Graphs as rotations

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Abstract

Using a notation of corner between edges when graph has a fixed rotation, i.e. cyclical order of edges around vertices, we define combinatorial objects - combinatorial maps as pairs of permutations, one for vertices and one for faces. Further, we define multiplication of these objects, that coincides with the multiplication of permutations. We consider closed under multiplication classes of combinatorial maps that consist of closed classes of combinatorial maps with fixed edges where each such class is defined by a knot. One class among them is special, containing selfconjugate maps.

1 Introduction

Usually, speaking about graphs on orientable surfaces [15], we suppose that edges incident with a vertex are cyclically ordered, or, in other words, the rotation of such graphs is fixed. Actually the objects that are fixed are ordered pairs of edges that follow one another in the cyclical order around a vertex. We name such pair of edges a *corner* due to a natural geometrical interpretation. Now thinking in terms of corners we may say, that just the corners around vertices are ordered cyclically. Moreover, the cyclical sequences of corners of different vertices do not overlap, so they form together one common permutation on the set of all corners. We look on this permutation as the rotation of this graph. Further, if we "forget" the other information about the graph except this one permutation, then the graph cannot be restored uniquely. Only if we take permutations both of the graph and its dual graph (on this surface), then the graph is restorable (up to isomorphism) from them, but, of course, together with its rotation, i. e. its imbedding in the surface. Generally, two arbitrary permutations p_1 and p_2, whenever all cycles of $p_1^{-1} \cdot p_2$ are transpositions, define some graph with fixed rotation(up to isomorphism). In this work we are dealing with combinatorial objects called *(combinatorial) maps*, which are pairs of permutations.

Combinatorial maps have been studied in many works by several authors, for example [14, 10, 11, 12, 13, 6, 2]. This work is done independently from them.

Many ground things considered in this work are well known for many years. But in some points we differ from the attitudes of the named authors. For example,

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we define a binary operation on the maps that corresponds to multiplication of permutations. This operation is defined only if the product $p_1^{-1} \cdot p_2$ is equal for both the graphs and so it is common for the set of graph that is closed under this operation. This set is as big as the set of all permutations (of some fixed degree). Consequently it gives possibility to establish one-one map between all maps and all permutations. We are arrived at the situation where one permutation corresponds to one graph, and, if we could consider this permutation as the rotation of this graph, then this graph is given, when just this rotation is given.

This insight gives us right to speak now about graphs as rotations in the place of graphs with rotation, when we consider them as a whole set, that is closed under one common operation.

All permutations of fixed order comprise the class of maps closed under multiplication of maps.

In chapter 6 we enter a notion of combinatorial knot that is uniquely connected with the edges of that map. It corresponds to zigzag walks in [2]. In [7] it is called a closed travel obtained by the T.T.Rule . Every knot defines some subclass of combinatorial maps these with fixed edges. Besides, we define selfconjugate map with as many links in its knot as edges in it. We prove that the subclass of selfconjugate maps is a subgroup of the fixed class of combinatorial maps and that every map can be expressed as the multiplication of the knot of a map with some selfconjugate map.

2 Permutations

Fixed elements in permutations are these that form cycles of length one. *Transpositions* are cycles of length two. Permutations with only fixed elements and transpositions we call *involutions*. Involutions without fixed elements we call *matchings*.

Our permutations act on a universal set C the elements of which we call sometimes corners (or simply elements). For a permutation P and $c \in C$ c^P denotes that element of C to which c goes over. We are multiplying permutations from left to right. For two permutations P and Q P^Q equals to $Q^{-1} \cdot P \cdot Q$ and is the conjugate permutation to P (with respect to Q). When S is a cycle $(c_1 c_2 \ldots)$ in P, then S^Q is the cycle $(c_1^Q c_2^Q \ldots)$ in P^Q. (For arbitrary permutations P, Q, R it holds: $(P^Q)^R = P^{Q \cdot R}$ and $(P \cdot Q)^R = P^R \cdot Q^R$.) I denotes identical permutation, i.e. that with all elements fixed. One way to express a cycle $(c_1 c_2 \ldots c_k)(k > 0)$ of some permutation as the multiplication of transpositions is this: $(c_k c_{k-1}) \ldots (c_2 c_1)$.

