Nearness, subfitness and sequential regularity

In honour of Bernhard Banaschewski on the occasion of his 70th birthday

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Abstract : In the point-free context, the structure of nearness has been so far studied in the regular case only. Here we answer the question as to how far beyond that one can go. It turns out that a frame (locale) admits a nearness iff it is subfit. Unlike in the case of spaces, where admitting nearness is a hereditary property, subfitness is not; therefore, also the hereditary subfitness (here called *sequential regularity* for reasons obvious from the properties presented) is studied. It is weaker than regularity and seems to be of some interest also in the spatial case.

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Nearness structures (see, e.g., [**5**], [**6**], T-uniformities in [**11**]) have been studied in the point-free context so far only in the regular case (see, e.g., [**3**]), where it has particularly natural properties. In this paper we are concerned with the question as to how far beyond the regularily one can go, that is, what is the weakest separation property under which an admissible nearness makes a reasonable sense. The answer is the *subfitness* (also studied under the name of *conjunctivity* by Simmons in [**15**], see also [**12**]) as we show in Section 2 below.

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Unlike the separation axiom of symmetry ($x \in \overline{\{y\}} \Longrightarrow y \in \overline{\{x\}}$), necessary for admitting a nearness in spaces, subfitness is not hereditary. Thus, the question naturally arises as to which locales (point-free spaces) have the property that all their sublocales admit a nearness structure. This question is treated in Sections 3 and 4, and a necessary and sufficient condition is found (Theorem 4.5), in a form of a "separation axiom" which we call *sequential regularity.* Finally, in Section 5 we compare the sequential regularity with more common separation axioms, and add a few further remarks.

1. Preliminaries

1.1. A *frame* is a complete lattice L satisfying the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for all $a \in L$ and $S \subseteq L$, and a *frame homomorphism* $\varphi\colon L \longrightarrow M$ is a map preserving finite meets including the top (usually denoted by 1), and arbitrary joins including the bottom (usually denoted by 0). The resulting category will be denoted by

$$\mathbf{Frm}.$$

Example: The frame $\mathfrak{O}(X)$ of open sets of a topological space X, the homomorphism $\mathfrak{O}(f)\colon \mathfrak{O}(Y) \longrightarrow \mathfrak{O}(X)$ obtained from a continuous function $f\colon X \longrightarrow Y$ by putting $\mathfrak{O}(f)(u) = f^{-1}(u)$. Thus a contravariant functor

$$\mathfrak{O}\colon\ \mathbf{Top} \longrightarrow \mathbf{Frm}$$

is obtained; the dual of **Frm** is called the *category of locales*, usually denoted by **Loc**, and $\mathfrak{O}$ becomes a covariant functor **Top** $\longrightarrow$ **Loc**.

For details on frames see [**8**], [**16**].

1.2. For $x, y \in L$ write

$$x \prec y$$

if there is a $z \in L$ such that $x \wedge z = 0$ and $z \vee y = 1$. L is said to be *regular* if

$$a = \bigvee \{x \mid x \prec a\}$$

for each $a \in L$. Note that $\mathfrak{O}(X)$ is regular iff X is a regular space.

1.3. A *sublocale* (*part* in [**7**]) of a frame L is a surjective frame homomorphism $\varphi\colon L \longrightarrow M$. We write

$$\varphi \sqsubseteq \psi$$

if there is a homomorphism γ such that $\gamma \circ \psi = \varphi$. Sublocales φ, ψ are considered to be equivalent if $\varphi \sqsubseteq \psi$ and $\psi \sqsubseteq \varphi$. The (equivalence classes of) sublocales can be represented by the associated congruences

$$xE_\varphi y \quad \equiv \quad \varphi(x) = \varphi(y)$$

and one has $\varphi \sqsubseteq \psi$ iff $E_\varphi \supseteq E_\psi$. In consequence, the preorder $\sqsubseteq$ is complete; the suprema and infima will be denoted by

$$\bigsqcup \varphi_i, \;\; \varphi_1 \sqcup \varphi_2, \;\; \varphi_1 \sqcap \varphi_2 \;\; \text{etc.}$$

In particular, the congruence associated with $\bigsqcup \varphi_i$ is $\bigcap E_{\varphi_i}$.

The elements $a \in L$ induce the *open* resp. *closed* sublocales

$$\hat{a} = (x \longmapsto a \wedge x)\colon L \longrightarrow{\downarrow}a \;\; \text{resp.} \;\; \check{a} = (x \longmapsto a \vee x)\colon L \longrightarrow{\uparrow}a$$

(where ${\downarrow}a = \{x \mid x \leq a\}$, ${\uparrow}a = \{x \mid x \geq a\}$), complementing each other.

