KAMAK 2025

Hotel ČESKÁ FARMA, Dolní dvůr

September 14 – 19

Charles University, Prague

Organizers:

David Mikšaník
Robert Šámal

Brochure of open problems,
Prague, 2025.

Contents

Program*

07:30	–	09:00	Breakfast
09:00	–	**10:30**	**Morning Problem-Solving Session I**
10:30	–	11:00	Coffee Break
11:00	–	**12:00**	**Morning Problem-Solving Session II**
12:30	–	13:30	Lunch
15:00	–	**16:30**	**Afternoon Problem-Solving Session I**
16:30	–	17:00	Coffee Break
17:00	–	**18:00**	**Afternoon Problem-Solving Session II**
18:00	–	**18:30**	**Problem Progress Reports**
19:00	–	20:00	Dinner

*From Tuesday lunch until Friday lunch.

OPEN PROBLEMS

Problem 1. *Untangling geometric graphs (suggested by Todor Antić)*

Source: Proposed by Jan Kynčl in the year of our lord 2023.

Definitions.
• *A* geometric graph *is a graph drawn in the plane with vertices as points and edges as straight line segments between the points.*
• *A geometric graph G is* k-plane *if every edge has at most k crossings with other edges of G.*
• *A geometric graph G is* h-quasiplane *if it contains no h pairwise crossing edges.*

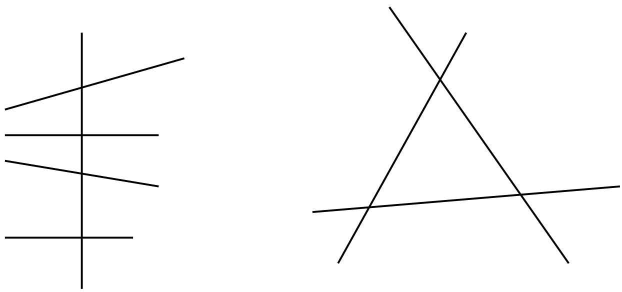


Figure 1: Forbidden configurations in 3-plane and 3-quasiplane geometric graphs.

Question: Can every k-plane geometric graph be redrawn so that it is $(k+1)$-quasiplane?

Related results:
• Every geometric k-plane graph can be redrawn to be geometric $(k+2)$-quasiplane (exercise).

• Every convex-geometric k-plane graph can be redrawn to be convex-geometric k-quasiplane [1].
• Every simple topological k-plane graph can be redrawn to be simple $(k+1)$-quasiplane [2].

References:

[1] Todor Antic. "Convex-Geometric k-Planar Graphs Are Convex-Geometric(k+1)-Quasiplanar". In: Combinatorial Algorithms. Ed. by Adele Anna Rescigno and Ugo Vaccaro. Cham: Springer Nature Switzerland, 2024, pp. 138–150.

[2] Patrizio Angelini et al. "Simple k-planar graphs are simple (k+1)-quasiplanar". In: Journal of Combinatorial Theory, Series B 142 (2020), pp. 1–35. ISSN: 0095-8956. DOI: https://doi.org/10.1016/j.jctb.2019.08.006. URL: https://www.sciencedirect.com/science/article/pii/S0095895619300838.

Problem 2. *Fast computation of Bézout polynomials (suggested by Juraj Belohorec)*

Source: Proposed by Alex Ozdemir.

Question: Let $\mathbb{F}$ be a field (in practice, a finite field of prime order). Let $a_1, \ldots, a_m$ be m distinct elements of $\mathbb{F}$. Let $Z(X) = \prod_{i=1}^{m}(X - a_i)$. Then (since the roots a_i are distinct) Z and its derivative Z' have no common factor, so there exist polynomials $s(X)$ and $t(X)$ such that $sZ + tZ' = 1$. Is there an algorithm to compute the coefficients of s and t, from $a_1, \ldots, a_m$, using at most $O(m \log m)$ field operations?

References:

`https://notes.0xparc.org/problems/compute-bezout/`

Problem 3. *Complexity of 3-coloring circle graphs (suggested by Petr Chmel)*

Source: Reopened by Bachmann, Rutter, and Stumpf in 2023.

Definitions.
• *A graph is called a* circle graph *if it can be represented as an intersection graph of chords in a circle. (That is, each vertex gets assigned a chord in a circle, and two vertices are adjacent iff their chords intersect.)*

Question: What is the complexity of 3-coloring circle graphs?

Related results:
• For k-coloring of circle graphs with $k \geq 4$, we know the problem is NP-complete.
• For most superclasses of circle graphs, 3-coloring is known to be NP-complete, usually because the superclasses also contain the class planar graphs.
• Unger gave an algorithm for 3-coloring circle graphs in time $\mathcal{O}(n \log n)$ using a backtracking algorithm for a 3-SAT instance with a specific structure based on the circle graph, however Bachmann, Rutter, and Stumpf have shown counterexamples to both the reduction to 3-SAT and the backtracking algorithm.

References:

Bachmann, P., Rutter, I., & Stumpf, P. (2023, September). On 3-coloring circle graphs. In *International Symposium on Graph Drawing and Network Visualization* (pp. 152-160). Cham: Springer Nature Switzerland.

Unger, W. (1988, February). On the k-colouring of circle-graphs. In *Annual Symposium on Theoretical Aspects of Computer Science* (pp. 61-72). Berlin, Heidelberg: Springer Berlin Heidelberg.

Unger, W. (1992, February). The complexity of colouring circle graphs. In *Annual Symposium on Theoretical Aspects of Computer Science* (pp. 389-400). Berlin, Heidelberg: Springer Berlin Heidelberg.

Problem 4. *A bicolored pseudoline arrangement has a bicolored triangle (suggested by Niloufar Fuladi)*

Source: Proposed in the book Oriented Matroids [1].

Definitions.

• *A* pseudoline *is a simple curved line in the plane. We consider this as a generalization of a straight line in the plane.*

• *A set of pseudolines in the plane that pairwise cross* exactly *once is called a* pseudoline arrangement*. This is a generalization of a line arrangement in the plane. A pseudoline arrangement is* simple *if no three pseudolines cross in a point.*

• *A pseudoline arrangement decomposes the plane into polygonal cells of degree $d \geq 3$ which we call d-*cells.