3 Combinatorial maps

We say that two permutations P and Q of equal order n are *differing* whenever $i^P \neq i^Q$ for every $i \in [1 \ldots n]$. A pair of permutations (P, Q) we call *combinatorial map* whenever P and Q are differing permutations and $P^{-1} \cdot Q$ is a matching.

We denote the product $P^{-1} \cdot Q$ with the letter π. It is convenient to choose one fixed π. In that case we are speaking about a map P, keeping Q and matching π in mind, or saying, that π is fixed. The definition of the combinatorial map is such, that we do not need to speak about the orders of the permutations. But sometimes we do, and then it is easy to see, that the order of the permutations P, Q, π is even, let us put it equal to $n = 2 \times m$.

We say, that P, Q, π act on $2m$ *corners* or elements. If C is the set of corners , then it is divided by π into m pairs. Let (c_1, c_2) be such a pair, i.e. $c_1^\pi = c_2$. Then there exists such pair of corners (c, c') that $(c, c') \cdot P = P \cdot (c_1, c_2)$. We call the pair (c, c') an *edge of the map* and (c_1, c_2) we call a *next edge of the map.* We call matching π the *next-edge-matching(n-matching) of a map.* For a map $(P, P \cdot \pi)$ there exists some other unique matching ϱ satisfying $\varrho \cdot P = P \cdot \pi$. We call matching ϱ *edge-matching(e-matching) of the map.* We say that π *contains* next edges and ϱ contains edges of the map P , n-matching π being fixed.

For a combinatorial map (P, Q) we call the map (Q, P) its *dual* combinatorial map. We use also denotation $\bar{P}$ for Q, saying that $\bar{P}$ is dual to P by π being fixed.

4 Closed classes of combinatorial maps

Let us define the multiplication of two maps S and T (with fixed the same n-matching π) as usual multiplication of permutations, i.e. we put

$$(S_1, S_2) \cdot (T_1, T_2) = (S_1 \cdot T_1, S_1 \cdot T_1 \cdot \pi)$$

by definition, where on the left side of the expression sign $'\cdot'$ stands for multiplication of maps.

It is easy to see that

$$(S_1, S_2) \cdot (T_1, T_2) = (S_1 \cdot S_2, S_1 \cdot T_2)$$

In practice, writing $S \cdot T$ we mean both multiplication of permutations S and T and multiplication of corresponding maps with common n-matching π.

It is easy to see that it hold

$$1) \quad (S, \bar{S}) \cdot (T, \bar{T}) = (S \cdot T, S \cdot \bar{T})$$

$$2) \quad (S, \bar{S}) \cdot (\bar{T}, T) = (S \cdot \bar{T}, S \cdot T)$$

$$3) \quad (\bar{S}, S) \cdot (T, \bar{T}) = (\bar{S} \cdot T, \bar{S} \cdot T)$$

$$4) \quad (\bar{S}, S) \cdot (\bar{T}, T) = (\bar{S} \cdot \bar{T}, \bar{S} \cdot T)$$

Multiplication for maps with different n-matchings is not defined.

For any fixed n-matching (with m next edges) the set of all permutations P_{2m} with $2m$ elements defines and forms a class of maps that is closed against multiplication of maps. For two different permutations we have correspondingly two different (not equal) maps. Hence, we have established one-one map between a set of permutations and a set of maps. So, we may think in terms of one fixed n-matching and a set of all permutations, under each permutation seeing a graph P that stands for the ordered pair $(P, \bar{P})$. So, for arbitrary permutation P we have also a graph $(P, \bar{P}))$. Let us take the map (I, π) as identity map. Graphically interpreted it consists of m isolated edges. Its dual map (π, I) consists of m isolated loops.

