Finite commutative monoids of open maps

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Abstract : A finite commutative semigroup is isomorphic to the semigroup of all open continuous mappings of a Hausdorff space into itself, resp. of all complete Heyting endomorphisms of a Hausdorff Heyting algebra if and only if it is a product of a group and several groups augmented by zero. Characterization of this type does not apply in the infinite case.

1991 Mathematics Subject Classification : Primary: 20M30. Secondary: 06D20, 54D10, 54C10

Keywords : endomorphism, open continuous map, complete Heyting homomorphism, strong graph homomorphism, Hausdorff property

In 1966, Paalman de Miranda proved that a certain finite monoid cannot be represented as the monoid $\mathfrak{Op}(X)$ of all open continuous mappings of any Hausdorff space X into itself [**9**]. Her discovery interestingly contrasts with these two facts:

- every monoid is isomorphic to the monoid of all *quasi-open* continuous mappings of a *metric* space into itself (this, together with similar results, can be found in [**15**], see also [**14**,16]),
- every monoid is isomorphic to the monoid $\mathfrak{Op}(X)$ for a T_1-space X, see [**14**,16] and, as observed in [**13**], even for a space X that is both T_1 and sober.

In [**13**], we have extended Paalman de Miranda's result to the considerably wider context of Hausdorff Heyting algebras and their complete Heyting homomorphisms (the word 'wider' is somewhat imprecise, for there exist marginal cases of Hausdorff spaces whose corresponding Heyting algebra is *not* Hausdorff, see Isbell [**5**]). In addition, we found that, apart from cyclic groups, only a single one-generated finite monoid is representable in either category.

*The authors gratefully acknowledge the support of the National Science and Engineering Research Council of Canada and of the Grant Agency of the Czech Republic under Grant 201/96/0119.

In the present paper we describe all finite commutative monoids isomorphic to the monoid $\mathfrak{Op}(X)$ for some Hausdorff space X, or to the monoid of all complete Heyting endomorphisms of some Hausdorff Heyting algebra in the sense of Isbell [**5**]. It turns out that these monoids are products

$$G_0 \times G_1^0 \times \cdots \times G_n^0$$

in which G_0 is a group and each monoid G_j^0 (if any) is a group augmented by a zero. This also characterizes finite commutative monoids formed by strong endomorphisms of symmetric graphs. These three categories strongly contrast their respective non-Hausdorf or non-symmentric counterparts: the categories of

- T_1-spaces and open maps,
- complete Heyting algebras and complete Heyting homomorphisms,
- directed graphs and strong homomorphisms

are algebraically universal (see [**14**,16], [**13**], [**14**]), and hence every monoid is representable as an endomorphism monoid in each of these wider categories.

We also show that the situation changes for infinite endomorphism monoids: for instance, the additive monoid of natural numbers is representable, and so are all its finite or countably infinite powers.

Only basic notions of category theory, those found in the initial chapters of [**1**] or [**8**], are needed here. The few facts on frames not explained below can be found in [**7**] or in [**13**]. A topological construction presented in Section 4 is based on a construction from Chapter VI of [**14**]; to aid the reader and to avoid unnecessary repetition, we refer to [**14**] in considerable detail and adopt the notation used there.

1. Preliminaries

1.1. A *concrete category* is, as usual, a category $\mathcal{C}$ equipped with a faithful functor $|\ | : \mathcal{C} \to$ **Set** into the category **Set** of all sets and mappings. We say that a finite coproduct

$$\{j_i : X_i \to X \mid i = 1, \ldots, n\}$$

in $\mathcal{C}$ is a *concrete sum* in a concrete category $(\mathcal{C}, |\ |)$ provided $|X|$ is a disjoint union $|X_1| \cup \cdots \cup |X_n|$ and $|j_i| : |X_i| \to |X|$ are the corresponding inclusion maps. If this is the case, we write

$$X = X_1 \cup \cdots \cup X_n = \bigcup X_i.$$

A $\mathcal{C}-$object Y is called a *summand* of a $\mathcal{C}$-object X if $X = Y \cup Z$ is a concrete sum for some Z. Then, of course, the $\mathcal{C}$-object Z is a summand of X as well and $|Z| = |X \smallsetminus Y|$.

A concrete category has *tame summands*, and is called a *TS-category*, if it has these three properties:

(TS1) for any two summands X_1 and X_2 of X there exists a summand Y of X with $|Y| = |X_1| \cap |X_2|$,

(TS2) all preimages of summands under $\mathcal{C}$-morphisms are summands,

(TS3) for any two concrete sums $X = \bigcup X_i$ and $X = \bigcup Y_j$ there exists a common refinement, that is, a concrete sum $X = \bigcup Z_{ij}$ with $|Z_{ij}| = |X_i| \cap |Y_j|$.

There is no danger of confusion in writing $Y = X_1 \cap X_2$ rather than $|Y| = |X_1| \cap |X_2|$ in (TS1). By (TS1), for any two summands X_1 and X_2 of X, there is a summand Z of X with $|Z| = |X_1| \smallsetminus |X_2|$, and we will simply write $Z = X_1 \smallsetminus X_2$ here again.

There are numerous examples of TS-categories. The category

$$\mathbf{Top}_{2,o}$$

of Hausdorff spaces and their open continuous maps is a TS-category and, together with categories related to it, will be of particular interest here.