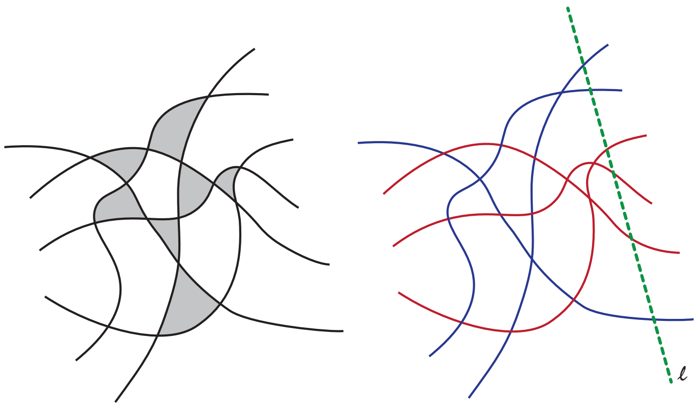


Figure 2: Left: a simple pseudoline arrangement with highlighted 3-cells; Right: A separated bicoloring of the pseudolines and a straight line l at infinity.

Question ([1]): A *bicoloring* of a pseudoline arrangement is to color the pseudolines with two colors. In a bicolored simple pseudoline arrangement, there exists a bicolored 3-cell (triangle).

Related results:

• A bicolored line arrangement contains a bicolored 3-cell [2].

• A bicolored arrangement with at most 5 red pseudolines has a bicolored 3-cell [3].

• Consider a bicoloring of a pseudoline arrangement with red and blue. Consider a straight pseudoline at infinity l that crosses all the pseudolines exactly once and that all the crossings of the arrangement appear on one side of it. If there exists a line l such that all the crossings between red (or blue) pseudolines appear consecutively on the line, we call the coloring a *separated coloring*. An arrangement with a separated coloring has a bicolored 3-cell [3]. See the right picture in Figure 2.

• Every bicolored arrangement contains a bicolored 3-cell or a bicolored 4-cell [3].

References:

[1] Oriented Matroids. Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter Ziegler. Cambridge University Press, Cambridge, 1993.

[2] Arrangements of Approaching Pseudo-Lines. Stefan Felsner, Alexander Pilz, and Patrick Schnider

[3] On Triangles in Colored Pseudoline Arrangements. Yan Alves Radtke, Balázs Keszegh, and Robert Lauff.

Problem 5. *Bounded-stability chromatic threshold (suggested by Tomáš Hons)*

Source: Inspired by the Hong Liu's talk at Eurocomb'25.

Definitions.
- *All graphs are finite. The minimum degree of G is denoted by $\delta(G)$.*
- *Graph G is F-*free *if F is not an induced subgraph of G.*
- *The half graph of size n is denoted by H_n and defined as*

$$V(H_n) = A \sqcup B = \{a_i : i \in [n]\} \sqcup \{b_j : j \in [n]\},$$
$$E(H_n) = \{a_i b_j : i, j \in [n], i \leq j\}.$$

- *A bipartite graph H with a fixed bipartition is a* semi-induced *subgraph of G if there is an injective mapping $f : V(H) \to V(G)$ which respects the edges and non-edges* between *the parts.*
- *A graph G is called k-*stable *if it does not contain a copy of H_k as a semi-induced subgraph. The minimum value of k such that G is k-stable is denoted by $\sigma(G)$ and referred to as the* stability of *G.*
- *We define the* bounded-stability chromatic threshold *of a graph F, denoted by $\delta^{\sigma}_{\chi}(F)$, as*

$$\delta^{\sigma}_{\chi}(F) = \inf\{\alpha \geq 0 : \forall\ s \in \mathbb{N}, \exists C = C(\alpha, F, s) \text{ s.t.}$$
$$\forall \text{ } n\text{-vertex } F\text{-free graph } G, \sigma(G) \leq s, \delta(G) \geq \alpha n \implies \chi(G) \leq C\}.$$

Examples:
- We have $\delta^{\sigma}_{\chi}(K_3) \leq 1/2$ as there are no K_3-free graphs with with $\delta(G) > 1/2$ is $K_{n,n}$. Similarly, we have $\delta^{\sigma}_{\chi}(K_t) \leq \frac{t-2}{t-1}$ by the Turán theorem.
- Moreover, by work on a related graph parameter $\delta_{\chi}(F)$, we know from [1] that $\delta^{\sigma}_{\chi}(K_t) \leq \delta_{\chi}(K_t) = \frac{t-3}{t-2}$. The value of $\delta_{\chi}(F)$ is defined analogously, but VC-dimension of G is used instead of stability, see below.

Question: Determine $\delta^{\sigma}_{\mathrm{hom}}(F)$. Start with $F = K_3$.

Motivation and related results:

• A well studied notion is the *chromatic threshold* of F defined as

$$\delta_\chi(F) = \inf\{\alpha \geq 0 : \exists C = C(\alpha, F) \text{ s.t.}$$
$$\forall\ n\text{-vertex } F\text{-free graph } G, \delta(G) \geq \alpha n \implies \chi(G) \leq C\}.$$

The values of $\delta_\chi(F)$ were classified in [2].

• In subsequent works [1, 3], the effect of bounded VC-dimension was studied, which was formalized as the notion of *bounded-VC chromatic threshold*:

$$\delta_\chi(F) = \inf\{\alpha \geq 0 : \forall\ d \in \mathbb{N}, \exists C = C(\alpha, F, d) \text{ s.t.}$$
$$\forall\ n\text{-vertex } F\text{-free graph } G, (G) \leq d, \delta(G) \geq \alpha n \implies \chi(G) \leq C\}.$$

We clearly have $\delta_\chi(F) \leq \delta_\chi(F)$.

• The parameter $\delta_\chi^\sigma(F)$ is a natural variant of $\delta_\chi(F)$ in the view that both bounded VC-dimension and bounded stability are related and increasingly important tameness notions coming from model theory. See e.g. [4, 5].