The closure of a sublocale $\varphi\colon L \longrightarrow M$ is

$$\overline{\varphi} = \check{c}\colon L \longrightarrow{\uparrow}c$$

where $c = \bigvee\{x \mid \varphi(x) = 0\}$, and it is easy to see that this is the smallest closed $\check{a}$ such that $\varphi \sqsubseteq \check{a}$.

The meet $\varphi \sqcap \hat{a}$ is obtained from the pushout

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d, "\hat{a}"'] & M \arrow[d, "(\varphi(a))\,\hat{}"] \\ {\downarrow}a \arrow[r] & {\downarrow}\varphi(a). \end{tikzcd}$$

In particular,

$$(1.3.1) \qquad \begin{aligned} \varphi \sqcap \hat{a} \neq 0 \quad &\text{iff} \quad \varphi(a) \neq 0, \\ \check{b} \sqcap \hat{a} \neq 0 \quad &\text{iff} \quad a \not\leq b. \end{aligned}$$

Another formula we will use is that

$$\check{b} \sqsubseteq \hat{a} \quad \text{iff} \quad a \vee b = 1. \tag{1.3.2}$$

1.4. A *cover* of a frame L is a subset $A \subseteq L$ such that $\bigvee A = 1$. We say that A refines B and write $A \leq B$ if

$$\forall a \in A \;\; \exists\, b \in B, \;\; a \leq b.$$

For $x \in L$ and a cover A we write

$$Ax = \bigvee \{a \in A \mid a \wedge x \neq 0\}$$

(see, e.g., [**13**], [**3**]). Obviously,

$$A \leq B \;\;\&\;\; x \leq y \Longrightarrow Ax \leq By$$

and

$$Ax \leq y \quad \Longrightarrow \quad x \prec y. \tag{1.4.1}$$

More generally, for a sublocale $\varphi \colon L \longrightarrow M$,

$$A\varphi = \bigvee \{a \in A \mid \hat{a} \sqcap \varphi \neq 0\} = \bigvee \{a \in A \mid \varphi(a) \neq 0\}.$$

In particular (recall (1.3.1))

$$A\hat{x} = Ax, \quad A\check{x} = \bigvee \{a \in A \mid a \not\leq x\}. \tag{1.4.2}$$

It is easy to check that

$$A\varphi = A\overline{\varphi}. \tag{1.4.3}$$

1.5. Congruences, nuclei and prenuclei. If $R \subseteq L \times L$ is a relation on L, an $s \in L$ is called *saturated* if

$$\forall a, b, c \in L \quad aRb \Longrightarrow (a \wedge c \leq s \text{ iff } b \wedge c \leq s).$$

The system of saturated elements is closed under meets. If we define $\kappa(a)$ as the least saturated $s \geq a$ we obtain the so called *nucleus* (associated with R)

$\kappa\colon L \longrightarrow L$, a monotone function satisfying the formulas $a \leq \kappa(a), \kappa\kappa(a) = \kappa(a)$ and $\kappa(a \wedge b) = \kappa(a) \wedge \kappa(b)$. This κ can be viewed as a sublocale

$$\kappa\colon L \longrightarrow L/R = \{a \mid \kappa(a) = a\}$$

and $\{(a,b) \mid \kappa(a) = \kappa(b)\}$ is the least congruence containing R.

If $aRb \Longrightarrow a \leq b$, the nucleus κ can be constructed from a *pre-nucleus*

$$\kappa_1(u) = \bigvee\{x \wedge b \mid x \wedge a \leq u \text{ for some } (a,b) \in R\}$$

by setting

$$\kappa_{\alpha+1}(u) = \kappa_1\big(\kappa_\alpha(u)\big), \quad \kappa_\alpha(u) = \bigvee_{\beta<\alpha} \kappa_\beta(u) \ \text{ for limit } \alpha,$$

and $\kappa(u) = \kappa_\alpha(u)$ where α is such that $\kappa_{\alpha+1}(u) = \kappa_\alpha(u)$. (See [**2**], [**8**], [**9**], [**10**].)

1.6. Convention: To unify the notation in the spatial and in the general point-free case, we will denote open sets of spaces by small case letters (a, u, x, etc.), open covers by capitals (A, B, etc.) and points by greek letters (ξ, η, etc.).

