KAMAK 2021

Penzion Jeloma, Jetřichovice

September 19 – 24

Charles University, Prague

Organizers:

Michal Opler

Pavel Dvořák

Robert Šámal

Brochure of open problems,
Prague, 2021.

Contents

Program

8:00 breakfast

9:00 morning session I

10:30 break

11:00 morning session II

12:30 lunch

15:00 afternoon session I

16:30 break

17:00 afternoon session II

18:30 progress reports

19:00 dinner

Wednesday is planned to be free to make an excursion in the neighborhood with everyone who would like to come.

OPEN PROBLEMS

Problem 1. *Eternal Domination on super-graphs of cacti (suggested by Václav Blažej)*

Source: Proposed by B., Křišťan, Valla 2021.

Definitions.
• *Graph is a* cacti *when every edge lies in at most one cycle.*
• *Eternal domination game is a 2-player game with one attacker and one defender. First, defender places k guards on the graph. Next, in each turn attacker chooses a vertex and defender may move each guard by at most one edge so that the chosen vertex is occupied by a guard. We play eternally, if at some point defender fails to defend a vertex, he looses, otherwise he wins.*
• *Our goal is for a given graph G find the minimum number of guards k so that defender wins the eternal domination game. (Ideally, find the strategy.)*

Example:
Observe that if we played exactly one turn, then the minimum number of necessary guards for the defender to win the game is the domination number.

Question: How hard is it to decide eternal domination on graph classes bigger than the class of cacti graphs (e.g. series-parallel).

Related results:
• There is an easy way to deduce an optimal defending strategy for trees.
• There is a linear time algorithm to get an optimal defending strategy for cacti graphs.

References:

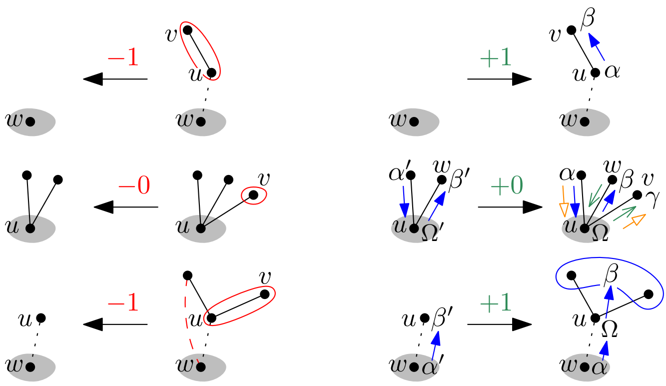


Figure 1: Sketch of reductions which lead to complete solution of the eternal domination on trees.

Andrei Braga, Cid C. de Souza, and Orlando Lee. The eternal dominating set problem for proper interval graphs. Information Processing Letters, 115(6):582–587, 2015.

Stephen Finbow, Margaret-Ellen Messinger, and Martin F. van Bommel. Eternal domination on 3 × n grid graphs. Australasian Journal of Combinatorics, 61:156–174, 2015.

Wayne Goddard, Sandra M. Hedetniemi, and Stephen T. Hedetniemi. Eternal security in graphs. Journal of Combinatorial Mathematics and Combinatorial Computing, 52:169–180, 2005.

Michael A. Henning, William F. Klostermeyer, and Gary MacGillivray. Bounds for the m-eternal domination number of a graph. Contributions to Discrete Mathematics, 12(2), 2017.

William F. Klostermeyer and Gary MacGillivray. Eternal dominating sets in graphs. Journal of Combinatorial Mathematics and Combinatorial Computing, 68:97–111, February 2009.

Christopher M. van Bommel and Martin F. van Bommel. Eternal domination numbers of 5 × n grid graphs. Journal of Combinatorial Mathematics and Combinatorial Computing, 97:83–102, 2016.

Problem 2. *Offline caching with page sizes 1 and 2 (suggested by Lukáš Folwarczný)*

Source: A problem left open in the paper by Folwarczný and Sgall [FS17]

Definitions.

• *We have a parameter C, called the cache size.*

• *We are given a set of pages $\mathcal{P}$. Each page $p \in \mathcal{P}$ has* $\mathrm{SIZE}(p) \in \{1, 2\}$.

• *We are also given a sequence of page requests $r_1, \ldots, r_m \in \mathcal{P}$.*

• *A* service *is a sequence $\emptyset = S_0, S_1, \ldots, S_m$ satisfying for every $i = 1, \ldots, m$: $r_i \in S_i$ and $\sum_{p \in S_i} \mathrm{SIZE}(p) \leq C$.*

• *The* cost *of the service is $\sum_{i=1}^{m} |S_i \setminus S_{i-1}|$.*

• *Explanation: We maintain a set of pages of total size at most C in the cache. We pay one for loading a page into the cache and we can erase pages for free. It is necessary to have a page in the cache when it is requested.*

• *The problem* 12Caching *is a decision problem. You are given the pages $\mathcal{P}$, request sequence $r_1, \ldots, r_m$, cache size C and cost limit L. It is necessary to decide whether a service of cost at most L exists.*

Example:

Consider the following instance: $C = 4$, $L = 3$, $\mathcal{P} = \{p_1, p_2, p_3\}$, $\mathrm{SIZE}(p_1) = 1$, $\mathrm{SIZE}(p_2) = \mathrm{SIZE}(p_3) = 2$. The request sequence is $p_1, p_2, p_3, p_1, p_2, p_3$. It is a no instance. The total size of all the pages is 5 which is more than the cache size. There is therefore at least one page which must be loaded twice and the total cost is at least 4. There is a service of cost 4, for example this one: $\emptyset$, $\{p_1\}$, $\{p_1, p_2\}$, $\{p_1, p_3\}$, $\{p_1, p_3\}$, $\{p_2, p_3\}$, $\{p_2, p_3\}$.

