = (x_1, x_2, \ldots, x_6)$ be a nonnegative integer vector, x_i corresponding to v_i. The maximum k such that $N(G)x \geq k\mathbf{1}$ is $k = \min_{i=1}^{6}\{x_{i-1} + x_i + x_{i+1}\}$ (addition in subscripts is modulo 6). We will show how to find k 0-1 vectors $w_i, i = 1, 2, \ldots, k$, so that $x \geq \sum_{i=1}^{k} w_k$ and each w_i is the characteristic vector of a dominating set in C_6.

Let $m_i = \min\{x_i, x_{i+3}\}$ for $i = 1, 2, 3$. We take m_1 vectors $(1, 0, 0, 1, 0, 0)$, m_2 vectors $(0, 1, 0, 0, 1, 0)$ and m_3 vectors $(0, 0, 1, 0, 0, 1)$. Each of these vectors corresponds to a perfect code in C_6. If we set $y = x-(m_1, m_2, m_3, m_1, m_2, m_3)$, y satisfies $N(G)y \geq (k - m_1 - m_2 - m_3)\mathbf{1}$ and it is enough to show that $y \geq \sum_{i=1}^{k-m_1-m_2-m_3} w_i$ for suitable w_i. Now y has at least 3 zeros, and in fact $y_i > 0$ implies $y_{i+3} = 0$ for $i = 1, 2, \ldots, 6$. It follows that either y contains 3 consecutive zeros, e.g., $y_1 = y_2 = y_3 = 0$, or every other coordinate of y is zero, e.g., $y_1 = y_3 = y_5 = 0$. In the former case $k = m_1 + m_2 + m_3$ and we are done, while in the latter case we set $k' = \min_{i\in\{2,4,6\}}\{y_2, y_4, y_6\}$ and add k' vectors $(0, 1, 0, 1, 0, 1)$ (each of them corresponding to a dominating set $\{v_2, v_4, v_6\}$).

The case of C_9 is analogous. We first reduce k and x by subtracting m_1 vectors $(1, 0, 0, 1, 0, 0, 1, 0, 0)$, m_2 vectors $(0, 1, 0, 0, 1, 0, 0, 1, 0)$ and m_3 vectors $(0, 0, 1, 0, 0, 1, 0, 0, 1)$. Thus we may assume that y is such that $N(G)y \geq k'\mathbf{1}$ and $\min\{y_i, y_{i+3}, y_{i+6}\} = 0$ for $i = 1, 2, 3$ (now addition in subscripts is modulo 9 and $m_i = \min\{x_i, x_{i+3}, x_{i+6}\}$ for $i = 1, 2, 3$). Then y has at least 3 zero coordintes, and one can show that either
(α) two of them are consecutive and the third one is distance 4 from these two (e.g., $y_1 = y_2 = y_6 = 0$), or
(β) two of them are distance 4 apart and the third one is distance 2 from both of them (e.g., $y_1 = y_3 = y_5 = 0$).

In case (α), $k' = \min\{y_3, y_4 + y_5, y_5 + y_7, y_7 + y_8, y_9\}$ and we set $m_1 = \min\{y_4, y_7\}, m_2 = \min\{y_5, y_8\}, m_3 = \min\{y_3, y_9\}$. We may assume that $m_2 \geq m_1$.

• If $m_1 \geq m_3$, we take $m_3 = k'$ vectors $(0, 0, 1, 1, 0, 0, 1, 0, 1)$ and we are done.
•• If $m_1 < m_3$ and $m_1 + m_2 \geq m_3$, we take m_1 vectors $(0, 0, 1, 1, 0, 0, 1, 0, 1)$, and $m_3 - m_1$ vectors $(0, 0, 1, 0, 1, 0, 0, 1, 1)$ (note that again $k' = m_3$ in this case).
••• If $m_1 < m_3$ and $m_1 + m_2 < m_3$, we take m_1 vectors $(0, 0, 1, 1, 0, 0, 1, 0, 1)$ and m_2 vectors $(0, 0, 1, 0, 1, 0, 0, 1, 1)$ and we are done, provided $k' = m_1 + m_2$, otherwise
••• if $k' > m_1 + m_2$, we add $k' - (m_1 + m_2)$ vectors $(0, 0, 1, 0, 1, 0, 1, 0, 1)$ and we are done (note that in this case, $m_1 = y_4, m_2 = y_8$ and both $y_5 - m_2 \geq k' - y_4 - m_2 = k' - (m_1 + m_2)$ and $y_7 - m_1 \geq k' - y_8 - m_1 = k' - (m_1 + m_2)$).