1.2. Proposition. *Let $X_1, \ldots, X_n$ be summands (not necessarily disjoint) of an object X in a TS-category $\mathcal{C}$, and let $|X| = |\bigcup X_i|$. If $f : |X| \to |Y|$ is a mapping such that each $f \restriction |X_i|$ carries a $\mathcal{C}$-morphism $X_i \to Y$, then f carries a $\mathcal{C}$-morphism $X \to Y$.*

Proof: Use the common refinement of the n concrete sums $X = X_i \cup (X \smallsetminus X_i)$. ∎

1.3. Let $(X, \leq)$ be a poset. We say that a subset $Y \subseteq X$ is *decreasing* if $x \leq y \in Y$ implies $x \in Y$, and *increasing* if $x \geq y \in Y$ implies $x \in Y$. Recall that an ordered topological space $X = (X, \tau, \leq)$ is a *Priestley space* if $(X, \leq)$ is a poset and (X, τ) is a compact topological space, and if for any closed decreasing set $Y \subset X$ and any $x \in X \smallsetminus Y$ there exists a clopen decreasing set B such that $Y \subseteq B$ and $x \notin B$. Any continuous, order preserving mapping $f : X \to X'$ between Priestley spaces X, X' is called a *Priestley map*. In what follows,

$$\mathbf{PSp}$$

will denote the category of all Priestley spaces and all Priestley maps.

Observation. *A subset Y of a Priestley space X is a summand of X if and only if it is increasing, decreasing and clopen in X. Hence* **PSp** *is a TS-category.* ∎

1.4. Recall that a *frame* is any complete lattice L in which $(\bigvee S) \wedge a = \bigvee\{s \wedge a \mid s \in S\}$ for any $S \subseteq L$ and $a \in L$, and that a frame homomorphism $h : L \to L'$ is any mapping preserving finite meets and arbitrary joins. A typical instance of a frame is the lattice $\mathfrak{O}(X)$ of all open sets of a topological space X; and if $f : X \to X'$ is a continuous map into a space X', then the inverse-image map $f^{-1} = \mathfrak{O}(f) : \mathfrak{O}(X') \to \mathfrak{O}(X)$ is a frame homomorphism.

For the so-called *sober* spaces, that is, spaces satisfying a certain condition weaker than the Hausdorff separation axiom, the mapping $f \mapsto \mathfrak{O}(f)$ is an invertible contravariant correspondence of continuous maps $f : X \to X'$ to frame homomorphisms $\mathfrak{O}(f) : \mathfrak{O}(X') \to \mathfrak{O}(X)$. Moreover, any sober space X can be reconstructed from its frame $\mathfrak{O}(X)$, and this is why a frame can be viewed as a natural generalization of a (sober) space.

A frame L is called *Hausdorff* (in the sense of Isbell, see [**5**,7]) if the codiagonal map $L \oplus L \to L$ of the free product $L \oplus L$ into L satisfies a certain condition, applied in [**13**]. The actual form of this condition is secondary, for we shall use only its consequence proved in [**13**], and use it only once. It should be noted that the frame $\mathfrak{O}(X)$ of a Hausdorff space X is not necessarily Hausdorff in the sense of Isbell, but that $\mathfrak{O}(X)$ is Hausdorff whenever X is regular.

1.5. The distributive law of 1.4 implies that any frame L is also a complete Heyting algebra, and that the Heyting operation $\to$ is uniquely determined by L. Moreover,

> for Hausdorff spaces X and X', complete Heyting homomorphisms $\mathfrak{O}(X') \to \mathfrak{O}(X)$ correspond to those continuous maps $f : X \to X'$ which are open, see [**2**,6].

In what follows,

$$\mathbf{HHeyt}$$

will denote the category of all Heyting algebras that are Hausdorff in the sense of Isbell, and of all their complete Heyting homomorphisms.

1.6. We now recall the celebrated *Priestley duality*, a contravariant natural isofunctor

$$\mathcal{D} : \mathbf{PSp} \to \mathbf{D}$$

of the category **PSp** of Priestley spaces onto the category **D** of distributive (0,1)-lattices and their (0,1)-homomorphisms. For any Priestley space X, the elements of $\mathcal{D}(X)$ are represented by clopen decreasing subsets of X and, for any Priestley map $f : Y \to X$, the homomorphism $\mathcal{D}(f) : \mathcal{D}(X) \to \mathcal{D}(Y)$ is the restriction of the inverse image map f^{-1} to the collection of these representing sets.

In what follows, the image of the category **HHeyt** under the inverse $\mathcal{D}^{-1}$ of $\mathcal{D}$ will be denoted as

HPSp

and referred to as the category of all *HP-spaces* and *HP-maps.* It is clear that **HPSp** is not a full subcategory of **PSp**.

2. Hausdorff categories

2.1. Definition. We say that a TS-category $\mathcal{C}$ is *Hausdorff* if, for any $\mathcal{C}$-object X, the image

$$h[X] = \{x \mid h(x) = x\}$$

of any endomorphism $h : X \to X$ with $h^2 = h$ is a summand of X.

2.2. Proposition. *The category* $\mathbf{Top}_{2,o}$ *is Hausdorff.*

Proof: If $h : X \to X$ is continuous and open, then its image $h[X]$ is open. Since $h[X] = \{x \mid h(x) = x\}$ and X is Hausdorff, the set $h[X]$ is also closed, and it is clear that any clopen subset of X is a summand of X in $\mathbf{Top}_{2,o}$. ∎

2.3. Proposition. *The category* **HPSp** *is Hausdorff.*

Proof: We already know that **HPSp** is a TS-category. Let X be a Hausdorff Priestley space, and let $h = h^2 \in \operatorname{End} X$. Since h is the Priestley dual of a complete Heyting endomorphism, the image $Y = h[X]$ is a closed subspace of X that is also decreasing, see [**12**]. Furthermore, the **HPSp**-map h has a decomposition $X \xrightarrow{h_1} Y \xrightarrow{h_2} X$ in which the Priestley map h_1 is surjective, and the inclusion map h_2 is such that $\mathcal{D}(h_2)$ is a complete surjective Heyting homomorphism. In particular, $\mathcal{D}(h_2)$ is a surjective frame homomorphism, and hence $\mathcal{D}(Y)$ is Hausdorff, see [**7**]. Therefore $Y \in$ **HPSp**.