• Since $(G) \leq \sigma(G)$, we have $\delta_\chi(F) \geq \delta_\chi^\sigma(F)$ for every F. Thus, we obtain upper bounds on $\delta_\chi^\sigma(F)$ from known results, while the lower bounds would require constructing of F-graphs with bounded stability, large minimum degree, and unbounded chromatic number.

References:

[1] Kim, J., Liu, H., Shangguan, C., Wang, G., Wu, Z., & Xue, Y. (2025). Stability with minuscule structure for chromatic thresholds.

[2] Allen, P., Böttcher, J., Griffiths, S., Kohayakawa, Y., & Morris, R. (2013). The chromatic thresholds of graphs. *Advances in Mathematics*, 235, 261-295.

[3] Xinqi Huang, Hong Liu, Mingyuan Rong, & Zixiang Xu. (2025). Interpolating chromatic and homomorphism thresholds.

[4] Malliaris, M., & Shelah, S. (2013). Regularity lemmas for stable graphs. *Transactions of the American Mathematical Society*, 366(3), 1551–1585.

[5] Nguyen, T., Scott, A., & Seymour, P. (2023). Induced subgraph density. VI. Bounded VC-dimension.

Problem 6. *Oriented trees in oriented graphs of large girth (suggested by Tereza Klimošová)*

Source: Proposed by Maya Stein in [1].

Definitions.
• *The* girth *of a graph G is the length of the shortest cycle in G,*
• $\Delta(T)$ *denotes the* maximum degree *of T, and*
• $\delta^0(D)$ *denotes the* minimum semidegree *of a digraph D, which is defined as the minimum over all the indegrees and all the outdegrees of the vertices of D.*

Question: Does every oriented graph D of girth at least $2\ell + 1$ with $\delta^0(D) \geq \max(k/\ell, \Delta(T))$ contain every orientation of each k-edge tree?

Related results:
• The undirected version of the question is known to be true;

Theorem 1 (Jiang [2])**.** *Every graph G of girth at least $2\ell + 1$ with $\delta(G) \geq \max(k/\ell, \Delta(T))$ contains any k-edge tree.*

• There is a version forbidding oriented 4-cycles, see [3].

References:

[1] M. Stein. Degree conditions for trees in undirected and directed graphs. In Women in Mathematics in Latin America, Trends in Mathematics. Springer-Birkhäuser. To appear.

[2] T. Jiang, On a conjecture about trees in graphs with large girth, Journal of Combinatorial Theory. Series B 83 (2001), no. 2, 221–232

[3] M. Stein and A. Trujillo-Negrete, Oriented trees in digraphs without oriented 4-cycles, arXiv preprint arXiv:2411.13483 (2024).

Problem 7. *Clique Dynamics of Whitney triangulations (suggested by Anna Margarethe Limbach)*

Source: Proposed by F. Larrión, V. Neumann-Lara, and M. A. Pizaña in 2002 [4].

Definitions.

• *Given a graph G, its* clique graph *kG is the intersection graph of its maximal complete subgraphs, called* cliques. *This means the set of cliques of G is the vertex set of kG, and two vertices of kG are adjacent if and only if they intersect non-trivially as cliques. The operator k that maps each graph to its clique graph is called the* clique graph operator, *and one can apply it iteratively to generate the sequence of iterated clique graphs $k^0G = G$, $k^1G = kG$, $k^2G = k(kG)$, k^3G, ...*

• *If the sequence of iterated clique graphs of a graph G consists of pairwise non-isomorphic graphs, G is called k-*divergent, *otherwise,* k-convergent, *and if it contains the graph that consists of a singleton (and then becomes stationary there) G is called k-*null.

• *A graph is called* locally cyclic, *if each open neighbourhood of a vertex induces a cycle.*

• *A Whitney-triangulation of a compact surface is a triangulation such that each triangle of its underlying graph is a face of the surface. Whitney triangulations of closed surfaces are locally cyclic with minimum degree ≥ 3 and the only one with minimum degree 3 is the K_4 that triangulates the sphere.*

From another perspective one can start with such a locally cyclic graph and obtain a closed surface by "filling in the triangles".

Questions:

1. Is the underlying graph of a finite Whitney triangulation of the disk always k-null?

2. Is K_4 the only underlying graph of a finite Whitney triangulation of the sphere that is k-convergent?

Related results:

• The K_4 is k-null.
• The octahedron graph is k-divergent.
• The icosahedron graph is k-divergent.
• All locally cyclic graphs with minimum degree ≥ 7 are k-convergent.
• Every closed surface admits a k-divergent Whitney triangulation.
• Every closed surface with negative Euler characteristic admits a k-convergent Whitney triangulation.
• Every Whitney triangulation of the disk such that every interior vertex has degree ≥ 6 is k-null.

References:

[1] Larrión, F., Pizaña, M.A. and Villarroel-Flores R.: Iterated clique graphs and bordered compact surfaces. Discrete Mathematics 313 (2013) 508–516 `https://doi.org/10.1016/j.disc.2012.11.017`

[2] Larrión, F., Neumann-Lara, V. and Pizaña, M.A.: Clique Convergent Surface Triangulations. Matemática Contemporânea, Vol. 25, 135–143 (2003) `http://doi.org/10.21711/231766362003/rmc2511`

[3] Larrión, F., Neumann-Lara, V. and Pizaña, M.A.: Graph relations, clique divergence and surface triangulations. J. Graph Theory, 51: 110-122 (2006) `https://doi.org/10.1002/jgt.20126`

[4] Larrión, F., Neumann-Lara, V., Pizaña, M.A.: Whitney triangulations, local girth and iterated clique graphs. Discrete Mathematics 258(1), 123–135 (2002) `https://doi.org/10.1016/S0012-365X(02)00266-2`

[5] Limbach, A.M., Winter, M.: Characterising clique convergence for locally cyclic graphs of minimum degree $\delta \geq 6$. Discrete Mathematics 347(11), 114144 (2024) `https://doi.org/10.1016/j.disc.2024.114144`

[6] Limbach, A. M., Winter, M.: When do graph covers preserve the clique dynamics of infinite graphs? arXiv preprint (2025) `https://doi.org/10.48550/arXiv.2503.22893`

Problem 8. *Large chromatic number implies long colorful path (suggested by David Mikšaník)*

Source: Proposed by Raphael Steiner at Eurocomb'25.