Question: Is the problem 12Caching NP-complete?

Related results:

• When all pages have size one, the problem is easily in P.

• When the page sizes are in the range $\{1, 2, 3\}$, the problem is NP-complete. [FS17] The proof is relatively involved and it seems to me that it would be better to start looking at the problem from scratch rather than spending time on this proof.

• If the cost of loading a page into the cache is not one for all pages, but it is the same as the page size, the problem is also open. And I believe it is of the same value.

• If the page sizes are in the set $\{1, 2, 4\}$, the problem is also open.

• The paper [FS17] contains all references to previous and related results I am aware of.

References: [FS17] Lukáš Folwarczný, Jiří Sgall: General Caching Is Hard: Even with Small Pages. Algorithmica 79(2): 319-339 (2017)

Problem 3. *Depth-three circuits for inner product (suggested by Lukáš Folwarczný)*

Source: A problem left open in the paper by Golovnev, Kulikov and Williams [GKW21]

Definitions.
• *An* OR $\circ$ AND $\circ$ OR *circuit is a depth-3 circuit; the single gate in the top layer is* OR*, all the gates in the second layer are* ANDs*, all the gates in the third layer are* ORs*. These gates in the third layer have as inputs literals (variables or their negations).*
• *An* OR $\circ$ AND $\circ$ OR$_k$ *circuit has the additional condition that the fan-in of the gates in the third layer is bounded by k.*
• *The size of a circuit is the total number of wires (edges in its graph).*
• $s_3^k(f)$ *denotes the minimum size of an* OR $\circ$ AND $\circ$ OR$_k$ *circuit computing the function f.*
• IP $=$ IP$_n$ *(for n even) is the function of inner product of n variables mod 2, that is* $\mathrm{IP}(x_1, \ldots, x_n) = x_1x_2 \oplus x_3x_4 \oplus \cdots x_{n-1}x_n$.

Example: $s_3^2(\mathrm{IP}) \leq 2^{n/2-o(n)}$. We follow the proof from [GKW21]: Observe that $\mathrm{IP}(x_1, \ldots, x_n) = 1$ iff there is an odd number of ones among $p_1 = x_1x_2, p_2 = x_3x_4, \ldots, p_{n/2} = x_{n-1}x_n$. We can then express

$$\mathrm{IP}(x_1, \ldots, x_n) \equiv \bigvee_{S \subseteq [\frac{n}{2}] : |S| \bmod 2 = 1} \left(\bigwedge_{i \in S} [p_i = 1] \wedge \bigwedge_{i \notin S} [p_i = 0] \right).$$

Conditions $[p_i = 1]$ and $[p_i = 0]$ depend only on two variables and can therefore be expressed as 2-CNFs.

Question: Determine $s_3^3(\mathrm{IP})$.

Related results:
• Lower bounds for the parity function imply $s_3^3(\mathrm{IP}) \geq 2^{n/6}$.
• A more involved modification of the example gives $s_3^3(\mathrm{IP}) \leq 3^{n/4}$.

• If the actual value of $s_3^3(\mathrm{IP})$ is close to the above upper bound, an interesting conjecture of [GKW21] would turn out to be true.
• On the other hand, inventing an interesting upper bound would, as far as I can tell, not have as strong consequences, but it would definitely be very interesting and may be not that difficult ...

References: [GKW21] Alexander Golovnev, Alexander S. Kulikov, R. Ryan Williams: Circuit Depth Reductions. ITCS 2021: 24:1-24:20. Full version arXiv:1811.04828

Problem 4. *Powers of Paths/Cycles in Randomly Perturbed Graphs (suggested by Eng Keat Hng)*

Definitions.

• *The* kth power *of a graph H, denoted by H^k, is the graph obtained from H by joining every pair of vertices at distance at most k in H.*

Question: For what values of α and p can we guarantee that for any graph G on n vertices with minimum degree αn, the graph $G \cup G(n,p)$ a.a.s. contains the kth power of a path/cycle on ℓ vertices?

Related results:

• Komlós, Sárközy and Szemerédi: $p = 0$, $\ell = n$.

• Allen, Böttcher and Hladký: $p = 0$, $k = 2$, ℓ linear in n.

• Antoniuk, Dudek, Reiher, Ruciński and Schacht: $\alpha > \frac{1}{2}$, $\ell = n$

References:

J. Komlós, G. N. Sárközy and E. Szemerédi. Proof of the Seymour conjecture for large graphs. Ann. Comb., 2(1):43–60, 1998.

P. Allen, J. Böttcher and J. Hladký. Filling the gap between Turán's theorem and Pósa's conjecture. J. Lond. Math. Soc. (2), 84(2):269–302, 2011.

S. Antoniuk, A. Dudek, C. Reiher, A. Ruciński and M. Schacht. High powers of Hamiltonian cycles in randomly augmented graphs. J. Graph Theory, 98(2):255–284, 2021.

Problem 5. *Clique chromatic number vs. clique-width (suggested by Lars Jaffke)*

Definitions.
• *A* clique coloring *of a graph is a vertex-coloring (not necessarily proper) without monochromatic maximal cliques.*
• *The* clique chromatic number *of a graph is the minimum number of colors in any of its clique colorings.*

Question: Is the clique chromatic number of every graph bounded by a function of its clique-width?

