Complexity of colored graph covers II. When 2-SAT helps.

Jan Kratochvíl*
Charles University
Prague, Czech Republic

Jan Arne Telle†
University of Bergen
Bergen, Norway

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Abstract

A covering projection from a graph G onto a graph H is a "local isomorphism": a mapping from the vertex set of G onto the vertex set of H such that, for every $v \in V(G)$, the neighborhood of v is mapped bijectively onto the neighborhood (in H) of the image of v. We continue the investigation of the computational complexity of the H-cover problem – deciding if a given graph G covers H. We present a characterization of a large class of parameter graphs H for which this problem is solvable in polynomial time.

1 Motivation and overview

For a fixed graph H, the H-cover problem admits a graph G as input and asks about the existence of a "local isomorphism": a labeling of the vertices of G by vertices of H so that the label set of the neighborhood of every $v \in V(G)$ is equal to the neighborhood (in H) of the label of v and each neighbor of v is labeled by a different neighbor of the label of v. Such a labeling is referred to as a *covering projection* from G onto H. Abello *et al.* [1] raised the question of computational complexity of H-cover problems, noting that there are both polynomial-time solvable (*easy*) and NP-complete (*difficult*) versions of this problem depending on the parameter graph H. We have studied the question of complexity of graph covering problems in [8], where one of our main results was a complete catalogue of the complexity of this problem for graphs on at most 6 vertices. In [10], we showed that in order to achieve a complete characterization of the complexity of graph covering problems for simple undirected graphs, it is necessary and sufficient to give a complete characterization of the H-cover problem for colored directed multigraphs H of minimum total degree ≥ 3. We continue the study of the complexity of the H-cover problem for cdm-graphs in this paper by describing a large class of graphs that allow polynomial-time solution. In [8], we have showed that in the case of simple undirected graphs the H-cover problem is polynomial if every block of

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the degree partition of H has at most two vertices. Our Theorem 1 provides a generalization of this theorem to cdm-graphs.

We first define the necessary notions in the next subsections. The main theorem is then presented in Section 2.

1.1 Directed multigraphs

A directed multigraph with vertex set V has edge set $E = D \cup F \cup L$, where D is the set of directed edges (including directed loops), F is the set of undirected edges and L is the set of undirected loops. A function $\mu : E \to (V \times V) \cup \binom{V}{2} \cup V$ describes the incidences, i.e., for an undirected edge $e \in F$, $\mu(e) \in \binom{V}{2}$ is the pair of vertices connected by e, for a loop $e \in L$, $\mu(e) \in V$ is the vertex hosting the loop e and for a directed edge $e \in D$, $\mu(e) \in V \times V$ is the ordered pair of vertices connected by e. Vertices and edges are colored by a coloring $C : V \cup E \to C(V \cup E)$ (this coloring need not be proper in the sense that adjacent vertices and/or edges may receive the same color). Since vertices, directed and undirected edges may be distinguished regardless the color, we may assume that $C(V)$, $C(D)$ and $C(F \cup L)$ are disjoint. A cdm-graph is uncolored if $C(V)$, $C(D)$ and $C(F \cup L)$ are one-element set each.

For an edge-color c, the *c-colored degree* of a vertex $x \in V(G)$ is defined as

$$deg_G^c(x) = |\{e : x \in \mu(e), e \in F(G), C(e) = c\}| + 2|\{e : x = \mu(e), e \in L(G), C(e) = c\}|$$

for $c \in C(F \cup L)$, and

$$deg_G^c(x) = (deg_G^{c-}(x), deg_G^{c+}(x))$$

where

$$deg_G^{c-}(x) = |\{e : \mu(e) = (x, u) \text{ for some } u, e \in D(G), C(e) = c\}|$$

and

$$deg_G^{c+}(x) = |\{e : \mu(e) = (u, x) \text{ for some } u, e \in D(G), C(e) = c\}|.$$

The *total degree* of a vertex u is

$$deg_G u = \sum_{c \in C(F \cup L)} deg^c(u) + \sum_{c \in C(D)} (deg^{c+}(u) + deg^{c-}(u)).$$

A *covering projection of a cdm-graph G onto a cdm-graph H* is a mapping $f : V(G) \cup E(G) \to V(H) \cup E(H)$ such that

(1) $f(u) \in V(H)$ and $C(f(u)) = C(u)$ for every $u \in V(G)$,

(2) $f(e) \in D(H)$, $C(f(e)) = C(e)$ and $\mu(f(e)) = (f(u), f(v))$ for every $e \in D(G)$ such that $\mu(e) = (u, v)$,

(3) $f(e) \in F(H) \cup L(H)$, $C(f(e)) = C(e)$ and $\mu(f(e)) = \{f(u), f(v)\}$ for every $e \in F(G) \cup L(G)$ such that $\mu(e) = \{u, v\}$,

(4) for every $u \in V(G)$ and every $e \in D(H)$ such that $\mu(e) = (f(u), w)$ ($\mu(e) = (w, f(u))$) for some w, there is exactly one arc $e' \in D(G)$ such that $\mu(e') = (u, w')$ ($\mu(e') = (w', u)$, respectively) for some w' and $f(e') = e$,

(5) for every $u \in V(G)$ and every $e \in F(H)$ such that $f(u) \in \mu(e)$, there is exactly one edge $e' \in F(G)$ such that $u \in \mu(e')$ and $f(e') = e$,

(6) for every $u \in V(G)$ and every $e \in L(H)$ such that $\mu(e) = f(u)$, there is either exactly one loop $e' \in L(G)$ such that $\mu(e') = u$ and $f(e') = e$, or there are exactly two edges $e', e'' \in F(G)$ such that $u \in \mu(e'), u \in \mu(e'')$ and $f(e') = f(e'') = e$.

