Hom-properties are uniquely factorizable into irreducible factors

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Abstract

For a simple graph H, a graph G is called *H-colourable* if there is a homomorphism from G to H (a mapping $f{:}V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$). The class $\to H$ of H-colourable graphs is an additive hereditary property of graphs, called a hom-property. For hereditary properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of a graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that each subgraph $G[V_i]$ induced by V_i has property $\mathcal{P}_i$, $i = 1, 2, \ldots, n$. The class of all vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs is denoted by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$. An additive hereditary property $\mathcal{P}$ is *reducible* if there exist additive hereditary properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$, it is *irreducible* otherwise. A graph is a *core* if it admits no homomorphism to any of its proper subgraphs.

We prove that for any core H the hom-property $\to H$ is reducible if and only if H is a join (the Zykov sum of two

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nonempty graphs). Moreover, we prove that the factorization of any hom-property $\to H$ into irreducible factors is unique.

1 Introduction

We consider finite undirected simple graphs (without loops or multiple edges). In general, we follow the notation and terminology of [3]. The *join* $G+H$ of two graphs G and H is the graph consisting of the disjoint union of G and H and all the edges between $V(G)$ and $V(H)$. If a graph G is a join we say that G it is *decomposable*. A graph that is not decomposable is called *indecomposable*. It is easy to see that a graph G is decomposable if and only if its complement $\overline{G}$ is not connected. Then G is the join of the complements of the components of $\overline{G}$. Thus every decomposable graph G can be expressed in a unique way as the join of indecomposable graphs.

We denote by $\mathcal{I}$ the class of all finite simple graphs. A *graph property* is a nonempty isomorphism-closed subclass of $\mathcal{I}$. (We also say that a graph has the property $\mathcal{P}$ if $G \in \mathcal{P}$.) A property $\mathcal{P}$ of graphs is called *hereditary* if it is closed under subgraphs, i.e., if $H \subseteq G$ and $G \in \mathcal{P}$ imply $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if it is closed under disjoint unions of graphs, i.e., if every graph has property $\mathcal{P}$ provided all of its connected components have this property. A hereditary property $\mathcal{P}$ is uniquely characterized by the set

$$\boldsymbol{M}(\mathcal{P}) = \{G \in \mathcal{P} |\text{ for any edge } e \in E(\overline{G}),\ \text{the graph } G+e \text{ does not belong to } \mathcal{P}\}$$

of its $\mathcal{P}$-*maximal graphs*.

Additive and hereditary properties of graphs, partially ordered by set-inclusion form an algebraic distributive lattice $(\mathbb{L}^a, \subseteq)$ with the least element $\mathcal{O} = \{G \in \mathcal{I} \mid E(G) = \emptyset\}$ (see [1],[7],[2]).

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n \geq 2$, be hereditary graph properties. A vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*partition* of a graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that each subgraph $G[V_i]$ induced by V_i has property $\mathcal{P}_i$, $i = 1, 2, \ldots, n$. We denote by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ the class of all vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs. An additive hereditary property $\mathcal{P}$ is called *reducible* in $\mathbb{L}^a$ if there exist additive hereditary properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$, and it is called *irreducible* otherwise.

The notion of reducible properties was introduced and studied in connection to the existence of uniquely partitionable graphs ([7, 8, 5]).

A *homomorphism* of a graph G to a graph H is an edge-preserving mapping f: $V(G) \to V(H)$, i.e., $e = uv \in E(G)$ implies $f(e) = f(u)f(v) \in E(H)$. In this case we say that G is *homomorphic* to H and we write $G \to H$. The image of G is denoted by $f(G)$, i.e., $f(G) = (\{f(v)|v \in V(G)\}, \{f(e)|e \in E(G)\})$.