Related results:
• In the CLIQUE COLORING problem, we are given a graph G and an integer k and the question is if G has a clique coloring with k colors. This problem is expressible in MSO_1, and therefore FPT parameterized by clique-width plus number of colors by the meta-theorem of Courcelle, Makowsky, and Rotics. In case of a positive answer to our questions, this gives an FPT-algorithm parameterized by clique-width alone. (An explicit XP-algorithm, $k^{2^{2^{\mathcal{O}(w)}}} n^{\mathcal{O}(1)}$ time where w is the clique-width, was also given recently.)
• Related question: Can 2-CLIQUE COLORING, the restriction of CLIQUE COLORING to $k = 2$, be solved in $2^{2^{2^{o(w)}}} n^{\mathcal{O}(1)}$ time where w is the clique-width, or would that refute the Exponential Time Hypothesis?
• Graphs of unbounded clique chromatic number are not so well understood. Since for triangle-free graphs, the clique chromatic number is equal to the chromatic number, constructions of triangle-free graphs of unbounded chromatic number seem like a good starting point to look for counterexamples to the conjecture. However, they cannot have bounded clique-width: Bonamy and Pilipczuk showed that graphs of bounded clique-width are χ-bounded, meaning their chromatic number is bounded by a function of size of the the maximum clique.

References:

- Bruno Courcelle, Johan A. Makowsky, and Udi Rotics. Linear Time Solvable Optimization Problems on Graphs of Bounded Clique-Width. *Theory of Computing Systems* 33, pages 125–150, 2000.
- Lars Jaffke, Paloma T. Lima, and Geevarghese Philip. Structural parameterizations of clique coloring. In: MFCS 2020, pages 49:1–49:15.
- Marthe Bonamy and Michał Pilipczuk. Graphs of bounded cliquewidth are polynomially χ-bounded. *Advances in Combinatorics* 8, 2020.

Problem 6. *Complexity of deciding whether given a set of vertices in a graph is eternally dominating (suggested by Matyáš Křišťan)*

Source: Proposed by William F. Klostermeyer in 2015.

Definitions.
• *m-Eternal domination is a game of two players, an attacker and a defender, played on a graph. The rules are as follows.*

- *Given a graph, the defender places guards on some vertices.*
- *In each turn, the attacker chooses one vertex to attack.*
- *In response, the defender can move each of the guards into their respective neighborhoods. After the movements, the attacked vertex must be occupied. Otherwise, the defender loses.*
- *Each vertex can be occupied by at most one guard.*

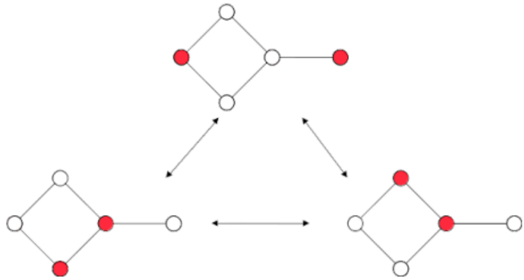
• *If a defender can use a set of vertices S as a starting configuration and defend indefinitely against any sequence of attacks, we say that S is an* m-eternal dominating set*.*

Example: For example, we may use the following three configurations to defend the given graph indefinitely using 2 guards. From each configuration, we can transition to any other.

Question: Consider the following problem: Given a graph G and a set of vertices S, can we eternally defend G with S as the starting configuration of guards?

Can this problem be solved in PSPACE? Is it EXPTIME-hard?

Related results:
• The problem is known to be NP-hard and can be solved in EXPTIME.
• It can be solved in linear time on trees and on proper interval graphs.



• Let $\gamma_m^\infty(G)$ be the minimum size of an m-eternal dominating set on G and γ be the minimum size of a dominating set on G. Then it holds $\gamma(G) \leq \gamma_m^\infty(G) \leq 2\gamma(G)$.

References:

- Protecting a Graph with Mobile Guards (`https://arxiv.org/abs/1407.5228`)
- Eternal Domination Numbers of $5 \times n$ Grid Graphs (`https://people.stfx.ca/mvanbomm/publicat/eternal5xn.pdf`)
- Bounds for the m-Eternal Domination Number of a Graph (`https://cdm.ucalgary.ca/article/view/62550`)
- The eternal dominating set problem for interval graphs (`https://arxiv.org/abs/1808.09591`)
- On the m-eternal Domination Number of Cactus Graphs (`https://arxiv.org/abs/1907.07910`)

Problem 7. *Surface Connectivity Variation on Albertson & Berman Conjecture (suggested by Tomáš Masařík)*

Source: Proposed in our paper [4] in 2021.

Famous Albertson & Berman conejcture [1] states that; if G is a planar graph on n vertices, then G contains an induced forest of size at least $n/2$. In [4], we provided a connection of related concepts of robust connectivity and surface connectivity.

Definitions.
- *τ_G is a set of all spanning trees of G and $\Lambda(T)$ is a set of all leafs of some tree T.*
- Robust connectivity*:* $\kappa_\rho(G) := \min_{\substack{R \subseteq V(G) \\ R \neq \emptyset}} \max_{T \in \tau_G} \frac{|R \cap \Lambda(T)|}{|R|}$.
- *$\tilde{G}$ is some embedding of G, where $\mathcal{G}_S$ is a set of all graph embeddings embedable on a fixed surface S.*
- *$m(\tilde{G})$ is maximum induced embedded subgraph $\tilde{G'} \subseteq \tilde{G}$ for which $S \text{ ✂ } \tilde{G'}$ is a connected surface, where $S \text{ ✂ } \tilde{H}$ is defined as cutting along the edges of $\tilde{H}$.*
- Surface connectivity*:* $\kappa_s(S) = \inf\left\{ \frac{m(\tilde{G})}{|\tilde{G}|} : \tilde{G} \in \mathcal{G}_S \right\}$.

