Counting bad bracketings

Martin Klazar

Department of Applied Mathematics of Charles University
Malostranské náměstí 25
118 00 Praha 1
Czech Republic
klazar@kam.ms.mff.cuni.cz

Abstract

Suppose a partition of $\{1, 2, \ldots, 2n\}$ into n 2-element blocks is given. By its crossing graph we mean the graph describing the intertwinings of the blocks. We use power series to find numbers of partitions whose crossing graph (1) is connected, (2) has no isolated vertex, (3) is a tree, or (4) is a path. We determine the parity of that number in case (2) and we show that the numbers in cases (1) and (2) do not form a P-recursive sequence. For (1) and (2) we count also the symmetric bracketings. We extend and simplify some results of Stein [8].

1 Introduction

A *bracketing* B with n *brackets* is a partition of $\{1, 2, \ldots, 2n\}$ into n 2-element sets. We use the symbol $|B|$ to refer to the number of brackets of B. The set $\mathcal{B}(n)$ of all bracketings with n brackets has $(2n-1)!! = (2n-1)\cdot(2n-3)\cdots 3\cdot 1$ elements. We shall investigate the cardinalities $a_i(n) = |\mathcal{B}_i(n)|$ of seven subfamilies $\mathcal{B}_i(n) \subset \mathcal{B}(n)$, $i \in \{0, 1, 2, 3, 4, 5, 6\}$, defined below. By $\mathcal{B}_i$ we denote $\bigcup_{n\geq 1} \mathcal{B}_i(n)$, the empty bracketing is denoted by $\emptyset$.

Two brackets b and c of a bracketing B *cross* if either $\min b < \min c < \max b < \max c$ or the same holds with b and c switched. The *crossing graph* $G(B)$ has B as its vertex set, two brackets are adjacent iff they cross. The *reflection* $\overline{B}$

of $B \in \mathcal{B}(n)$ is $\overline{B} = \{\{2n-i+1, 2n-j+1\} : \{i,j\} \in B\}$. A bracketing is *symmetric* if it coincides with its reflection.

Crossing graphs are known in graph theory as *circle graphs* — intersection graphs of circle chords — and are intesively studied. See, for instance, Bouchet [1]. Here, however, we are interested in the representations of crossing graphs by bracketings and not in the graphs themselves.

Good bracketings $\mathcal{B}_0(n)$ are the ones whose crossing graph has no edge. *Connected bracketings* $\mathcal{B}_1(n)$ have connected crossing graph. *Symmetric connected bracketings* $\mathcal{B}_2(n)$ form the subset of symmetric bracketings in $\mathcal{B}_1(n)$. *Bad bracketings* $\mathcal{B}_3(n)$ have crossing graph with no isolated vertex. Those which are symmetric form the set $\mathcal{B}_4(n)$ of *symmetric bad bracketings*. Crossing graphs of *tree bracketings* $\mathcal{B}_5(n)$ are trees. Finally, *path bracketings* $\mathcal{B}_6(n)$ have path as their crossing graph.

The family of good bracketings is a classical one. The families $\mathcal{B}_1(n)$ and $\mathcal{B}_2(n)$ have been investigated before but the remaining families seem new. The families $\mathcal{B}_0(n)$, $\mathcal{B}_5(n)$, and $\mathcal{B}_6(n)$ can be counted more easily and nicely than the other families.

Our results are as follows. We start Section 2 with some more definitions and then we review briefly the properties of the numbers $a_0(n)$ and some papers concerning counting arrangements of circle chords. We prove in a simpler way the known recurrence for $a_1(n)$ and derive a new simpler recurrence for $a_2(n)$. We derive a recurrence for $a_3(n)$ and a relation for $a_4(n)$. In Section 3 it is shown that $a_1(n)$ and $a_3(n)$ satisfy no homogeneous linear recurrence with polynomial coefficients (in fact, we prove something stronger). The parity patterns of the numbers $a_i(n), i \leq 4$, are determined. Tree and path bracketings are enumerated. The first nine values of any $a_i(n)$ are given in the table. We finish by mentioning some questions which might be worth of further investigation.