P^{-1} being reverse permutation of P, $(P^{-1}, P^{-1} \cdot \pi)$ is called *reversed* map of $(P, \bar{P})$. So, both map and its reverse map belong to the same class of maps closed under multiplication. Hence, because all permutations with $2m$ elements comprise the symmetric group S_{2m} [1, 16], so similarly, all maps with one fixed n-matching form a group that is isomorphic to S_{2m}.

If (a, b) is a transposition, $(a, b) \cdot P$ can be graphically interpreted as a union of two vertices at the given corners a and b (or split of the vertex which contains both corners)in the graph corresponding to the map P. Remembering that each permutation can be expressed as a sequence of multiplications of simple transpositions, each map can be generated by this sequence of the operations of union of two corners (or split of vertex): if $P = (a_l b_l)(a_{l-1} b_{l-1})...(a_1 b_1)$ for some $l > 0$ then $P_0 = I$, $P_k = (a_k, b_k) \cdot P_{k-1}$, $0 < k \leq l$, $P = P_l$. Graphically, this shows l unions of vertices (or splits)at given corners giving the graph P.

It is convenient to choose one special class of maps with n-matching equal to $\pi = (12)(34)\ldots$
$(2i-1\ \ 2i)\ldots(2m-1\ \ 2m)$. The maps of this class we call *normal.* It is convenient for practice to get used to some particular chosen n-matchings and this one named normal is sufficiently natural.

The multiplication of maps gives right to the following graphical interpretation. The right member of multiplication is the map is the given one, i.e. that is subjected to changes, but the left comprise these transpositions that must affect the right one. Consequently, a map can be considered as move from one map to some other. In category language a map is both the object and the arrow.

In applications it is convenient to choose one big natural number, say M, and consider it as the bound for the possible number of edges in maps . Taking this number M as the order of all permutations, defining maps, then formally all maps have M edges, but most of them should be isolated. Such view is convenient also because it can be used directly in computer application. The only pay for this approach is that we loose possibility to have isolated edges, when we actually need them, from some graph theoretical point of view. In the graphical interpretation then we have, say, at the beginning M isolated edges (being initialized), and each permutation, multiplied to the currant graph, is some set of transpositions, that union and split vertices at distinct corners, giving in this way new graphs.

5 Classes of maps with fixed edges

For a map P with n-matching π its e-matching ϱ is equal to $P \cdot \pi \cdot P^{-1} (= \pi^{P^{-1}})$. Except P there exist other maps with the same e-matching. Truly, every permutation Q against which matchings π and ϱ are conjugate, defines a map with the same e-matching ϱ.

Let us define a set K_ϱ of maps as class of all maps whose e-matching is ϱ. For different ϱ these classes K_ϱ are subclasses of maps of all class of maps with fixed π, under which one this subclass is special, namely, K_π, that comprise the maps with e-matching equal to π. This subclass is not empty, because the maps (π, I) and (I, π) belong to it. Really, $\varrho_I = \pi^I = \pi$.

Theorem 1 *For two maps S, T with n-matching π e-matching of their multiplication $S \cdot T$ is equal to $\varrho_T^{S^{-1}}$, i.e. $\varrho_{S \cdot T}^S = \varrho_T$.*

Proof

$$\varrho_{S \cdot T} = S \cdot T \cdot \pi \cdot T^{-1} \cdot S^{-1} = S \cdot \varrho_T \cdot S^{-1} = \varrho_T^{S^{-1}}.$$

Let us denote by $P \cdot K_\sigma$ class of maps $\{P \cdot Q | Q \in K_\sigma\}$. From theorem 1 we know that $P \cdot K_\sigma$ goes into K_ϱ, where $\varrho = \sigma^{P^{-1}}$. We are going to prove the equality of these classes. First we start with this.

Theorem 2 *K_π is a group.*

Proof If $\varrho_S = \varrho_T = \pi$ then $\varrho_{S \cdot T} = S \cdot T \cdot \pi \cdot T^{-1} \cdot S^{-1} = S \cdot \pi \cdot S^{-1} = \pi$. Further, $\varrho_{S^{-1}} = S^{-1} \cdot \pi \cdot S = \pi$. Identity map also belongs to this subclass , and, so, it forms a subgroup.