Theorem 5 in [4] proved for a fixed surface S the following equivalence: Every graph G with an edge-maximal embedding on S satisfies $\kappa_\rho(G) \geq k$ if and only if $\kappa_s(S) \geq k$.
Question: If S is torus how large is $\kappa_s(S)$? What about projective plane or klein bottle.

Related results: For upper bound we know that $\kappa_s(S) \leq \frac{1}{2}$ for any S because of K_4 and $\kappa_s(S) \leq \frac{3}{7}$ when S is torus as demonstrated by K_7, see Figure 1. I am not aware of any better resutls than straightforward lower bounds given by *acyclic colouring* (Any two color classes does not induce any cycle): In [3] it was shown that for S projective plane $\kappa_s(S) \geq \frac{2}{7}$. For S torus $\kappa_s(S) \geq \frac{2}{8}$ [2].

[1] Michael O. Albertson and David M. Berman. A conjecture on planar graphs. *Graph theory and related topics* 357:1, 1979.

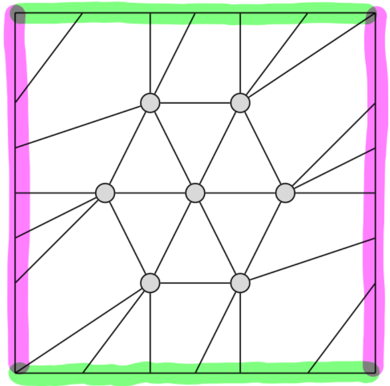


Figure 2: Embedding of K_7 on torus.

[2] Michael O. Albertson and David M. Berman. The acyclic chromatic number. *Congressus Numerantium* 17, 51–69, 1976.

[3] Noga Alon, Bojan Mohar, Daniel P. Sanders. On acyclic colorings of graphs on surfaces. *Israel Journal of Mathematics* 94:1, 273–283 1996.

[4] Peter Bradshaw, Tomáš Masařík, Jana Novotná, and Ladislav Stacho. Robust Connectivity of Graphs on Surfaces. *Trends in Mathematics*, 848–854, 2021. `arXiv:2104.12030`.

Problem 8. *Filling the gaps: Complexity of (4-)coloring in classes defined by small forbidden induced subgraphs (suggested by Jana Novotná)*

Source: Question 2 proposed by Chinh T. Hoàng at Dagstuhl Seminar 19271 —Graph Colouring: from Structure to Algorithms in 2019.

Definitions.
• *Let H be a graph, G is H-free if it does not contain a copy of H as an* induced *subgraph. Let $\mathcal{H} = \{H_1, \ldots, H_s\}$ be a set of graphs, G is $\mathcal{H}$-free (or alternatively $(H_1, \ldots, H_s)$-free) if it is H_1-, H_2-, $\ldots$, and H_s-free.*
• *Let P_t be the path on t-vertices, C_s be the cycle on s vertices, and* claw *denote the graph $K_{1,3}$. Let $H + G$ denote the disjoint union of graphs H and G.*

Question:

Filling the gaps:

1 Is 4-coloring of $(P_7, C_7)-$, $(P_8, C_7)-$, or (P_t, C_3)-free graphs, for $7 \leq t \leq 21$, NP-hard or polynomial?

2 Is coloring of $(claw, 4P_1)-$, $(claw, 4P_1, P_2+2P_1)-$, or $(C_4, 4P_1)$-free graphs polynomial or NP-hard?

Related results:
• The problem of 4-coloring is polynomial in P_6-free graphs and NP-complete in P_7-free graphs. It stays NP-complete even when some (P_t, C_s)-free graphs are considered when $t \geq 7$. Hell and Huang [1] and Huang et al. [2] settled many NP-complete cases of this type. These results, in combination with the polynomiality of P_6-free case, leave open only the following cases: $(P_7, C_7)-$, $(P_8, C_7)-$, and (P_t, C_3)-free graphs, for $7 \leq t \leq 21$.
• Let F_4 denote a set of graphs with at most four vertices. The complexity of coloring is determined for F_4-free graphs, up to three

cases: $F_4 = \{claw, 4P_1\}$, $F_4 = \{claw, 4P_1, P_2 + 2P_1\}$ (coloring of this class is polynomialy equivalent to $F_4 = \{claw, P_2 + 2P_1\}$), and $F_4 = \{C_4, 4P_1\}$. Lozin and Malyshev conjectured in [4] the polynomiality for the last case. All these classes have unbounded clique-width. Moreover, atoms (graphs without clique-cutsets) of these classes have unbounded clique-width. There is a polynomial result for subclass of $(claw, 4P_1)$-free graphs, namely $(claw, 4P_1, hole - twin)$-free graphs [5] where hole-twin are holes with twin of one vertex.

References:

1 Pavol Hell and Shenwei Huang. Complexity of coloring graphs without paths and cycles. Discrete Applied Mathematics, 216:211–232, January 2017

2 Shenwei Huang, Matthew Johnson, and Daniël Paulusma. Narrowing the complexity gap for colouring (Cs, Pt)-free graphs. The Computer Journal, 58(11):3074–3088, June 2015

3 Dallas J.Fraser, Angèle M. Hamel, Chính T. Hoàng, Frédéric Maffray. A coloring algorithm for $4K1$-free line graphs. Discrete Applied Mathematics, 234:76-85, January 2018.

4 V. V. Lozin, D. S. Malyshev. Vertex coloring of graphs with few obstructions. Discrete Applied Mathematics, 216:273-280, January 2017.

5 Yingjun Dai Angele, M. Foley Chính, T. Hoàng. On Coloring a Class of Claw-free Graphs. Electronic Notes in Theoretical Computer Science, 346:369-377, August 2019.