2 Recurrences

By an *arc* we mean here any plane semicircle, i.e., any intersection $H \cap C$ where H is a closed halfplane with the borderline going through the center of a circle C. A bracketing $B \in \mathcal{B}(n)$ can be depicted in the following way. We draw a horizontal line l and mark on it from left to right $2n$ distinct points $p_1, p_2, \ldots, p_{2n}$, called the *endpoints* of B. For each bracket $b = \{i, j\} \in B$ we join p_i and p_j by an arc $A_b = A_{ij}$ lying above l. Any p_i lies on exactly one arc and A_b and A_c intersect iff b and c cross. We call this a *picture* $p(B)$ of B. We prefer this way of visualization (it is used also in [4], [9], [8]), the transition to circle chords is obvious. The points p_i split l into $2n+1$ *gaps of* B, the two unbounded ones are called the *outer gaps* and the $2n-1$ remaining ones are called the *inner gaps*. From now on we will work with pictures rather than with partitions. The words as component, vertex, path etc. used in connection with a picture $p(B)$ refer to the corresponding crossing graph $G(B)$.

Let $B_1, B_2 \subset p(B)$ be two *separated* — that is, disjoint and connected by no edge in $G(B)$ — subbracketings. In the case that all endpoints of B_1 are localized in one inner gap of B_2 we write $B_1 < B_2$. In the case they lie in one outer gap we write $B_1 <> B_2$ (then also $B_2 <> B_1$). If B_1 and B_2 are two different components of $G(B)$ then clearly $B_1 < B_2$ or $B_2 < B_1$ or $B_1 <> B_2$.

The numbers $a_0(n)$ are the well known *Catalan numbers*

$$a_0(n) = \frac{1}{n+1}\binom{2n}{n} \tag{1}$$

following the basic recurrences $a_0(0) = 1$,

$$a_0(n) = \sum_{i=1}^{n} a_0(i-1)a_0(n-i), \tag{2}$$

and

$$(n+1)a_0(n) - (4n-2)a_0(n-1) = 0. \tag{3}$$

Their bracket representation is also classical. Catalan numbers pop out in many situations in combinatorics. The probability that $n+1$ random points chosen uniformly and independently in the unit square form a convex $n+1$-gon is $(a_0(n)/n!)^2$. The conditional probability that they form a convex chain, provided they form a convex $n+1$-gon, is $1/a_0(n)$. See Valtr [13] and [14] for details.

The problem to count all nonintersecting arrangements of n circle chords with $2n$ distinct (and labeled) ends — the anwer is, of course, that there are $a_0(n)$ of them — was proposed and solved by Errera [2] in 1931. Touchard enumerated in the papers [10], [11], and [12] general arrangements of circle chords by the number of intersections (in our terminology by the number of edges of

$G(B)$). Riordan [5] continued in this direction. The families $\mathcal{B}_1(n)$ and $\mathcal{B}_2(n)$ were enumerated recursively by Stein in [8]. There they are called *irreducible diagrams* and *symmetric diagrams*. Stein and Everett [9] proved that

$$a_1(n) \sim \frac{(2n-1)!!}{e}.$$

Much shorter proof of this asymptotics was given by Kleitman [4]. A generalization of this result to a wider class of diagrams is in the paper [3] of Hsieh.

The nice relation $a_1(1) = 1$,

$$a_1(n) = (n-1)\sum_{i=1}^{n-1} a_1(i)a_1(n-i), \tag{4}$$

is proved in [8] in a circumvential way via another, more complicated recurrence. We give now a simple direct argument.

Let $B \in \mathcal{B}_1(n)$ for $n \geq 2$. We delete from $p(B)$ the leftmost arc A_{1i} and obtain a bracketing with $n-1$ arcs. Its components form a chain $B_1 > B_2 > \ldots > B_k$ and p_i lies in some, say the jth one, inner gap of B_k. Thus B decomposes into (B^*, B_k, j) where B^* arises from B by removing B_k. Therefore $\mathcal{B}_1(n)$ corresponds bijectively to the triples (A, B, j) where A and B are connected bracketings, $|A| + |B| = n$, and $1 \leq j \leq 2|B| - 1$. Hence

$$a_1(n) = \sum_{i=1}^{n-1} (2i-1)a_1(i)a_1(n-i),$$

and (4) follows by pairing $a_1(i)a_1(n-i)$ with $a_1(n-i)a_1(i)$.