A *core of a graph* G is any subgraph G' of G such that $G \to G'$ while G fails to be homomorphic to any proper subgraph of G'. It is known that up to isomorphism every graph G has a unique core, denoted by $C(G)$. A graph G is called a *core* if G is a core for itself, i.e. $G \cong C(G)$. A graph G homomorphic to a given graph H is also said to be H-colourable. Graph homomorphisms and their structures are extensively investigated (see e.g. [4], [11], [5], [13], more references may be found in the survey [10]).

Let us denote by $\to H$ the set of all H-colourable graphs $G \in \mathcal{I}$. The properties $\to H$, $H \in \mathcal{I}$, are called *hom-properties* or *colour classes* (see [10]).

We shall show in Section 2 that for every $H \in \mathcal{I}$, the property $\to H$ is additive and hereditary. As it was first pointed out by P. Cameron [personal communication], the hom-properties form a sublattice of $\mathbb{L}^a$ isomorphic to the lattice of the cores partially ordered by the existence of homomorphisms (cf. [12, 10, 6]).

In this paper we deal with factorization of hom-properties into irreducible factors in $\mathbb{L}^a$. The problem:

Is the factorization of every property into irreducible factors unique in $\mathbb{L}^a$*?*

seems to be very difficult and only partial results are known (cf. [5], Problem 17.9, p.266, [9]). Our main results can be formulated as follows:

Theorem 1 *Let a graph H be a core. A hom-property $\to H$ is irreducible if and only if H is indecomposable.*

Since the decomposition of a decomposable graph into the join of indecomposable graphs is unique, the factorization of reducible hom-properties into irreducible hom-properties is unique as well. We shall prove that this factorization is unique also in $\mathbb{L}^a$:

Theorem 2 *Let a core $H = H_1 + H_2 + \ldots + H_n$ be the join of indecomposable graphs H_i, $i = 1, 2, \ldots, n$. Then $\to H = (\to H_1)\circ(\to H_2)\circ\ldots\circ(\to H_n)$ is the unique factorization of $\to H$ into irreducible factors in $\mathbb{L}^a$, apart from the order of the factors.*

As a corrolary of Theorem 2 we obtain the characterization of decomposable cores:

Theorem 3 *Let a graph $H = H_1 + H_2 + \ldots + H_n$ be the join of indecomposable graphs H_i, $i = 1, 2, \ldots, n$. Then H is a core if and only if each H_i, $i = 1, 2, \ldots, n$, is a core.*

We prove some basic results on hom-properties in Section 2, the proofs of our main results are presented in Section 3.

2 Preliminaries

In this Section we present some basic results on hom-properties which will be used in the proofs of our main results.

Proposition 2.1 *For any graph $H \in \mathcal{I}$, the hom-property $\to H$ is hereditary and additive.*

Proof For every $G \in \to H$, there exists a homomorphism $f : V(G) \to V(H)$. For any $G^* \subseteq G$, the mapping f restricted to $V(G^*)$ is a homomorphism of G^* to H. Therefore $G^* \in \to H$ and $\to H$ is hereditary.

Let G be the disjoint union of graphs $G_1, G_2 \in \to H$, and let f_1, f_2 be homomorphisms of G_1 and G_2 to H, respectively. Then the mapping $f : V(G_1 \cup G_2) \to H$ defined in the following way

$$f(v) = \begin{cases} f_1(v) & \text{for } v \in V(G_1), \\ f_2(v) & \text{for } v \in V(G_2) \end{cases}$$

is a homomorphism of G to H. Hence, $\to H$ is also additive. $\square$

The following technical lemma will be used further on.

Lemma 2.2 *Let F_1, F_2, F be graphs and let $G_1 = F_1 + F$ and $G_2 = F_2 + F$. Then $G_1 \to G_2$ implies $F_1 \to F_2$.*

Proof We will proceed by induction on $|V(F)|$. The statement is clear for $V(F) = \emptyset$.