We have proved in [10] that a cdm-graph G covers a cdm-graph H if and only if there exists a *vertex-covering projection* of G onto H, i.e., a mapping $g : V(G) \to V(H)$ such that

(1') $C(g(u)) = C(u)$ for every $u \in V(G)$,

(4') for every $u \in V(G)$, $w \in V(H)$ and every edge color $c \in C(D)$,

$$|\{e \in D(G) : \mu(e) = (u, x), g(x) = w, C(e) = c\}| = |\{e' \in D(H) : \mu(e') = (g(u), w), C(e') = c\}|$$

and

$$|\{e \in D(G) : \mu(e) = (x, u), g(x) = w, C(e) = c\}| = |\{e' \in D(H) : \mu(e') = (w, g(u)), C(e') = c\}|,$$

(5') for every $u \in V(G)$, $w \in V(H)$ such that $w \neq g(u)$, and every $c \in C(F \cup L)$,

$$|\{e \in F(G) : \mu(e) = \{x, u\}, g(x) = w, C(e) = c\}| = |\{e' \in F(H) : \mu(e') = \{w, g(u)\}, C(e') = c\}, |$$

(6') for every $u \in V(G)$ and every $c \in C(F \cup L)$,

$$|\{e \in F(G) : \mu(e) = \{x, u\}, g(x) = g(u), C(e) = c\}| + 2|\{e \in L(G) : \mu(e) = u, C(e) = c\}| =$$

$$2|\{e' \in L(H) : \mu(e') = g(u), C(e') = c\}|.$$

We have also presented reductions in [10] that show that it suffices to consider connected cdm-graphs of minimum total degree ≥ 3.

1.2 Degree partitions and refinements

The *degree partition* of a cdm-graph G is the coarsest partition of $V(G)$ into monochromatic equivalence classes $B_1, \ldots, B_k$ such that there exist numbers $r_{ij}^c, d_{ij}^{c+}, d_{ij}^{c-}$ $(i, j = 1, 2, \ldots, k$, $c \in C(D) \cup C(F) \cup C(L))$ such that

(i) for every i, j and every $u \in B_i$,

$$|\{e \in D(G) : \mu(e) \in \{u\} \times B_j, C(e) = c\}| = d_{ij}^{c+},$$

(ii) for every i, j and every $u \in B_i$,

$$|\{e \in D(G) : \mu(e) \in B_j \times \{u\}, C(e) = c\}| = d_{ij}^{c-},$$

(iii) for every $i \neq j$ and every $u \in B_i$,

$$|\{e \in F(G) : u \in \mu(e), \mu(e) \setminus \{u\} \in B_j, C(e) = c\}| = r_{ij}^c,$$

(iv) for every i and every $u \in B_i$,

$$|\{e \in F(G) : u \in \mu(e), \mu(e) \setminus \{u\} \in B_i, C(e) = c\}| + 2|\{e \in L(G) : \mu(e) = u, C(e) = c\}| = r_{ii}^c.$$

The collection $r^c_{ij}, d^{c+}_{ij}, d^{c-}_{ij}$ $(i,j=1,2,\ldots,k,\ c\in C(D)\cup C(F)\cup C(L))$ is then called the *degree refinement* of G. The equivalence classes of the degree partition are refered to as *blocks* of the partition.

As in the case of simple undirected graphs, the degree partition of a cdm-graph is unique and can be determined in polynomial time. It is also clear that if a cdm-graph G covers a cdm-graph H then G and H have the same degree refinements. Moreover, if $B_1,\ldots,B_k$ are the blocks of the degree partition of G and $B'_1,\ldots,B'_k$ are the blocks of the degree partition of H (indexed so that $r^c_{ij}=r'^c_{ij}$, $d^{c+}_{ij}=d'^{c+}_{ij}$ and $d^{c-}_{ij}=d'^{c-}_{ij}$ for all $i,j=1,2,\ldots,k$ and $c\in C(D)\cup C(F)\cup C(L)$) then $f(B_i)=B'_i$ for every i and every covering projection $f:G\to H$.

2 The polynomial theorem

For a cdm-graph H and a color c, we denote by H^c the subgraph induced by the edges of color c. We call H^c a *color specification* of H. Since vertices of different blocks can be distinguished in the covering graph as well as in H, we may assume without loss of generality that colors used on edges within distinct blocks and between distinct pairs of blocks are distinct. For colors used on directed edges, we introduce one more simplification. We may assume without loss of generality that edges directed into a block have different colors than edges directed out from that block. The orientation of such edges is then meaningless, and so we assume that all edges between vertices of *different* blocks are undirected.

Theorem 1 *The H-cover problem is polynomial if each block of the degree partition of H has 1, 2 or 4 vertices, every color specification of each block is of one of the types described in Table 1, and every color specification of every bipartite graph that expresses the connections between two blocks is of one of the types described in Table 2.*

Note that no edges are allowed between blocks of size 1 and 4. It may seem somewhat surprising that block size 3 does not appear in the table. The reason is that 4 vertices may be split naturally into two groups of two vertices each, which does help for certain block types (see the proof below). And indeed, while 2 undirected cycles of different colors on the same 4-vertex set yield a polynomial instance, 2 undirected cycles of different colors on the same 3-vertex set generate an NP-complete graph (equivalent to the 3-starfish, cf. [8]). The graphs from Tables 1 and 2 are depicted in Figures 1, 2 and 3.