We introduce the (formal) power series $F_1(x) = \sum_{n\geq 1} a_1(n)x^n$. The above combinatorial decomposition reads in terms of F_1 as $F_1 = F_1 \cdot (2x\frac{d}{dx}F_1 - F_1) + x$. Hence

$$F_1' = \frac{F_1^2 + F_1 - x}{2xF_1}. \tag{5}$$

We proceed to enumerate symmetric connected bracketings. We use decomposition techniques similar to those in [8]. The recurrence we obtain is more transparent and compact than the one in [8].

Theorem 2.1 *We define $a_2(0) = -1$. We have $a_2(1) = 1$ and, for $n \geq 2$,*

$$a_2(n) = \sum_{i=1}^{n-2} a_2(i)a_2(n-i) + \sum_{i=1}^{n/2} (2n-4i-1)a_1(i)a_2(n-2i). \tag{6}$$

Proof. Let $B \in \mathcal{B}_2(n)$ for $n \geq 2$. We delete from $p(B)$ the first and last arcs $X = A_{1i}$ and $Y = A_{2n-i+1,2n}$ and obtain a symmetric bracketing with $n-2$ arcs. Its components split uniquely in three groups $\{B_1, \ldots, B_{k-1}\}$, C_k, and D_k (C_k and D_k are chains of components) where $B_1 > B_2 > \ldots > B_{k-1} > C_k, D_k$ and $C_k <> D_k$. The following holds.

- $B_j \in \mathcal{B}_2$, B_{j+1} lies in the central inner gap of B_j.
- C_k and D_k are reflections of one another, $C_k \in \mathcal{B} \cup \{\emptyset\}$, C_k and D_k lie in the symmetric pair of (not necessarily distinct) inner gaps of B_{k-1}.
- The point p_i lies in an inner gap of C_k and p_{2n-i+1} lies in an inner gap of D_k or vice versa. In the former case we put $C_k^* = C_k \cup \{X\}$ and $D_k^* = D_k \cup \{Y\}$, in the latter case we switch X and Y. Then $C_k^* = \overline{D_k^*}$ and C_k^* is a general nonempty connected bracketing.
- In the case that X and Y do not intersect we have $k \geq 2$, otherwise $k \geq 1$.

This decomposition expresses in terms of the power series $F_2(x) = \sum_{n\geq 1} a_2(n)x^n$ and $F_1(x)$ as

$$F_2(x) = 2xF_1(x^2)\sum_{k\geq 0} F_2'(x)F_2^k(x) + F_1(x^2) + x.$$

Here the coefficient 2 accounts for the two cases of crossing and noncrossing X and Y, the derivative accounts for the locations of C_k and D_k, and the first occurrence of $F_1(x^2)$ accounts for $C_k^* \cup D_k^*$. The second occurrence of $F_1(x^2)$ accounts for the case when there is no B_j.

Using the geometric series formula and simplifying we can recast this as

$$F_2^2 - (1+x)F_2 + F_1(x^2)(2xF_2' - F_2 + 1) + x = 0.$$

Equating the coefficient at x^n to zero we obtain (6). □

Now we enumerate bad bracketings.

Theorem 2.2 *We set $a_3(-1) = a_3(0) = 1$. Then, for $n \geq 1$,*

$$a_3(n) = \sum (2m+1)a_3(k)a_3(l)a_3(m), \tag{7}$$

we sum over integer triples (k, l, m) satisfying $-1 \leq k, l, m \leq n-1$, $k \neq -1$, and $k + l + m = n - 2$.

Proof. We claim that the power series $F_3(x) = \sum_{n\geq 0} a_3(n)x^n$ satisfies the equation

$$F_3 = \sum_{k\geq 0}(x(2xF_3' + F_3))^k - xF_3^2 \tag{8}$$

that can be written as

$$F_3 \cdot (1 + xF_3) \cdot (1 - xF_3 - 2x^2 F_3') = 1. \tag{9}$$

Or, to isolate the derivative, as

$$F_3' = \frac{-x^2F_3^3 + F_3 - 1}{2x^3F_3^2 + 2x^2F_3}. \tag{10}$$

To see this we delete from $B \in \mathcal{B}_3(n), n \geq 1$, the first arc A_{1i} and consider the structure of the resulting bracketing. It is as follows.