Suppose $V(F) \neq \emptyset$ and let ϕ be a homomorphism of G_1 to G_2. Since ϕ preserve edges, $\phi(G_1) = \phi(F_1 + F) = \phi(F_1) + \phi(F)$. Let us denote by A,B,C and D the subsets of vertices of the graph G_2 defined as follows:

$$A = \phi(V(F_1)) \cap V(F_2);$$

$$B = \phi(V(F_1)) \cap V(F);$$

$$C = \phi(V(F)) \cap V(F_2);$$

$$D = \phi(V(F)) \cap V(F).$$

If the set B is empty, we have $\phi(F_1) \to F_2$ ad hence $F_1 \to \phi(F_1) \to F_2$. Thus we may assume $B \neq \emptyset$, whence $|D| < |V(F)|$. Since ϕ restricted to $F[B] + F[D]$ is a homomorphism to $F_2[C] + F[D]$, it follows from the induction hypothesis that $F[B] + F[D] \to F_2[C] + F[D]$ implies $F[B] \to F_2[C]$. This yields $F_1 \to F_2[A] + F[B] \to F_2[A] + F_2[C] \subseteq F_2$. Hence in both cases $F_1 \to F_2$. □

We further investigate graphs which are maximal with respect to additive hereditary properties.

Proposition 2.3 *Let G be a $\mathcal{P}_1 \circ \mathcal{P}_2$-maximal graph. Then every vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition (V_1, V_2) of G satisfies*

$$G = G[V_1] + G[V_2],$$

and the graphs $G[V_i]$ are $\mathcal{P}_i$-maximal ($i = 1, 2$).

Proof Let $G \in \boldsymbol{M}(\mathcal{P}_1 \circ \mathcal{P}_2)$ be a $\mathcal{P}_1 \circ \mathcal{P}_2$-maximal graph and let (V_1, V_2) be an arbitrary vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G. Suppose, to the contrary, that there exists an edge $e = xy$ such that $x \in V_1$ and $y \in V_2$ and $e \notin E(G)$. Then $G + e \in \mathcal{P}_1 \circ \mathcal{P}_2$ contradicting the assumption of $\mathcal{P}_1 \circ \mathcal{P}_2$-maximality of G. This proves that $G = G[V_1] + G[V_2]$.

Assume that for some $i = 1, 2$, $G[V_i]$ is not $\mathcal{P}_i$-maximal. Then we can add an edge e to the edge set of $G[V_i]$ so that $G[V_i] + e \in \mathcal{P}_i$ and, consequently, $G + e \in \mathcal{P}_1 \circ \mathcal{P}_2$. This is again a contradiction with the assumed $\mathcal{P}_1 \circ \mathcal{P}_2$-maximality of G. □

For any graph $G \in \mathcal{I}$ with vertex set $V(G) = \{v_1, v_2, \dots v_n\}$, we define a *multiplication* $G^{::}(W_1, W_2, \dots, W_n)$ of G in the following way:

1. $V(G^{::}) = W_1 \cup W_2 \cup \dots W_n$,
2. for each $1 \le i \le n : |W_i| \ge 1$,
3. for any pair $1 \le i < j \le n$: $W_i \cap W_j = \emptyset$,
4. for any $1 \le i \le j \le n$, $u \in W_i, v \in W_j$: $\{u, v\} \in E(G^{::})$ if and only if $\{v_i, v_j\} \in E(G)$.

The sets $W_1, W_2, \dots, W_n$ are called the *multivertices* corresponding to vertices $v_1, v_2, \dots, v_n$, respectively. In the case $|W_1| = |W_2| = \dots = |W_n| = k$, we denote the graph $G^{::}(W_1, W_2, \dots, W_n)$ by $G(k)$. Obviously, $G(1) \cong G$.