Proof. Suppose H is as described in the assumption and we are given a cdm-graph G subject to the question if G covers H. We first check if G and H have the same degree refinement, and let us assume that this is the case. Let $B_i, i=1,2,\ldots,n$ be the blocks of the degree partition of H and let $B'_i, i=1,2,\ldots,n$ be the corresponding degree partition of G.

Assume $f:V(G)\to V(H)$ is a mapping such that $f^{-1}(B_i)=B'_i$ for every block B_i. It follows that $f(x)$ is uniquely determined if $x\in B'_i$ and B_i has only one vertex.

For every two-element block B_i, denote the vertices of B_i by L_i, R_i. For every $x\in B'_i$, we introduce a variable $\alpha(x)$. These variables will attain values from $GF(2)=\{0,1\}$ so that $\alpha(x)=1$ if and only if $f(x)=L_i$.

block size	undirected colors	directed colors
1	$K_1(k)$ - k loops around a vertex	$DK_1(k)$ - k directed loops around a vertex
2	$2K_1(k)$ - k loops around each of the two vertices	$2DK_1(k)$ - k directed loops around each of the two vertices
2	$K_2(0,m)$ - m parallel edges between the two vertices	$DK_2(0,m)$ - m parallel directed edges in each direction between the two vertices
2		$DK_2(1,1)$ - one directed loop around each of the two vertices and one directed edge in each direction between the two vertices
4	$4K_1(k)$ - k loops around each of the 4 vertices	$4DK_1(k)$ - k directed loops around each of the 4 vertices
4	$2K_2(0,m)$ - two bundles of m parallel edges between distinct pairs of vertices	$2DK_2(0,m)$ - disjoint union of two $DK_2(0,m)$'s
4	$K_2(0,2m)+2K_1(m)$ - a bundle of $2m$ parallel edges between two vertices and m loops around each of the remaining two vertices	$DK_2(0,m)+2DK_1(m)$ - disjoint union of a $DK_2(0,m)$'s and two copies of $DK_1(m)$
4		$2DK_2(1,1)$ - disjoint union of two $DK_2(1,1)$'s
4	C_4 - a simple undirected 4-cycle	$DC_4(1,1)$ - symmetric orientation of C_4
4		$DC_4(m,0)$ - directed 4-cycle with each edge replaced by m parallel edges

Table 1: Color specifications of blocks.

block sizes	connecting graphs
1 - 1	$K_{1,1}(k)$ - k parallel edges between the vertices
1 - 2	$K_{1,2}$ - single edges between vertices from different blocks
2 - 2	$2K_{1,1}(k)$ - disjoint union of two bundles of k parallel edges each
2 - 2	$K_{2,2}$ - complete bipartite graph between the two pairs of vertices
2 - 4	$2K_{1,2}$ - disjoint union of two copies of $K_{1,2}$
4 - 4	$4K_{1,1}(k)$ - disjoint union of four bundles of k parallel edges each
4 - 4	$2K_{2,2}$ - disjoint union of two copies of $K_{2,2}$

Table 2: Color specifications of block connections.

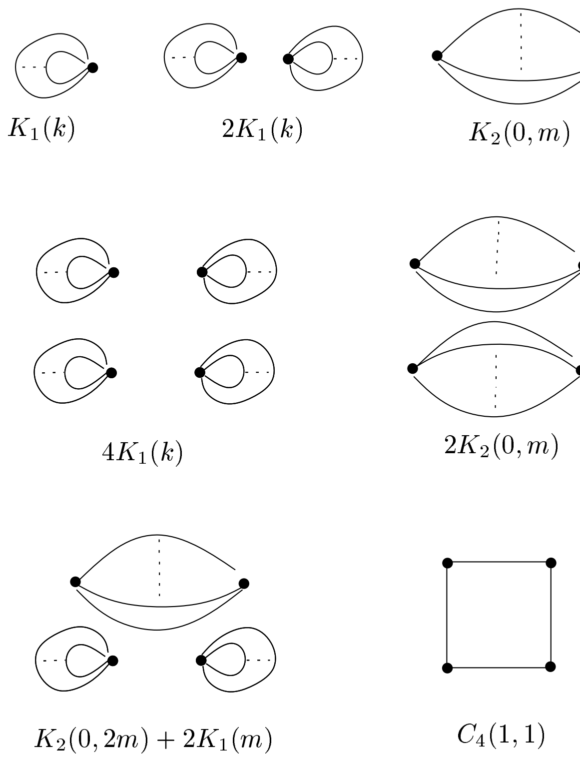

Figure 1: Undirected blocks.

For every four-element block B_i, denote its vertices by S_i, T_i, X_i, Y_i. For every $x \in B_i'$, we introduce variables $\phi(x), \psi(x)$ and $\xi(x)$ so that $\phi(x) = 1$ if and only if $f(x) \in \{S_i, T_i\}$, $\psi(x) = 1$ if and only if $f(x) \in \{S_i, X_i\}$ and $\xi(x) = 1$ if and only if $f(x) \in \{S_i, Y_i\}$.

We build up a system of linear equations modulo 2, which is solved by these variables if and only if f is a vertex-covering projection of G onto H. Since equations modulo 2 can be solved in polynomial time via Gauss elimination, a polynomial algorithm for our covering problem will thus follow. The description of this system is somewhat tedious but cannot be avoided.