- The point components must form a chain $O_1 > O_2 > \ldots > O_k$ (any O_j is a single arc).
- The rest splits uniquely into separated bracketings $B_j \in \mathcal{B}_3 \cup \{\emptyset\}$, $j = 1, \ldots, k+1$, such that $O_j > B_{j+1}$ and $B_j > O_j$ or $B_j <> O_j$.
- If $k > 0$ the point p_i lies in one of the $2|B_{k+1}| + 1$ gaps of B_{k+1} and in the inner gap of O_k.
- The case $k = 0$ is different, the gaps splitting B_1 into $C_1 <> C_2$, $C_j \in \mathcal{B}_3 \cup \{\emptyset\}$, are for p_i forbidden.

It is not difficult to translate this to generating functions, then (8) is obtained. The coefficient extraction and indices shift transform (9) to (7). □

Finally, we enumerate symmetric bad bracketings. A combination of arguments of Theorems 2.1 and 2.2 yields an equation involving $F_3(x)$ and $F_4(x)$. A recurrent relation can be obtained but is not too pleasant to be stated. For the sake of brevity we omit the details.

Theorem 2.3 *Let $P(x) = 1 - 2x^4F_3'(x^2) - x^2F_3(x^2)$ and $a_4(0) = 1$. The power series $F_4(x) = \sum_{n\geq 0} a_4(n)x^n$ satisfies the equation*

$$x(1 + x^2F_3(x^2))PF_4^2 + (2x^2 - (1 + x + x^2F_3(x^2))P)F_4 + 2x^3F_4' + P = 0.$$

Proof. (Sketch) Considering a symmetric bad bracketing and deleting from it the first and last arcs we obtain

$$F_4 = 2x^2(xF_4' + F_4)\sum_{k\geq 0}(xF_4)^k \cdot \sum_{k\geq 0}(x^2(xF_3(x^2)' + F_3(x^2)))^k - x^2F_3(x^2)F_4 + 1.$$

$\square$

3 P-recursivity, Parity, Trees, and Paths

A sequence of complex numbers $\{a(n)\}_{n\geq 0}$ is called *P-recursive* (short for polynomially recursive) if there are $j+1$ polynomials $P_0(x), \ldots, P_j(x) \in \mathbf{C}[x]$, $P_j(x) \not\equiv 0$, such that for any integer $n \geq 0$

$$P_0(n)a(n) + P_1(n)a(n+1) + \ldots + P_j(n)a(n+j) = 0. \tag{11}$$

P-recursive sequences are certainly "nice" sequences because (11) provides us with an easy way to calculate quickly their values. Therefore for any counting sequence s it is of some interest to know whether s is P-recursive. The formula (3) shows that Catalan numbers form a P-recursive sequence. We shall see that $\{a_5(n)\}_{n\geq 0}$ and $\{a_6(n)\}_{n\geq 0}$ are P-recursive as well. On the other hand we prove now that $\{a_1(n)\}_{n\geq 0}$ and $\{a_3(n)\}_{n\geq 0}$ are not P-recursive.

The basic reference for P-recursivity is Stanley [7]. There it is proved that the P-recursivity of $\{a(n)\}_{n\geq 0}$ is equivalent to the fact that $F(x) = \sum_{n\geq 0} a(n)x^n$ satisfies, for some $m+1$ polynomials $R_0(x), \ldots, R_m(x) \in \mathbf{C}[x]$, $R_m(x) \not\equiv 0$, the differential equation

$$R_m(x)F^{(m)}(x) + \ldots + R_1(x)F'(x) + R_0(x)F(x) = 0. \tag{12}$$

This property of $F(x)$ is called *D-finiteness*.

We define now a property weaker than P-recursivity and we prove that the sequences $\{a_{1,3}(n)\}_{n\geq 0}$ still do not have it. We say that a sequence of complex numbers $\{a(n)\}_{n\geq 0}$ is *approximately P-recursive* if there is a P-recursive sequence $\{b(n)\}_{n\geq 0}$ and a constant $c > 0$ such that $|a(n)-b(n)| < c^n$ for any $n \geq 0$. A power series $F(x) = \sum_{n\geq 0} a(n)x^n$ is *analytic* if it has a nonzero radius of convergence, i.e. if $|a(n)| < c^n$ for any $n \geq 0$ for some constant $c > 0$.

Theorem 3.1 *Let $P(x,y)$ and $Q(x,y)$ be two polynomials satisfying*

$$Q(x,0) \equiv 0 \text{ and } P(x,0) \cdot (Q/y)(x,0) \not\equiv 0,$$

and let $F(x) = \sum_{n \geq 0} a(n)x^n$ be a nonanalytic power series satisfying

$$F' = \frac{P(x,F)}{Q(x,F)}.$$

Then $\{a(n)\}_{n\geq 0}$ is not approximately P-recursive.