In case (β), $k' = \min\{y_2, y_4, y_6 + y_7, y_8 + y_9\}$. We set $m_1 = \min\{y_6, y_8\}$, $m_2 = \min\{y_7, y_9\}$, $m_3 = \min\{y_2, y_4\}$. Again we suppose $m_2 \geq m_1$.

• If $m_1 \geq m_3$, we take $m_3 = k'$ vectors $(0, 1, 0, 1, 0, 1, 0, 1, 0)$ and we are done.
•• If $m_1 < m_3$ and $m_1 + m_2 \geq m_3$, we take m_1 vectors $(0, 1, 0, 1, 0, 1, 0, 1, 0)$ and $m_3 - m_1$ vectors $(0, 1, 0, 1, 0, 0, 1, 0, 1)$ and we are done as $k' = m_3$ in this subcase.
••• If $m_1 + m_2 < m_3$, we take m_1 vectors $(0, 1, 0, 1, 0, 1, 0, 1, 0)$ and m_2 vectors $(0, 1, 0, 1, 0, 0, 1, 0, 1)$ and we are done, provided $k' = m_1 + m_2$, otherwise
••• (if $m_1 + m_2 < k'$), we add $k' - (m_1 + m_2)$ vectors $(0, 1, 0, 1, 0, 1, 0, 0, 1)$ (when $m_1 = y_8$ and $m_2 = y_7$), or we add $k' - (m_1 + m_2)$ vectors $(0, 1, 0, 1, 0, 0, 1, 1, 0)$

(when $m_1 = y_6$ and $m_2 = y_9$). $\square$

5 Complexity

In this section we present the NP-completeness proof. Apart from the decision problems defined above, we consider the following one:

Unfolding
Input: A graph G.
Question: Is G unfolding?

The problem k-unfold as defined in Section 2 is a search problem and so it makes sense to ask for its complexity under a promise that a given k-fold dominating set does allow a partition into k dominating sets. Being of this flavor, our next result is slightly stronger:

Theorem 5.1 *The problem* 2-**unfold** *is NP-hard for chordal graphs, even when the input graph is promised to be unfolding.*

Proof: We reduce the problem of bicolorability of 3-uniform hypergraphs to our problem. If $H = (V, E)$ is a 3-uniform hypergraph (i.e., every edge $e \in E$ is a 3-element subset of V), we construct a graph G with vertex set $V \cup V' \cup E$ where $V' = \{v' : v \in V\}$, and edge set $\{vv' : v \in V\} \cup \{ve : v \in e \in E\} \cup \{uv : u, v \in V\}$. Then $C = V$ is a 2-fold dominating set and it can be split into 2 dominating sets if and only if the vertices of H can be colored black and white so that every edge of H contains at least one black and at least one white vertex.

Note also, that C is the only inclusion-wise minimal 2-fold dominating set in G, and since G contains vertices of degree 1, G is unfolding if and only if C can be unfolded. Thus the existence of an unfolding algorithm that would only work properly under the promise of unfoldingness of the input graph, could be used to decide bicolorability of H. $\square$

As the graph constructed in the previous proof had only one minimal 2-fold dominating set (and no 3-fold dominating set), it follows that it is unfolding if and only if this 2-fold dominating set unfolds. Therefore we have:

Corollary 5.2 *The problem* **Unfolding** *is NP-hard for chordal graphs.*

It is also plausible to ask

Problem 5. Does k-unfold remain NP-hard when the input graph is promised to be hereditary unfolding (or hereditary strong unfolding)?