1.7. A *nearness on a topological space* X (this has appeared, in essence, as T-*uniformity* [**11**], for a detailed study of this structure see, e.g., [**5**], [**6**]) is a non-void system $\mathfrak{N}$ of open covers such that

(a) if $A \in \mathfrak{N}$ and A refines B then $B \in \mathfrak{N}$, and
(b) for any $A, B \in \mathfrak{N}$ there is a common refinement $C \in \mathfrak{N}$.

Similarly, a *nearness on a frame* L (cf., e.g., [**3**]) is a non-void system $\mathfrak{N}$ of covers of L such that

(a) $A \in \mathfrak{N}$ & $A \leq B \Longrightarrow B \in \mathfrak{N}$,
(b) $A, B \in \mathfrak{N} \Longrightarrow A \wedge B = \{a \wedge b \mid a \in A,\ b \in B\} \in \mathfrak{N}$.

A nearness on a space induces a new topology by declaring u to be an $\mathfrak{N}$-neighbourhood of a point ξ if $A\xi = \bigcup\{a \in A \mid \xi \in A\} \subseteq u$ for some $A \in \mathfrak{N}$. We say that the given $\mathfrak{N}$ is *admissible* if this new topology coincides with the original one. The admissibility (also in the point-free case) will be discussed in Section 2; so far we will only prepare the notation. We write $a \lhd_{\mathfrak{N}} b$ if there is an $A \in \mathfrak{N}$ such that $Aa \leq b$ and, more generally, for sublocales, $\varphi \lhd_{\mathfrak{N}} \psi$ if $(A\varphi)\hat{}\sqsubseteq \psi$ for an $A \in \mathfrak{N}$. If clear from the context, the subscript in $\lhd_{\mathfrak{N}}$ may be omitted.

Obviously (recall (1.4.1))

$$a \lhd b \quad \Longrightarrow \quad a \prec b$$

and in case of an $\mathfrak{N}$ containing all the finite covers,

$$a \lhd b \quad \text{iff} \quad a \prec b$$

(as $\{x, b\}a \le b$ when $x \wedge a = 0$ and $x \vee b = 1$).

1.8. Sobriety and T_D: A space X is said to be *sober* if

(Sob) X is T_0 and each meet-irreducible open $u \neq X$ (i.e., a u such that $u = u_1 \cap u_2 \Longrightarrow u = u_j$ for some j) is of the form $X \smallsetminus \overline{\{\xi\}}$ for some $\xi \in X$.

In particular,
if X is sober then each frame homomorphism $\varphi: \mathfrak{O}(Y) \longrightarrow \mathfrak{O}(X)$ is of the form $\mathfrak{O}(f)$ for a uniquely defined continuous map $f: X \longrightarrow Y$
(see, e.g., [**8**]).

A space X is said to be a T_D-*space* (first defined in [**1**]) if

(T_D) for each $\xi \in X$ there is an open $u \subseteq X$ such that $\xi \in u$ and $u \smallsetminus \{\xi\}$ is open.

In particular,
noindent*a space is T_D iff distinct subspaces $Y \subseteq X$ generate non-equivalent sublocales*

$$(u \longmapsto u \cap Y)\colon \ \mathfrak{O}(X) \longrightarrow \mathfrak{O}(Y).$$

see [**4**], [**14**])

(Sob) is between T_0 and T_2 and incomparable with T_1. T_D is between T_0 and T_1 and incomparable with *(Sob)*.

1.9. Subfit frames: A frame is said to be *subfit* (see [**7**]) if the following condition holds:

(Sf1) each open sublocale is a join of closed ones.

This is well-known to be equivalent to the statement

(Sf) whenever $a \not\le b$ there is a c such that

$$a \vee c = 1 \neq b \vee c.$$

(The equivalence *(Sf1)* $\equiv$ *(Sf)* is very easy to see: in view of (1.3.2), *(Sf1)* can be rewritten as

$$(Sf2) \qquad (x \wedge a \neq y \wedge a \implies (\exists c,\ a \vee c = 1 \text{ and } x \vee c \neq y \vee c).$$

Now if $a \not\leq b$ consider in *(Sf2)* $x = a, y = b$ to obtain *(Sf)*. On the other hand, if $x \wedge a \neq y \wedge a$ consider in *(Sf)* $x \wedge a \not\leq x \wedge y \wedge a$ to obtain *(Sf2)*.)

It should be noted that *(Sf)* was called *conjunctivity* in [**15**] for frames and *weak regularity* for spaces [**11**].