Next we note that h_2 is an equalizer of h and id_X. Indeed, $h \circ h_2 = \mathrm{id}_X \circ h_2$ follows from $h \circ h_2 \circ h_1 = h \circ h = h = h_2 \circ h_1$ because h_1 is surjective. On the other hand, if $h \circ k = k$, then $h_2 \circ (h_1 \circ k) = k$, and the factorizing map $h_1 \circ k$ is unique because h_2 is one-to-one. This implies that the surjective complete Heyting homomorphism $\mathcal{D}(h_2) : \mathcal{D}(X) \to \mathcal{D}(Y)$ is a coequalizer of $\mathcal{D}(h)$ and $\mathrm{id}_{\mathcal{D}(X)}$. But then, by Lemma 4.4 of [**13**], there exists a complemented $a \in \mathcal{D}(X)$ such that, for any $b, c \in \mathcal{D}(X)$,

$$\mathcal{D}(h_2)(b) = \mathcal{D}(h_2)(c) \iff b \wedge a = c \wedge a.$$

Therefore $\mathcal{D}(h_2)(b) = 1_{\mathcal{D}(Y)}$ if and only if $b \geq a$ in $\mathcal{D}(X)$. Simultaneously, since h_2 is the inclusion map of Y into X, it follows that $h_2^{-1}(b) = b \cap Y$ for any clopen decreasing $b \subseteq X$. In particular, $Y = 1_{\mathcal{D}(Y)} = \mathcal{D}(h_2)(a) = h_2^{-1}(a) = a \cap Y$, and hence $Y \subseteq a$. If there exists an $x \in a \setminus Y$ then, by 1.6, there exists a clopen decreasing set $b \subseteq X$ with $x \notin b$ and $Y \subseteq b$. But then $\mathcal{D}(h_2)(b) = h_2^{-1}(b) = Y = 1_{\mathcal{D}(Y)}$ while $b \not\geq a$ – a contradiction. Therefore $Y = a$. Since the element a is complemented in $\mathcal{D}(X)$, the set $Y = h[X]$ is increasing, decreasing and clopen in X, and this is obviously true also for the set $X \setminus Y$. Therefore X is a summand of X, see 1.3. ∎

2.4. Recall that a pair (X, R) is a directed graph if X is a set and $R \subseteq X \times X$ is a binary relation on X, and that a mapping $f : X \to Y$ is a graph homomorphism from (X, R) to a directed graph (Y, S) if $(f(x), f(x')) \in S$ for every $(x, x') \in R$. If $\circ$ denotes also the composition of relations, then the condition defining a graph homomorphism is equivalent to the inclusion

$$f \circ R \subseteq S \circ f.$$

A map $f : X \to Y$ is called a *strong homomorphism* from (X, R) to (Y, S) if

$$f \circ R = S \circ f.$$

The category of all directed graphs and their strong homomorphisms is algebraically universal, and hence every monoid is isomorphic to the monoid of all strong endomorphisms of some directed graph, see [**14**].

As we will see, this is no longer true for strong homomorphisms of *symmetric* directed graphs, that is, graphs (X, R) satisfying $(x, x') \in R$ if and only if $(x', x) \in R$ (see 3.5 and 4.6 below). Thus, in a sense, the graph symmetry plays a role similar to the Hausdorff topological axiom.

Observation. *The category*

$$\mathsf{SymGraph}_S$$

of all symmetric directed graphs and their strong homomorphisms is Hausdorff.

Proof: It is easy to verify that $Y \subseteq X$ is a summand of a symmetric directed graph (X, R) exactly when

(s) $\quad xRy$ and $y \in Y$ imply $x \in Y$,

and that $\mathsf{SymGraph}_S$ is a TS-category. Let f be any (not necessarily idempotent) strong endomorphism of (X, R). To show that $f[X]$ is a factor, suppose that xRy and $y \in f[X]$. Then $y = f(t)$ for some $t \in X$, and hence $x(R \circ f)t$. But then $x(f \circ R)t$ because f is strong, and hence $x = f(u)$ for some u with uRt. Therefore $f[X]$ satisfies (s), and hence it is a summand in (X, R). ∎

2.5. Our main interest is the behaviour of endomorphism monoids in **HHeyt** or **HPSp**, and now we return to these categories. Since the latter category does not contain all of $\mathbf{Top}_{2,o}$ and because proofs for both categories are essentially identical, we work in the more general setting of Hausdorff categories that covers both of them. Moreover, this notion applies to other categories, such as $\mathbf{SymGraph}_S$.

3. Endomorphisms in Hausdorff categories

3.1. With the exception of Theorem 3.6 and its corollary, the symbol X will always denote an object in a Hausdorff TS-category. In what follows,

$\operatorname{End} X$ is the endomorphism monoid of X,
$\operatorname{Aut} X$ is the automorphism group of X, and
$\operatorname{Idp} X = \{f \in \operatorname{End} X \mid f^2 = f\}$.