Motivation. Since in any proper vertex coloring of G, vertices in each clique receive distinct colors, a large clique number $\omega(G)$ implies a large chromatic number $\chi(G)$. It is well-known, however, that the reverse is not true in general—there exist triangle-free graphs (i.e., graphs with $\omega(G) = 2$) with arbitrarily large chromatic number (see e.g. [1] for a recent survey on this and related topics). Thus, the size of the largest clique is not always the main reason for a graph's large chromatic number, but perhaps other "large structures" are inherently present in such graphs. For example, Erdős and Hajnal asked: does a graph with large chromatic number always contain a Hamiltonian (induced) subgraph with large chromatic number [2]? Here, we consider a simplified version of this problem.

Question: Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph G of chromatic number at least $f(k)$ contains a path P that induces a subgraph of chromatic number at least k?

Raphael Steiner at Eurocomb'25 explicitly mentioned the case $k = 3$ (stated contrapositively).

Question: Does there exists $c \in \mathbb{N}$ such that the following holds? Every graph G such that every path induces a 3-colorable subgraph has $\chi(G) \leq c$.

Remark: Proving that $\chi(G) \leq O(\log |G|)$ should be easy. Any asymptotic improvement would be new and interesting.

References:

[1] A. Scott: *Graphs of large chromatic number*. Proceedings of the International Congress of Mathematicians, Vol. 6, pp. 4660—4681, (2022)

[2] P. Erdős: *Some recent problems and results in graph theory*. Discrete Mathematics, Vol. 164 (1–3), pp. 81–85 (1997)

Problem 9. *Signed Network Coordination Games (suggested by Lluís Sabater)*

Source: Proposed by Vanelli et al. (2025).

Definitions.
• *A* signed network *is a graph $G = (V, E)$ where each edge has a weight $w_{uv} \in \mathbb{R}$ (can be negative). Each vertex $i \in V$ is a player who chooses an* action $x_i \in \{-1, +1\}$.
• *The* utility *of $i \in V$ is given by a function that depends on the weights and actions. Typical examples:*

- *pure anti-coordination* (*all weights* $= -1$)*:*

$$u_i(x_i) = |j \in N(i) : x_j \neq x_i|$$

- *signed payoff* (*binary actions* $x_i \in \{\pm 1\}$)*:*

$$u_i(x_i) = \sum_{j \in N(i)} w_{ij} \cdot x_i \cdot x_j$$

• *An strategy X is a* pure Nash equilibrium *(or* stable coloring*) if no vertex can change its action and strictly increase its utility.*

Example:

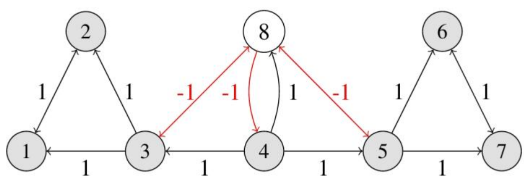


Figure 3: Example from Vanelli et al. paper

Question: Identify graphs that guarantee the existence of a pure Nash equilibrium in signed coordination games.

Related results:
• Vanelli et al. (2025) give sufficient conditions for existence: if there is a cohesive subset or a structurally balanced region that can be "frozen", and the remainder admits an equilibrium under those boundary conditions.

References:

Vanelli, Arditti, Como, Fagnani, "On Signed Network Coordination Games", arXiv preprint (May 2025).

Jeremy Kun, Brian Powers, Lev Reyzin, "Anti-Coordination Games and Stable Graph Colorings", SAGT 2013.

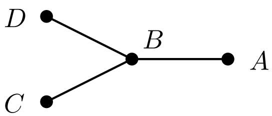
Problem 10. *Reachable Pareto-efficient allocation on a tree (suggested by Lluís Sabater)*

Source: Model introduced[1] by Gourvès, Lesca & Wilczynski (IJCAI 2017).

Definitions.

- **Model.** *Tree* $T = (V, E)$*. Each agent* $v \in V$ *initially holds object* h_v *and has a strict preference* $\succ_v$ *over all objects.*
- **Allowed move.** *A* pairwise swap *on edge* $\{u, w\}$ *is allowed only if both agents strictly prefer the object they would receive.*
- **Reachable / Pareto-efficient.** *Allocation* π *is* reachable *if obtained by a finite sequence of allowed swaps from the initial allocation.* π *is Pareto-efficient* (PE) *if no allocation (global) Pareto-dominates it.*

Example:



Agent	**Object**	**Preference**
A	a	$d \succ c \succ b \succ a$
B	b	$a \succ d \succ c \succ b$
C	c	$b \succ a \succ d \succ c$
D	d	$c \succ b \succ a \succ d$

Figure 4: Social graph and Initial Setting.

Example solution: sequence of swaps $(B, C), (B, D), (B, A)$.

Question: Given T, initial allocation and preferences. Either produce a reachable Pareto-efficient allocation or report that none exists (TREE-REACHABLE-PE). Is this decidable in *polynomial time on trees*?

[1]The tree case is explicitly listed as open in Plaxton et al., 2021; MFCS 2021.

Related results:

• For general graphs, the reachable/PE problems are NP-hard.

• Paths and stars accept polynomial algorithms.

• Object variant (moving people instead of objects) has similar results.

References:

Gourvès, Lesca & Wilczynski, "Object Allocation via Swaps along a Social Network", IJCAI 2017.

Plaxton, Li & Sinha, "Object Allocation Over a Network of Objects", AAMAS 2021.

Problem 11. *Matching–Pseudoforest Planar Decomposition Problem (suggested by Vibha Sahlot)*

Question: Does every finite planar graph G with girth $g(G) \geq 7$ admit an edge-2-colouring $E(G) = M \cup P$ such that

- M is a matching (every vertex incident with at most one edge of M), and
- P is a pseudoforest (every connected component of P contains at most one cycle)?