0. If B_i is a 1-element block and $x \in B_i'$ then the fact that H and G have the same degree refinement implies that x has the right number of neighbors in B_i' (if $B_i^c \simeq K_1(k)$ then $r_{ii}^c = 2k$, and hence $r_{ii}'^c = |\{e \in F(G) : x \in \mu(e), \mu(e) \setminus \{x\} \in B_i', C(e) = c\}| + 2|\{e \in L(G) : \mu(e) = x, C(e) = c\}| = 2k$ and (6') of the definition of vertex-coverings is fulfilled for x; if $B_i^c \simeq DK_1(k)$ then $d_{ii}^{c+} = d_{ii}^{c-} = k$, and hence $d_{ii}'^{c+} = |\{e \in D(G) : \mu(e) \in \{x\} \times B_i, C(e) = c\}|, d_{ii}'^{c-} = |\{e \in D(G) : \mu(e) \in B_i \times \{x\}, C(e) = c\}|$ and (4') of the definition of vertex-coverings is fulfilled for x). The following three cases describe two-element blocks.

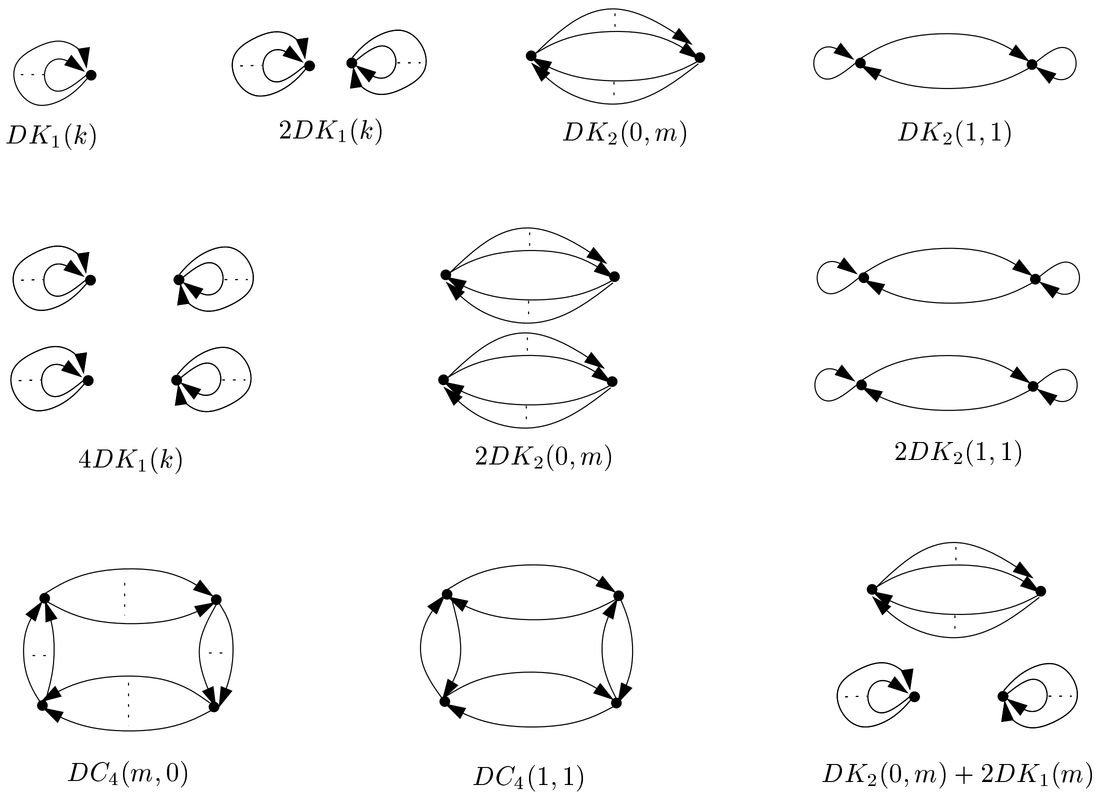

Figure 2: Directed blocks.

1. For every edge color c such that $B_i^c \cong 2K_1(k)$ or $B_i^c \cong 2DK_1(k)$ and vertices $x, y \in B_i'$ connected by an edge of color c (note that $deg_G^c(x) = 2k$ resp. $deg_G^c(x) = (k,k)$ for each vertex $x \in B_i'$), we introduce the equation

$$\alpha(x) + \alpha(y) = 0. \tag{1}$$

Indeed, if f is a vertex-covering projection then x, y must map on the same vertex of B_i (i.e., either both of them map on L_i or both of them on R_i) and $\alpha(x) = \alpha(y)$, which is the same as equation (1). On the other hand, if $\alpha(x) = \alpha(y)$ for each pair x, y connected by an edge of color c, then f maps all vertices connected to a vertex $x \in B_i'$ by an edge of color c onto the same vertex, namely onto $f(x)$, and the fact that G and H have the same degree refinement implies that (4') resp. (6') is fulfilled for vertex x.