2. Admissible nearnesses

2.1. Recall 1.7. Explicitly, a nearness $\mathfrak{N}$ is admissible on a space X if

$$\text{for each open } u \subseteq X,\ u = \{\xi \mid A\{\xi\} \subseteq u \text{ for an } A \in \mathfrak{N}\},$$

or, in a reformulation more handy for a pointfree treatment:

$$(2.1.1)\quad \text{for each open } u \subseteq X,\ u = \bigcup\{\varphi \subseteq X \mid A\varphi \subseteq u \text{ for an } A \in \mathfrak{N}\}.$$

The *admissibility in the point-free case* has been so far studied as the condition (see [**3**], cf. [**13**])

$$\forall u \in L,\ u = \bigvee\{x \in L \mid Ax \leq u \text{ for an } A \in \mathfrak{N}\}.$$

This is correct - that is, in accordance with the admissibility in spaces - for regular L. As the formula implies regularity (see (1.4.1)), however, it cannot be used in a more general case. Thus, we will use, extending the formula (2.1.1) above, the following

Definition: A nearness $\mathfrak{N}$ is *admissible on a frame* L if for each $u \in L$,

$$(2.1.2) \qquad \hat{u} = \bigsqcup\{\varphi \mid \varphi \text{ sublocale of } L,\ A\varphi \leq u \text{ for an } A \in \mathfrak{N}.$$

2.2. As $A\varphi = A\overline{\varphi}$ (recall (1.4.3)), the formula (2.1.2) can be rewritten to

$$(2.2.1) \qquad \forall\ u \in L, \quad \hat{u} = \bigsqcup\{\breve{a} \mid \exists\ \forall \in \mathfrak{N},\ A\breve{a} \leq u\},$$

and since the inclusion $\sqsupseteq$ is trivial, the inclusion $\sqsubseteq$ is what has to be checked in a concrete case.

More explicitly, by 1.3, the condition in (2.2.1) can be expressed by the requirement that

if for all $c \in L$ such that there is an $A \in \mathfrak{N}$ with $A\breve{c} = \bigvee\{a \in A,\ a \not\leq c\} \leq u$ one has $c \vee x = c \vee y$, then $x \wedge u = y \wedge u$.

We have a trivial

Observation:. *If $\mathfrak{N}$ is admissible and $\mathfrak{N} \subseteq \mathfrak{M}$ then $\mathfrak{M}$ is admissible.*

Thus, L admits a nearness iff the conditions above hold for $\mathfrak{N} = \{\text{all covers}\}$.

2.3. THEOREM. *A frame L admits a nearness iff it is subfit.*

PROOF: ($\Longrightarrow$): follows immediately from (2.2.1).

($\Longleftarrow$): We will use the formula *(Sf)* from 1.9 for prove (2.2.1) for the system of all covers. Thus, let $x \vee a = y \vee a$ for any a such that there is a cover A with $A\check{a} = \bigvee\{b \in A \mid b \not\leq a\} \leq u$. We want to prove that then $x \wedge u = y \wedge u$. Let, say, $x \wedge u \not\leq y \wedge u$. By *(Sf)* there is a c with $1 = (x \wedge u) \vee c \neq (y \wedge u) \vee c$ (in particular, $u \vee c = 1$). But $\{x \wedge u, c\}\check{c} \leq x \wedge u \leq u$ and hence $x \vee c = y \vee c$ and we get a contradiction

$$(y \wedge u) \vee c = y \vee c = x \vee c = (x \vee c) \wedge (u \vee c) = (x \wedge u) \vee c = 1. \ \square$$

2.4. Recall that a nearness $\mathfrak{N}$ is said to be *regular* ([**6**]; *strong* in [**3**]) if for each $A \in \mathfrak{N}$ also

$$\widetilde{A} = \{a \mid a \lhd_{\mathfrak{N}} b \text{ for some } b \in A\} \in \mathfrak{N}.$$

We have

THEOREM. *L admits a regular nearness iff it is regular.*

PROOF: ($\Longleftarrow$): The system of all covers of a regular frame is a regular nearness.