Problem 9. *Vertex deletion problem for poset-related graph classes (suggested by Karolina Okrasa)*

Source: Proposed by Bożyk et al. in 2020.

Definitions.
• comparability graphs*: graphs whose edges correspond to the pairs of vertices comparable in some fixed partial order $<$ on the vertex set (such an order is called a* transitive orientation *of the graph),*
• co-comparability graphs*: the complements of comparability graphs.*
• permutation graphs*: intersection graphs of segments whose endpoints lie on two parallel lines ℓ_1 and ℓ_2, one on each.*
• Vertex deletion to the class $\mathcal{G}$ of graphs*:*

Input:	*A graph G on n vertices and a number k*
Question:	*Can G be transformed into a graph of the class $\mathcal{G}$ by deleting at most k vertices?*

Question: Is the vertex deletion problem to the class of comparability/co-comparability/permutation graphs fixed-parameter tractable (FPT)? In other words, does there exist an algorithm solving the problem in time $f(k) \cdot n^{\mathcal{O}(1)}$?

Related results:
• The three mentioned graph classes are *hereditary*, i.e., they admit a characterization by forbidden induced subgraphs. In all three cases the family of forbidden induced subgraphs is known.
• Vertex deletion to bipartite permutation graphs (a subclass of all three mentioned graph classes) is FPT.
• Vertex deletion to perfect graphs (a superclass of all three mentioned graph classes) is W[1]-hard.

References:
• Bożyk, Ł., Derbisz, J., Krawczyk, T., Novotná, J., Okrasa, K. (2020, December). Vertex Deletion into Bipartite Permutation Graphs. In *15th International Symposium on Parameterized and Exact Computation IPEC 2020, December 14-18, 2020, Hong Kong, China (Virtual Conference).*

• Heggernes, P., Van't Hof, P., Jansen, B. M., Kratsch, S., Villanger, Y. (2013). *Parameterized complexity of vertex deletion into perfect graph classes. Theoretical Computer Science, 511*, 172-180.

Problem 10. *Narrow Dots & Boxes (suggested by Michal Opler)*

Source: Proposed by Buchin et al. [1].

Definitions.
• *Dots & Boxes is a game played by two players on an $m \times n$ grid of dots. The players take turns connecting two adjacent dots. If a player completes the fourth side of a unit box, the player is awarded a point and an additional turn. When no more moves can be made, the player with the highest score wins the game*
• *The problem* DOTS & BOXES *is to decide which player wins given an initial game state.*

Question:
• Does restricting DOTS & BOXES to a $k \times n$ grid for a small k make the game easier? In particular, the question is not understood even for $1 \times n$ grids.

Related results:
• Recently, Buchin et al. [1] resolved a long-standing open question by showing that DOTS & BOXES is PSPACE-complete.
• Collette et al. [2] analyzed the behaviour of the so-called Misère Dots & Boxes where the player with fewer points wins on $1 \times n$ and $2 \times n$ grids.

References:
[1] Buchin, K., Hagedoorn, M., Kostitsyna, I., and van Mulken, M. Dots & Boxes Is PSPACE-Complete *MFCS 2021*, Vol. 202, 25:1–25:18 (2021).
[2] Collette, S., Demaine, E. D., Demaine, M. L., and Langerman, S. Narrow Misère Dots-and-Boxes *Games of No Chance 4*, 57–64 (2015).

Problem 11. *Superpatterns of permutation classes (suggested by Michal Opler)*

Source: Proposed by Engen and Vatter [4], Bannister et al. [3].

Definitions.
• *A permutation π is a sequence $\pi = \pi_1, \pi_2, \ldots, \pi_n$ in which each number from the set $[n] = \{1, 2, \ldots, n\}$ appears exactly once. A permutation diagram of π is the point set $\{(i, \pi) \mid i \in [n]\}$.*
• *A permutation π contains a permutation σ, if π has a subsequence of length k whose elements have the same relative order as the elements of σ, otherwise we say that π avoids σ, or π is σ-avoiding.*

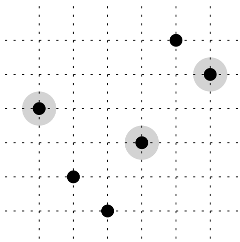


Figure 3: Pattern 213 contained in permutation 421365.

• *$Av(\sigma)$ denotes the class of all σ-avoiding permutations.*
• *A permutation π is said to be n-universal (or a superpattern) for a class $\mathcal{C}$ if π contains all permutations of length n from $\mathcal{C}$. A permutation is said to be simply n-universal if it contains all permutations of length n. An n-universal permutation π for a class $\mathcal{C}$ is called proper if additionally $\pi \in \mathcal{C}$.*
• *The skew-merged permutations are all permutations that can be obtained as a union of one increasing and one decreasing sequence.*

Question:
• What is the size of the smallest (proper) n-universal permutations for the skew-merged permutations?

• Is there σ such that any n-universal permutation for $\mathrm{Av}(\sigma)$ has to be of size $\Omega(n^2)$? Optimistic aim is the class $\mathrm{Av}(4321)$.

Related results:
• The best bounds on the size of the smallest n-universal permutation are that it lies between n^2/e^2 (a consequence of Stirling's Formula) and $\lceil \frac{n^2+1}{2} \rceil$ [5].
• Banninster et al. [3] constructed n-universal permutation for $\mathrm{Av}(213)$ of size $n^2/4 + \Theta(n)$ and n-universal permutation of size $O(n \log^{O(1)} n)$ for any proper subclass of $\mathrm{Av}(213)$.
• There is a proper n-universal permutation for $\mathrm{Av}(321)$ while Banninster et al. [4] constructed a non-proper n-universal permutation for $\mathrm{Av}(321)$ of size $O(n^{3/2})$.
• On the other hand, any proper n-universal permutation for $\mathrm{Av}(321)$ must be of size $\Omega(n^\alpha)$ for any $\alpha < 2$ [2].