2. For every edge color c such that $B_i^c \cong K_2(0,m)$ or $B_i^c \cong DK_2(0,m)$ and vertices $x, y \in B_i'$ connected by an edge of color c (note that $deg_G^c(x) = m$ resp. $deg_G^c(x) = (m,m)$ for each vertex

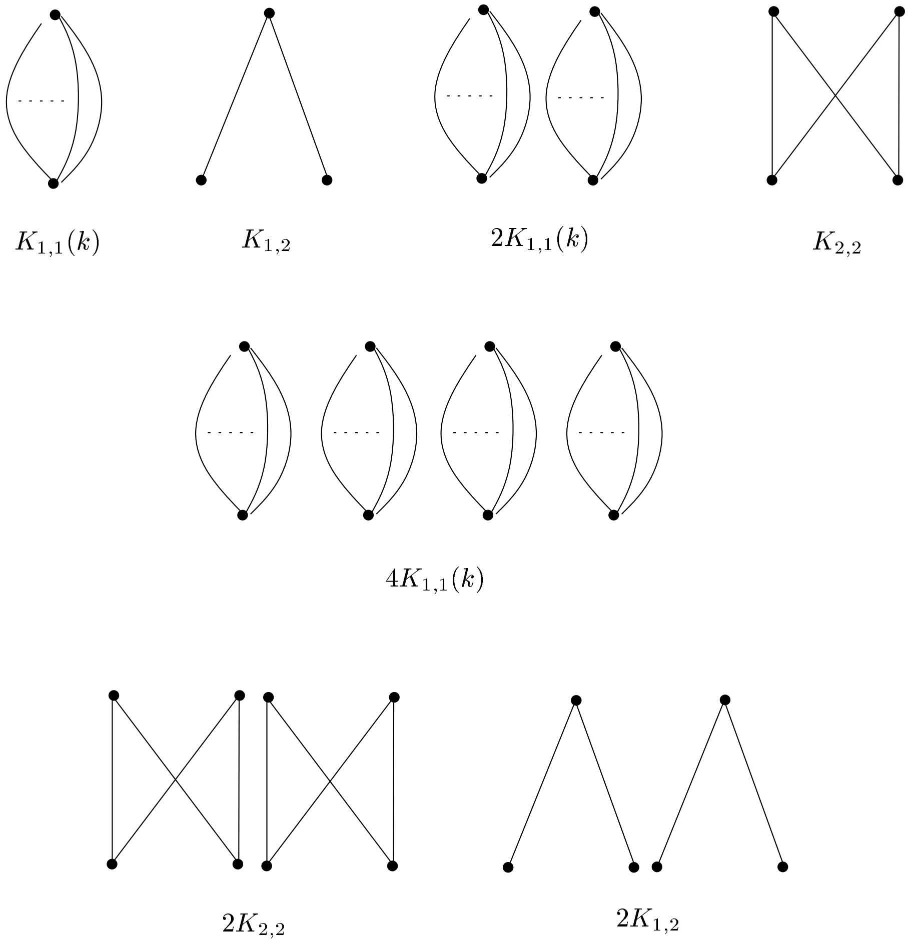

Figure 3: Block connections.

$x \in B_i'$), we introduce the equation

$$\alpha(x) + \alpha(y) = 1. \tag{2}$$

If f is a vertex-covering projection then x, y must be mapped onto different vertices of B_i, i.e., one of them onto L_i and the other one onto B_i. This means that $\alpha(x) \neq \alpha(y)$, which is equivalent to equation (2). On the other hand, if equation (2) is fulfilled for every two vertices x, y connected by an edge of color c, none of such edges is a loop and the equality of degree refinements of G and H implies that (4') resp. (5') is fulfilled for x.

3. For every edge color c such that $B_i^c \cong DK_2(1,1)$ and edges $e, e' \in D(G)$ of color c such that $\mu(e) = (x,z), \mu(e') = (y,z)$ (or $\mu(e) = (z,x), \mu(e') = (z,y)$) for vertices $x, y, z \in B_i'$ (note that $deg_G^c(x) = (2,2)$ for each vertex $x \in B_i'$), we introduce the equation

$$\alpha(x) + \alpha(y) = 1. \tag{3}$$

If f is a vertex-covering projection then one of x, y is mapped onto L_i and the other one onto R_i (note that necessarily $x \neq y$) and hence $\alpha(x) \neq \alpha(y)$, i.e., $\alpha(x) + \alpha(y) = 1$. On the other hand, if equation (3) is fulfilled for each such triple x, y, z, then $f(x) \neq f(y)$. Since for every $z \in B_i'$, there are exactly two vertices, say $x, y \in B_i'$, connected to z by edges of color c directed to z (and exactly two vertices, say $x', y' \in B_i'$ connected to z by edges of color c directed from z), one of the vertices x, y maps onto L_i and the other one onto R_i (and the same is true for x', y'), and (4') is fulfilled for z. The argumentation is analogous in the forthcoming cases. Therefore we always give the full description of the equations, but only sketch the proofs.

Before we proceed to the equations based on particular 4-vertex blocks, we should note that the variables $\phi(x), \psi(x)$ and $\xi(x)$ are not independent. It is readily seen that $\phi(x) = \psi(x)$ if and only if $\xi(x) = 1$, i.e., these 3 variables are tied by the equation

$$\phi(x) + \psi(x) + \xi(x) = 1 \tag{4}$$

and such equation is added to our system for every $x \in B_i'$, whenever $|B_i| = 4$. The reason why we introduce three dependent variables is that this enables a more uniform description of the equations. We purposedly include dependent equations, since we are not interested in the minimum possible number of them, but rather favor the clarity of the argumentation in the proof below.

4. For every edge color c such that $B_i^c \cong 4K_1(k)$ or $B_i^c \cong 4DK_1(k)$ and vertices $x, y \in B_i'$ connected by an edge of color c, we introduce the equations

$$\phi(x) = \phi(y) \tag{5}$$
$$\psi(x) = \psi(y) \tag{6}$$
$$\xi(x) = \xi(y). \tag{7}$$

These equations express $f(x) = f(y)$ for such pair of vertices x, y.