It is not difficult to see the following (the details and more results about hom-maximal graphs can be found in [6]):

Proposition 2.4 *Let $H = H_1 + H_2$ be a core. Then the following statements hold true:*

(i) the graphs H_1 and H_2 are cores;

(ii) $\to H = (\to H_1) \circ (\to H_2)$;

(iii) for every $k \ge 1$, the graph $H(k) = H_1(k) + H_2(k)$ is $(\to H)$-maximal;

(iv) for any graph $G \in \to H$, there exists a $k \ge 1$ such that $G \subseteq H(k)$.

3 The proofs of the main results

Proof of Theorem 1 If the core H is decomposable, then the property $\to H$ is reducible according to Proposition 2.4(ii).

Since the graph H is a core, $H = H(1)$ is $(\to H)$-maximal. Then for $\to H = \mathcal{P}_1 \circ \mathcal{P}_2$, Proposition 2.3 would imply that H is a join. Hence $\to H$ is irreducible, provided H is indecomposable. $\square$

Proof of Theorem 2 Let $H = H_1 + H_2 + \ldots + H_n$ be a core, expressed as the join of indecomposable graphs H_i, $i = 1, 2, \ldots, n$. Assume that $\to H = \mathcal{P} \circ \mathcal{Q}$. We prove that then $\mathcal{P}$ and $\mathcal{Q}$ are hom-properties.

For every $k \geq 1$ and any vertex $(\mathcal{P},\mathcal{Q})$-partition (V_1, V_2) of the $\to H$-maximal graph $H(k)$, each of the subgraphs $H_i(k) = H(k)[W_i]$ induced by $W_i, i = 1, 2, \ldots, n$, is either contained in V_1 or in V_2 (in general, if (V_1, V_2) is a vertex $(\mathcal{P},\mathcal{Q})$-partition of a $(\mathcal{P} \circ \mathcal{Q})$-maximal graph G and F is an indecomposable induced subgraph of G , then either $F \subseteq G[V_1]$ or $F \subseteq G[V_2]$). Therefore,

$$H(k)[V_1] = \sum_{i \in I_1} H_i(k); \quad H(k)[V_2] = \sum_{i \in I_2} H_i(k);$$

for some $I_1, I_2 \subset \{1, 2, \ldots, n\}$ such that $I_1 \cup I_2 = \{1, 2, \ldots, n\}$, $I_1 \cap I_2 = \emptyset$.

Let (I_1^*, I_2^*) be a partition of the set $\{1, 2, \ldots, n\}$ such that the vertex $(\mathcal{P}, \mathcal{Q})$-partitions (V_1^*, V_2^*) of $H(k)$ induce

$$H(k)[V_1^*] = \sum_{i \in I_1^*} H_i(k); \quad H(k)[V_2^*] = \sum_{i \in I_2^*} H_i(k) \tag{1}$$

for infinitely many $k \geq 1$. Let us remark that then (1) holds true for every k, because of $\to H$ being hereditary ($H(k_1) \subseteq H(k_2)$ for $k_1 \leq k_2$).

Let us denote $F_1 = \sum_{i \in I_1^*} H_i$ and $F_2 = \sum_{i \in I_2^*} H_i$. Then for every $k \geq 1$, $F_1(k) \in \mathcal{P}$ and $F_2(k) \in \mathcal{Q}$. Hence by Proposition 2.4.(iv), $\to F_1 \subseteq \mathcal{P}$ and $\to F_2 \subseteq \mathcal{Q}$.

On the other hand, consider a graph $G \in \mathcal{P}$. Then $G + F_2 \in \mathcal{P} \circ \mathcal{Q}$ and hence $G + F_2 \to F_1 + F_2$ (as $\to H = \mathcal{P} \circ \mathcal{Q}$). It follows from

Lemma 2.2 that $G \to F_1$. Thus $\mathcal{P} \subseteq\to F_1$, and analogously $\mathcal{Q} \subseteq\to F_2$.