($\Longrightarrow$): We have

$$\widetilde{A}\varphi = \bigvee\{a \in \widetilde{A} \mid \varphi(a) \neq 0\} \leq \bigvee\{a \mid a \lhd_{\mathfrak{N}} A\varphi\}$$

and hence

$$u = \bigvee\{A\varphi \mid A \in \mathfrak{N},\ A\varphi \leq u\} \leq \bigvee\{a \mid a \lhd_{\mathfrak{N}} u\}.$$

$\square$

2.5. For the spatial case, the equivalence from 2.3 is analoguous to that from [**11**], Theorem 1. Another well-known necessary and sufficient condition for a space X to admit a nearness (see [**5**], [**6**]) is that it satisfied the symmetry axiom

$$\xi \in \overline{\{\eta\}} \quad \text{iff} \quad \eta \in \overline{\{\xi\}}$$

(which is the same as saying that the T_0-modification of X is T_1, and hence we will refer to this property as

$$T_{1-0}.)$$

Now in view of 2.3 one might expect that this T_{1-0} is equivalent to the subfitness of $\mathfrak{O}(X)$. But this is false, and the discrepancy should be explained.

Recall 1.8: If X were not T_D, two subsets may be distinct although the associated congruence coincides. In particular, this can happen with u and $\bigcup\{\varphi \text{ closed} \mid \varphi \subseteq u\}$. On the other hand, under T_D, T_{1-0} is indeed equivalent to subfitness. More exactly, one has

$$\begin{aligned} T_1 &\equiv \textit{(Sf)} \;\&\; T_D, \quad \text{and} \\ T_{1-0} &\equiv \textit{(Sf)} \;\&\; T_{D-0} \end{aligned}$$

where T_{D-0} stands for the condition that the T_0-modification of the space is T_D.

3. Sobriety as a partly hereditary subfitness

3.1. Example: Consider the following spaces:

X: the ordinal ω with the cofinite topology,

Y: $\omega + 1$ with open $\emptyset$ and $Y \smallsetminus K$ where K is finite and $\subseteq \omega$,

S: the Sierpiński space represented as the subspace of Y generated by the subset $\{0, \omega\}$,

and the mappings:

j: the embedding $S \subseteq Y$,

k: the embedding $X \subseteq Y$,

f, f': the mappings $f\colon X \longrightarrow X$ resp. $f'\colon Y \longrightarrow Y$ sending 0 to 0, $2n+1$ to $2n+2$, $2n+2$ to $2n+1$, and ω to ω,

$$\iota = \mathfrak{O}(j), \kappa = \mathfrak{O}(k),\ \varphi = \mathfrak{O}(f).$$

Obviously

(3.1.1) κ is an isomorphism,

(3.1.2) j is the equalizer of f' and id,

(3.1.3) $\iota \circ \kappa^{-1}$ is the coequalizer of φ and id.

((3.1.3) does not follow from (3.1.2) but it is easy to check).

Now one sees that:

Subfitness is not inherited by subspaces (see $S \subseteq Y$), *and not even by equalizers with subfit range* (see (3.1.2)). *A T_1-space* (although all of its subspaces are T_1, and hence subfit) *can have a spatial sublocale that is not subfit and, again, even a coequalizer with a subfit domain* (see (3.1.3)).

In this section we will investigate the question of subfitness of spaces hereditary with respect to the spatial sublocales. In the next section, the general hereditarity question will be treated.

3.2. The following is certainly folklore and can be left to the reader:

LEMMA. *A space Y is T_0 iff the embeddings $f: X \longrightarrow Y$ are exactly those continuous maps f for which $\mathfrak{O}(f): \mathfrak{O}(Y) \longrightarrow \mathfrak{O}(X)$ are onto.*

3.3 Recall that an element $a \in L$ is *prime* if the following holds:

$$a \neq 1 \quad \text{and} \quad (x \wedge y \leq a \Longrightarrow x \leq a \text{ or } y \leq a).$$

LEMMA. *Let $a, b \in L$ be prime and let $a \lneqq b$. Then the Sierpiński space is a sublocale of L.*

PROOF: Represent the Sierpiński frame ($\mathfrak{O}(S)$ of a Sierpiński space S) as

$$\mathbb{S} = \{0 < \frac{1}{2} < 1\}.$$

Define $\varphi: L \longrightarrow \mathbb{S}$ by setting

$$\varphi(x) = \begin{cases} 0 & \text{if } x \leq a, \\ \frac{1}{2} & \text{if } x \not\leq a \text{ and } x \leq b, \\ 1 & \text{if } x \not\leq b. \end{cases}$$

It is easy to check that φ is a frame homomorphism. □

3.4 THEOREM. *Let X be a T_1-space. Then the following statements are equivalent:*

(i) *X is sober,*
(ii) *each T_0-spatial sublocale is T_1,*
(iii) *each spatial sublocale is subfit,*
(iv) *X has no Sierpiński sublocale.*

PROOF: (i) $\Longrightarrow$ (ii): Let X be sober. Suppose we have a surjective homomorphism φ: $\mathfrak{O}(X) \longrightarrow \mathfrak{O}(Y)$, with Y a T_0-space. Then, by 3.2, Y is a subspace and hence it is T_1.