References:
[1] Albert, M. H., Engen, M. T., Pantone, J. T., and Vatter, V. Universal layered permutations. *Electron. J. Combin.*, Vol. 25, 3 (2018).
[2] Alecu, B., Lozin, V., and Malyshev, D. Critical properties of bipartite permutation graphs. arXiv:2010.14467 [math.CO]..
[3] Bannister, M. J., Cheng, Z., Devanny, W. E., and Eppstein, D. A. Superpatterns and universal point sets. *J. Graph Algorithms Appl.*, Vol. 18, 2 (2014).
[4] Bannister, M., Devanny, W., and Eppstein, D. Small superpatterns for dominance drawing. In *ANALCO14 — Meeting on Analytic Algorithmics and Combinatorics*, (2014).
[5] Engen, M., and Vatter, V. Containing all permutations. *The American Mathematical Monthly*, Vol. 128, 4 – 24 (2021)

Problem 12. *Implicit graph conjecture (suggested by Robert Šámal)*

Source: Proposed by Kannan, Naor, and Rudich (1988)

An *adjacency labelling scheme* for a graph class $\mathcal{F}$ is a function $A : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ such that for any graph $G \in \mathcal{F}$ there is a mapping $\ell : V(G) \to \{0,1\}^*$ such that $A(\ell(u), \ell(v)) = 1$ iff $uv \in E(G)$.

We say that the scheme A has labels of size k if for any $G \in \mathcal{F}$ and any $v \in V(G)$ we have $|\ell(v)| \leq k$.

A class $\mathcal{F}$ is *hereditary*, if for any $G \in \mathcal{F}$ any induced subgraph of G is also in $\mathcal{F}$.

For a graph class $\mathcal{F}$ we let $\mathcal{F}_n$ be the set of n-vertex graphs in $\mathcal{F}$.

Conjecture: Let $\mathcal{F}$ be a hereditary class such that $|\mathcal{F}_n| \leq 2^{O(n \log n)}$. Then $\mathcal{F}_n$ has an adjacency labelling scheme with labels of size $O(\log n)$.

Related results:

• [KNR] Easily, n-vertex trees have an adjacency labelling scheme of size $2\lceil \log n \rceil$.

• [KNR] Also easy: if $\mathcal{F}$ has an adjacency labelling scheme of size k then it has a induced-subgraph universal graph with 2^k vertices.

• [BEGC] Let $\mathcal{F}$ be a hereditary class. Then $\mathcal{F}_n$ has an adjacency labelling scheme with labels of size $\frac{1}{n} \log |\mathcal{F}_n| + o(n)$.

• [DEGJMM] Planar graphs of size n have an adjacency labelling scheme with labels of size $\log n + o(\log n)$ bits per vertex.

References:

• V. Dujmović, L. Esperet, G. Joret, C. Gavoille, P. Micek, P. Morin: Adjacency Labelling for Planar Graphs (and Beyond), arXiv:2003.04280, FOCS 2020

• S. Kannan, M. Naor, S. Rudich: Implicit representation of graphs. SIAM J. Discrete Math., 5(4):596–603, 1992. doi:10.1137/0405049.

• M. Bonamy, L. Esperet, C. Groenland, A. Scott: Optimal labelling schemes for adjacency, comparability, and reachability, arXiv:2012.01764,

STOC 2021

• L. Esperet: Universal graphs and applications (talk at Eurocomb 2021)

Problem 13. *Perfect Code in c-closed graphs (suggested by Roohani Sharma)*

Source: Proposed by Koana, Komusiewicz, Sommer at ESA 2020.

Definitions.

The Perfect Code *problem is defined as follows.*

Perfect Code **Input:** *An undirected graph G and a positive integer k* **Question:** *Does there exists a set of at most k vertices of G, say S, such that for each $v \in V(G)$, $\|N[v] \cap S\|=1$?*

In other words, the solution of the Perfect Code *problem is an independent set and each vertex that is not in the solution is dominated exactly once by the solution. The problem is NP-hard, and in fact, it is W[1]-complete [1]. This means the problem cannot exhibit an algorithm running in $f(k) \cdot n^{\mathcal{O}(1)}$ time, where $f(\cdot)$ is a function that depends only on k, and n is the number of vertices in the input graph.*

For any positive integer c, a graph G is called a c-closed graph if any pair of vertices in G, with at least c common neighbours, are adjacent. That is, for any $u, v \in V(G)$, $u \neq v$, if $|N(u) \cap N(v)| \geq c$, then $(u, v) \in E(G)$.

Question: Is Perfect Code FPT parameterized by $k + c$ on c-closed graphs? That is, does there exists an algorithm that solves Perfect Code in c-closed graphs in $f(k, c) \cdot n^{\mathcal{O}(1)}$ time, where $f(\cdot)$ is a function that depends only on k and c, and n is the number of vertices in the input graph?

Also does Perfect Code admit a polynomial kernel parameterized by k, for every fixed c? That is, does there exists a polynomial time algorithm that takes as input an instance of Perfect Code on c-closed graphs, and outputs an equivalent instance of Perfect Code on c-closed graphs, such that the number of vertices in the output instance is $\mathcal{O}(k^{f(c)})$ for some function $f(\cdot)$ that depends only on c.

Related results:
• The notion of c-closed graphs was introduced by Fox et al. in [2].
• Koana et al. in [3] gave kernelization results for various hard problems, including the Dominating Set problem, on c-closed graphs.