We have proved: If a core H is the join $\sum_{i=1}^{n} H_i$ of indecomposable cores H_i and $\to H = \mathcal{P}\circ\mathcal{Q}$, then

$$\mathcal{P} =\to \sum_{i\in I_1^*} H_i$$

and

$$\mathcal{Q} =\to \sum_{i\in I_2^*} H_i$$

for a partition (I_1^*, I_2^*) of the set $\{1, 2, \ldots, n\}$. The proof of Theorem 2 follows immediately by induction on n. $\square$

Proof of Theorem 3 According to (i) of Proposition 3 it suffices to show that if $H = H_1 + H_2 + \ldots + H_n$ is the join of indecomposable cores H_i, $i = 1, 2, \ldots, n$, then H is a core as well. Suppose $C(H) = \widetilde{H_1} + \widetilde{H_2} + \ldots + \widetilde{H_k}$, where $\widetilde{H_i}$, $i = 1, 2, \ldots, k$ are indecomposable. Since $\to H =\to C(H)$, it follows from Theorem 2 that

$$(\to H_1)\circ(\to H_2)\circ\ldots\circ(\to H_n) = (\to \widetilde{H_1})\circ(\to \widetilde{H_2})\circ\ldots\circ(\to \widetilde{H_k})$$

is the unique factorization of $\to H =\to C(H)$ into hom-properties. Hence, $k = n$ and $\to H_i =\to \widetilde{H_i}$ for $i = 1, 2, \ldots, n$, for suitable permutation of $\widetilde{H_i}$'s. Each H_i is a core, and thus $|V(\widetilde{H_i})| \geq |V(H_i)|$ for $i = 1, 2, \ldots, n$. Therefore, $|V(C(H))| \geq |V(H)|$ and it follows that $H \cong C(H)$, i.e., H is a core. $\square$

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References

[1] M.Borowiecki and P.Mihók, *Hereditary Properties of Graphs*, in: Advances in Graph Theory, Vishwa International Publications, (1991), 41 – 68.

[2] M.Borowiecki, I.Broere, M.Frick, P.Mihók and G.Semanišin, *Survey of Hereditary Properties of Graphs*, (to appear)

[3] R.L.Graham, M.Grötschel and L.Lovász, *Handbook of combinatorics*, Elsevier Science B.V. Amsterdam, 1995

[4] P.Hell, J.Nešetřil, *The core of a graph*, Discrete Mathematics 109(1992), pp. 117 – 126

[5] T.R.Jensen, B.Toft, *Graph Colouring problems*, Wiley–Interscience Publications (New York), 1995

[6] J.Kratochvíl, P.Mihók and G.Semanišin, *Graphs maximal with respect to hom-properties*, (to appear)

[7] P.Mihók, *Additive hereditary properties and uniquely partitionable graphs*, in: Graphs, Hypergraphs and Matroids (1985), Zielona Góra, pp. 49 – 58.

[8] P.Mihók and G.Semanišin, *Reducible properties of graphs*, Discussiones Math., Graph Theory, **15** Num.1 (1995), pp. 11 – 18.

[9] P.Mihók and R.Vasky, *On the factorization of reducible properties of graphs into irreducible factors*, Discussiones Math., Graph Theory, **15**,Num.2 (1995), pp. 195 – 204.

[10] J.Nešetřil, *Graph homomorphisms and their structures*, Proc. Seventh Quadrennial International Conference on the Theory and Applications of graphs Vol.2 (1995), pp. 825 – 832.

[11] F.S.Roberts, *New directions in graph theory (emphasizing applications)* in: Quo Vadis Graph Theory ? J.Gimbel,J.W.Kennedy and L.V.Quintas (eds), Annals of Discrete Math., **55**, (1993), pp. 13 – 44.

[12] E. Welzl, *Color Families are Dense*, Theoretical Computer Sci. **17**, (1982), pp. 29 – 41.

[13] X. Zhu, *Uniquely H-colourable graphs with large girth*, J. Graph Theory, **23** No.1 (1996), pp. 33 – 41.