6 Generalized unfolding

In the proof of Theorem 5.1, the vertices of degree one were introduced only to guarantee that G has only one minimal 2-fold dominating set and no k-fold dominating set for $k > 2$. However, if we perform the same reduction without these additional vertices of degree one (i.e., G would be the bipartite incidence graph of H plus the edges of the clique on V), the set V would be a 3-fold dominating set which would still allow a partition into 2 disjoint dominating sets if and only if the hypergraph H were bicolorable. Thus we may conclude:

Theorem 6.1 *Given a 3-fold dominating set in a graph, it is NP-complete to decide if this set can be partitioned into 2 dominating sets.*

One may ask if this completeness result could be pushed further or whether there is a bound, say k, such that in any graph any k-fold dominating set can be partitioned into 2 dominating sets. Though this is not true in general, it is maybe slightly surprising that such a theorem holds true if we assume that the maximum degree of the graph is bounded:

Theorem 6.2 *For every m there is a $k = k(m)$ such that in every graph G of maximum degree $\Delta(G) < \frac{1}{2\sqrt{m}}(1 + \frac{1}{m-1})^{k/2}$, every k-fold dominating set can be partitioned into m dominating sets.*

Proof: We will prove the statement using the well known Lovász Local Lemma [4]. Given a graph G and a k-fold dominating set S, we color its vertices randomly using m colors. This coloring is a partitioning of S into m dominating sets if every vertex of the graph is adjacent to at least one vertex of each color. Let B_v be the event that vertex v does not have all m colors in its closed neighborhood. If $d = |S \cap N[v]|$, we have m^d possible colorings of the closed S-neighborhood of v, and out of them at most $m(m-1)^d$ do not contain all m colors. Since $d \geq k$, the probability of B_v is

$$Prob(B_v) \leq \frac{m(m-1)^d}{m^d} \leq m(\tfrac{m-1}{m})^k.$$

For every vertex v, B_v is totally independent of B_w for all w of distance at least 3 from v, so the second power of G serves as a dependence graph. The degree in the dependence graph is at most $\Delta + \Delta(\Delta - 1) = \Delta^2$ for every vertex B_v. It follows that

$$Prob(B_v) \cdot \Delta^2 < \frac{1}{4},$$

and by the Lovász Local Lemma,

$$Prob(\bigwedge_{v \in V(G)} \overline{B_v}) > 0,$$

which means that there is a particular coloring of S such that none of B_v occurs, i.e., a coloring which is a partitioning of S into m dominating sets. $\square$

In reverse, the preceding theorem says that every k-fold dominating set in a given graph can be partitioned into m disjoint dominating sets, provided $k > \frac{\log(4m\Delta^2)}{\log m - \log(m-1)}$. Of course, a graph of maximum degree Δ can only have k-folding dominating sets for $k \leq \Delta + 1$, so the theorem gives nontrivial results only for Δ large enough with respect to m. In particular, for $m = 2$ (i.e., when we are interested in partitioning into two disjoint dominating sets), our result says that every k-fold dominating set can be partitioned if the maximum degree is at most $2^{(k-3)/2}$. Let us mention that exponentional bound is the best possible we may have hoped for, as shown by the following example:

Observation 6.3 *For every k there is a graph of maximum degree $\binom{2k-1}{k} + 2k-2$ and a k-fold dominating set that cannot be decomposed into two disjoint dominating sets.*

Proof: Take a set X of $2k-1$ pair-wise adjacent vertices and for every k-element subset $A \subset X$, add a vertex v_A adjacent to the elements of A. In this graph, X is a k-fold dominating set, but being partitioned into two parts $X = X_1 \cup X_2$, one of them, say X_1 has at most $k-1$ vertices and therefore does not dominate the vertices v_A for $A \subset X_2$. $\square$

Let us mention that from bicolorability of k-uniform hypergraphs, one can prove that deciding if a k-fold dominating set can be partitioned into 2 disjoint dominating sets is an NP-complete problem for every $k \geq 2$.

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