5. Consider $x, y \in B_i'$ connected by an edge of color c and suppose $B_i^c \cong 2K_2(0,m)$ or $B_i^c \cong 2DK_2(0,m)$. If the edges of B_i^c are between S_i and T_i (and between X_i and Y_i), we add equations

$$\phi(x) + \phi(y) = 0 \tag{8}$$
$$\psi(x) + \psi(y) = 1 \tag{9}$$
$$\xi(x) + \xi(y) = 1. \tag{10}$$

Equation (8) expresses $f(x) \in \{S_i, T_i\}$ iff $f(y) \in \{S_i, T_i\}$, while equations (9) and (10) express $f(x) \in \{S_i, X_i\}$ iff $f(y) \in \{T_i, Y_i\}$, i.e., altogether $f(x) = S_i$ iff $f(y) = T_i$ and $f(x) = X_i$ iff $f(y) = Y_i$.

If the edges of B_i^c are between S_i and X_i (and between T_i and Y_i), we add equations

$$\phi(x) + \phi(y) = 1 \tag{8'}$$
$$\psi(x) + \psi(y) = 0 \tag{9'}$$

$$\xi(x) + \xi(y) = 1. \tag{10'}$$

Finally, if these edges are between S_i and Y_i (and between T_i and X_i), we add equations

$$\phi(x) + \phi(y) = 1 \tag{8''}$$

$$\psi(x) + \psi(y) = 1 \tag{9''}$$

$$\xi(x) + \xi(y) = 0. \tag{10''}$$

6. Let $x, y \in B_i'$ be connected by an edge of color c and suppose $B_i^c \cong K_2(0, 2m) + 2K_1(m)$ or $B_i^c \cong DK_2(0, m) + 2DK_1(m)$. Let $\tau \in \{\phi, \psi, \xi\}$ be such that the two vertices forming $K_2(0, 2m)$ (resp. $DK_2(m)$) in B_i^c have the same value of τ and let this value be $\varepsilon \in \{0, 1\}$. Then we add equations

$$\rho(x) + \sigma(y) = \varepsilon \tag{11}$$

for $\rho, \sigma \in \{\phi, \psi, \xi\} \setminus \{\tau\}, \rho \neq \sigma$. For the sketch of the proof, consider e.g. the case when X_i and Y_i are adjacent in B_i^c, i.e., $\tau = \phi$ and $\varepsilon = 0$. Then equation (11) becomes

$$\psi(x) + \xi(y) = 0,$$

$$\psi(y) + \xi(x) = 0,$$

which is solved exactly by the following 4-tuples

$$(\psi(x), \xi(x), \psi(y), \xi(y)) \in \{(0,0,0,0), (1,1,1,1), (0,1,1,0), (1,0,0,1)\},$$

i.e., these equations express

$$f(x) = f(y) = S_i \text{ or } f(x) = f(y) = T_i \text{ or } \{f(x), f(y)\} = \{X_i, Y_i\}.$$

7. For every edge color c such that $B_i^c \cong 2DK_2(1,1)$ and edges $e, e' \in D(G)$ of color c such that $\mu(e) = (x, z), \mu(e') = (y, z)$ (or $\mu(e) = (z, x), \mu(e') = (z, y)$) for vertices $x, y, z \in B_i'$, we add equations depending again on the actual placement of $2DK_2(1,1)$ on B_i^c: If the non-loop edges of B_i^c are between S_i and T_i (and between X_i and Y_i), we add equations

$$\phi(x) + \phi(z) = 0 \tag{12}$$

$$\psi(x) + \psi(y) = 1 \tag{13}$$

$$\xi(x) + \xi(y) = 1. \tag{14}$$

These equations express that $f(x), f(y) \in \{S_i, T_i\}$ (or $f(x), f(y) \in \{X_i, Y_i\}$) provided $f(z) \in \{S_i, T_i\}$ (resp. $f(z) \in \{X_i, Y_i\}$), and $f(x) \neq f(y)$. I.e., if z maps into $\{S_i, T_i\}$ then one of x, y maps onto S_i and the other one onto T_i, and if z maps into $\{X_i, Y_i\}$ then one of x, y maps onto X_i and the other one onto Y_i.

If the non-loop edges of B_i^c are between S_i and X_i (and between T_i and Y_i), we add equations

$$\phi(x) + \phi(y) = 1 \tag{12'}$$

$$\psi(x) + \psi(z) = 0 \tag{13'}$$

$$\xi(x) + \xi(y) = 1. \tag{14'}$$

Finally, if these edges are between S_i and Y_i (and between T_i and X_i), we add equations

$$\phi(x) + \phi(y) = 1 \tag{12''}$$

$$\psi(x) + \psi(y) = 1 \tag{13''}$$
$$\xi(x) + \xi(z) = 0. \tag{14''}$$

Next we pay attention to those blocks B_i whose color specifications are based on a 4-cycle. For the sake of brevity we assume that this 4-cycle is $S_i \to T_i \to X_i \to Y_i \to S_i$ (the other possibilities are similar and would be derived from Equations (15-27) similarly as Equations (12'-14') and (12"-14") were derived from Equations (12-14).