3.2. Proposition. *Let* $\operatorname{End} X$ *be finite and commutative. Then every* $f \in \operatorname{End} X$ *has a decomposition*

$$f = g \circ h \text{ with } g \in \operatorname{Aut} X \text{ and } h \in \operatorname{Idp} X,$$

in which the idempotent h *is uniquely determined.*

Proof: Since $\operatorname{End} X$ is finite, there is a least integer $n \geq 0$ such that $f^{n+k} = f^n$ for some $k > 0$.

If $n = 0$, then $f \in \operatorname{Aut} X$ and we are done.

Suppose that $n > 0$. Then

$$X \supset f[X] \supset \cdots \supset f^n[X] = f^{n+i}[X] \text{ for all } i > 0.$$

For any integer $kr \geq n$ we thus have $f^{kr}[X] = f^n[X]$. From $f^{2kr} = f^{kr}$ it then follows that $F = f^n[X]$ is a summand. But then the preimage $f^{-1}(F) \supseteq F$ of F is also a summand, and hence the map $\varphi : X \to X$ defined by

$$\varphi(x) = \begin{cases} f(x) & \text{for } x \in f^{-1}(F), \\ x & \text{for } x \notin f^{-1}(F) \end{cases}$$

is an endomorphism of X. For any $x \in f^{-2}(F) \smallsetminus f^{-1}(F)$ we then have

$$\varphi f(x) = f^2(x) \in F \text{ and } f\varphi(x) = f(x) \notin F,$$

which is impossible. From $f^{-2}(F) \supseteq f^{-1}(F)$ it then follows that

$$f^{-2}(F) = f^{-1}(F)$$

and hence, by induction,

$$X = f^{-n}(F) = f^{-1}(F).$$

But then $f[X] = F$ and, consequently,

$$f^{k+1} = f.$$

Therefore the restriction of f^k to F is the identity on F, and hence the mapping $g : X \to X$ given by

$$g(x) = \begin{cases} f(x) & \text{for } x \in F, \\ x & \text{for } x \notin F \end{cases}$$

belongs to $\mathrm{Aut}\, X$. Furthermore,

(3.2.1) $\qquad$ if $x \notin F$ then $gf(x) = fg(x) = f(x)$.

Next we define $h \in \mathrm{End}\, X$ by

$$h(x) = \begin{cases} x & \text{for } x \in F, \\ f(x) & \text{for } x \notin F. \end{cases}$$

Since $h^2(x) = hf(x) = f(x) = h(x)$ for $x \notin F$, we have $h \in \mathrm{Idp}\, X$. If $x \notin F$, then $gh(x) = gf(x) = f(x)$ by (3.2.1) and, if $x \in F$, then $gh(x) = g(x) = f(x)$ again. Therefore $f = g \circ h$ as claimed.

Finally, suppose that $f = g_1 h_1 = g_2 h_2$ with $g_i \in \mathrm{Aut}\, X$ and $h_i \in \mathrm{Idp}\, X$. Then $g_1 h_1 = g_2 h_2 h_2 = g_1 h_1 h_2$ and hence $h_1 = h_1 h_2$. By symmetry $h_2 = h_2 h_1$, and then $h_1 = h_2$ follows from $h_1 h_2 = h_2 h_1$. ∎

3.3. For any $f \in \mathrm{End}\, X$ define

$$\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\} \text{ and } \mathrm{Mov}(f) = \{x \in X \mid f(x) \neq x\}.$$

Since X belongs to a Hausdorff category, for any $h \in \mathrm{Idp}\, X$,

(3.3.1) $\qquad$ $\mathrm{Fix}(h) = h[X]$ and $\mathrm{Mov}(h)$ are summands of X.

3.4. Lemma. *Let* $\mathrm{End}\, X$ *be commutative. Then for any* $h, k \in \mathrm{Idp}\, X$ *and* $f \in \mathrm{End}\, X$

(3.4.1) $\qquad$ $h(\mathrm{Mov}(h)) \subseteq \mathrm{Fix}(f)$, *and*

(3.4.2) $\qquad$ *if* $x \in \mathrm{Mov}(h) \cap \mathrm{Mov}(k)$ *then* $h(x) = k(x)$.

Proof: By (3.3.1), there is an endomorphism

$$\varphi(x) = \begin{cases} x & \text{for } x \in \mathrm{Mov}(h), \\ f(x) & \text{for } x \in \mathrm{Fix}(h). \end{cases}$$

For $x \in \mathrm{Mov}(h)$ we thus have $fh(x) = \varphi h(x) = h\varphi(x) = h(x)$, and this proves (3.4.1). Suppose that $x \in \mathrm{Mov}(h) \cap \mathrm{Mov}(k)$. Then $h(x) \in \mathrm{Fix}(k)$ and $k(x) \in \mathrm{Fix}(h)$, by (3.4.1), and hence $h(x) = kh(x) = hk(x) = k(x)$. ∎

3.5. THEOREM. *Let X be an object of a Hausdorff category, and let* $\operatorname{End} X$ *be finite and commutative. Then there is an* $n \geq 0$ *such that*

$$\operatorname{End} X \cong G_0 \times G_1^0 \times \cdots \times G_n^0$$

where G_0 *is a group and* $G_1^0, \ldots, G_n^0$ *are groups augmented by an external zero.*

PROOF: The monoid $\operatorname{Idp} X$ is a semilattice. For every $h \in \operatorname{Idp} X$ we write $\mu(h) = \operatorname{Mov}(h)$. By (3.4.1), for any $k \in \operatorname{Idp} X$ we have $\operatorname{Mov}(h) \cup \operatorname{Mov}(k) \subseteq \operatorname{Mov}(hk)$ and, since the reverse inclusion is obvious, we conclude that

$$\mu(hk) = \mu(h) \cup \mu(k) \text{ for any } h, k \in \operatorname{Idp} X.$$

Secondly, if $h, k \in \operatorname{Idp} X$ and $\mu(h) = \mu(k)$ then, by (3.4.2), for every $x \in \operatorname{Mov}(h) = \operatorname{Mov}(k)$ we have $h(x) = k(x)$, and then $h = k$ follows. Therefore μ is an injective homomorphism onto a join semilattice $(J, \cup)$ of subsets of X, and $\emptyset = \mu(\mathrm{id}_X) \in J$.