References:
[1] Perfect Code is W[1]-complete: Marco Cesati: IPL 2002.
[2] Finding Cliques in Social Networks: A New Distribution-Free Model: Jacob Fox, Tim Roughgarden, C. Seshadhri, Fan Wei, Nicole Wein: SICOMP 2020.
[3] Exploiting c-Closure in Kernelization Algorithms for Graph Problems: Tomohiro Koana, Christian Komusiewicz, Frank Sommer: ESA 2020.

Problem 14. *Long cycles in the random graphs (suggested by Matas Šileikis)*

Source: Proposed by P. Condon, A. Espuny Díaz, A. Girão, D. Kühn and D. Osthus.

Definitions.
• *For* $n = 1, 2, \ldots,$ *let* Q^n *be the* discrete hypercube of dimension n*, that is, a graph with vertex set* $\{0,1\}^n$ *and two vertices adjacent if and only if the corresponding vectors differ in exactly one entry.*
• *Let* Q^n_p *denote the random subgraph of* Q^n *obtained by independently removing each edge with probability* $1-p$*.*

Conjecture Suppose that $p = p(n)$ satisfies $pn \to \infty$, as $n \to \infty$. Then for some sequence $\delta_n = o(1)$,

$$\mathbb{P}\left(Q^n_p \text{ contains a cycle of length at least } (1-\delta_n)2^n\right) \to 1, \quad n \to \infty.$$

Related results:
• In 2020, Condon, Espuny Díaz, Girão, Kühn and Osthus showed that

- for every fixed $\delta, p \in (0,1]$ the graph Q^n_p contains a cycle of length at least $(1-\delta)2^n$ with probability tending to 1, as $n \to \infty$.
- for every fixed $\varepsilon > 0$, as $n \to \infty$,

$$\mathbb{P}\left(Q^n_p \text{ contains a Hamilton cycle}\right) \to \begin{cases} 0, & p \le 1/2 - \varepsilon \\ 1, & p \ge 1/2 + \varepsilon \end{cases}$$

References:
Condon, Padraig, Alberto Espuny Díaz, António Girão, Daniela Kühn, and Deryk Osthus. "Hamiltonicity of random subgraphs of the hypercube." In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 889-898. Society for Industrial and Applied Mathematics, 2021.

Problem 15. *Uniqueness of House Assignment Mechanism (suggested by David Sychrovský)*

Source: Proposed by Abdulkadiroglu and Sönmez in 1998.

Definitions.
• *House assignment mechanism is a (stochastic) assignment of houses to agents based on their ordered preferences.*
• *It needs to satisfy*

- *Equal treatment of equals: if two agents have same preferences, then they must have the same odds of obtaining a given house.*
- *Ex-post efficiency: agents cannot exchange assigned houses in a way that would make everyone happier.*
- *Strategy proofness: no agent may gain by lying about his preferences.*

• *Random serial dictatorship is a mechanism which uniformly samples the ordering of agents and then assigns each their to house among those which are left.*

Example:
If agents have different first preferences, the assignment is deterministic

Question: Is Random serial dictatorship the only mechanism which satisfies all axioms of the house assignment problem?

References:

[1] Abdulkadiroglu, A. and Sonmez, T., *Random serial dictatorship and the core from random endowments in house allocation problems*, Econometrica 66 (1998) 689-701.

[2] Bogomolnaia, A. and Moulin, H., *A new solution to the random assignment problem*, Journal of Economic Theory 100 (2001) 295-328.

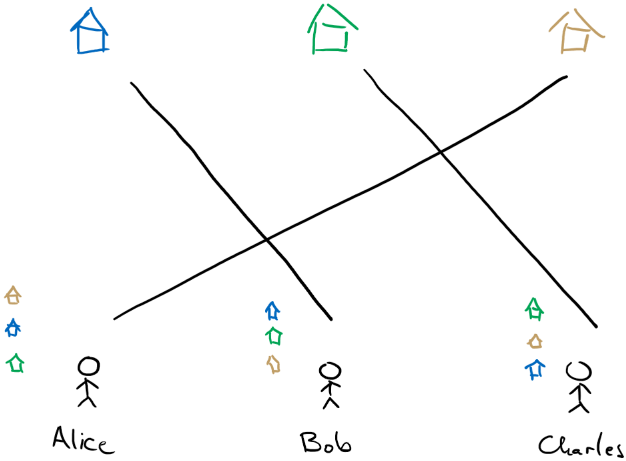


Figure 4: Example of all agents having different first preferences.

[3] Nesterov, A., *Fairness and efficiency in strategy-proof object allocation mechanisms*, Journal of Economic Theory 170 (2017) 145-168.

Problem 16. *b*-Coloring *Parameterized by Treewidth (suggested by Paloma Thome de Lima)*

Definitions.
• *A* b-coloring *of a graph G is a proper vertex coloring $(V_1, \ldots, V_k)$ such that for every color class $i \in \{1, \ldots, k\}$, there is a vertex $v_i \in V_i$ that has a neighbor in all other color classes; i.e. for all $j \neq i$, $N(v_i) \cap V_j \neq \emptyset$.*
• *The* b-Coloring *problem asks, given a graph G and an integer k, if G has a b-coloring with k colors.*

Example:

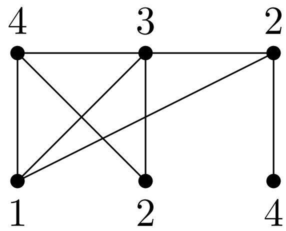


Figure 5: A graph with a b-coloring with four colors.