Related results:
• Already proven true for girth greater than seven and girth six is impossible.

Problem 12. *The Forest–Diameter-3 Decomposition Conjecture (suggested by Vibha Sahlot)*

Question: Does every finite planar graph G with girth $g(G) \geq 5$ admit an edge-2-colouring $E(G) = F \cup D$ such that

- F is a forest, and
- P has every connected component of diameter at most 3?

Related results:
• True for diameter 4.

Problem 13. *Hunters and rabbit (suggested by Sasha Sami)*

Source: Taken verbatim from `https://a3nm.net/work/research/questions/#HuntersRabbit`. But is originally credited to `https://cstheory.stackexchange.com/q/30592`.

Definitions.
• *The hunters and rabbit problem is a pursuit-evasion game played on an undirected graph. A strategy for k hunters is a sequence $s_1, s_2, \cdots, s_n$ of subsets of k vertices. A strategy is winning if, for every walk $v_1, v_2, \cdots, v_n$ in the graph (not necessarily simple), there is i such that $v_i \in s_i$. Intuitively, the evader is walking on the graph (it has to move at each turn by following exactly one edge) and must avoid the k vertices that the pursuers examine at each time step.*
• *The evasion number of a graph G is the smallest k for which there is a winning strategy with k hunters.*

Question: Is it the case that, for any constant k , we can recognize the graphs of the evasion number k in polynomial time? It is, in fact, not even known whether the problem is in NP.

Related results:
• Paper [1] characterizes the graphs with evasion number $k = 1$ (they are the unions of lobster graphs, and can be recognized in linear time), and determines explicit strategies for them. In this paper, the pursuer is called a prince and the evader is called a princess. See also this Reddit discussion [2], where the pursuer is a vampire hunter and the evader is a vampire.
• Paper [3] (2025), which shows that it is NP-hard to determine the evasion number of a graph given as input. However, the paper does not determine the complexity of identifying graphs of evasion number k for a fixed k.
• More papers:

  – How to Hunt an Invisible Rabbit on a Graph [4].

- Catching a Mouse on a Tree [5], where the pursuers are called cats and the evader mouse.
- Hunting Rabbits on the Hypercube [6], which follows the hunter/rabbit terminology.
- Recontamination Helps a Lot to Hunt a Rabbit [7].

References:

[1] `https://arxiv.org/abs/1204.5490`

[2] `https://www.reddit.com/r/mathriddles/comments/30tmoh/vampire_hunter/`

[3] `https://arxiv.org/abs/2502.15982`

[4] `https://arxiv.org/abs/1502.05614`

[5] `https://arxiv.org/abs/1502.06591`

[6] `https://arxiv.org/abs/1701.08726`

[7] `https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.MFCS.2023.42`

Problem 14. *Are the induced chordal graphs edge-Ramsey? (suggested by Maximilian Strohmeier)*

Source: Proposed by Nešetřil in 2025.

Let K be any category. (Imagine a class of finite graphs with some notion of homomorphisms.)

Definitions (Ramsey degree)**.**
• *K is* Ramsey *(or has the* Ramsey property*) if*

$$\left.\begin{array}{l} \forall A, B \in K, r \in \omega : \\ \exists C \in K : \\ \forall \chi \in r^{K(A,C)} : \\ \left.\begin{array}{l} \exists b \in K(B,C), k \in r : \\ \forall a \in K(A,B) : \\ \chi(b \circ a) = k. \end{array}\right\} \begin{array}{l} \exists b \in K(B,C) : \\ \Big| \chi\,(b \circ -)\,[K(A,B)] \Big| = 1 \end{array} \end{array}\right\} C \to (B)^A_{r,1}$$

• *$A \in K$ has* small Ramsey degree *d if d is minimal such that*

$$\forall B \in K, r \in \omega : \exists C \in K : C \to (B)^A_{r,d}.$$

Definitions (Induced chordal graphs)**.**
• *For a class of graphs K, the category of* induced K *has as morphisms the inclusions of subgraphs in K induced by a vertex set.*
• *A* chord *of a cycle in a graph is a edge in the graph between vertices which are not adjacent in the cycle.*
• *A graph is* chordal *(sometimes called* triangulated*) if every cycle of size ≥ 4 has a chord. So a graph is chordal if it has no induced C_n for $n \geq 4$.*

Question: What is the small Ramsey degree of the edge in the class of induced linearly-ordered chordal graphs?

Related results:

• (The chordal graphs are vertex-Ramsey; [1, Prop. 4.13]) The vertex has small Ramsey degree 1 in the class of induced chordal graphs.

• (Classification of chordal graphs; Folklore / [2, Thm. 5.3.17]) A finite graph is chordal if and only if there exists a linear ordering of its vertices such that, for every vertex, its neighborhood restricted to smaller vertices induces a complete graph.

General facts from structural Ramsey theory:

• (Partite construction) Let L be a language of finite structures (without any other restrictions or axioms). Then the class of linearly ordered L-structures has the Ramsey property.

• (Ramsey degree arithmetic) The Ramsey degree of an object is a multiple of its number of automorphisms.

• (Ramsey's Theorem) $\forall a, b, r \in \omega : \exists c \in \omega : c \to (b)^a_r$ in $(\omega, <)$. Equivalently we can consider order-preserving embeddings between linearly-ordered finite sets.

• (Generalized Hales-Jewett) For every finite set S, $\forall a, b, r \in \omega : \exists c \in \omega : c \to (b)^a_r$ in $\mathrm{HJ}(S)$.

References:

[1] Vince Guingona, Felix Nusbaum, Zain Padamsee, Miriam Parnes, Christian Pippin and Ava Zinman (2025), Indivisibility for Classes of Graphs, `https://arxiv.org/abs/2312.01466`

[2] Douglas B. West (2000) Introduction to Graph Theory, 2nd ed., Prentice Hall

Serving as reference aside from that is almost any paper in structural Ramsey theory. A recent survey is:

[3] Jan Hubička and Matěj Konečný (2025), Twenty years of Nešetřil's classification programme of Ramsey classes, doi: `10.48550/arXiv.2501.17293`

Problem 15. *Flow-related conjecture (suggested by Robert Šámal)*

Source: DeVos (with Ghanbari, Šámal).