Proof. We show that for any $k \geq 1$

$$F^{(k)} = \frac{P_k(x,F)}{Q(x,F)^{2k-1}} \tag{13}$$

where P_k is a bivariate polynomial such that $P_k(x,0) \not\equiv 0$. For $k=1$ it is true. Assuming the claim for $k \geq 1$ and taking the derivative of $F^{(k)}$ we obtain

$$F^{(k+1)} = \frac{P_k' Q^2 - (2k-1)P_k Q' Q}{Q^{2k+1}}.$$

Substituting the expression for F' we see that $P_{k+1} = P_k' Q^2 - (2k-1)P_k Q' Q$ is a polynomial in x and F. By our asumption $(Q'Q)(x,0) \not\equiv 0$, so by induction $P_{k+1}(x,0) \not\equiv 0$.

Now assume that $\{a(n)\}_{n\geq 0}$ is approximately P-recursive. Thus for some analytic power series $G(x)$ the sum $H(x) = F(x) + G(x)$ is D-finite. The derivatives $H^{(k)}(x) = F^{(k)}(x) + G^{(k)}(x)$ satisfy an equation of the type (12). Substituting the expressions (13) and clearing out the denominators we obtain the relation

$$Q^{2m-1}(x,F)\, R_0(x)\, F + \sum_{i=1}^{m} Q^{2m-2i}(x,F)\, R_i(x)\, P_i(x,F) +$$

$$+ Q^{2m-1}(x,F) \sum_{i=0}^{m} R_i(x)\, G^{(i)}(x) = 0.$$

The left side does not vanish because the specialization $F=0$ yields the nonzero polynomial $R_m(x)\, P_m(x,0)$. Thus F satisfies a nontrivial algebraic equation

$$S_0(x) + S_1(x)\, F(x) + \ldots + S_r(x) F^r(x) = 0$$

where the coefficients $S_i(x)$ are analytic power series. By M. Artin's approximation theorem, see Ruiz's book [6], pp. 94 and 106,, $F(x)$ itself must be analytic. But this contradicts our assumption. Hence $\{a(n)\}_{n\geq 0}$ is not approximately P-recursive. □

Corollary 3.2 *The sequences* $\{a_1(n)\}_{n\geq 1}$ *and* $\{a_3(n)\}_{n\geq 0}$ *counting connected bracketings and bad bracketings are not approximately P-recursive.*

Proof. The formal differential equations (5) and (10) are in the right form. The recurrences (4) and (7) show that the sequences $\{a_{1,3}(n)\}_{n\geq 0}$ grow faster than any exponential function. Thus F_1 and F_3 are nonanalytic. $\square$

It is a well known and neat property of Catalan numbers that $a_0(n)$ is odd iff $n = 2^m - 1$. Although $a_1(n)$ and $a_3(n)$ are not P-recursive, their parity follows a P-recursive pattern. To describe it we use the symbol (a mod b) meaning, for $a, b \in \mathbf{N}$, the least $c \in \mathbf{N}$ congruent mod b with a.

Theorem 3.3 *The number* $a_1(n)$ *is odd iff* $n = 2^m$. *The numbers* $a_3(2m+1)$ *are even. Finally,* $a_3(2m)$ *is even iff there is an* $r \in \mathbf{N}$ *such that*

$$(m \bmod 2^r) > \frac{2^{r+1}}{3}.$$

Proof. For $n > 1$ odd, by (4), $a_1(n)$ is even. For n even we have, by (4), $a_1(n) \equiv a_1(n/2) \pmod 2$ and the claim follows by induction. This is, of course, exactly the same proof as for Catalan numbers by means of (2).