Theorem 3 *$P \cdot K_\sigma = K_{\sigma^{P^{-1}}}$. $(P \cdot K_{T^P} = K_T.)$*

Proof If $\sigma = \pi$ then $P \cdot K_\sigma$ is a left coset of K_σ equal to $K_{\pi^{P^{-1}}}$ and theorem is right.

Let $\sigma \neq \pi$ and Q be such map that $Q \cdot K_\pi = K_\sigma$, i.e. $K - \sigma$ is a left coset of K_π and Q is some of its elements: $\varrho_Q = \pi^{Q^{-1}}$ and Q belongs to K_σ by theorem 1. If $P \cdot Q = R$ then we have $P \cdot K_\sigma = P \cdot Q \cdot K_\pi = R \cdot K_\pi = K_{\pi^{R^{-1}}} = K_{\sigma^{P^{-1}}}$. As a conclusion of this theorem we have that the class K_π and its left cosets are classes with fixed edges.

Corollary 1 *Left cosets of K_π are classes of maps with fixed edges.*

So arbitrary map P with e-matching $\varrho = \pi^{P^{-1}}$ belongs to $K_\varrho = P \cdot K_\pi$.

For two involutions σ and τ $(\sigma^2 = \tau^2 = I)$ we write $\sigma \subseteq \tau$, if every transposition of σ is also a transposition in τ.

Lemma 1 *$P^\pi = P$ iff $P \in K_\pi$.*

Proof $\varrho_P = \pi \equiv \pi = P \cdot \pi \cdot P^{-1} \equiv P = \pi \cdot P \cdot \pi \equiv P = P^{\pi}$.

Let c be a cycle of P. Cycle c^{π} is called its *conjugate* cycle (with respect to π). If $c = c^{\pi}$ then it is called a *selfconjugate* cycle. If every cycle in P has its conjugate cycle in P or is selfconjugate then P is called *selfconjugate map*. Lemma says that K_{π} is the class of selfconjugate maps with respect to π.

We reformulate this in theorem.

Theorem 4 *K_{π} is the class of selfconjugate maps with respect to π.*

We can reveal the structure of these maps as follows.

Theorem 5 *K_{π} is isomorphic to $S_m \cdot S_2^m$.*

Proof Let $P \in K_{\pi}$ and P work on C and $C_1 \cup C_2$ be one partition of C induced by π. Then there is such involution $\sigma \subseteq \pi$ that $P = Q \cdot \sigma$ and cycles of Q belong completely to C_1 or C_2. i.e. if a cycle c goes into C_1 then c^{π} goes into C_2 or reversely. Q can be expressed as $Q_1 \cdot Q_2$ where Q_1 has corners of C_1 and Q_2 corners of C_2. Then Q_1 and Q_2 are both isomorphic between themselves and to some permutation in S_m and σ isomorphic to some permutation of S_2^m.

Theorem 6 $\mid K_{\pi} \mid = m! \times 2^m$.

Proof $\mid S_m \mid = m!$; $\mid S_2^m \mid = 2^m$.

Theorem 7 *K_{π} has (together with itself) $(2m-1)!!$ left cosets.*

Proof We have as many left cosets of K_{π} as many e-matchings can be generated. Their number is $(2m-1)!!$.

One more way to convince oneself that all left cosets of K_{π} have the same number of elements, i.e. $(2m-1)!!$, and that they really do not overlay each other, is to see that the equality $(2m-1)!! \times m!2^m = (2m)!$ holds.

6 Combinatorial knot

Let $(P, P \cdot \pi)$ be a map on set of corners C and $\varrho = \pi^{P^{-1}}$. Let $C_1 \cup C_2$ be some partition of C such that it is induced both by π and ϱ, i.e. for every both edge and next edge one of its corner belongs to C_1 and other to C_2. In this case we say that C is *well* partitioned or *well colored in two colors* or we say that $C_1 \cup C_2$ is *well coloring* of C induced by this map $(P, P \cdot \pi)$.