If S is a summand of X and $h \in \operatorname{Idp} X$, then $\operatorname{Mov}(h) \cap S$ is a summand of X as well, see (3.3.1), and hence the mapping $h^{(S)} : X \to X$ given by

$$h^{(S)}(x) = \begin{cases} h(x) & \text{for } x \in \operatorname{Mov}(h) \cap S, \\ x & \text{for } x \notin \operatorname{Mov}(h) \cap S \end{cases}$$

is an idempotent endomorphism with $\operatorname{Mov}(h^{(S)}) = \operatorname{Mov}(h) \cap S$. Since $S = \operatorname{Mov}(k)$ and $S = X \smallsetminus \operatorname{Mov}(k)$ are summands of X, see (3.3.1), we conclude that μ maps $\operatorname{Idp} X$ isomorphically onto the join reduct $(J, \cup)$ of a finite Boolean algebra J of subsets of X. If $J = \{\emptyset\}$, then $\operatorname{Idp} X = \{\mathrm{id}_X\}$ and hence $\operatorname{End} X = \operatorname{Aut} X$. From now on, we assume that J is non-trivial, and let $h_1, \ldots, h_n \in \operatorname{Idp} X$ denote those endomorphisms for which $\{\mu(h_i) \mid i = 1, \ldots, n\}$ is the set of atoms of J. Since any finite Boolean algebra J with more than two elements is atomistic, for every $h \in \operatorname{Idp} X$ there is a (possibly empty) unique subset $A \subseteq \{1, \ldots, n\}$ such that h is the composite of the 'atomic' idempotents h_i with $i \in A$, and we write $h = h_A$. Thus the sets

$$M_i = \operatorname{Mov}(h_i) \text{ with } i = 1, \ldots, n$$

are nonvoid, pairwise disjoint minimal summands of X, and

$$\operatorname{Mov}(h_A) = \bigcup \{M_i \mid i \in A\}.$$

From (3.4.2) it now follows that

$$h_A \restriction M_i = \begin{cases} h_i \restriction M_i & \text{for } i \in A, \\ \mathrm{id}_{M_i} & \text{for } i \notin A. \end{cases} \tag{3.5.1}$$

Finally, write $M_0 = X \smallsetminus (M_1 \cup \cdots \cup M_n)$. Then M_0 is a nonvoid summand of X because of (TS3), and hence

(3.5.2) $$X = M_0 \cup M_1 \cup \cdots \cup M_n \text{ is a concrete sum.}$$

It is clear that

(3.5.3) $$M_0 = \bigcap \{\mathrm{Fix}(h) \mid h \in \mathrm{Idp}\, X\}.$$

Let $f \in \mathrm{End}\, X$ and $h \in \mathrm{Idp}\, X$. If $x \in \mathrm{Mov}(h)$, then $fh(x) = h(x)$ by (3.4.1), while, for any $x \in \mathrm{Fix}(h)$, we obviously have $fh(x) = f(x)$. In particular,

(3.5.4)
$$\text{if } g \in \mathrm{Aut}\, X \text{ and } h \in \mathrm{Idp}\, X, \text{ then } gh(x) = \begin{cases} h(x) & \text{for } x \in \mathrm{Mov}(h), \\ g(x) & \text{for } x \in \mathrm{Fix}(h). \end{cases}$$

If $g \in \mathrm{Aut}\, X$ and $h = h_i$ for some $i = 1, \ldots, n$, then $h_i g(x) = g h_i(x) = h_i(x)$ for every $x \in M_i = \mathrm{Mov}(h_i)$, that is,

(3.5.5) $$h_i g \restriction M_i = h_i \restriction M_i \text{ for any } g \in \mathrm{Aut}\, X \text{ and } i = 1, \ldots, n.$$

Since g is invertible, we have $h_i g(x) = g h_i(x) \neq g(x)$ for every $x \in M_i$. Thus $g[M_i] \subseteq M_i$ for every $i = 1, \ldots, n$, and it follows that

(3.5.6) $$g[M_j] = M_j \text{ for any } g \in \mathrm{Aut}\, X \text{ and } j = 0, 1, \ldots, n$$

Now let $f \in \mathrm{End}\, X$ be arbitrary. Then $f = g \circ h$ for some $g \in \mathrm{Aut}\, X$ and a uniquely determined $h \in \mathrm{Idp}\, X$, by Proposition 3.2. Since $h = h_A$ for a unique $A \subseteq \{1, \ldots, n\}$, from (3.5.4) and (3.5.1) it then follows that

(3.5.7) $$f(x) = \begin{cases} h_j(x) & \text{if } x \in M_j \text{ and } j \in A, \\ g(x) & \text{if } x \in M_j \text{ and } j \in \{0, 1, \ldots, n\} \smallsetminus A. \end{cases}$$