(ii) $\Longrightarrow$ (iii): Consider a surjective $\varphi: \mathfrak{O}(X) \longrightarrow \mathfrak{O}(Y)$ and the T_0-modification $p: Y \longrightarrow \widetilde{Y}$ of Y . Then $\mathfrak{O}(p)$ is an isomorphism. Considering $\mathfrak{O}(p)^{-1} \circ \varphi$ we see that $\widetilde{Y}$ is T_1, hence $\mathfrak{O}(\widetilde{Y})$ is subfit, and hence $\mathfrak{O}(Y)$ is subfit.

(iii) $\Longrightarrow$ (iv) is trivial.

(iv) $\Longrightarrow$ (i): Let X not be sober. Then there is a prime $u \in \mathfrak{O}(X)$ unequal to $X \smallsetminus \{\xi\}$ for any ξ. Choosing $\xi \in X \smallsetminus u$ we obtain prime

$$u \lneqq v = X \smallsetminus \{\xi\}$$

and hence there is a Sierpiński sublocale by 3.3. $\square$

4. Hereditary subfitness

4.1. In this section we will characterize the frames L such that each sublocale of L admits a nearness. In view of 2.2, this amounts to the *hereditary subfitness*, that is,

(HSf) for every sublocale $\varphi: L \longrightarrow M$, M is subfit.

4.2 LEMMA. *A frame L is hereditarily subfit iff for each sublocale $\varphi: L \longrightarrow M$ the M satisfies*

(WSf) if $a \neq 0$ in M then there is $x \in L$ such that $x \neq 1$ and $x \vee a = 1$.

PROOF: Indeed, if L has the property indicated then, by force of the sublocales $\check{b} = (x \longmapsto x \vee b): L \longrightarrow \uparrow b$ it is, first, subfit, and as the property is hereditary, it is hereditarily subfit. $\square$

Note: Obviously, *(WSf) can be reformulated as the implication*

(4.2.1) $$(x \vee a = 1 \iff x = 1) \implies a = 0.$$

4.3. Sequential regularity: Let α be an ordinal. We say that L is *α-sequentially regular* if

(SRα) for each $a \in L$ there is a non-decreasing sequence $(a_\beta)_{\beta \leq \alpha}$ such that $a_0 = 0$, $a_{\beta+1} \leq \bigvee\{u \mid \exists\, x,\ x \vee a = 1\ \&\ u \wedge x \leq a_\beta\}$, $a_\beta = \bigvee\limits_{\gamma<\beta} a_\gamma$ for limit ordinals β, and $a = a_\alpha$.

L is *sequentially regular* if

(SR) L is α-sequentially regular for some α.

Note: One has $a_1 \le \bigvee\{u \mid \exists\, x,\ x \vee a = 1\ \&\ u \wedge x = 0\}$. Thus, 1-regularity is the same as regularity.

4.4. Lemma. *(SRα) is hereditary and implies (hereditary) subfitness.*

Proof: Consider a sublocale $\varphi\colon L \longrightarrow M$ and a $\varphi(a) \in L$. Let $(a_\beta)_{\beta\le\alpha}$ be the sequence from *(SRα)*. Then it is easy to check that $\big(\varphi(a_\beta)\big)_{\beta\le\alpha}$ is the required sequence for $\varphi(a)$.

As for the second statement, by 4.2 it suffices to show that *(SRα)* implies *(WSf)*. Let $a \ne 0$ and let $(a_\beta)_{\beta\le\alpha}$ be the associated sequence. Then there is a β such that

$$a_\beta \lneqq a_{\beta+1} \le \bigvee\{u \mid \exists\, x,\ x \vee a = 1\ \&\ u \wedge x \le a_\beta\}.$$

In particular there has to be a u and an x such that

$$x \vee a = 1, \quad u \wedge x \le a_\beta \quad \text{and} \quad u \not\le a_\beta.$$

Then x cannot be 1. □

4.5. Theorem. *The following statements on a frame L are equivalent:*

(i) *Each sublocale of L admits a nearness,*
(ii) *L is hereditarily subfit,*
(iii) *L is sequentially regular,*
(iv) *each congruence on L is uniquely determined by the class containing* 1.