A bisection of a graph is a decomposition of the vertex set in two sets of the same size. We will call subsets of one of these sets *monochromatic.*

Conjecture: Every 4-edge-connected 5-regular graph G has a bisection so that every monochromatic component has at most one cycle.

• For integer $k \geq 3$, a *nowhere-zero k-flow* (k-nzf) in a digraph is a mapping $\varphi : E(G) \to \mathbb{Z}$ that is a *flow* – so that at every vertex, flow in equals flow out, formally, for every v

$$\sum_{u \in N^+(v)} \varphi(vu) = \sum_{u \in N^-(v)} \varphi(uv).$$

Moreover, for every edge $e \in E(G)$ we have $1 \leq |\varphi(e)| \leq k - 1$.
• For a real $r \geq 2$, a *circular nowhere-zero r-flow* (r-cnzf for short) in a digraph is an assignment $f : E \to [-(r-1), -1] \cup [1, r-1]$ such that f is a flow in G.
• A *k-weak bisection* of a cubic graph G is a partition of the vertex-set of G into two parts V_1 and V_2 of equal size, such that each connected component of the subgraph of G induced by V_i $(i = 1, 2)$ is a tree of at most $k - 2$ vertices.
• A bisection (V_1, V_2) of a graph G is an *r-strong bisection* if for every set of vertices $X \subseteq V$ the number of edges leaving X is $\geq \frac{r}{r-2} ||V_1 \cap X| - |V_2 \cap X||$
• Note that an r-strong bisection is also a $\lfloor r \rfloor$-weak bisection.
• [EMT, J] Let $r \geq 3$ be a real number. A cubic graph $G = (V, E)$ admits an circular r-nowhere-zero flow if and only if there exists a r-strong bisection.
• So if a cubic bridgeless graph G has a 5-nowhere-zero flow (as Tutte conjectures is true for all such graphs), it also has a 5-strong bisection, and thus also a 5-weak partition.

• [EMT] prove this conclusion directly.
• The above mentioned conjecture is related in similar way to Tutte's 3-flow conjecture (every 4-edge-connected graph G has a 3-nzf).

References:

[EMT] Louis Esperet, Giuseppe Mazzuoccolo, Michael Tarsi: Flows and bisections in cubic graphs, Journal of Graph Theory 86(2) (2017), 149-158 `https://arxiv.org/abs/1504.03500`

[J] F. Jaeger: Balanced valuations and flows in multigraphs, Proceedings of the American mathematical society Volume 55, Number 1, February 1976 `https://www.ams.org/journals/proc/1976-055-01/S0002-9939-1976-0427156-5/S0002-9939-1976-0427156-5.pdf`

Problem 16. *Slice-wise complexity of scramble number (suggested by Josse van Dobben de Bruyn)*

Source: Posed by Echavarria et. al. [2, Question 3.6]; see also [3, Question 5.5], [4, Question 6.4].

Throughout this problem, we assume that all graphs are finite, undirected, simple, and connected, with n vertices and m edges.

We say that a vertex set $E \subseteq V(G)$ is *connected* if the induced subgraph $G[E]$ is connected. We follow the common convention that the empty set is not connected.

Recall that a *cut* (T, T^c) in G is a partition of $V(G)$ into two disjoint non-empty vertex sets $T, T^c \subseteq V(G)$. The *size* of a cut (T, T^c), denoted $|E(T, T^c)|$, is the number of edges between T and T^c.

Definitions ([1])**.**

• *A* scramble *in a graph G is a non-empty collection $\mathcal{S} = \{E_1, \ldots, E_s\}$ of connected vertex sets $E_1, \ldots, E_s \subseteq V(G)$. The sets $E_1, \ldots, E_s$ are called the* eggs *of the scramble.*[2]

• *Let $\mathcal{S} = \{E_1, \ldots, E_s\}$ be a scramble. A* hitting set *for $\mathcal{S}$ is a vertex set $C \subseteq V(G)$ that contains at least one vertex from each egg E_i of $\mathcal{S}$. The* hitting set number $h(\mathcal{S})$ *of $\mathcal{S}$ is the minimum size of a hitting set for $\mathcal{S}$.*

• *Let $\mathcal{S} = \{E_1, \ldots, E_s\}$ be a scramble. An* egg cut *for $\mathcal{S}$ is a cut (T, T^c) such that there exists eggs $E_i \subseteq T$ and $E_j \subseteq T^c$. The* egg cut number $e(\mathcal{S})$ *of $\mathcal{S}$ is the minimum size of an egg cut (∞ if no egg cut exists).*

• *The* order *of a scramble is defined as $\|\mathcal{S}\| := \min(h(\mathcal{S}), e(\mathcal{S}))$. Note that $\|\mathcal{S}\| \leq h(\mathcal{S}) \leq n$.*

• *The* scramble number $\mathrm{sn}(G)$ *of a graph G is the maximum order of a scramble in G.*

[2]The terminology, introduced in [1], is a bad pun on the notion of *brambles* and *bags*.

Examples:

• For every (connected) graph G, the set $\mathcal{S} = \{V(G)\}$ is a scramble with $h(\mathcal{S}) = 1$ and $e(\mathcal{S}) = \infty$. Thus, $\mathrm{sn}(G) \geq 1$ for all G.

• For every graph G, the set $\mathcal{E}(G) = \{\{u, v\} \mid uv \in E(G)\}$ is a scramble in G, called the *edge scramble* of G [2, §3]. The hitting sets for this scramble are precisely the vertex covers of G, so we have $h(\mathcal{E}(G)) = n - \alpha(G)$. The egg cut number of $\mathcal{E}(G)$ is the minimum size of a cut (T, T^c) such that both T and T^c contain at least one edge, which is typically smaller than $n-\alpha(G)$. If $\delta(G) \geq \lfloor \frac{n}{2} \rfloor + 1$, then this scramble has order $n - \alpha(G)$ and is optimal [2, Corollary 3.2].