For $j = 0, 1, \ldots, n$, denote $f_j = f \restriction M_j$. Then $f_j = h_j$ for $j \in A$ and $f_j \in \mathrm{Aut}\, M_j = G_j$ for all $j \in \{0, 1, \ldots, n\} \smallsetminus A$. Since h_i is an external zero of G_i for all $i = 1, \ldots, n$, see (3.5.5), this shows that

$$\varphi(f) = (f_0, f_1, \ldots, f_n)$$

is a well-defined mapping of $\mathrm{End}\, X$ into the Cartesian product

$$P = G_0 \times G_1^0 \times \cdots \times G_n^0$$

of the group G_0 and monoids $G_i^0 = G_i \cup \{0_i\}$ obtained from the groups G_i by the addition of an external zero 0_i for each $i = 1, \dots n$. Using (3.5.7), it is easy to verify that $\varphi : \operatorname{End} X \to P$ is a monoid homomorphism.

Conversely, let $(f_0, f_1, \dots, f_n) \in P$. Then $f_0 \in \operatorname{Aut} M_0$ and $f_i \in \operatorname{Aut} M_i \cup \{0_i\}$, and the mapping $f = \psi(f_0, f_1, \dots, f_n)$ given by

$$f(x) = \begin{cases} f_0(x) & \text{for } x \in M_0 \\ f_i(x) & \text{for } x \in M_i \text{ with } f_i \neq 0_i, \\ h_i(x) & \text{for } x \in M_i \text{ with } f_i = 0_i \end{cases}$$

belongs to $\operatorname{End} X$ because of (3.5.2). It is clear that the mapping ψ is the inverse of the homomorphism φ. ∎

Corollary 3.6. *Let X be an object of a Hausdorff category. Let* $\operatorname{End} X$ *be finite and commutative, and let* $\operatorname{Aut} X$ *be trivial. Then* $\operatorname{End} X$ *is isomorphic to a free join semillatice with zero, or, equivalently, to a semilattice* $(2^n, \cup)$. ∎

4. Monoid representations

In this section, we shall use certain spaces and graphs constructed in [**14**] as building blocks for representations of the finite commutative monoids described by Theorem 3.5 as endomorphism monoids in the categories $\mathbf{Top}_{2,o}$ (and hence also in **HHeyt** and **PSp**) and $\mathbf{SymGraph}_S$. First we turn to the topological construction.

4.1. For $j = 0, 1, \dots$, let H^j be non-trivial metric continua with distinguished points $a_0^j, a_1^j, a_2^j \in H^j$ constructed in Section 6 of Chapter VI in [**14**]. We divide up these continua into two families

$$\mathcal{H}^i = \{H^{2k+i} \mid k = 1, 2, \dots\} \text{ with } i = 0, 1,$$

and denote

$$H_0^j = H^j \smallsetminus \{a_0^j, a_1^j, a_2^j\}$$

Further, we recall the metric spaces $\mathcal{M}(X, R, \varphi)$ amalgamated from these metric continua H^j elsewhere in Chapter VI of [**14**], and consider only those spaces $\mathcal{M}(X, R, \varphi)$ such that

$$(*) \qquad \text{only } H^j \in \mathcal{H}^1 \text{ occur in } \mathcal{M}(X, R, \varphi).$$

Thus we begin with

- non-trivial metric continua H^j with distinguished points $a_0^j, a_1^j, a_2^j \in H^j$ for $j = 0, 1, \dots,$
- a countably infinite collection $\mathcal{P}_1$ of metric spaces $P = \mathcal{M}(X, R, \varphi)$ subject to $(*)$ above.

The proof of Lemma 6.9 in Section 6 of Chapter VI in [**14**] shows that $\mathcal{P}_1$ has these properties:

(4.1.1) if H^j is embedded into a space $P \in \mathcal{P}_1$ in such a way that H_0^j is open, then every continuous map $f : H_0^k \to P$ with $f[H_0^k] \cap H_0^j \neq \emptyset$ is either a constant or else $j = k$ and f is the embedding,

(4.1.2) if $H^j \in \mathcal{H}^0$ and $P \in \mathcal{P}_1$ then any continuous $f : H^j \to P$ is constant.

Furthermore, we recall that for any monoid S there exists a space $P(S) \in \mathcal{P}_1$ such that all non-constant continuous maps $f : P(S) \to P(S)$ form a monoid isomorphic to S. Thus, in particular,

(4.1.3) for any group G there exists some $P(G) \in \mathcal{P}_1$ such that the monoid of all open continuous mappings $f : P(G) \to P(G)$ is isomorphic to G.

4.2. Construction. For a metric space P and a non-trivial metric continuum H^j, we define a space

$$P \star H^j$$

on the Cartesian product $P \times H^j$ by requiring that

- the neighbourhoods of (x, a_0^j) are the sets containing $(U \times \{a_0^j\}) \times (\{x\} \times V)$, where U is a neighbourhood of x in P and V is a neighbourhood of a_0^j in H^j, while
- the neighbourhoods of any (x, y) with $y \neq a_0^j$ are the sets containing $\{x\} \times V$, where V is a neighbourhood of y in H^j.

Having renamed each (x, a_0^j) as x, we observe that P is a subspace of $P \star H^j$. The space $P \star H^j$ may thus be visualized as the space obtained by attaching separate copies of H^j to members of P by their distinguished element a_0^j.

The claim below is immediate.