Reducing (9) mod 2 we obtain the equation

$$x^2F^3 - F + 1 = 0$$

whose unique power series solution $F(x) = \sum_{n\geq 0} a(n)x^n$ is coefficientwise congruent mod 2 with $F_3(x)$. Substituting $G = xF$ we obtain the equation

$$G = \frac{x}{1 - G^2}$$

that can be easily resolved by the Lagrange inversion formula (see, for instance, the book of Wilf [15]). By the formula the coefficient at x^n in G equals to the coefficient at y^{n-1} in $(1 - y^2)^{-n}/n$. We obtain

$$a_3(n) \equiv a(n) = \begin{cases} 0 & \text{for } n \text{ odd} \\ \frac{1}{n+1}\binom{3n/2}{n} & \text{for } n \text{ even.} \end{cases}$$

To obtain for $n = 2m$ an explicit parity criterion we have to determine the parity of $\binom{3m}{m}/(2m+1)$. This is a standard task for elementary number theory. The argument uses the formula

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots$$

for the highest power in which a prime p divides $n!$ and we omit it. $\square$

For example, $a_3(304)$ is even because 152 mod 32 is 24. On the other hand $a_3(296)$ is odd because 148 mod 2 and 4 is 0, mod 8 and 16 is 4, mod 32, 64, and 128 is 20, and mod 256 is 148.

Obviously $a_2(n)$ has the same parity as $a_1(n)$ for it differs from $a_1(n)$ by the number of nonsymmetric connected bracketings which come in pairs. The same for $a_3(n)$ and $a_4(n)$.

The Catalan numbers (1) enumerate also *rp* (short for rooted plane) *trees* with $n-1$ vertices. These are unlabeled trees with a specified vertex called a *root*, with all edges directed away from the root, and with a linear order on any set of children of a vertex. *Ternary rp trees* are rp trees in which each vertex has either 3 or 0 children. It is not difficult to prove by the Lagrange inversion formula that there are

$$\frac{1}{2n+1}\binom{3n}{n} \tag{14}$$

ternary rp trees with $3n+1$ vertices.

Theorem 3.4 *The number of tree bracketings with n brackets is given by the formula*

$$a_5(n) = \frac{1}{2n-1}\binom{3n-3}{n-1}.$$

Proof. We present a bijection between $\mathcal{B}_5(n)$ and the set of ternary rp trees with $3n-2$ vertices. Let $B \in B_5(n)$, consider the picture $p(B)$. We define a graph $H(B)$ whose vertices are the endpoints of the arcs, except for the rightmost one, and the arc intersections. The root is the counterclockwise first vertex on the rightmost arc. Two vertices form an edge iff they are connected by an arc segment with no vertex inbetween.

Obviously $H(B)$ is connected and has no cycle because $G(B)$ is a tree. Directing the edges away from the root we turn $H(B)$ into a ternary rp tree. $H(B)$ has $(2n-1)+(n-1)$ vertices. The transformation of B in $H(B)$ is the desired bijection. The inverse reconstruction of B from $H(B)$ is left to the reader. Hence, by (14), the formula for $a_5(n)$ follows. $\square$

We finish enumerating the path bracketings.

Theorem 3.5 *The ordinary generating function $F_6(x) = \sum_{n\geq 1} a_6(n)x^n$ counting path bracketings is*

$$F_6(x) = \frac{1}{2x}\left(1 - x - \sqrt{\frac{1-3x-x^2-x^3}{1-x}}\right) - \frac{x^2}{1-x}. \tag{15}$$

Proof. We call an arc X in the picture $p(B)$ of a bracketing B *free* if $Y > X$ for no other arc Y. It is easy to see that there is only one bracketing in $\mathcal{B}_6(n)$, $n > 1$, having free both of the arcs corresponding to the endvertices of the path $G(B)$. We call this exceptional bracketing a *chain*.

Now delete from a path bracketing $B \in \mathcal{B}_6(n)$, $n > 1$, the first arc $X = A_{1i}$. The crossing graph $G(B^*)$ of the remaining bracketing B^* is a path or has two path components. Suppose first $G(B^*)$ is a path. The point p_i must lie in the inner gap which is covered by the free endvertex of $G(B^*)$ and which is not covered by any other arc. So we have only one way how to add X back to obtain B, except for B^* a chain when we have two ways. Such B's are accounted for by the coefficient in $x(F_6 - x^2/(1-x) + 2x^2/(1-x))$.

Suppose $G(B^*)$ has two path components $C > D$. As in the previous case, C has only one gap D can lie in, except for C a chain when there are two such gaps. The same for the location of the endpoint p_i in D. Therefore, such B's are accounted for in $x(F_6 + x^2/(1-x))^2$.