8. Consider first a color of undirected edges, i.e., $B_i^c \cong C_4$. Let $x \in B_i'$ and let $e, e' \in D(G)$ be distinct edges of color c such that $\mu(e) = \{x, y\}, \mu(e') = \{x, z\}$ for vertices $y, z \in B_i'$. Then we add equations

$$\phi(z) + \phi(y) = 1 \tag{15}$$
$$\psi(z) + \psi(y) = 0 \tag{16}$$
$$\xi(z) + \xi(y) = 1, \tag{17}$$

and for every pair of vertices x, y connected by an edge of color c, we add equation

$$\psi(x) + \psi(y) = 1. \tag{18}$$

Equation (18) expresses that vertices connected by edges of color c have distinct value of ψ, while equations (15-17) express that vertices connected to a common neighbor by edges of color c have the same ψ but map onto distinct vertices of B_i. I.e., if $f(x) = S_i$ then one of y, z is mapped onto T_i and the other one onto Y_i, etc.

9. Next consider a color specification $B_i^c \cong DC_4(m, 0)$. Let $x \in B_i'$ and let $e, e' \in D(G)$ be edges of color c such that $\mu(e) = (x, y), \mu(e') = (x, z)$ (or $\mu(e) = (y, x), \mu(e') = (z, x)$) for vertices $y, z \in B_i'$. Then we add equations

$$\phi(z) + \phi(y) = 0 \tag{19}$$
$$\psi(z) + \psi(y) = 0 \tag{20}$$
$$\xi(z) + \xi(y) = 0, \tag{21}$$

and for every pair of vertices x, y connected by an edge of color c (in this order, i.e., $\mu(e) = (x, y)$ for some edge $e \in D(G)$ of color c), we add equations

$$\psi(x) + \psi(y) = 1 \tag{22}$$
$$\psi(y) + \xi(x) + \xi(y) = 1. \tag{23}$$

Equations (19-21) express that all vertices connected to a common neighbor by edges of color c directed all towards (or all from) this common neighbor map onto the same vertex of B_i. Equations (22-23) are solved exactly by values corresponding to

$$(f(x), f(y)) \in \{(S_i, T_i), (T_i, X_i), (X_i, Y_i), (Y_i, S_i)\}.$$

10. Finally let $B_i^c \cong DC_4(1, 1)$. Let $x \in B_i'$ and let $e, e' \in D(G)$ be the two edges of color c that start in x (or that end in x) and let these edges end (resp. start) in vertices $y, z \in B_i'$. Then we

add equations

$$\phi(z) + \phi(y) = 1 \tag{24}$$
$$\psi(z) + \psi(y) = 0 \tag{25}$$
$$\xi(z) + \xi(y) = 1. \tag{26}$$

For every pair of vertices x, y connected by an edge of color c, we add equation

$$\psi(x) + \psi(y) = 1. \tag{27}$$

Similarly as in case 8., these equations guarantee that if $f(x) = S_i$ then one of x, y is mapped onto T_i and the other one onto Y_i, etc.

Now we describe the equations arising from interblock connections. We consider blocks B_i, B_j such that $|B_i| \leq |B_j|$. Let BC_{ij}^c be the color c specification of the bipartite graph between B_i and B_j (we consider only colors such that the edge set of BC_{ij}^c is nonempty).

11. When $|B_i| = |B_j| = 1$, no equations are needed (the values of $f(x)$ and $f(y)$ are uniquely determined and the right number of neighbors is guaranteed by the equality of the degree refinements of G and H).

12. If $|B_i| = 1$ and $|B_j| = 2$ then $BC_{ij}^c \cong K_{1,2}$. Let $x \in B_i'$ and let $e, e' \in F(G)$, $\mu(e) = \{x, y\}, \mu(e') = \{x, z\}$, be the two edges of color c incident to x that end in B_j'. Then y, z should be mapped onto different vertices of B_j, which is expressed by the equation

$$\alpha(y) + \alpha(z) = 1. \tag{28}$$

13. Next suppose $|B_i| = |B_j| = 2$ and $BC_{ij}^c \cong 2K_{1,1}(m)$. Let $x \in B_i'$ and $y \in B_j'$ be connected by an edge of color c in G. If the edges of BC_{ij}^c connect the vertices L_i and L_j (and R_i and R_j), we add the equation

$$\alpha(x) + \alpha(y) = 0, \tag{29}$$

if the edges of BC_{ij}^c connect the vertices L_i and R_j (and R_i and L_j), we add the equation

$$\alpha(x) + \alpha(y) = 1. \tag{29'}$$

These equations express that all neighbors of a vertex mapped onto L_i (or R_i) are mapped onto L_j (resp. onto R_j) in the former case, and vice versa in the latter case.

14. If $|B_i| = |B_j| = 2$, $BC_{ij}^c \cong K_{2,2}$, $x \in B_i'$ and $y \in B_j'$ are connected by an edge of color c in G and $u \in B_j'$ (resp. $v \in B_i'$) is the endpoint of the other edge of color c incident to x (resp. y), we add the equations

$$\alpha(y) + \alpha(u) = 1, \tag{30}$$
$$\alpha(x) + \alpha(v) = 1, \tag{31}$$

which express that each vertex of B_i' is connected by an edge of color c to exactly one vertex mapped onto L_j and to one vertex mapped onto R_j (and vice versa).

15. Let $|B_i| = 2$, $|B_j| = 4$ and $BC_{ij}^c \cong 2K_{1,2}$. Let $x \in B_i'$ and $y \in B_j'$ be connected by an edge of color c in G, and let $z \in B_j'$ be the endpoint of the other edge of color c incident to x.