LEMMA. *The mapping $g \times id : P \star H^j \to Q \star H^j$ is continuous and open for any open continuous $g : P \to Q$.* ∎

4.3. LEMMA. *Let Q be such that every continuous mapping of H^k to Q is constant, and let a mapping*

$$f : P \star H^k \to Q \star H^j$$

be open continuous. Then $k = j$ and $f = g \times id$ for some open continuous $g : P \to Q$.

PROOF: Let $x \in P$. Since f is open, the set $f[\{x\} \times H^k]$ is not a singleton, and hence $f[\{x\} \times H^k] \not\subseteq Q$ by the hypothesis. But then (4.1.1) implies that $k = j$ and there is a unique $y \in Q$ such that $f(x, z) = (y, z)$ for all $z \in H^k$.

This gives rise to a unique mapping $g : P \to Q$ for which $f = g \times \mathrm{id}$. It is routine to verify that g is open. ∎

4.4. Recall that, for any space P, the symbol $\mathfrak{Op}(P)$ denotes the monoid of all open continuous selfmaps of P.

LEMMA. *Let $\{P_a \mid a \in A\}$ be a countable collection of metric spaces from $\mathcal{P}_1$. Then there exists a metric space P such that*

$$\mathfrak{Op}(P) \cong \prod\{\mathfrak{Op}(P_a \mid a \in A\}.$$

PROOF: Select distinct metrizable continua $H^{j(a)} \in \mathcal{H}^0$ and then apply (4.1.2), Lemma 4.3 and Lemma 4.2. ∎

4.5. Now we turn to the category $\mathbf{SymGraph}_S$ of strong homomorphisms od symmetric directed graphs.

Let A be an arbitrary set, and let $Z = (\{0\}, \{(0,0)\})$ denote the singleton graph in the alg-universal category $\mathbf{Gra}_c$ of all connected directed graphs and their homomorphisms. From Theorem 3 of [**4**] and from the results of Chapter II of [**14**] it follows that for every $a \in A$ there is a full embedding Φ_a of $\mathbf{Gra}_c$ into its full subcategory formed by connected *symmetric* graphs such that

(4.5.1) the homomorphism $\Phi_a : (X, R) \to \Phi_a(Z)$ belongs to $\mathbf{SymGraph}_S$ for any (X, R),

(4.5.2) there is no homomorphism $\Phi_a(X, R) \to \Phi_{a'}(X', R')$ for distinct $a, a' \in A$.

Since $\mathbf{Gra}_c$ is an alg-universal category, for every group G there exists a graph (X, R) with $\mathrm{End}(X, R) \cong G$. But then $\mathrm{End}\,\Phi_a(X, R) \cong G$ as well and, since every $g \in \mathrm{End}\,\Phi_a(X, R)$ is invertible, the monoid of all strong endomorphisms of $\Phi_a(X, R)$ is isomorphic to the group G. Therefore

(4.5.3) for any $a \in A$, every group is isomorphic to the monoid of all strong endomorphisms of some $\Phi_a(X, R)$.

4.6. The category $\mathbf{Metr}_o$ of all metric spaces and all their open continuous maps is a subcategory of $\mathbf{Top}_{2,o}$. Since metric spaces are regular, the contravariant functor $\mathfrak{D}$ from 1.4 maps $\mathbf{Metr}_o$ onto a full subcategory of $\mathbf{HHeyt}$. Accordingly, the positive, constructive claim of the theorem below for categories other than $\mathbf{SymGraph}_S$ will be established through a construction of appropriate metric spaces.

THEOREM. *Let $\mathcal{C}$ be any one of the categories $\mathbf{Top}_{2,o}$, $\mathbf{HHeyt}$, $\mathbf{PSp}$, or $\mathbf{SymGraph}_S$, and let S be a finite commutative monoid. Then $S \cong \mathrm{End}\,X$ for some object $X \in \mathcal{C}$ if and only if, for some $n \geq 0$,*

$$S = G_0 \times G_1^0 \times \cdots \times G_n^0$$

where G_0 is a group and each G_j^0 is a group augmented by zero.

PROOF: To complete the proof for the three 'topological' categories, in view of Theorem 3.5, we need only represent a group G and any zero extension G_j^0 of a group G_j by a metric space satisfying the hypothesis of Lemma 4.4. To represent a given group G, we simply use the metric space $P(G) \in \mathcal{P}_1$ of (4.1.3). A proper extension $P(G_j) \cup \{p\}$ of the space $P(G_j) \in \mathcal{P}_1$ from (4.1.3) by an open singleton $\{p\}$ obviously represents G_j^0 and satisfies the hypothesis of Lemma 4.4.

To complete the proof, we need to represent the monoid $G_0 \times G_1^0 \times \cdots \times G_n^0$ in the category $\mathbf{SymGraph}_S$. But this is also easy. For $i = 0, 1, \ldots, n$, we select $(X_i, R_i) \in \mathbf{Gra}_c$ so that the strong endomorphisms of $\Phi_i(X_i, R_i)$ form a monoid isomorphic to G_i, see (4.5.3), and then we form a disjoint union

$$X = \Phi_0(X_0, R_0) \cup \bigcup_{i=1}^{n} \Phi_i(X_i, R_i) \cup \Phi_i(Z).$$

Then X belongs to $\mathbf{SymGraph}_S$ and, since all graphs $\Phi_j(\ldots)$ are connected, from (4.5.2) it follows that every strong endomorphism g of X preserves $\Phi_0(X_0, R_0)$ and also every $\Phi_i(X_i, R_i) \cup \Phi_i(Z)$ with $i = 1, \ldots, n$. Therefore

$$\operatorname{End} X \cong G_0 \times \prod_{i=1}^{n} \operatorname{End}\big(\Phi_i(X_i, R_i) \cup \Phi_i(Z)\big).$$

Furthermore, any endomorphism g either preserves $\Phi_i(X_i, R_i)$ or else maps this graph to $\Phi_i(Z)$. From (4.5.1) it now follows that $\operatorname{End}\big(\Phi_i(X_i, R_i) \cup \Phi_i(Z)\big) \cong G_i^0$ for all $i = 1, \ldots, n$. Therefore $\operatorname{End} X \cong S$, as claimed. ∎

4.7. By Theorem 4.6, all join reducts $(2^n, \cup)$ of finite Boolean algebras are isomorphic to $\mathfrak{Op}(P)$ for some metric space P. This result can be extended to the infinite case.