• If G is a tree, then we claim that every scramble has order 1. If $h(\mathcal{S}) = 1$, then we are done, so assume $h(\mathcal{S}) > 1$. Let D be the directed graph obtained from G by replacing every edge $uv \in E(G)$ by two arcs $(u, v), (v, u) \in A(D)$. Consider the directed subgraph $D' \subseteq D$ consisting of all arcs (u, v) such that the connected component of $G - uv$ containing v contains an egg E_i. Since $h(\mathcal{S}) > 1$, every vertex $u \in V(G)$ misses at least one egg, so the outdegree of every vertex of D' is at least 1. Therefore, D' has at least n arcs. Since G has $n - 1$ edges, it follows that there is an edge $uv \in E(G)$ such that both (u, v) and (v, u) belong to D'. Now the edge uv forms an egg cut of size 1, so we have $e(\mathcal{S}) = 1$. Hence, $\|\mathcal{S}\| = 1$.

• If G contains a cycle C, then $\mathrm{sn}(G) \geq 2$. Indeed, choose three distinct vertices u, v, w on the cycle C, and edge-partition the cycle as $C = P_{[u,v]} \cup P_{[v,w]} \cup P_{[w,u]}$, where $P_{[x,y]}$ denotes the path along the cycle from x to y. Then the scramble

$$\mathcal{S} = \{V(P_{[u,v]}), V(P_{[v,w]}), V(P_{[w,u]})\}$$

satisfies $h(\mathcal{S}) = 2$ and $e(\mathcal{S}) = \infty$, so $\|\mathcal{S}\| = 2$. This is illustrated in Figure 5 below.

• Combining the preceding two examples, we see that $\mathrm{sn}(G) = 1$ if and only if G is a tree. Using more advanced properties of the scramble number, a one-line proof of this fact can be given; see [1, Corollary 4.2].

• For a more advanced example, let $G_{m,n}$ denote the $m \times n$ grid graph with rows $R_1, \ldots, R_m \subseteq V(G_{m,n})$ and columns $C_1, \ldots, C_n \subseteq$

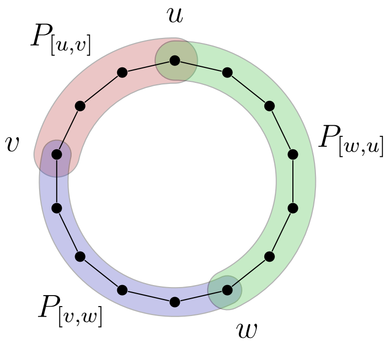


Figure 5: Every cycle gives rise to a scramble of order 2.

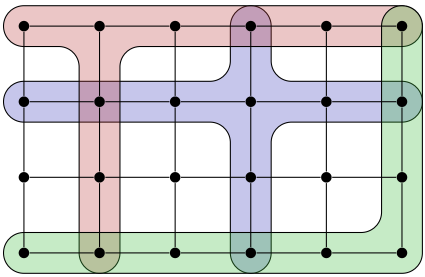


Figure 6: Three eggs of the scramble $\mathcal{S}$ in the grid graph $G_{4,6}$. (The scramble contains all possible unions of a row and a column.)

$V(G_{m,n})$. Consider the scramble $\mathcal{S} = \{R_i \cup C_j \mid (i,j) \in [m] \times [n]\}$, illustrated in Figure 6 below. Then $h(\mathcal{S}) = \min(m,n)$ and $e(\mathcal{S}) = \infty$, so $\|\mathcal{S}\| = \min(m,n)$. This can be shown to be optimal [1, Proposition 5.2].

• A *bramble* is a scramble $\mathcal{B}$ with the additional property that $E_i \cup E_j$ is connected for all $E_i, E_j \in \mathcal{B}$. The (*bramble*) *order* of a bramble $\mathcal{B}$ is defined to be the hitting set number $h(\mathcal{B})$. If $\mathcal{B}$ is a bramble of order k, then the scramble order of $\mathcal{B}$ is either $k-1$ or k [1, Lemma 3.6]. If $\mathcal{B}$ is a bramble such that $E_i \cap E_j \neq \varnothing$ for all $E_i, E_j \in \mathcal{B}$, then the bramble order of $\mathcal{B}$ equals the scramble order of $\mathcal{B}$ [1, Lemma 3.5]. (Brambles play an important role in the theory of treewidth.)

For this problem, we focus on the (slice-wise) computational complexity of the scramble number.

Definitions.
• *The decision problem* Scramble Number *takes as input a graph G and an integer k and asks whether or not G admits a scramble of order at least k; that is, whether or not* $\mathrm{sn}(G) \geq k$.
• *For a fixed integer $k \geq 1$, the decision problem* k-Scramble *takes as input a graph G and asks whether or not G admits a scramble of order at least k; that is, whether or not* $\mathrm{sn}(G) \geq k$.

Note the subtle but important difference between Scramble Number and k-Scramble: the first considers k to be part of the input, whereas the second considers k to be a constant. By comparison, the problems Vertex Cover, Independent Set, Dominating Set, and Chromatic Number are all NP-hard when k is part of the input, but the first three become polynomial time solvable when k is fixed, whereas k-Colouring is still NP-hard when $k \geq 3$ is fixed.

In [2], the authors prove that Scramble Number is coNP-hard, using a polynomial-time reduction from Independent Set. The more interesting problem of determining the complexity of k-Scramble is left open.

Main Question ([2, Question 3.6]; see also [3, Question 5.5], [4, Question 6.4])**:** For fixed k, can the problem k-Scramble be solved in polynomial time, or is it still NP-hard (or in between)?