Question: Is b-Coloring parameterized by the treewidth of the input graph W[1]-hard?

Related results:
• b-Coloring parameterized by clique-width is in XP and W[1]-hard.
• b-Coloring parameterized by vertex cover is in FPT.

References:

- Lars Jaffke, Paloma T. Lima, and Daniel Lokshtanov. *b*-Coloring parameterized by clique-width. In: STACS 2021, pages 43:1–43:15.

Problem 17. *Colouring edges between grid points. (suggested by Misha Tyomkyn)*

Let G be the complete graph on the vertex set $[n] \times [n]$. Define the colouring c on $E(G)$ as follows. For an edge $e = v_1v_2$, where $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ to define $c(e)$ we first determine the *type* of e. Different types receive different colours.

- e is of *type* 1 if $(a_1 - a_2)(b_1 - b_2) > 0$,
- e is of *type* 2 if $(a_1 - a_2)(b_1 - b_2) < 0$,
- e is of *type* 3 if $a_1 = a_2$, and
- e is of *type* 4 if $b_1 = b_2$.

Now, for edges of type $t \in \{1, 2\}$ put

$$c(e) := (t, \min\{a_1, a_2\}, \min\{b_1, b_2\}).$$

For edges of type 3 and 4 put

$$c(e) := (3, \min\{b_1, b_2\})$$

and

$$c(e) := (4, \min\{a_1, a_2\}),$$

respectively.
Question: Prove that for any vertex set $U \subseteq V(G)$ with $|U| \geq 2$ there exist $u_1, u_2 \in U$ such that $c(u_1u_2)$ does not appear on any other edge of the induced subgraph $G[U]$.

Question: Solve the analogous problem in $[n]^d$ for $d \geq 3$.

The problem has interesting consequences in Ramsey theory.

Problem 18. *Two-Class (r, k)-Coloring (suggested by Tung Anh Vu)*

Definitions.
• *A* k-coloring *c of a graph G is a mapping $c : V(G) \to \{1, \ldots, k\}$.*
• *The assignment of the same color to a pair of adjacent vertices is called a* conflict*, and the edge connecting these two vertices is a* conflict edge*. Edges that are not conflicting are* covered edges*.*
• *A color of a coloring is* proper *if the vertices mapped to this color are required to form an independent set. Remaining colors are called* relaxed *colors.*
• *In the* Two-Class (r, k)-Coloring *problem (with $r \leq k$), a feasible solution is a k-coloring of the input graph such that we can divide the k colors into a group of r relaxed colors and a group of $k - r$ proper colors. The cost of such a coloring is the amount of conflicts, and the goal is to minimize the number of conflicts. Equivalently, the cost of a coloring is the number of covered edges, and the goal is to maximize the number of covered edges.*

Known facts:
• $(0, k)$-Coloring is classical k-Coloring.
• The family of (r, k)-Coloring problems is NP-complete except for the special case when $(r, k) = (0, 2)$.
• $(1, k)$-Coloring cannot be approximated to any constant factor within polynomial time for $k \geq 2$.
• (r, k)-Coloring for $2 \leq r \leq k$ is APX-complete, i.e., it can be approximated to some constant in polynomial time but not an arbitrary constant.

Question: Let us explore this problem from the perspective of parameterized complexity. Additionally, since polynomial approximation schemes are not possible, can we develop a parameterized approximation scheme? At least for some special graph classes?

References:

• Papp, P., Schmid, R., Stoppiello, V., & Wattenhofer, R. (2021). Two-Class (r, k)-Coloring: Coloring with Service Guarantees. ArXiv, abs/2108.03882.

Problem 19. *Generalized Conflict Coloring (suggested by Tung Anh Vu)*

Source: Proposed by Papp et al. in 2021.

Definitions.
• *A* k-coloring c *of a graph* G *is a mapping* $c : V(G) \to \{1, \dots, k\}$.
• *The assignment of the same color to a pair of adjacent vertices is called a* conflict, *and the edge connecting these two vertices is a* conflict edge.
• *Given a graph* $G = (V, E)$ *and a* k-coloring *of* G, *let us denote the number of conflict edges adjacent to a vertex* $u \in V$ *by* $\kappa(v)$.
• *Given a real parameter* $p > 0$, *we define the* Generalized Conflict Coloring *problem, where the cost of a coloring* c *is defined as*

$$\mathrm{cost}(c) = \sum_{u \in V} \kappa(u)^p.$$

The goal of the problem is to find a k*-coloring that minimizes the cost.*

Examples:
• For $p = 1$, the cost is twice the number of conflict edges. Thus the problem is equivalent to Two-Class (k, k)-Coloring or Max-k-Cut.
• As $p \to \infty$, the cost is dominated by the highest value $\kappa(u)$. This is known as the Defective Coloring problem, where the goal is to find a coloring which minimizes the maximum number of conflicts adjacent to a vertex.
• As $p \to 0$, the difference between the distinct $\kappa(u)$ values diminishes for all $\kappa(u) > 0$, that is

$$\lim_{p \to 0} \mathrm{cost}(c) = \sum_{\substack{u \in V \\ \kappa(u) > 0}} 1.$$

.

Thus the goal is to minimize the number of conflict vertices.

Question: What is the precise complexity of this problem? If it is NP-hard, can we develop (parameterized) approximation algorithms?

Related results:
• Defective Coloring is NP-hard.
• Two-Class (r, k)-Coloring is NP-hard except when $(r, k) = (0, 2)$.

References:
• Papp, P., Schmid, R., Stoppiello, V., & Wattenhofer, R. (2021). Two-Class (r, k)-Coloring: Coloring with Service Guarantees. ArXiv, `abs/2108.03882`.