Our equations now depend on the routing of the edges of BC_{ij}^c. Let $\pi \in \{\phi, \psi, \xi\}$ be such that the neighbors of L_i (in BC_{ij}^c) have the same π, and let the value of their π be ε. Then we add the equations ($\sigma \in \{\phi, \psi, \xi\} \setminus \{\pi\}$)

$$\pi(y) + \pi(z) = 0, \tag{32}$$

$$\sigma(y) + \sigma(z) = 1, \tag{33}$$

$$\alpha(x) + \pi(y) = 1 + \varepsilon \tag{34}$$

(note that the first and last of these equations yield $\alpha(x) + \pi(z) = 1 + \varepsilon$ as well).
E.g., if L_i is adjacent to T_j, X_j, we have $\pi = \xi$ and $\varepsilon = 0$ and the equations are

$$\begin{aligned}\xi(y) + \xi(z) &= 0,\\ \phi(y) + \phi(z) &= 1,\\ \psi(y) + \psi(z) &= 1,\\ \alpha(x) + \xi(y) &= 1.\end{aligned}$$

The first three of these equations guarantee that y, z map onto distinct vertices having the same value of ξ, and the last equation expresses how this value of ξ depends on $f(x)$. Namely, if $f(x) = L_i$ then $\xi = 0$ and one of y, z maps onto T_j and the other one onto X_j, and if $f(x) = R_i$ then one of y, z maps onto S_j and the other one onto Y_j.

16. Suppose $|B_i| = |B_j| = 4$ and $BC_{ij}^c \cong 2K_{2,2}$. Let $x \in B_i'$ and $y \in B_j'$ be connected by an edge of color c in G, and let $u \in B_j'$ (resp. $v \in B_i'$) be the endpoint of the other edge of color c incident to x (resp. y). Our equations depend again on the routing of the edges of BC_{ij}^c. Let $\pi, \rho \in \{\phi, \psi, \xi\}$ be such that the vertices of B_i with $\pi = 1$ are adjacent (in BC_{ij}^c) to vertices of B_j of the value of $\rho = \varepsilon$. Then we add the equations ($\sigma \in \{\phi, \psi, \xi\} \setminus \{\pi\}$, $\tau \in \{\phi, \psi, \xi\} \setminus \{\rho\}$)

$$\pi(x) + \pi(v) = 0, \tag{35}$$

$$\sigma(x) + \sigma(v) = 1, \tag{36}$$

$$\rho(y) + \rho(u) = 0, \tag{37}$$

$$\tau(y) + \tau(u) = 1, \tag{38}$$

$$\pi(x) + \rho(y) = 1 + \varepsilon. \tag{39}$$

Similarly as in cases 14. and 15., these equations expresses that f is a vertex-covering projection in the neighborhood of x.

17. Finally let $|B_i| = |B_j| = 4$ and $BC_{ij}^c \cong 4K_{1,1}(m)$. Let $x \in B_i'$ and $y \in B_j'$ be connected by an edge of color c in G. Our equations again depend on the routing of edges in BC_{ij}^c, but this time the actual dependence is less simple to describe. Therefore we first describe the simplest case when S_i is adjacent to S_j, T_i to T_j, X_i to X_j and Y_i to Y_j. In this case the neighbors have to have the same ϕ, ψ and ξ, which is expressed by the equations (here we assume $x \in B_i', y \in B_j'$ connected by an edge of color c)

$$\phi(x) + \phi(y) = 0, \tag{40}$$

$$\psi(x) + \psi(y) = 0, \tag{41}$$

$$\xi(x) + \xi(y) = 0. \tag{42}$$

For the general case, we exploit the fact that the vertices of B_i and B_j are uniquely determined by their ϕ and ψ. We will express them as elements of $GF(2)^2$ according to their (ϕ, ψ), i.e.

$$\begin{array}{ccc} S & \sim & (1,1), \\ T & \sim & (1,0), \\ X & \sim & (0,1), \\ Y & \sim & (0,0). \end{array}$$

Given the edges of BC^c_{ij}, we define a bijection $g : GF(2)^2 \to GF(2)^2$ so that

$$g(p,q) = (r,s) \text{ if and only if } (p,q) \text{ is adjacent to } (r,s) \text{ in } BC^c_{ij}$$

$(p,q,r,s \in \{0,1\})$. Our separate Lemma 2.1 says that then there exists a system of linear equations (Eq) over the variables p,q,r,s such that solution set is exactly $\{(p,q,g(p,q)) : p,q \in GF(2)\}$. If $x \in B'_i$ and $y \in B'_j$ are connected by an edge of color c, we add the system of equations (Eq) with variables renamed $\phi(x), \psi(x), \phi(y), \psi(y)$ instead of equations (39-41). $\square$

Lemma 2.1 *For every bijection $g : GF(2)^2 \to GF(2)^2$, there exists a system of linear equations over variables p,q,r,s whose solution set is exactly $\{(p,q,g(p,q)) : p,q \in GF(2)\}$.*

Proof. Set $\mathcal{L} = \{(p,q,g(p,q)) : p,q \in GF(2)\}$ and $\mathcal{U} = \{(p,q,g(p,q) + g(0,0)) : p,q \in GF(2)\}$, i.e., $\mathcal{L} = (0,0,g(0,0)) + \mathcal{U}$. We claim that $\mathcal{U}$ is a linear subspace of $GF(2)^4$. Indeed, in $GF(2)^2$ the sum of *any* two non-zero vectors equals the third non-zero vector, and the same is true for $\mathcal{U}$, since the zero vector belongs to $\mathcal{U}$. Hence $dim_{GF(2)}\mathcal{U} = 2$ and $\mathcal{L}$ is the solution set of a system of two independent linear equations. $\square$

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