Finally, the single arc bracketing is accounted for by x. Altogether we have

$$F_6 = x\left(F_6 + \frac{x^2}{1-x}\right)^2 + x\left(F_6 + \frac{x^2}{1-x}\right) + x.$$

This leads to the quadratic equation $(1-x)xG^2 - (1-x)^2G + x = 0$ where $G = F_6 + x^2/(1-x)$. Solving the equation we obtain the formula (15). $\square$

The reader may wonder what is the P-recurrence for $\{a_6(n)\}_{n\geq 1}$. Writing $H(x)$ for the radical in (15) we find that

$$H'(x) \cdot (x^4 + 2x^2 - 4x + 1) - H(x) \cdot (x^3 - x^2 - x - 1) = 0.$$

Using this relation we obtain after some additional calculations the recurrence ($n \geq 6$)

$$(n+1)a_6(n) - (4n-1)a_6(n-1) + (2n-1)a_6(n-2) + a_6(n-3) + (n-4)a_6(n-4) = 2.$$

The initial values are: $a_6(2) = 1$, $a_6(3) = 3$, $a_6(4) = 8$, $a_6(5) = 21$. We could easily make the recurrence homogeneous but for the cost of enlarging its span.

In the table below we list first few values of the numbers $a_i(n)$.

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	1	1	1	0	0	1	1
2	2	1	1	1	1	1	1
3	5	4	2	4	2	3	3
4	14	27	7	31	9	12	8
5	42	248	22	288	26	55	21
6	132	2830	96	3272	122	273	56
7	429	38232	380	43580	466	1428	153
8	1430	593859	1853	666143	2299	7752	428
9	4862	10401712	8510	11491696	10316	43263	1222

The obvious question is whether $\{a_2(n)\}_{n\geq 0}$ and $\{a_4(n)\}_{n\geq 0}$ are P-recursive. We think that the anwer is negative. One can eliminate F_1 from the equation for F_2 but the resulting relation contains the second derivative of F_2. A theorem more involved than Theorem 3.1 is necessary. The situation for F_4 is even more complicated.

One can consider, besides the approximate P-recursivity, another way of approximating a non P-recursive sequence by a P-recursive one. We say that a sequence of integers $\{a(n)\}_{n\geq 0}$ is *P-recursive mod p*, p being prime, if there is a P-recursive sequence of integers $\{b(n)\}_{n\geq 0}$ such that $b(n) \equiv a(n) \bmod p$ for any $n \geq 0$. We have shown that any of the four sequences $\{a_{1,2,3,4}(n)\}_{n\geq 0}$ is P-recursive mod 2. We conjecture that this is not the case for any $p > 2$.

References

[1] A. Bouchet, Reducing prime graphs and recognizing circle graphs, *Combinatorica* **7** (1987), 243–254.

[2] A. Errera, *Mem. Acad. Bruxelles* 8° (2) **11** (1931).

[3] W. N. Hsieh, Proportions of irreducible diagrams, *Studies in Appl. Math.* **52** (1973), 277–283.

[4] D. J. Kleitman, Proportions of irreducible diagrams, *Studies in Appl. Math.* **49** (1970), 297–299.

[5] J. Riordan, The distribution of crossings of chords joining pairs of $2n$ points on a circle, *Math. Comp.* **29** (1975), 215–222.

[6] J. M. Ruiz, *The basic theory of power series*, Vieweg, Braunschweig/Wiesbaden 1993.

[7] R. P. Stanley, Differentiably finite power series, *Europ. J. Combinatorics* **1** (1980), 175–188.

[8] P. R. Stein, On a class of linked diagrams, I. Enumeration, *J. of Combinatorial Th.* Series A **24** (1978), 357–366.

[9] P. R. Stein and J. A. Everett, On a class of linked diagrams, II. Asymptotics, *Disc. Math.* **21** (1978), 309–318.

[10] J. Touchard, Sur une problème de configurations, *C. R. Acad. Sci. Paris* **230** (1950), 1997–2008.

[11] J. Touchard, Contribution à l'étude du problème des timbres-poste, *Canad. J. Math.* **2** (1950), 385–398.

[12] J. Touchard, Sur une problème de configurations et sur les fractions continues, *Canad. J. Math.* **4** (1952), 2–25.

[13] P. Valtr, Probability that n random points are in convex position, *Discrete Comput. Geom.* **13** (1995), 637–643.

[14] P. Valtr, Catalan numbers via random planar point sets, submitted.

[15] H. S. Wilf, *Generatingfunctionology*, Academic Press, New York 1994.