THEOREM. *For any set C, there exists a metric space P such that $\mathfrak{Op}(P)$ is isomorphic to $(2^C, \cup)$.*

PROOF: Since the category of metric spaces is almost universal, see Chapter VI of [**14**], there exists a colelction $\{P_c \mid c \in C\}$ of metric connected spaces such that a non-constant continuous $f : P_c \to P_{c'}$ exists only when $c = c'$, and then it is the identity mapping of P_c. For the disjoint union

$$P = \{p\} \cup \bigcup_{c \in C} P_c$$

with the union topology we then obviously have $\mathfrak{Op}(P) \cong (2^C, \cup)$. ∎

Remark. From 4.5 it follows that a similar claim holds true for the Hausdorff category $\mathsf{SymGraph}_S$ of symmetric graphs and their strong homomorphisms.

4.8. Infinite commutative monoids with no nontrivial invertible elements need not contain idempotents, and this is also true for endomorphism monoids in Hausdorff categories.

THEOREM. *Let* $\mathbb{N}$ *denote the additive monoid of all natural numbers. Then, for any* $n = 1, 2, \dots, \omega$, *there is a metric space* P *with* $\mathfrak{Op}(P) \cong \mathbb{N}^n$.

PROOF: For $H^1 \in \mathcal{H}^1$ from 4.1, denote $H = H^1 \setminus \{a_0^1\}$ and set

$$P = H \times \{0, 1, \dots\},$$

where the neighbourhoods of any (x, k) with $x \neq a_1^1$ are all sets containing the set $U \times \{k\}$ for some neighbourhood U of x in H, while the neighbourhoods of (a_1^1, k) are all sets containing the set $(U \times \{k\}) \cup ((V \setminus \{a_0^1\}) \times \{k+1\})$ in which U is a neighbourhood of a_1^1 in H and V is a neighbourhood of a_0^1 in H^1. In other words, P is obtained from

$$H \cup (H^1 \times \{1, 2, \dots\})$$

by means of identifying each (a_1^1, k) with $(a_0^1, k+1)$.

An open map $f : P \to P$ is not constant, and hence, by (4.1.1), it sends each $H \times \{j\}$ identically onto some $H \times \{k\}$, and it is not difficult to see that, in fact, there is some $n \geq 0$ such that $f(x, j) = (x, j + n)$ for all $j \geq 0$. Since any such map f is open, this proves that

$$\mathfrak{Op}(P) \cong \mathbb{N}.$$

Higher powers of $\mathbb{N}$ can be represented because Lemma 4.4 applies here, see (4.1.1). ∎

REFERENCES

1. J. Adámek, H. Herrlich and G. E. Strecker, "Abstract and Concrete Categories," J. Wiley and Sons, New York, 1990.
2. B. Banaschewski, *On pushing out frames*, Comment. Math. Univ. Carolin. **31** (1990), 13–21.
3. B. Banaschewski and A. Pultr, *Variants of openness*, Applied Categorical Structures **2** (1994), 331-350.

4. Z. Hedrlín and J. Sichler, *Any boundable binding category contains a proper class of mutually disjoint copies of itself*, Algebra Universalis **1** (1971), 97–103.
5. J. R. Isbell, *Atomless parts of spaces*, Math. Scand. **31** (1972), 5–32.
6. A. Joyal and M. Tierney, *An Extension of the Galois Theory of Grothendieck*, Mem. Amer. Math. Soc. **51** (1984).
7. P. T. Johnstone, "Stone Spaces," Cambridge University Press, Cambridge, 1982.
8. S. MacLane, "Categories for the Working Mathematician," Springer-Verlag, New York, 1971.
9. A. B. Paalman de Miranda, *Topological representations of semigroups*, in "in General Topology and its Relations to Modern Analysis and Algebra II, Proc. 2nd Prague Top. Symposium," Academia, Prague, 1967, pp. 276–282.
10. H. A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. **2** (1970), 186–190.
11. H. A. Priestley, *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. **24** (1972), 507–530.
12. H. A. Priestley, *Ordered sets and duality for distributive lattices*, Ann. Discrete Math. **23** (1984), 36–90.
13. A. Pultr and J. Sichler, *Endomorphisms of complete Heyting algebras*, Semigroup Forum **54** (1997), 364–374.
14. A. Pultr and V. Trnková, "Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories," North-Holland, Amsterdam, 1980.
15. V. Trnková, *Non-constant continuous mappings of metric or compact Hausdorff spaces*, Comment. Math. Univ. Carolin. **13** (1972), 283–295.
16. V. Trnková, *Strong embeddings of the categories of graphs into topological categories*, in "in Recent Advances in Graph Theory (Proc. Symp. Prague 1974)," Academia, Prague, 1975, pp. 511–515.