Theorem 8 *There always exists well coloring of C induced by arbitrary map $(P, P \cdot \pi)$.*

Proof Let $c_1c_2\ldots c_{2k}$ be a cycle of corners such that $c_2 = c_1^\pi$ and $c_3 = c_2^\varrho$ and so on in alternating way, i.e. $c_{2i} = c_{2i-1}^\pi$ and $c_{2i+1} = c_{2i}^\varrho$,for $i = 1,\ldots k$, where $c_{2k}^\varrho = c_{2k+1} = c_1$. Let us suppose for an instant that $c_{2k-1}^\pi = c_{2k} = c_1$. Then $c_{2k-1} = c_1^\pi = c_2$ and $c_{2k-2}^\varrho = c_2$ and $c_{2k-2} = c_2^\varrho = c_3$ and so on, until $c_{k+1} = c_k$, but it isn't possible. It follows, that these cycles may have only even number of elements.

Then we may put odd elements of cycle in C_1 and even elements in C_2 . If this cycle runs through all corners, then we have only one possibility to color all corners. Otherwise we choose arbitrary non colored corner taking it as c_1 and proceed as before. In the end we get well partition $C_1 \cup C_2$ induced by the map $(P, P\cdot\pi)$.

Let us define permutation μ having cycles as in the previous proof, i.e. if $C_1 \cup C_2$ is well coloring of the set C, then $c^\mu = c^\pi$ if $c \in C_1$ and $c^\mu = c^\varrho$ if $c \in C_2$. Permutation μ is called *combinatorial knot.* In place we use shorter name *knot* for μ. In a graphical 'corner' interpretation μ really is (alternating) knot of the graph [3]. In [2] it is called zigzag walk.

Lemma 2 *If μ is a knot then if μ' is obtained with some cycle of μ changed in the opposite direction then μ' is also a knot.*

Proof Starting a new cycle in the proof of 8 we can choose a corner arbitrary . Consequently, cycle in a knot can go in one or another direction. It follows that if μ is a knot then also μ^{-1} is a knot. It is easy to see that a knot μ depends only on π and ϱ, so it is common for all class K_ϱ. The following theorem shows that, if we consider a map (π, μ) then $\varrho_\mu = \varrho$.

Theorem 9 $\pi^\mu = \varrho$.

Proof Taking $(cc^\pi) \in \pi$, $(cc^\pi)^\mu$ equals to $(c^\pi c^{\pi\varrho})$ or $(c^\varrho c^{\pi\pi}) = (c^\varrho c)$ both cases giving $(cc^\pi)^\mu \in \varrho$.

By $\mu(\pi, \varrho)$ we denote arbitrary knot of some map belonging to K_ϱ . Previous theorem and 3 gives what follows .

Theorem 10 $K_\varrho = \mu(\pi, \varrho) \times K_\pi$.

This gives right for what follows.

Corollary 2 *Every map can be expressed as a knot of this map multiplied by some selfconjugate map.*

7 Isomorphism

Let us notice that $A \in K_\pi$ if and only if $\pi^A = \pi$, because $\pi^{A^{-1}} = \varrho_A$. Two maps $(P, P\cdot\pi_1)$ and $(Q, \cdot\pi_2)$ are isomorphic if and only if there exist such permutation A that $\pi_2^A = \pi_1$ and $Q^A = P$. We write $P \simeq Q$, saying that P is *isomorphic* to Q because they are conjugate with respect to A.

Theorem 11 *Let* $(\pi_1, P) \simeq (\pi_2, Q)$*, i.e. they are conjugate with respect to* A*. Both maps belong to one closed against multiplication class of maps iff* $A \in K_\pi$ *with* $\pi = \pi_1$*.*

Proof If $A \in K_{\pi_1}$ then π_1^A is equal to π_1 ,but because of conjugacy of π_1 and π_2 against A also equal to π_2. So $\pi_1 = \pi_2$ and P and Q are maps in one class with $\pi = \pi_1 = \pi_2$. Conversely, if $\pi_1 = \pi_2$, then $\pi_1^A = \pi_2 = \pi_1$ and $A \in K_\pi$.

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