The problems 1-Scramble and 2-Scramble are trivially in P, since we have $\mathrm{sn}(G) \geq 1$ for all graphs G, and $\mathrm{sn}(G) \geq 2$ if and only if G has a cycle. It was proved in [3] that a graph G satisfies $\mathrm{sn}(G) \geq 3$ if and only if G contains one (or several) of the four graphs depicted in Figure 7 as a topological minor. Therefore, 3-SCRAMBLE is also in P. For $k > 3$, it was proved that no such finite list of forbidden topological minors exists [3, Theorem 1.2]. The Main Question is currently still open for all $k > 3$.

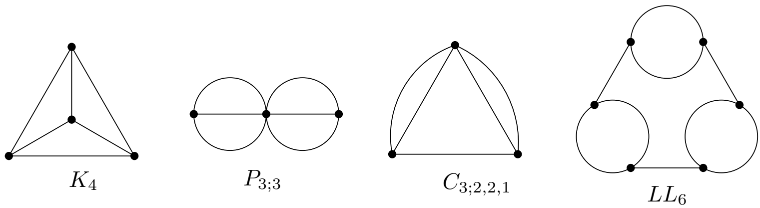


Figure 7: The four forbidden topological minors for $\mathrm{sn}(G) < 3$.

If k-SCRAMBLE is in P for all k, then a natural follow-up question would be whether or not SCRAMBLE NUMBER is fixed parameter tractable when parameterized by k [4, Question 6.4].

Related results:

- If H is a subgraph of G, then $\mathrm{sn}(H) \leq \mathrm{sn}(G)$ [1, Proposition 4.5].
- If G' is a subdivision of G, then $\mathrm{sn}(G') = \mathrm{sn}(G)$ [1, Proposition 4.6].
- The scramble number is not minor-monotone [1, Example 4.4].
- If $\mathrm{sn}(G) \leq 3$, then an optimal scramble exists whose eggs are pairwise disjoint [3, Proposition 5.2], but this is not true for larger scramble number.
- The scramble number is lower bounded by the *treewidth* and upper bounded by the *divisorial gonality* of the graph: $\mathrm{tw}(G) \leq \mathrm{sn}(G) \leq \mathrm{dgon}(G)$ [1]. This is the main motivation for studying the scramble number: it is currently the best known lower bound on the divisorial gonality, a graph parameter that has its origins in algebraic geometry [5,6] and has connections with structural graph theory [7] and parameterized complexity [8].

• For all $k \geq 2$, there exist graphs G_k such that $\mathrm{tw}(G_k) = 2$ and $\mathrm{sn}(G_k) = k$ [1, Example 4.8], as well as graphs H_k such that $\mathrm{sn}(H_k) = 2$ and $\mathrm{dgon}(H_k) = k$ [1, Example 4.9]. In other words, treewidth, scramble number, and divisorial gonality can be arbitrarily far apart.
• For fixed k, it can be decided in linear time whether or not $\mathrm{tw}(G) \leq k$ [9], and in polynomial time whether or not $\mathrm{dgon}(G) \leq k$ [10]. But scramble number seems to be somehow harder.
• I have a proof that k-Scramble is in NP for all k, using a polynomial time reduction to an instance of SAT (unpublished). Note: I am reducing to SAT, not from SAT, so this does not prove hardness.
• Treewidth is both the largest order of a bramble (minus one) and the smallest width of a tree decomposition. A similar width parameter has been defined for the scramble number, called *screewidth* [11]. However, one only has $\mathrm{sn}(G) \leq \mathrm{sw}(G)$; equality does not always hold. It is an open problem whether or not $\mathrm{sw}(G) \leq \mathrm{dgon}(G)$ for all graphs G [11, Question 5.1].

References:

[1] Michael Harp, Elijah Jackson, David Jensen, and Noah Speeter. "A new lower bound on graph gonality". *Discrete Appl. Math.* **309**:172–179, 2022. doi: 10.1016/j.dam.2021.11.003.

[2] Marino Echavarria, Max Everett, Robin Huang, Liza Jacoby, Ralph Morrison, and Ben Weber. "On the scramble number of graphs". *Discrete Appl. Math.* **310**:43–59, 2022. doi: 10.1016/j.dam.2021.12.009.

[3] Robin Eagleton and Ralph Morrison. "Graphs of scramble number two". *Discrete Math.* **346**(10):#113539, 2023. doi: 10.1016/j.disc.2023.113539.

[4] Seamus Connor, Steven DiSilvio, Sasha Kononova, Ralph Morrison, and Krish Singal. "On the size and complexity of scrambles". Preprint, 2023. arXiv: 2311.17342

[5] Matthew Baker. “Specialization of linear systems from curves to graphs”. *Algebra Number Theory* **2**(6):613–653, 2008. doi: 10.2140/ant.2008.2.613

[6] Matthew Baker and Serguei Norine. “Riemann-Roch and Abel–Jacobi theory on a finite graph”. *Adv. Math.* **215**(2):766–788, 2007. doi: 10.1016/j.aim.2007.04.012.

[7] Josse van Dobben de Bruyn and Dion Gijswijt. “Treewidth is a lower bound on graph gonality”. *Algebr. Comb.* **3**(4):941–953, 2020. doi: 10.5802/alco.124.

[8] Hans L. Bodlaender, Gunther Cornelissen, and Marieke van der Wegen. “Problems hard for treewidth but easy for stable gonality”. *Graph-Theoretic Concepts in Computer Science (WG 2022)*, Springer Lecture Notes in Computer Science 13453, pp. 84-97. Springer, Cham, 2022. doi: 10.1007/978-3-031-15914-5_7.

[9] Hans L. Bodlaender. “A linear-time algorithm for finding tree-decompositions of small treewidth”. *SIAM J. Comput.* **25**(6):1305–1317, 1996. doi: 10.1137/S0097539793251219.

[10] Josse van Dobben de Bruyn. *Reduced divisors and gonality in finite graphs.* Bachelor thesis, Leiden University, 2012. HDL: 1887/3596668.

[11] Lisa Cenek, Lizzie Ferguson, Eyobel Gebre, Cassandra Marcussen, Jason Meintjes, Ralph Morrison, Liz Ostermeyer, Shefali Ramakrishna, and Ben Weber. “Scramble number and tree-cut decompositions”. Preprint, 2022. arXiv: 2209.01459.