Proof: (i) $\equiv$ (ii) by 2.2.

(ii) $\Longrightarrow$ (iii): In view of 4.2 and 4.3 it suffices to prove that if each sublocale of L satisfies *(WSf)* then L is *(SR)*. Consider an $a \in L$ and the sublocale $L/_\sim$ where $\sim$ is the congruence generated by

$$R = \{(x,1) \mid x \vee a = 1\}.$$

Recall 1.5. $L/_\sim$ consists of all the R-saturated elements of L.
Claim: *Each $y \ge a$ is saturated.*
(Indeed, if $x \vee a = 1$ and $u \wedge x \le y$ we have

$$u = (x \vee a) \wedge u = (x \wedge u) \vee (a \wedge u) \le y \vee a = y.)$$

By (4.2.1), if $L/_\sim$ satisfies *(WSf)* we have

$$a \sim 0$$

and since by the above Claim a is saturated, this amounts to

$$\kappa(0) = a$$

where κ is the nucleus associated with R. The prenucleus of R, κ_1, is defined by

$$\kappa_1(b) = \bigvee\{u \mid \exists\ x,\quad x \vee a = 1,\ u \wedge x \leq b\}.$$

Set $a_0 = 0, \quad a_\beta = \kappa_\beta(0)$ for $\beta > 1$.

(iii) $\Longrightarrow$ (iv): Let $E_1,\ E_2$ be congruences on L, let $\varphi_j\colon\ L \longrightarrow L/E_j$ be the associated sublocale homomorphisms. Let $xE_1 1$ iff $xE_2 1$ and let $aE_1 b$. We have $\varphi_2(a_0) = 0 \leq \varphi_2(b)$. Let $\varphi_2(a_\beta) \leq \varphi_2(b)$. If u is such that there is an x with $x \vee a = 1$ and $u \wedge x \leq a_\beta$, we have $\varphi_1(x \vee b) = \varphi_1(x \vee a) = 1$, hence $\varphi_2(x \vee b) = 1$ and

$$\varphi_2(u) \quad = \quad \varphi_2(u \wedge (x \vee b)) \quad \leq \quad \varphi_2(a_\beta) \vee \varphi_2(b) \quad \leq \quad \varphi_2(b).$$

Thus, $\varphi_2(a_{\beta+1}) \leq \varphi_2(b)$ and since the limit step is trivial, $\varphi_2(a) \leq \varphi_2(b)$. Similarly, $\varphi_2(b) \leq \varphi_2(a)$ and hence $\varphi_2(b) = \varphi_2(a)$.

(iv) $\Longrightarrow$ (ii): It is easy to see that the property in (iv) is hereditary and hence, by 4.2, it suffices to prove that (iv) $\Longrightarrow$ *(WSf)*. Recall (4.2.1). If $x \vee b = 1 \ \equiv\ x = 1$, $\check{b} = (x \mapsto x \vee b)$ is the identity and hence $b = 0$. $\square$

5. More on sequential regularity

5.1. In this section all the concrete examples will be spaces. In these we will forget the convention 1.6 and use the more common notation (small case roman letters for points, capitals for subsets).

The topologies in the examples will be indicated as τ, and the closures by overline.

$\mathbb{R}$ (resp. $\mathbb{Q}$, resp. $\mathbb{N}$) is the *set* of real (resp. rational, resp. natural) numbers, and the intervals in $\mathbb{R}$ will be denoted by

$$[a, b], \quad]a, b[\quad \text{etc.}$$

The Urysohn separation axiom (requiring separation of distinct points by *closed* neighbourhoods will be indicated by

$$(U),$$

and regularity by *(R)*.

5.2. Let $(a_\beta)_{\beta\le\alpha}$ be the sequence from the definition of *(SRα)*. Define

$$a'_0 = 0, \quad a'_{\beta+1} = \bigvee\{u \mid \exists\ x,\ x\vee a = 1,\ u\wedge x \le a'_\beta\}$$
$$a'_\beta = \bigvee_{\gamma<\beta} a'_\gamma \text{ in the limit case.}$$

Thus $(a'_\beta)_{\beta<\alpha}$ has again the required property. Thus we can reformulate *(SRα)* by *defining*, for any $a\in L$, by induction

$$a_0 = 0, \quad a_{\beta+1} = \bigvee\{u \mid \exists\ x,\ x\vee a = 1,\ u\wedge x \le a_\beta\} \text{ etc.,}$$

and stating that L has *(SRα)* if for each $a\in L$, $a = a_\alpha$.

In a space X, the formula (for $A\subseteq X$ open)

$$A_{\beta+1} = \bigcup\{U \mid \exists\ V,\ A\cup V = X\ \&\ V\cap U\subseteq A_\beta\}$$

is equivalent to a handier

(5.2.1) $$A_{\beta+1} = \bigcup\{U \mid \overline{U\smallsetminus A_\beta}\subseteq A\}$$

(think of $V = X\smallsetminus(\overline{U\smallsetminus A_\beta})$), and this way it will be viewed in the constructions to follow.

5.3 Theorem. *If $\alpha<\beta$ then (SRα) is strictly stronger than (SRβ).*

Proof: We will produce spaces (X_α,τ_α) and open sets $A\in\tau_\alpha$ such that for $\beta<\alpha$, $A_\beta\ne A$, and $A_\alpha = A$.

Set

$$X_\alpha = \{(\beta,n) \mid \beta<\alpha,\ n<\omega\}$$

with $B\subseteq X_\alpha$ in τ_α iff

$$(\beta,n)\in B \Longrightarrow \exists\ m\in\omega,\ \{(\gamma,k) \mid \gamma<\beta,\ k>m\}\subseteq B.$$

Thus in particular any neighbourhood of (β,n) meets any $\{(\gamma,m) \mid m\in\omega\}$ with $\gamma<\beta$ while it does not need to meet $\{(\beta,m) \mid m\in\omega\}$ in anything but the (β,n) itself. Hence, if we set $A = \{(\beta,n)\in X_\alpha \mid n\ne 0\}$ we get $A_\gamma = \{(\beta,n) \mid \beta<\gamma,\ n\in\omega\}$ for $\gamma<\alpha$, and $A_\alpha = A$. $\square$

5.4. Theorem. *Between the mentioned separation axioms the following implications and (besides compositions) no others hold:*

$$
\begin{tikzcd}
 & \textit{(SR2)}\ \&\ T_1 \arrow[r] & \textit{(SR)}\ \&\ T_1 \arrow[dr] & \\
\textit{(R)}\ \&\ T_1 \arrow[ur] \arrow[dr] & & & \textit{(Sob)}\ \&\ T_1 \\
 & \textit{(U)} \arrow[r] & T_2 \arrow[ur] &
\end{tikzcd}
$$

Proof: The only less standard implication in the scheme *(SR)* & $T_1 \Longrightarrow$ *(Sob)* & T_1 follows from 4.5 and 3.4. Thus, it remains to be proved that *(SR2)* & T_1 does not imply T_2 and that *(U)* does not imply *(SR)*.

(SR2) & $T_1 \;\not\Longrightarrow\;$ (T_2): Consider $X = \mathbb{N} \cup \{a,b\}$ with $a \neq b,\ a,b \notin \mathbb{N}$, and the topology τ,

$$A \in \tau \quad \text{iff} \quad (A \cap \{a,b\} \neq 0 \Longrightarrow \exists n \in \mathbb{N},\ \{m \mid m \geq n\} \subseteq A)$$

Obviously, (X,τ) is not Hausdorff. Using the trivial observation that the $\{n\}$ with $n \in \mathbb{N}$ are clopen we easily deduce that

(1) if $A \subseteq \mathbb{N}$ or $\{a,b\} \subseteq A$, $\quad A = A_1$,
(2) if $A \cap \{a,b\} = \{x\}$ $\ (x = a$ or $b)$ then $A_1 = A \smallsetminus \{x\}$ and $A_2 = A$.

Thus, (X,τ) satisfies *(SR2)*.

$(U) \;\not\Longrightarrow\;$ *(SR)*: Consider $(\mathbb{R},\tau)$ with

$$A \in \tau \quad \text{iff} \quad \begin{cases} \forall a \in A \cap (\mathbb{R} \smallsetminus \mathbb{Q}) \ \ \exists\, \varepsilon > 0 \text{ s.t. }]a-\varepsilon, a+\varepsilon[\ \subseteq A, \text{ and} \\ \forall a \in A \cap \mathbb{Q} \ \ \exists\, \varepsilon > 0 \text{ s.t. } Q \cap]a-\varepsilon, a+\varepsilon[\ \subseteq A. \end{cases}$$

Obviously $(\mathbb{R},\tau)$ is Urysohn; but it is not *(SR)* as the closure of any $\mathbb{Q} \cap\,]a,b[$ equals $[a,b]$ and hence, in particular, $\mathbb{Q}_\alpha = \emptyset$ for all α. $\square$

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